Chapter 16

Logarithms

Contents

118. Getting Straight Into It!

Without any introduction, discussion, or fuss, can you figure out what is being asked of you in this "worksheet"? Are you able to complete it?

Give this serious consideration before reading on.

$\text{power}_5\,(25) = \underline{\ 2\ }$	$\text{power}_{10}(1000) = \underline{\ 3\ }$
$\text{power}_3\,(27) = \underline{\ 3\ }$	$\text{power}_4\,(4) = \underline{\ 1\ }$

$\text{power}_2\,(8) = \underline{\qquad}$	$\text{power}_2\,(\sqrt{2}) = \underline{\qquad}$	$\text{power}_1\,(5) = \underline{\qquad}$
$\text{power}_{10}(100) = \underline{\qquad}$	$\text{power}_{10}(\text{million}) = \underline{\qquad}$	$\text{power}_2\,(0) = \underline{\qquad}$
$\text{power}_4\,(16) = \underline{\qquad}$	$\text{power}_{73}(1) = \underline{\qquad}$	$\text{power}_{-2}(-8) = \underline{\qquad}$
$\text{power}_4\,(64) = \underline{\qquad}$	$\text{power}_{100}(0.1) = \underline{\qquad}$	$\text{power}_{-2}(8) = \underline{\qquad}$
$\text{power}_7\,(\frac{1}{7}) = \underline{\qquad}$	$\text{power}_{\sqrt{6}}(\frac{1}{36}) = \underline{\qquad}$	$\text{power}_0\,(0) = \underline{\qquad}$

The first example

$$\text{power}_5\,(25) = \underline{\;2\;}$$

is stating that

the power of 5 *that gives the answer* 25 *is* 2

Or you might prefer to phrases this backwards as

2 *is the power of* 5 *that gives* 25

Did you come to this understanding?

Filling out the table is a task of figuring out the correct power of a give base number to give a certain answer for fifteen examples.

I could do this for all but four of them.

$\text{power}_5\,(25) = \underline{\;2\;}$	$\text{power}_{10}(1000) = \underline{\;3\;}$
$\text{power}_3\,(27) = \underline{\;3\;}$	$\text{power}_4\,(4) = \underline{\;1\;}$

$\text{power}_2\,(8) = \underline{\;3\;}$	$\text{power}_2\,(\sqrt{2}) = \underline{\;\frac{1}{2}\;}$	$\text{power}_1\,(5) = \underline{\;?\;}$
$\text{power}_{10}(100) = \underline{\;2\;}$	$\text{power}_{10}(\text{million}) = \underline{\;6\;}$	$\text{power}_2\,(0) = \underline{\;?\;}$
$\text{power}_4\,(16) = \underline{\;2\;}$	$\text{power}_{73}(1) = \underline{\;0\;}$	$\text{power}_{-2}(-8) = \underline{\;3\;}$
$\text{power}_4\,(64) = \underline{\;3\;}$	$\text{power}_{100}(0.1) = \underline{\;\frac{-1}{2}\;}$	$\text{power}_{-2}(8) = \underline{\;?\;}$
$\text{power}_7\,(\frac{1}{7}) = \underline{\;-1\;}$	$\text{power}_{\sqrt{6}}(\frac{1}{36}) = \underline{\;-4\;}$	$\text{power}_0\,(0) = \underline{\;?\;}$

Just as a side note:

I computed the power of 100 that gives the answer 0.1 by writing $100^a = \frac{1}{10}$ and seeing that this is the same as $10^{2a} = 10^{-1}$.

I computed the power of $\sqrt{6}$ that gives $\frac{1}{36}$ by looking at $\left(6^{\frac{1}{2}}\right)^b = 6^{-2}$.

Here are the four troublesome examples.

$$\text{power}_1(5) = \underline{\ ?\ }$$

$$\text{power}_2(0) = \underline{\ ?\ }$$

$$\text{power}_{-2}(8) = \underline{\ ?\ }$$

$$\text{power}_0(0) = \underline{\ ?\ }$$

Every power of 1 is 1. (See Musing 117.8.)

$$\mathbf{1^x = 1 \text{ for all numbers } x}$$

There is no power of 1 that gives the answer 5.

Every power of 2 is a positive number. In fact we saw in the last chapter that

$$\boldsymbol{a^x} \textbf{ is a positive number for all numbers } \boldsymbol{x}\textbf{, for } \boldsymbol{a} \textbf{ a positive base number.}$$

There is no power of 2 that gives the answer 0.

Again, as we saw in the last chapter, powers of negative numbers are dangerous! At the very least, I can't think of a power of -2 that gives the answer 8.

There are many powers of 0 that give the answer 0. For example, $0^2 = 0 \times 0 = 0$ and $0^5 = 0 \times 0 \times 0 \times 0 \times 0 = 0$. There are too many possible answers to the final example.

$$\text{power}_1(5) = \underline{\text{does not exist}}$$

$$\text{power}_2(0) = \underline{\text{does not exist}}$$

$$\text{power}_{-2}(8) = \underline{\text{does not exist}}$$

$$\text{power}_0(0) = \underline{\text{indeterminant}}$$

Every other example was fine: they each asked for a power of a positive number to give a positive number. As we saw last chapter, the mathematics of powers is for positive values is all safe and secure.

Now let's do something very strange and cross out each instance of the word "power" in our worksheet and replace it with the letters *log*, short for **logarithm**.

Of course, "logarithm" is the word invented by Napier in the 1600s as the tool that saved scientific progress. With Napier's logarithms, scholars could conduct complicated arithmetic with the simpler task of addition.

$\overset{\text{log}}{\cancel{\text{power}}}_5\,(25) = \underline{\;2\;}$ $\qquad$ $\overset{\text{log}}{\cancel{\text{power}}}_{10}\,(1000) = \underline{\;3\;}$

$\overset{\text{log}}{\cancel{\text{power}}}_3\,(27) = \underline{\;3\;}$ $\qquad$ $\overset{\text{log}}{\cancel{\text{power}}}_4\,(4) = \underline{\;1\;}$

$\overset{\text{log}}{\cancel{\text{power}}}_2\,(8) = \underline{\;3\;}$ $\qquad$ $\overset{\text{log}}{\cancel{\text{power}}}_2\,(\sqrt{2}) = \underline{\;\frac{1}{2}\;}$ $\qquad$ $\overset{\text{log}}{\cancel{\text{power}}}_1\,(5) = \underline{\text{does not exist}}$

$\overset{\text{log}}{\cancel{\text{power}}}_{10}\,(100) = \underline{\;2\;}$ $\qquad$ $\overset{\text{log}}{\cancel{\text{power}}}_{10}\,(\text{million}) = \underline{\;6\;}$ $\qquad$ $\overset{\text{log}}{\cancel{\text{power}}}_2\,(0) = \underline{\text{does not exist}}$

$\overset{\text{log}}{\cancel{\text{power}}}_4\,(16) = \underline{\;2\;}$ $\qquad$ $\overset{\text{log}}{\cancel{\text{power}}}_{73}\,(1) = \underline{\;0\;}$ $\qquad$ $\overset{\text{log}}{\cancel{\text{power}}}_{-2}\,(-8) = \underline{\;3\;}$

$\overset{\text{log}}{\cancel{\text{power}}}_4\,(64) = \underline{\;3\;}$ $\qquad$ $\overset{\text{log}}{\cancel{\text{power}}}_{100}\,(0.1) = \underline{\;\frac{-1}{2}\;}$ $\qquad$ $\overset{\text{log}}{\cancel{\text{power}}}_{-2}\,(8) = \underline{\text{does not exist}}$

$\overset{\text{log}}{\cancel{\text{power}}}_7\,(\frac{1}{7}) = \underline{\;-1\;}$ $\qquad$ $\overset{\text{log}}{\cancel{\text{power}}}_{\sqrt{6}}\,(\frac{1}{36}) = \underline{\;-4\;}$ $\qquad$ $\overset{\text{log}}{\cancel{\text{power}}}_0\,(0) = \underline{\text{indeterminant}}$

Why change the word "power" to "logarithm"? To bring us to the next mathematical breakthrough on this topic, which as this:

In the 1700s, mathematicians finally started making sense of exponents (powers) beyond just whole number exponents. They could see that, for any positive number a, it is possible start making sense of

$$a^x$$

for all real numbers x. (They developed the superscript notation for exponents at this time too.) Moreover, they saw that Napier's logarithms are really just powers in disguise! His method of converting an multiplication arithmetic problem into an addition problem as just a matter of working with powers, backwards!

The word *logarithm* is a scary word for many students, chiefly because the notion is typically presented in classrooms with no story or context. The school topic of logarithms would be far less scary if we used the word *power* for them. (After all, you managed to do the opening worksheet seeing only the work "power.")

But for the historical reasons, the name logarithm stuck and we still use it to this day—even though we can see now what they are.

Here's the formal definition of a logarithm in its full scariness—almost. (The box at the bottom of this page is less scary.)

For a number b (called the **base**) and a number N, the base b **logarithm** on N,

$$\log_b(N)$$

is the power of b that gives the value N.

For example,

$\mathbf{log_4(16)}$ is the power of 4 that gives the answer 16.

$$\log_4(16) = 2$$

$\mathbf{log_{10}(\mathit{million})}$ is the power of 10 that gives the answer one million.

$$\log_{10}(million) = 6$$

$\mathbf{log_{\sqrt{6}}(\frac{1}{36})}$ is the power of $\sqrt{6}$ that gives the answer $\frac{1}{36}$.

$$\log_{\sqrt{6}}(\frac{1}{36}) = -4$$

Whenever you see the word logarithm, I suggest crossing it out and writing the word power in its stead.

$$\boxed{\overset{\text{power}}{\cancel{\log}}_b(N) = \text{the power of } b \text{ that gives the answer } N}$$

Practice 118.1 Please complete the following worksheet.

$\log_2(8) =$ _____ $\qquad \log_{0.1}(100) =$ _____

$\log_5(125) =$ _____ $\qquad \log_{402}(1) =$ _____

$\log_{125}(5) =$ _____ $\qquad \log_1(8) =$ _____

$\log_{\frac{1}{4}}(4) =$ _____ $\qquad \log_2(-4) =$ _____

a

Again, we are seeing that a base number of $b = 1$ is problematic, and negative values are usually problematic too. To obviate such woes, people usually add some caveats to the formal definition of a logarithm. (Now this is the FULL, and scary, definition.)

For positive numbers b and N, with $b \neq 1$, the base b **logarithm** on N,

$$\log_b(N)$$

is the power of b that gives the value N.

We have

$\log_4(1024) = 5$ because $4^5 = 1024$

$\log_{\frac{1}{3}}\left(\frac{1}{9}\right) = 2$ because $\left(\frac{1}{3}\right)^2 = \frac{1}{9}$

$\log_{10}(0.001) = -3$ because $10^{-3} = \frac{1}{1000} = 0.001$

Practice 118.2 The third example shows that a logarithm of a number can give a negative value as its answer. Does this violate the caveats of the formal definition?

$$b^x = N$$

x is the power of b that gives the answer N

$$x = \log_b(N)$$

Example: Rewrite $7^3 = 343$ as a logarithmic statement.

Answer: We see that 3 is the power of 7 giving the answer 343. That is,

$$3 = \log_7(343)$$

Practice 118.3 Is the statement $16 = (0.5)^{-4}$ correct?
If so, rewrite it as a logarithmic statement.

Practice 118.4 Rewrite $\log_{0.1}(10) = -1$ as a statement about powers.

Practice 118.5 Give an example of two positive numbers b and N such that

$$\log_b(N) = -billion$$

Example: Please evaluate $\log_{\frac{1}{a^2}}\left(a^{\frac{2}{3}}\right)$.

Answer: This question is icky!

Deep breath.

We're looking for the power of $\frac{1}{a^2}$ that gives $a^{\frac{2}{3}}$.

So, we are looking for the value x so that

$$\left(\frac{1}{a^2}\right)^x = a^{\frac{2}{3}}$$

Thus, we need

$$a^{-2x} = a^{\frac{2}{3}}$$

and see that $x = -\frac{1}{3}$ does the trick.

So,

$$\log_{\frac{1}{a^2}}\left(a^{\frac{2}{3}}\right) = -\frac{1}{3}$$

Practice 118.6: Please evaluate $\log_{(b^{0.5})}\left(\frac{1}{\sqrt{b}}\right)$.

Practice 118.7: Kindly fill in the two blanks.

$\log_b(b) =$ ______ $\qquad \log_b(1) =$ ______

Here's a tautological example.

Practice 118.8 a) What is the value of $\log_{11}(11^{4505})$?

b) In general, what can you say about the value of

$$\log_b(b^x)$$

for a positive number b and real number x?

Here's another tautological mind twister!

Practice 118.9 a) What is the value of $9^{\log_9(777)}$?

b) In general, what can you say about the value of

$$b^{\log_b(N)}$$

for positive numbers b and N?

Playing with the definition of $\log_b(N)$ for positive numbers b and N is all well and good, but two deep questions about them are hanging over our heads.

1. Who cares about logarithms today?

2. Even if I cared (because I like to imagine I am in the 1700s), what has our definition of logarithms got to do with Napier's work from last chapter? How do logarithms (as defined here) convert multiplication problems into addition problems?

We shall attend to both questions.

MUSINGS

Musing 118.10
a) Use computer software to sketch the graph of $y = 1^x$. What do you observe?

b) Why is the value of 1^6 equal to 1? What is the value of 1^n for any positive whole number n?
c) Why is the value of 1^{-6} equal to 1? What is the value of 1^{-n} for any positive whole number n?
d) Why is the value of $1^{\frac{1}{6}}$ equal to 1? What is the value of $1^{\frac{1}{n}}$ for any positive whole number n?
e) Why is the value of $1^{\frac{5}{6}}$ equal to 1? What is the value of $1^{\frac{a}{b}}$ for any fraction $\frac{a}{b}$?
f) Why is the value of $1^{\sqrt{2}}$ equal to 1?

Extra: Use computer software to sketch a graph of $y = 0^x$. What do you notice this time?

MECHANICS PRACTICE

Practice 118.11 Please rewrite each of these statements in logarithmic form.

a) $3^5 = 243$ b) $625^{\frac{1}{4}} = 5$ c) $2^0 = 1$

Practice 118.12 Please rewrite each of these statements as a statement about powers.

a) $2 = \log_7(49)$ b) $\frac{1}{2} = \log_5(\sqrt{5})$ c) $a = \log_3(b)$

Practice 118.13 Kindly evaluate each of these expressions.

a) $\log_2(8^{303})$ b) $\log_{\frac{1}{2}}\left(\frac{1}{16}\right)$ c) $\log_m((m^3)^2)$

Practice 118.14 If you are still here doing practice problems … For each statement, kindly find the value of b that makes the statement true.

a) $\log_b(125) = 3$ b) $\log_b(125) = -3$ c) $\log_b(8) = -0.25$

Practice 118.15 Since you have come this far … For each statement, please consider finding a value of x that makes the statement true.

a) $\log_3(2x + 1) = 4$ b) $\log_{10}(x + 2) = -2$ c) $\log_x(4) = x$

119. Properties of Logarithms we Care About

Logarithms are just powers in disguise. And since we know how exponents work—their properties and their rules of arithmetic—we, technically, know the same for logarithms.

Let's just have some fun looking at exponents rules and translating them into the languagen of logarithms. One caveat: We have to keep in mind that our theory of logarithms presumes the base numbers involved at positive numbers different from 1

1. We know $b^1 = b$ for any number base number b.

This is saying

$$\log_b(b) = 1$$

That is, 1 is the power of b that gives the answer b.

2. We know $b^0 = 1$ for any number base number b.

This is saying

$$\log_b(1) = 0$$

That is, 0 is the power of b that gives the answer 1.

Now for a mind-bendy one. It's the one that solves Napier's 16th-centure challenge.

3. We know $b^n \times b^m = b^{n+m}$ for any number base number b and real numbers n and m.

This is saying

$n + m$ is the power of b that gives the answer $b^n \times b^m$.

$$\log_b(b^n \times b^m) = n + m$$

If you are curious … Here's how this observation fits with Napier's dream of turning multiplication problems into addition problems.

Suppose we wish to compute the product $N \times M$ of two big numbers N and M without doing any multiplication. Do the following:

- Think of N and M each as a power of b

  $N = b^n$ for some value n

  $M = b^m$ for some value m

- Find the values of n and m by noticing

  $$n = \log_b(N)$$
  $$m = \log_b(M)$$

  Use a table of logarithm values to do this.

- Add the two logarithm values to get $n + m$.

- We've just seen that $n + m$ is $\log_b(b^n \times b^m)$.

  But b^n is N and b^m is M.

  So, the value of $n + m$ is $\log_b(N \times M)$.

- Go back to the log table and look up which number has logarithm value $n + m$.
  That number is the product $N \times M$!

Practice 119.1 Your calculator has a "log" button. To match our base-ten number system it assumes the base number is 10. (Your button might even show $\log_{10}$.)

a) Check that $\log(100)$ is 2 on your calculator.
b) On your calculator, what is the value of $n = \log(387)$?
c) On your calculator, what is the value of $m = \log(5092)$?
d) What is the value of $n + m$? (You should get a number close to 6.2946.)
e) What value V has $\log_{10}(V) = 6.2946$? (Think about how you can use your calculator to find this.)
f) Does V match the product 387×5092?

Schoolbooks present the observation

$$\log_b(b^n \times b^m) = n + m$$

in terms of the numbers $N = b^n$ and $M = b^m$ and so write

$$\log_b(N \times M) = \log_b(N) + \log_b(M)$$

"The log of a product is the sum of the logs."

Practice 119.2 On your calculator compute $\log_{10}(75)$ and $\log_{10}(7500)$.

a) Explain why the two values differ by 2.
b) What do you predict for the value of $\log_{10}(0.75)$? Why?

Practice 119.3 a) Explain why the following is true.

$$\log_b\left(\frac{b^n}{b^m}\right) = n - m$$

b) Explain why it leads to the log rule

$$\log_b\left(\frac{N}{M}\right) = \log_b(N) - \log_b(M)$$

We now know the 16th-century reason why people cared about logarithms. But why do people care about logarithms today?

It's because of this next property.

4. We know $(b^n)^m = b^{n\times m}$ for any number base number b and real numbers n and m.

This is saying

$n \times m$ is the power of b that gives the answer$(b^n)^m$.

$$\log_b((b^n)^m) = n \times m$$

Let's rewrite this in the schoolbook way.

Let's write $N = b^n$ and consequently that $n = \log_b(N)$.
Let's use the letter x instead of the letter m.

Then our statement reads

$$\log_b(N^x) = \log_b(N) \times x$$

People prefer to write this as

$\log_b(N^x) = x \log_b(N)$

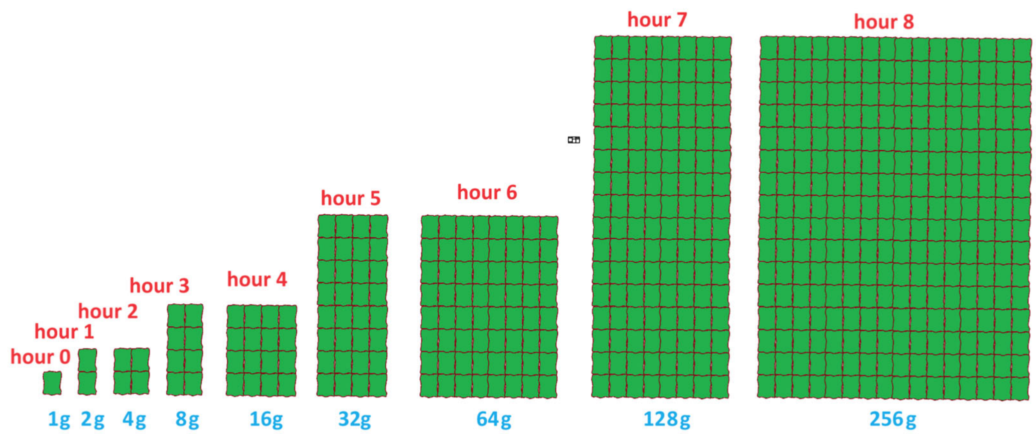
"Hit an exponential expression with a log, shake down the exponent"

Practice 119.4 Explain why $\log_b(\frac{1}{N})$ equals $-\log_b(N)$.

Example: We are green, gooey biological culture grows continuously and in way that its mass doubles every hour.

At time 0 hours we start with 1 gram of culture. So, at time t hours, the mass of the culture is 2^t grams.

When will we have 1.5 kilogram of mass?



This is the example from last chapter that motivated us to want to make sense of the expression 2^t even if t is not a nice whole number.

Now, in this scenario we can use our calculators to find the mass of the culture at any given time.

Check: Do you see an "x^y" on your calculator?

Test it: Work out the value of $2^{2.5}$ to see that we will have about 5.66 grams of culture at time $t = 2.5$ hours.

But is not clear how to calculato the reverse, to find the time that gets us to a given mass.
This example question is asking for the value of t such that

$$2^t = 1500$$

(expressed the mass in terms of grams).

Sure, we can say that $t = \log_2(1500)$, but there is no $\log_2$ button on our calculators.

We want to solve an equation with the unknown "stuck upstairs" as an exponent.

But our fourth log rule helps us out. We can shake that exponent down by hitting both sides with a log!

And since there is a $\log_{10}$ button on our calculators, let's hit it with $\log_{10}$!

Answer: From

$$2^t = 1500$$

we have

$$\log_{10}(2^t) = \log_{10}(1500)$$

Shaking down the exponent turns this into

$$t \log_{10}(2) = \log_{10}(1500)$$

We can work out approximate values on our calculator.

$$t \times 0.301 \approx 3.176$$

and so

$$t \approx \frac{3.176}{0.301} \approx 10.551$$

We have 1500 grams of culture at about time $t = 10.6$ hours.

Comment: One can leave all the numerical calculations to the very end and give an exact answer to this problem. From $t \log_{10}(2) = \log_{10}(1500)$ we see that

$$t = \frac{\log_{10}(1500)}{\log_{10}(2)}$$

is the exact number of hours solving this problem.

Often leaving all numerical calculations to the very end reduces the amount of rounding error.

Textbooks are very good at presenting exercises that are not likely to ever come up in real life. They are, of course, designed just to test students' understanding of the topic at hand.

For instance:

Example: Please solve $5 \cdot 3^x = 7 \cdot 4^{x+1}$

Answer: The unknowns are all stuck upstairs.
To bring them down, let's hit each side of this equation with a log.

$$\log_{10}(5 \cdot 3^x) = \log_{10}(7 \cdot 4^{x+1})$$

We have the logarithm of products here. Recall: **"The log of a product is the sum of the logs."**

$$\log_{10}(5) + \log_{10}(3^x) = \log_{10}(7) + \log_{10}(4^{x+1})$$

Now we're ready to "**shake down exponents**" (catching that one of our exponents is all of $x + 1$).

$$\log_{10}(5) + x\log_{10}(3) = \log_{10}(7) + (x+1)\log_{10}(4)$$

This is visually confusing, but most everything here is just a number. According to my calculator this equation is (approximately):

$$0.699 + 0.477x = 0.845 + 0.602(x+1)$$

I am going to multiply through by 1000 to avoid all the decimals.

$$699 + 477x = 845 + 602(x+1)$$

$$699 + 477x = 845 + 602x + 602$$

$$125x = -748$$

$$x = -\frac{748}{125} \approx -5.984$$

Icky!

Practice 119.5 Show that the exact solution to the previous example is

$$x = \frac{\log_{10}(5) - \log_{10}(7) - \log_{10}(4)}{\log_{10}(4) - \log_{10}(3)}$$

Extra: Can you see that this can also be written as

$$x = \frac{\log_{10}(\frac{5}{28})}{\log_{10}(\frac{4}{3})}$$

Practice 119.6 Kindly solve $2^x \cdot 3^x = 4 \cdot 5^x$.
(If you are game, feel free to give the exact solution as well as an approximate one.)

Practice 119.7

a) Please solve $\left(\frac{6}{5}\right)^x = 4$ with your answer in terms of base ten logrithms.
b) Why is the solution to this problem exactly the same as the solution to the previous one?

This example shows why calculators don't have logarithm buttons for every possible base.

Example: Compute the value of $\log_3(7)$ with a calculator.

Answer: The key is to give the quantity a name. Let's call it F for Frederica.

$$F = \log_3(7)$$

So, Frederica is the power of 3 that gives the answer 7.

$$3^F = 7$$

Let's hit this with our $\log_{10}$ button.

$$\log_{10}(3^F) = \log_{10}(7)$$

$$F\log_{10}(3) = \log_{10}(7)$$

$$F = \frac{\log_{10}(7)}{\log_{10}(3)} \approx \frac{0.845}{0.477} \approx 1.771$$

That's it:

$$\log_3(7) \approx 1.771$$

Practice 119.8 a) Compute $\log_{37}(500)$ if doing so seems fun to you.

b) If you are game, show that $\log_b(N)$ can be computed as

$$\frac{\log_{10}(N)}{\log_{10}(b)}$$

Some curriculums want students to know this **change of base formula**.

Curriculum can really “dig deep” into properties of logarithms and ask students to derive (and memorize) all sorts of complicated-looking formulas.

In the end, each formula is just an application of three basics.

1. **$\log_b(N)$ is the power of b that gives the answer N.**

$$x = \log_b(N) \text{ means } b^x = N$$

(Replace the word “log” with the word “power”)

2. **The log of a product is the sum of the logs**.

$$\log_b(N \times M) = \log_b(N) + \log_b(M)$$

3. **Hitting an exponential expression with a log shakes down the exponent.**

$$\log_b(N^x) = x \log_b(N)$$

For instance, the rule

$$\log_b\left(\frac{N}{M}\right) = \log_b(N) - \log_b(M)$$

can be thought through as follows:

Regard $\frac{N}{M}$ as $N \cdot \frac{1}{M} = N \cdot M^{-1}$ and so

$$\begin{aligned}\log_b\left(\frac{N}{M}\right) &= \log_b(N \cdot M^{-1})\\ &= \log_b(N) + \log_b(M^{-1})\\ &= \log_b(N) + (-1)\log_b(M)\\ &= \log_b(N) - \log_b(M)\end{aligned}$$

MUSINGS

Musing 119.9 Why must $\log_b(3) + \log_b(9)$ equal $\log_b(27)$?

Musing 119.10 Explain why $10^{\log_{10}(N)}$ is sure to equal N.
(Say the meaning of $\log_{10}(N)$ out loud to yourself!)

Musing 119.11 Can you see why $\log_{\frac{1}{b}}(N)$ equals $-\log_b(N)$?

Musing 119.12 Can you see why $\log_{100}(N)$ equals $\frac{1}{2}\log_{10}(N)$?

Musing 119.13 Here's an annoying, made-up question just for the sake of testing students on logarithms and quadratics at the same time:

What value(s) of x make this equation true?

$$(\log_{15}(x))^2 + \log_{15}(x^2) = 15$$

MECHANICS PRACTICE

Practice 119.14 Please evaluate each of these quantities.

a) $\log_4(16)$ b) $\log_5(625)$ c) $\log_9(9)$ d) $\log_{96}(96^{107.3})$ e) $\log_3(\frac{1}{9})$

f) $\log_4(\sqrt{2})$ g) $\log_{\frac{1}{2}}(\frac{1}{2})$ h) $\log_{\frac{1}{2}}(\frac{1}{4})$ i) $\log_{\frac{1}{2}}(1)$ j) $\log_{\frac{1}{2}}(4)$

k) $\log_{\frac{1}{2}}(\sqrt{2})$

Practice 119.15 Please solve each of these equations

a) $5 \cdot 4^x = 3^x$

b) $2^x + 2^{x+1} = 24$

c) $2^x + 2^{x+1} = 25$

d) $10 = (1.05)^x$

Practice 119.16 Some biological gloop grows continuously in such a manner that its mass doubles every hour. How long does it take for the mass to triple?

Practice 119.17 What value of x makes this statement true?

$$\log_x(7) = 3$$

Practice 119.18 Why does $\log_b(\sqrt{N})$ equal $\frac{\log_b(N)}{2}$?

120. Compounding Growth

The human population is growing.

According to an internet search at the time of me writing this, we seem to be averaging about 17.3 births for every 1000 people each year, and 8.9 deaths for every 1000 people each year.

So, we're growing at a rate of an increase of 8.4 additional Earth inhabitants for every 1000 people per annum.

> **Practice 120.1** These figures are continually changing. What are global birth and mortality rates at the time of you reading this?
>
> Can you find some data showing how these rates seem to have changed over the decades?
>
> **Comment**: The data I found is presented in terms of average counts per 1000 people. Some data might be presented instead as a percentage, an average count per 100 people.
>
> For example, the fraction $\frac{17.3}{1000}$ is the same as $\frac{1.73}{100} = 1.73\%$, so our data corresponds to a birth rate of 1.73%.

Just to be clear, we're speaking of growth *rates*, not absolute figures. The total numbers of actual births and deaths depends on the total number of people there are. We've just been told, for every group of 1000, we can expect an average number of 17.3 births, and so on.

For instance, my internet search also tells me there are currently about 8 billion people on the planet. If we trust the figures I've shared, over the coming year there will be

$$8{,}000{,}000{,}000 \times \frac{17.3}{1000} = 138{,}400{,}000 \text{ babies born}$$

$$8{,}000{,}000{,}000 \times \frac{8.9}{1000} = 71{,}200{,}000 \text{ deaths}$$

for a total increase of $67{,}200{,}000$ inhabitants (which is indeed 0.84% of $8{,}000{,}000{,}000$).

We're thus predicting a new human population figure of $8{,}067{,}000{,}000$ by the end of one year.

> **Practice 120.2** Suppose the birth rate of 1.73% and death rate of 0.89% stay the same for next few years. What do you predict then for the Earth's population after a second year? We'll grow from 8.067 billion humans to how many?
>
> And after a third year? A fourth?

We're seeing that if the Earth's population is a count of P people, then after a year there will be

$P \times 0.0173$ babies born

$P \times 0.0089$ deaths

for an increase $P \times 0.0173 - P \times 0.0089 = P \times 0.0084$ in the count of people on the planet.

Over a year, the human population changes from P people to $P + P \times 0.0084$ people.
We have

$$P \;+\; P \times 0.0084 = P \times (1 + 0.0084)$$
$$= P \times 1.0084$$

For an overall growth rate of 0.84% per year (birth and death rates combined), the population changes by a factor of 1.084 from year to year.

Let's play with this.

We started with a population of $P = 8{,}000{,}000{,}000$ people.

After 1 year the population grows to

$$P_1 = 8{,}000{,}000{,}000 \times 1.084 = 8{,}067{,}000{,}000$$

After another year, by year 2, the population grows to

$$P_2 = 8{,}000{,}000{,}000 \times 1.084 \times 1.084 = 8{,}744{,}628{,}000$$

By year 3 the population is

$$P_3 = 8{,}000{,}000{,}000 \times 1.084 \times 1.084 \times 1.084 = 9{,}479{,}176{,}752$$

By year 4 the population is

$$P_4 = 8{,}000{,}000{,}000 \times 1.084 \times 1.084 \times 1.084 \times 1.084 = 11{,}046{,}052{,}825$$

If P_n denotes the population by the end of year n, then our (likely unrealistic) reasoning predicts that the Earth's population will be given by

$$P_n = 8{,}000{,}000{,}000 \times (1.084)^n$$

Practice 120.3 Let's continue to believe that the Earth's population will grow without bound at a constant rate of 0.84% per annum.

According to our model:

a) What will the Earth's population by the end of ten years?

b) In how many years will the Earth's population be one-hundred billion?

Practice 120.4 Assume that the birth and death rates of fruit flies living in my compost bin are each constant. Yesterday there were 160 fruit flies in the bin. Today there are 200.

Create simple mathematical model that predicts the number of fruit flies I can expect living in my compost bin each day.

On which day will I have just over a million fruit flies according to your model?

Practice 120.5 There are fruit flies in my recycling bin too, but the population of them there is not thriving. I've observed they are breeding at a constant birth rate of 10% per day, but dying at a rate of 12% per day.

There are currently 100 fruit flies in my recycling bin.

Create simple mathematical model that predicts the number of fruit flies I can expect living in my recycling bin each day.

On which day will I just one fruit fly in my bin according to your model?

Banking

If you think about it for a moment, it seems somewhat curious that banks pay you for the honor of conducting a service for you, namely, to securely house your money. Ten thousand dollars in cash is safer in the protection of a bank than in your pockets or under your mattress.

And how do banks pay you? By giving you interest on the money you have stored with them.

Of course, banks do make a profit despite regularly adding money to your and all their customers' accounts. With the large cash sums the accrue from having many customers, banks can invest in high-paying financial opportunities beyond what any one individual can typically do. They make a profit and thank you for this by sharing some of the profit with you.

The total amount of money you earn from an interest payment of course depends on the amount of money you have in your account. An interest payment of 2% say on a balance of \$10,000 gives you $\frac{2}{100} \times 10{,}000 = \$200$, but on a balance of \$1,000,000 it gives you $\frac{2}{100} \times 1{,}000{,}000 = \$20{,}000$.

If you let a balance sit in an account untouched, you'll also earn interest on the interest payments awarded to you, repeatedly, and your account balance will grow.

Example: I decide to invest \$1200 with *Grimy Hands Money Market.* They offer 3.5% annual interest calculated at the end of each calendar year.

It's the start of the year, and I am willing to keep my money in the account untouched for many years.

a) What will my balance be after ten years?

b) I'd like to be a millionaire. By which year will I have a million dollars?

Answer: My starting balance is

$$B = 1200$$

dollars.

At the end of the first year, my balance will grow to

$$B_1 = 1200 \;+\; \frac{3.5}{100} \times 1200 = 1242$$

dollars. But let's write this answer as

$$B_1 = 1200 \times 1.035$$

At the end of the second year my balance will grow to

$$B_2 = 1242 \quad + \quad 1242 \times 0.035 = 1285.47$$

dollars. But this really

$$\begin{aligned} B_2 &= 1242 \times 1.035 \\ &= 1200 \times 1.035 \times 1.035 = 1200 \times (1.035)^2 \end{aligned}$$

Each year my balance grows by a factor of 1.035 and after n years will have

$$B_n = 1200 \times (1.035)^n$$

dollars.

a) By the end of ten years my balance is

$$B_{10} = 1200 \times (1.035)^{10} \approx 1692.71$$

(Grimy Hands always rounds <u>down</u> to the nearest penny!)

b) To be a millionaire I need

$$1200 \times (1.035)^n = 1{,}000{,}000$$

So,

$$1.035^n = \frac{1000000}{1200} = \frac{2500}{3}$$

"Hitting with a log" gives

$$n \log_{10}(1.035) = \log_{10}\left(\frac{2500}{3}\right)$$

And so

$$n = \frac{\log_{10}\left(\frac{2500}{3}\right)}{\log_{10}(1.035)}$$

According to my calculator, this gives $n \approx 195.5$. I'll need to wait 196 years to be a millionaire if I start with $1200 and just sit back! (Maybe I should be a little more pro-active if this truly is my goal!)

The truth is banks don't assign interest just once a year, they assign it at every instant!

Let's make sense of this intriguing claim.

Example: I have $\$10{,}000$ I'd like to invest for a year. Two respectable institutions are offering 5% interest per annum but calculated over smaller time periods.

- *Buckets-o-Cash Bank* calculates their interest payments monthly, meaning that they assign $\frac{5}{12}\%$ of interest each month for 12 months.
- *Cash Flux Bank* calculates their interest payments weekly, meaning that they assign $\frac{5}{52}\%$ of interest each week for 52 weeks.

Which bank will leave me with the bigger balance by the end of the year?

Answer: Let's examine each bank in turn.

Buckets-o-Cash: They pay $\frac{5}{12} = 0.4166\ldots\%$ interest each month. So, a balance of B dollars at the start of a month becomes

$$B + \frac{0.41666\ldots}{100} \times B = B \times (1 + 0.0041666\ldots) = B \times (1.0041666\ldots)$$

dollars at the end of the month. That is, my balance grows by a factor of $1.0041666\ldots$ each month.

Let's write our a "spread-sheet" of my balance month-per-month.

Start: $\$10{,}000$
Month 1: $\$10{,}000 \times (1.0041666\ldots)$
Month 2: $\$10{,}000 \times (1.0041666\ldots) \times (1.0041666\ldots) = \$10{,}000 \times (1.0041666\ldots)^2$
Month 3: $\$10{,}000 \times (1.0041666\ldots)^3$

$\vdots$

Month 12: $\$10{,}000 \times (1.0041666\ldots)^{12}$

My calculator says this final balance amount is $\mathbf{\$10,511,62.}$

Cash Flux Bank: They pay $\frac{5}{52} = 0.0961 \ldots \%$ interest each week. So, a balance of B dollars at the start of a week becomes

$$B \;+\; \frac{0.0961 \ldots}{100} \times B = B \times (1 + 0.000961 \ldots) = B \times (1.000961 \ldots)$$

dollars at the end of the week. That is, my balance grows by a factor of $1.000961 \ldots$ each week.

Start: $\$10{,}000$
Week 1: $\$10{,}000 \times (1.000961 \ldots)$
Week 2: $\$10{,}000 \times (1.000961 \ldots)^2$
Week 3: $\$10{,}000 \times (1.000961 \ldots)^3$

⋮

Week 52: $\$10{,}000 \times (1.000961 \ldots)^{52}$

My calculator says this final balance amount is $\$\mathbf{10{,}512{,}46.}$

That's 84 cents better! I'll go with *Cash Flux Bank*.

Example Continued: I have since learned of some more banks paying the same interest rate per annum, but calculated over even shorter time periods.

- *Swimming-In-It Bank* calculates their interest payments daily, meaning that they assign $\frac{5}{365}\%$ of interest each day for365 days.

- *Cash Galore Bank* calculates their interest payments every hour, meaning that they spread the 5% interest payment over each and every hour of the year.

- *Bank Bonanza* calculates their interest payments every minute, meaning that they spread the 5% interest payment over each and every minute of the year.

- *No-Messing-About-Bank* calculates their interest payments every second, meaning that they spread the 5% interest payment over each and every second of the year.

And there are additional banks that spread the 5% interest payment over every milli-second of the year, over every nano-second of the year, and so on.

For the four banks named, what would my end-of-year balance be if I invested my $\$10{,}000$ with each of them?

We saw the following general structure from the first example.

If a bank pays $r\%$ interest each period and my balance at the beginning of a period is B dollars, then my new balance at the end of that period will be $B(1+\frac{r}{100})$ dollars. (This is $B\ +\frac{r}{100}\times B$.)

After n such periods my balance will be

$$B\left(1+\frac{r}{100}\right)^n$$

dollars.

Let's now analyze the next four banks.

Answer:

Swimming-In-It Bank: There are 365 days in a year. We have $r=\frac{5}{365}=0.01369\ldots\%$ amount of interest paid each day for 365 days. My balance after a year will be

$$10{,}000\times(1.0001369\ldots)^{365}\approx\$\mathbf{10,512.67}$$

Cash Galore Bank: There are $365\times 24=8{,}760$ hours in a year.

Now $r=\frac{5}{8760}=0.00057\ldots\%$ and we have 8760 periods.
My balance after a year will be

$$10{,}000\times(1.0000057\ldots)^{8760}\approx\$\mathbf{10,512.70}$$

Bank Bonanza: There are $365\times 24\times 60=525{,}600$ minutes in a year.

Now $r=\frac{5}{525600}=0.00000951\ldots\%$ and we have 525,600 periods.
My balance after a year will be

$$10{,}000\times(1.0000000951\ldots)^{525600}\approx\$\mathbf{10,512.71}$$

No-Messing-About Bank: There are $365\times 24\times 60\times 60=31{,}536{,}000$ seconds in a year.

Now $r=\frac{5}{31{,}536{,}000}=0.000000158\ldots\%$ and we have 31,536,000 periods.
My balance after a year will be

$$10{,}000\times(1.00000000158\ldots)^{31536000}\approx\$\mathbf{10,512.71}$$

Computing interest over finer and finer time periods at this point seems to create balance increases only in fractions of pennies. I don't think it is worth going through the work of calculating final balances for banks that compute interest every micro-second or finer. We won't see the effect at the levels of pennies.

But we do see that there does seem to be some kind of "ultimate" balance value if banks were to approach computing interest for us at each and every instant.

And here is something lovely. Banks today do provide interest at these ultimate balance values for their customers, that is, as though they were providing interest each and every instant.

And this is kind. We all know that we are constantly depositing and withdrawing cash from our accounts, and it would be sad if interest were assigned to our balances on a day after we happened to have made a large withdrawal.

> **Practice 120.6** Suppose I kept my account open with *No-Messing-About Bank* for double the amount of time, 2 years instead of just 1.
>
> What would my balance be after two years?
>
> (After one year I will earn $512.71 in interest, but over two years my interest earned will be more than double this. This is because during the second year I'll be earning interest on even higher interest payments.)

MUSINGS

Musing 120.7 I borrowed $\$15,600$ to buy a new car, promising to pay $\$1,200$ each month towards completing the loan.

The bank charges me 9% interest per annum, but charges it monthly ($\frac{9}{12}\%$ each month) and applies that interest charge just before my $\$1,200$ payment is made for that month.

How many months will it take me to pay off the loan?
How much will I have actually paid at the end of this loan?

Hint: Write out a "spread sheet" for this.

MECHANICS PRACTICE

Practice 120.8 I would like to open a bank that makes every customer a millionaire by the end of ten years if they invest just $\$1,000$ with us. I'll offer an annual interest rate of $r\%$ per year for some value r and pay $\frac{r}{12}\%$ interest at the end of each month for 120 months.

What value for r should I offer?