Chapter 17

Infinity

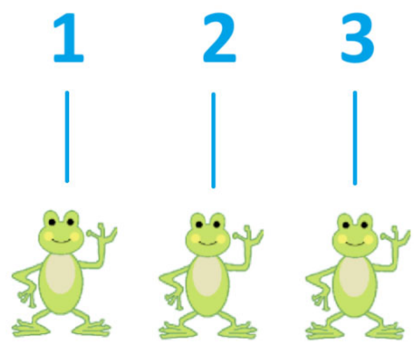
Contents

Introduction to Part Three

We started our entire mathematical journey together with the numbers that arise from the natural act of counting, namely, the **counting numbers**, $1, 2, 3, 4, \ldots$.

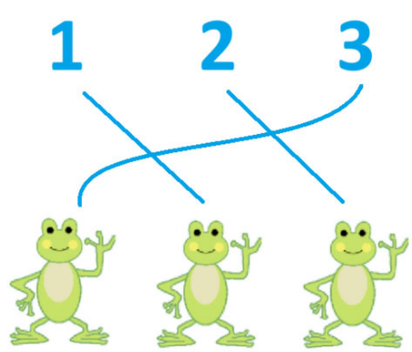
As a human collective we have agreed upon an ordered list of words and matching symbols—*one* (1), *two* (2), *three* (3), …, *twelve* (12), … *ninety-seven* (97), … *one-million-and-two* $(1{,}000{,}002)$, …—and use this list to communicate to each other the sizes of sets of objects.

For example, in this collection of frogs, we can associate with each in turn a number from our memorized list, find that we stop at the word *three*, and so declare "There are three frogs."



(We are literally counting the frogs "one, two, three.")

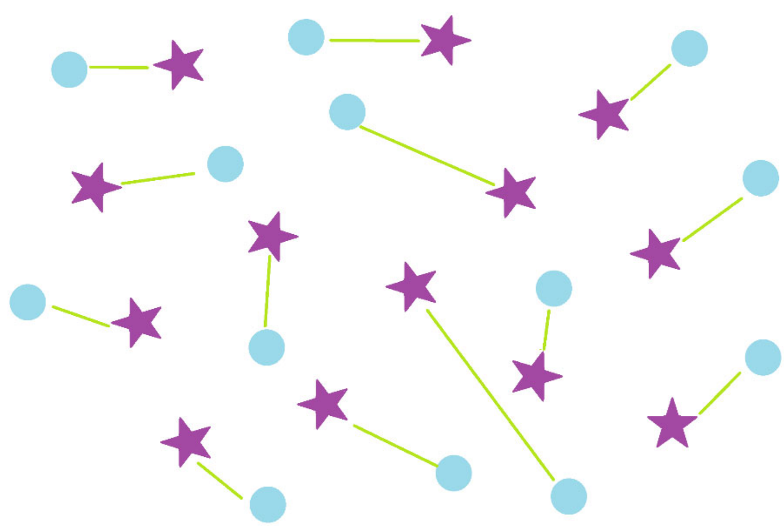
The order in which we assign the symbols 1, 2, and 3 to frogs does not matter. We'll still stop at "three" with a different matching of number-symbols and frogs and declare that there are three frogs.



This idea of matching can be both powerful and confusing.

For example, here's a collection of dots and stars. It's hard to tell if there is an equal number of each.

But this picture makes it clear that there is an equal number of them (and we can say this without ever counting *thirteen*.)



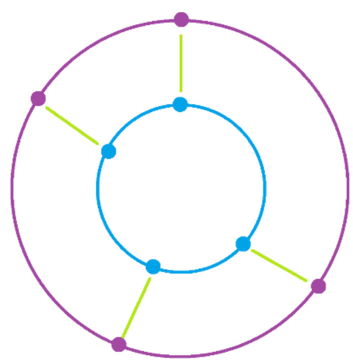
This shows the power of matching.

But 400 years ago, Italian scientist and mathematician Galileo Galilei (1564-1642) observed that each counting number can be matched with a square number in just the same manner.

1	2	3	4	5	6	...
1	4	9	16	25	36	...

Concluding that there are "just as many" square numbers as there are counting numbers didn't seem right to him.

He also thought it was strange that there should be "just as many" points on a small circle as there are on a bigger circle.



This is the paradox of matching.

The idea of matching elements of two different collections of objects is a powerful idea in mathematics, but it took a long time for scholars to recognize this idea, to really "lean into" it, and to make sense of it.

That is the basis of this third volume. We'll focus on a special kind of matching, now called a **function**, but the idea to keep in mind is that we're really defining a link between elements of one set with elements of a second set.

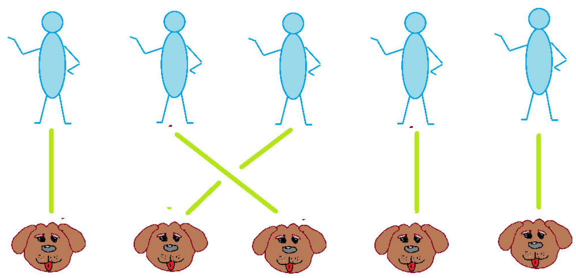
But let's start off with the fun of seemingly paradoxical matching. We'll first explore the work of German mathematician Georg Cantor (1845-1918) who tried to make sense of the infinite with matchings.

121. Just One Type of Infinity?

Did you read the introduction?

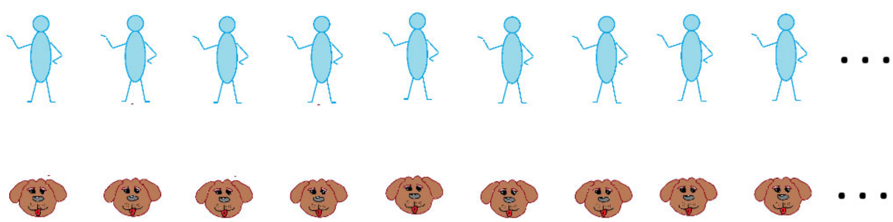
That essay is the actually the start of this chapter as it illustrates how we can identify two collections of objects as the “same size” without explicitly counting.

For example, this picture shows some people, some dogs, and some leashes. Each dog is leashed to one person and each person is leashed to one dog. Without ever counting the number of people or the number of dogs, we can be sure that the set of people and the set of dogs are each the same size.

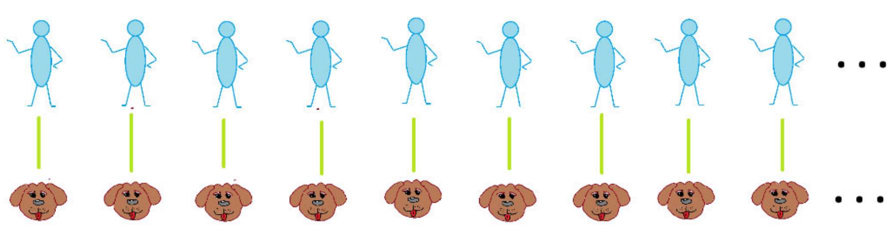


We can push this idea of leashing!

Here’s an infinite line of people going infinitely far to the right and an infinite line of dogs also going infinitely far to the right. Are the set of people and the set of dogs in this picture “the same size”?



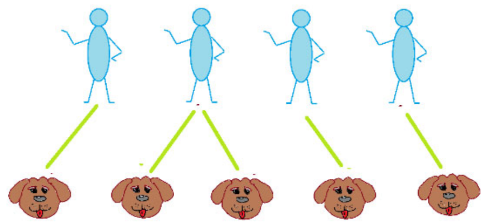
The answer is yes if we follow this leashing idea. We can certainly envision a way to draw leashes so that each and every dog is leashed to a person and each and every person is leashed to a dog.



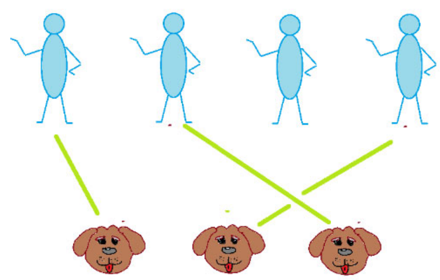
Even with the "...", we can see how this leashing pattern will continue: the tenth dog and the tenth person will be leashed together, the thousandth dog and the thousandth person will be leashed together, the googolth dog and the googolth person will be leashed together, and so on.

Describing (or just visually exhibiting a leashing pattern that can be continued) is enough to say that two different sets are the same size.

Practice 121.1 a) What is fundamentally "wrong" with this leashing pattern to stop us from concluding that the set of people and the set of dogs are the same size?



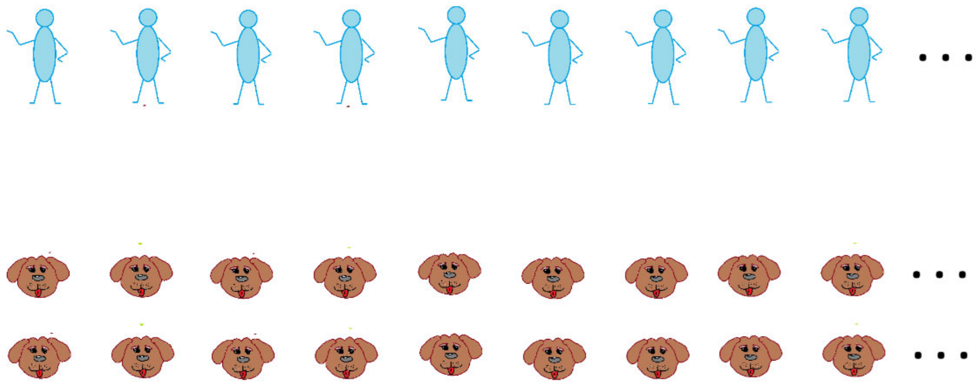
b) What is fundamentally "wrong" with this leashing pattern to stop us from concluding that the set of people and the set of dogs are the same size?



To be clear, let's make this definition:

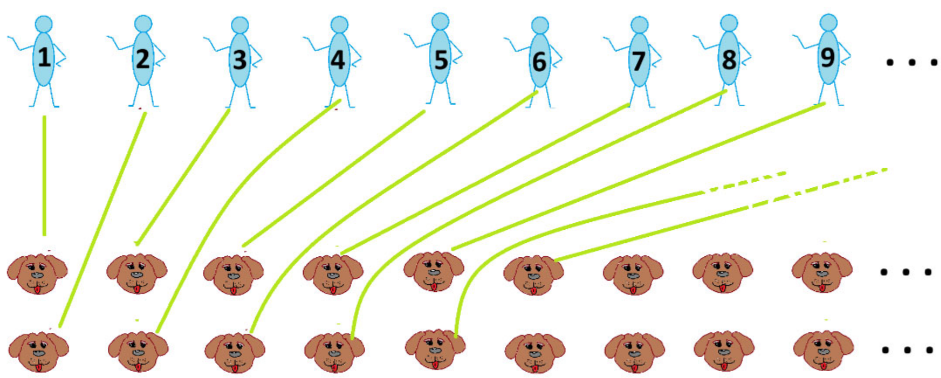
Two sets of objects are said to be the **same size** if it is possible to describe a leashing pattern between objects so that each item of the first set is leashed to exactly one item of the second set and each item of the second set is leashed to exactly one item of the first set.

Here’s an infinite row of people and two infinite rows of dogs.



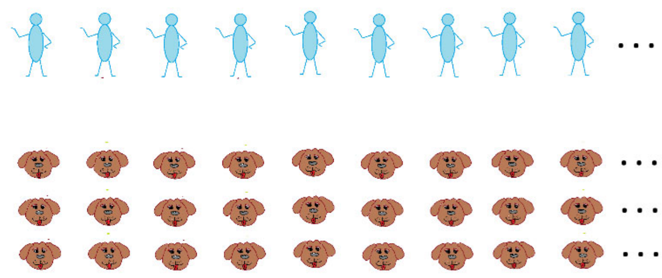
Are these two sets the same size? Or is the count of dogs “double the infinity” of the count of people?

Surprisingly, our leashing idea shows that the are just as many people as there are dogs! Here’s a way to show a valid leashing pattern that can clearly be extended.

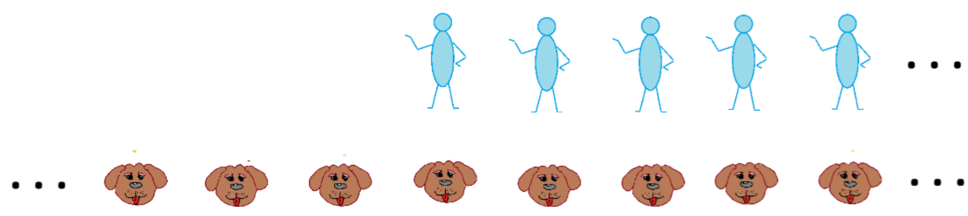


If we number the people 1, 2, 3, 4, 5, … , we can match people 1, 3, 5, 7, … with the dogs in the first row and people 2, 4, 6, 8, with the dogs in the second row.

Practice 121.2 Show that a "triple infinity" of dogs is the same size as a "single infinity" of people.



How about a single infinity of people and a "double-ended infinity" of dogs?
Are these two sets the same size?

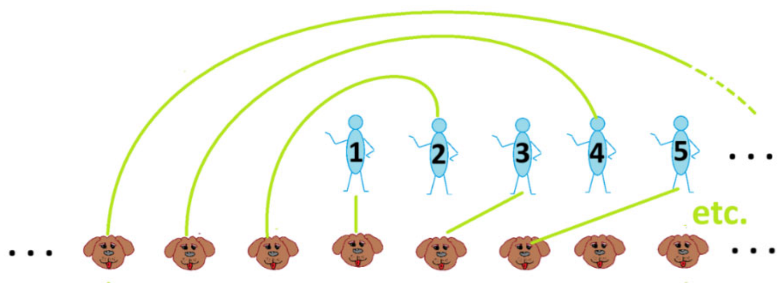


What do you think?

What does your instinct tell you?

Is there a leashing pattern between people and dogs that works, or no possible pattern?

Again, perhaps surprisingly, there is a leashing pattern that shows a "single infinity" and a "double-ended infinity" being the same size. Number the people and match person number 1 to a dog, and then people 3, 5, 7, 9, … to dogs to its right and people 2, 4, 6, 8,… to dogs to its the left.

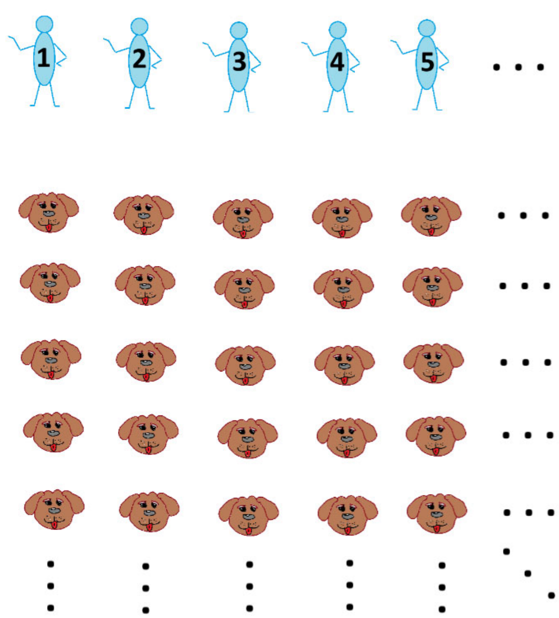


We've now shown that double- and triple- and double-ended-infinitely big sets of dogs are all the same size as a single infinity of people numbered 1, 2, 3, 4, … .

People call this these infinite sets **countably infinite** because we are matching elements of those sets (the dogs) with the set of counting numbers (labeled people). All the infinite sets we've seen thus far are the same, countably infinite size.

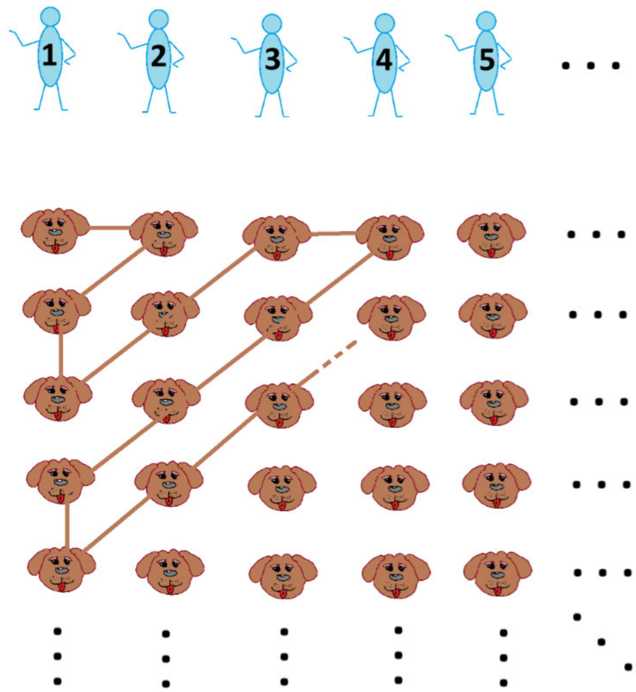
But surely a two-dimensional array of dogs—infinitely many rows of infinitely many dogs—is "more infinite" that than a single countable infinity? There just can't be a leashing pattern between the people and dogs in this picture.

What do you think?

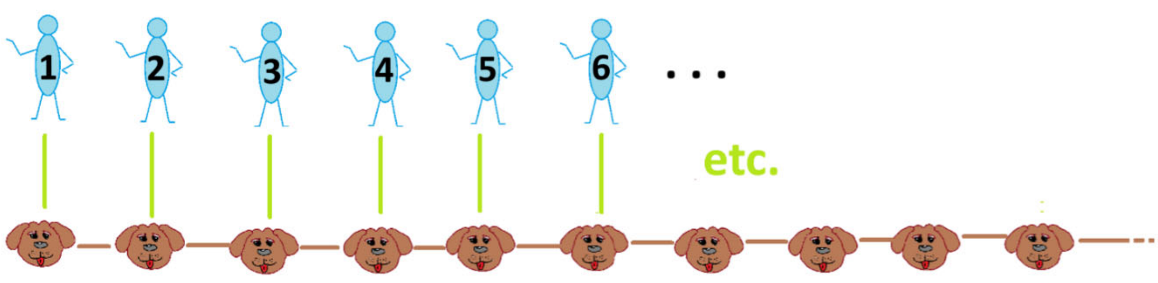


You might be surprised to see that there is a leashing pattern that works!

Start by drawing a zig-zag line that weaves through the whole two-dimensional array of dogs.

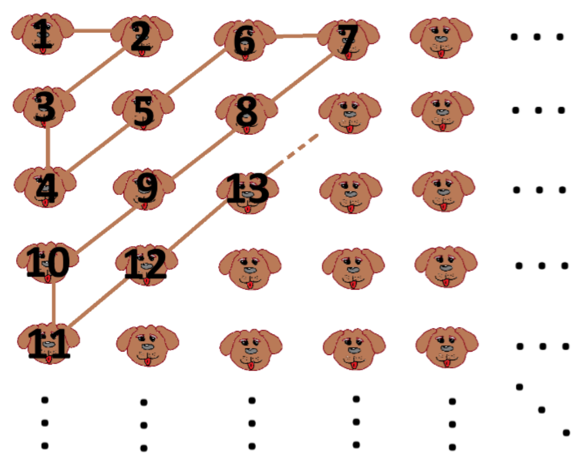


Then we can straighten out that line of dogs and display a leashing pattern.



Actually … simply demonstrating a counting scheme 1, 2, 3, 4, … that weaves its way through an infinite set without ever missing an element is enough to illustrate a leashing pattern.

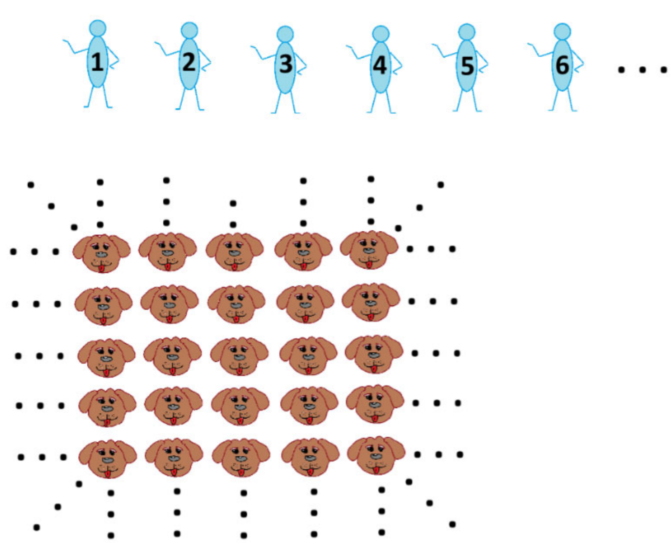
For example, this picture tells us to which person each dog is leashed.



We have:

An infinite collection of objects is countably infinite if we can label the elements of the set with the counting numbers 1, 2, 3, 4, … without missing an element.

Practice 121.3 Show that a full two-dimensional array of dogs is the same size as a "single infinity" of people.

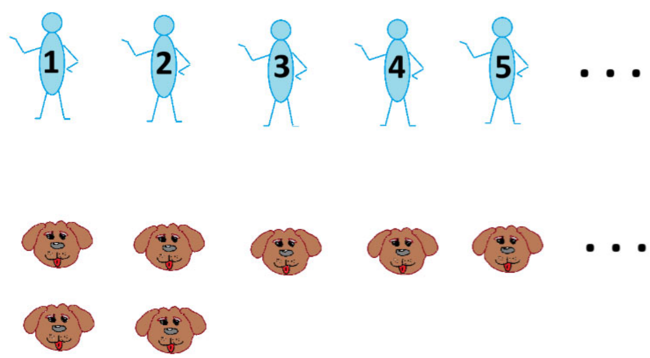


The concept of using leashing patterns to demonstrate that two infinitely large sets are the same size—beyond simply stating that "they are both infinite"—was developed by the German mathematician Georg Cantor (1845-1918). At the time, this idea was considered controversial because it seemed like a mental game: one cannot physically draw all the leashes involved and so must rely on the perception that a pattern persists.

While this may not seem overly concerning based on our current work, Cantor shocked the mathematical community by providing examples of sets he could prove that were "more infinite" than the countable infinity we've seen multiple times so far. This brought distrust to his leashing idea.

We'll construct a "more infinite" set in the next section.

Practice 121.4 Is a single infinity of dogs plus two more dogs just as infinite as a single infinity of people?



MUSINGS

Musing 121.5 As we saw in the introduction, Italian scientist and mathematician Galileo was perplexed by the idea that there are “just as many” square numbers as there are counting numbers.

1	2	3	4	5	6	...
|	|	|	|	|	|	
1	4	9	16	25	36	...

As philosophically disturbing as this might be, Cantor would say that Galileo demonstrated a valid leashing pattern between the set of counting numbers and the set of square numbers, and so , yes, these two infinite sets are the same size.

a) Is the set of just the **even counting numbers** the same size the set of all counting numbers?
b) Is the set of just the **odd counting numbers** the same size the set of all counting numbers?
c) Is the set of all the **integers** (positive and negative counting numbers, and zero) the same size the set of all counting numbers?

Musing 121.6 There are an infinite number of houses numbered 1, 2, 3, 4, 5, … along an infinitely long street. The family in house 1 has one dog, the family in house 2 has two dogs, the family in house 3 has three dogs, and so on. (And yes, the family in house 1,000,000 owns a million dogs, etc.)

Now consider the set of all dogs to be found on this street in totality. Is this set countably infinite? Could all the dogs be appropriately leashed to a single-infinity line of people numbers 1, 2, 3, 4, …?

Extra Challenge: Suppose instead each house along this street has a countable infinity of dogs. (That is, in each house there are an infinite number of dogs that can be numbered 1, 2, 3, 4, 5, … .) Explain why the set of all dogs along this street is still countably infinite.

Musing 121.7 In 1925, German mathematician David Hilbert presented a paradox now called **Hilbert's Hotel Paradox**. It proceeds as follows:

A grand hotel has an infinite number of rooms numbered 1, 2, 3, 4, 5, … . The hotel is full—every room is occupied—but one night an extra guest arrives. Is it possible for the hotel manager to accommodate the guest?

The answer is yes. Despite the inconvenience to the current guests, she asks them to each move to the neighboring room of the next highest room number: the guest in room 1 moves to room 2, the guest in room 2 moves to room 3, and so on. This then frees up room 1 for the new guest and the matter is resolved.

a) How could the hotel manage resolve the matter if seven new guests arrived, not just one?

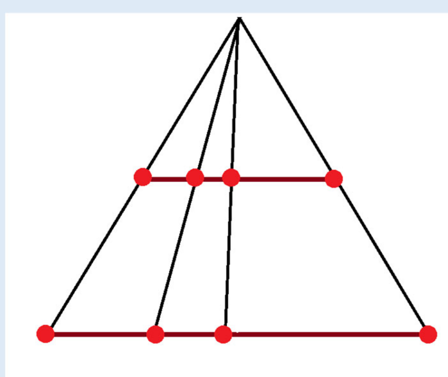
b) How could the hotel manage resolve the matter if a countably infinite number of guests arrived? (That is, assume here that new guests can be numbered 1, 2, 3, 4, … .)

c) **CHALLENGE:** An infinite number of busses arrive (the busses are numbered 1, 2, 3, 4, …) and each bus contains an infinite number of new guests (numbered 1, 2, 3, 4,, ….). Explain how the hotel manager can indeed accommodate this overwhelming number of new guests.

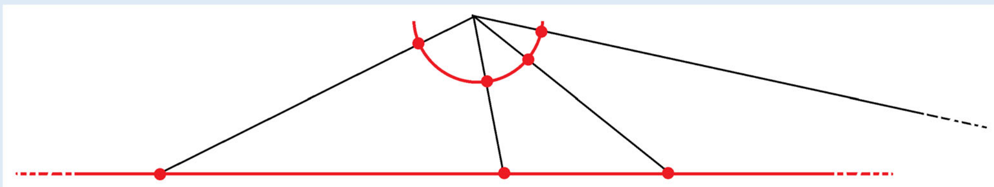
d) **CHALLENGE:** A double-ended bus arrives with guests numbered $\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots$ in it. How can the hotel manage accommodate these new arrivals?

Musing 121.8 This picture shows that there are just as many points on a line segment 1 inch long as there are on a line segment 2 inches long.

Would you agree?



And would you agree that this picture shows that there are just as many points in the bottom portion of a circle (a finite length) as there are on an infinite line?



Musing 121.9 Here are all the positive fractions arranged in an array. (I've crossed out the fractions that are repeats of previously appearing fractions. For instance, I deleted $\frac{2}{2}$ as this is the same as the number 1, which already appears as $\frac{1}{1}$. And I deleted $\frac{2}{4}$ as it already appears as $\frac{1}{2}$.

$\frac{1}{1}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	…
$\frac{2}{1}$	~~$\frac{2}{2}$~~	$\frac{2}{3}$	~~$\frac{2}{4}$~~	$\frac{2}{5}$	…
$\frac{3}{1}$	$\frac{3}{2}$	~~$\frac{3}{3}$~~	$\frac{3}{4}$	$\frac{3}{5}$	…
$\frac{4}{1}$	~~$\frac{4}{2}$~~	$\frac{4}{3}$	~~$\frac{4}{4}$~~	$\frac{4}{5}$	…
⋮	⋮	⋮	⋮	⋮	⋱

Explain how this picture shows that there are just as many (positive) fractions as there are counting numbers.

122. There is More the One Type of Infinity

It is tricky to devise an infinite set that is "more infinite" than the set of counting numbers. But Cantor succeeded in doing so (and thus shocked the mathematics community to then instilled doubt about his methods).

Here's an example of such a set. It's a variation of the example Cantor presented.

Consider two different types of dogs: brown dogs and white dogs. But rather than focus on individual dogs, we're now going to focus on infinitely long lines of dogs.

For example, here's one line of dogs. It alternates brown and white dogs.

Here's another line with all the dogs brown.

And here's one with every dog in a prime-numbered position white.

But there doesn't have to be a structure to the line of brown or white dogs. Here's a line that has no pattern to the color of the dogs in each place.

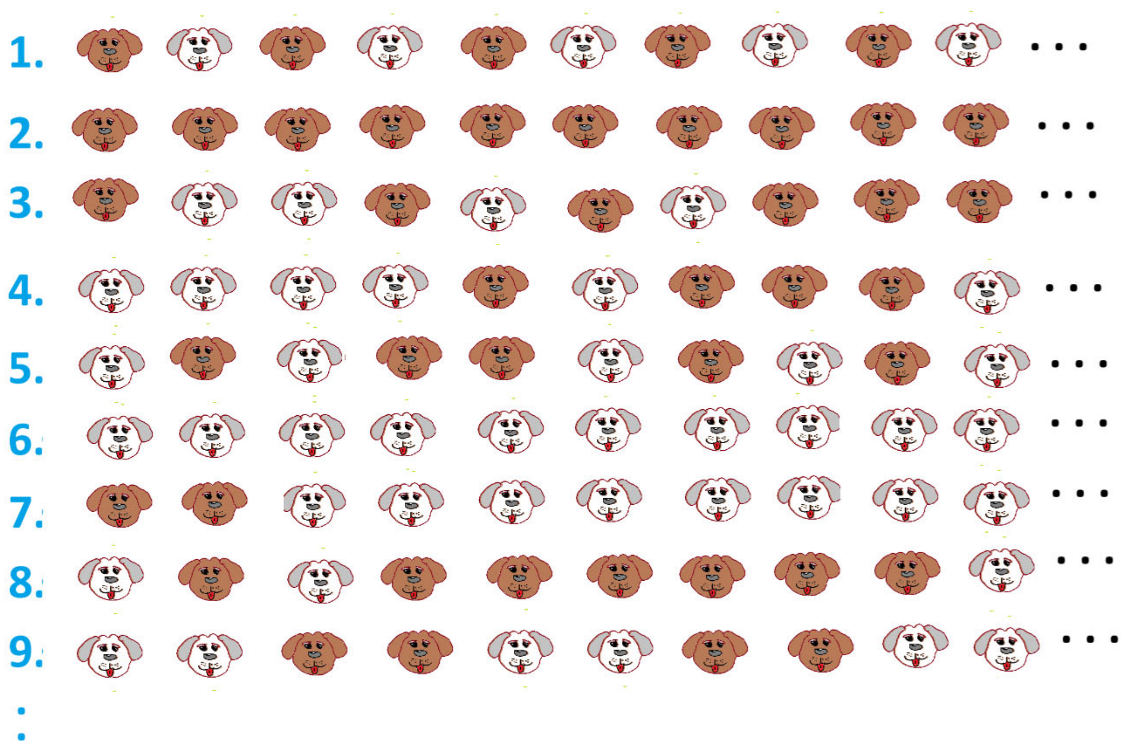
Let's consider the collection of all possible lines of brown and white dogs.

We'll prove that this set is "bigger" than the countable infinite sets we've seen so far.

And we'll do this by showing that something goes wrong if we claim that this set of all possible lines of dogs is countable.

Suppose we can draw all the possible lines of dogs and label the lines 1, 2, 3, 4, 5, … without missing a line. (Recall from the last section, this is enough to demonstrate that a set is countably infinite.) The key point is that no line of dogs is missed in our labeling.

I drew four lines of dogs on the previous page and labeled them 1, 2, 3, and 4 here. But we are assuming that we can keep going and assign a number to each and every possible line of dogs without missing any possibilities.

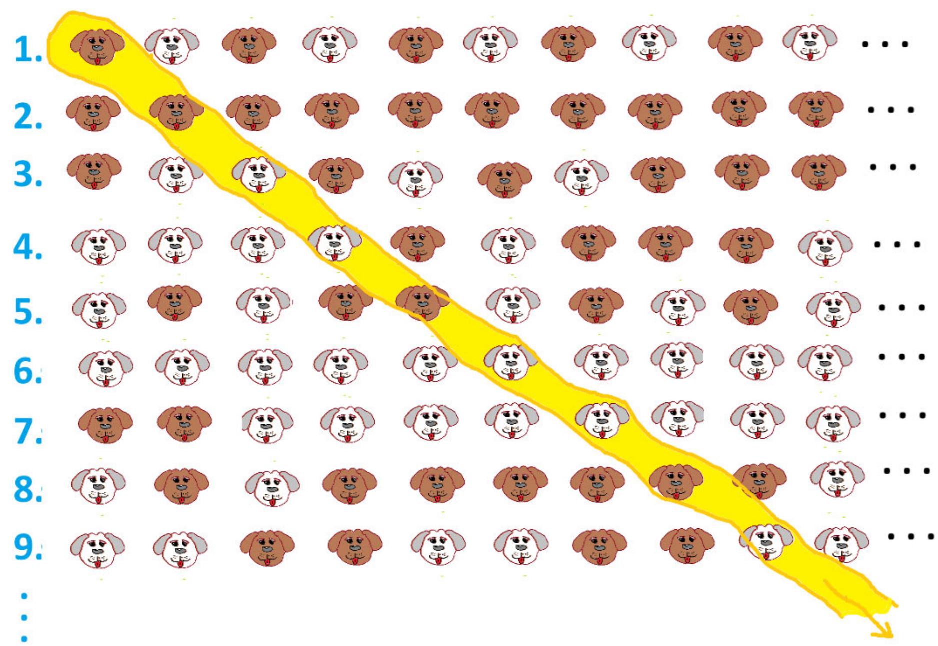


Here's what goes wrong with our thinking. Whatever numbering system we come up for supposedly all the possible lines of dogs will necessarily miss a line of dogs despite ourselves!

What line of dogs did we miss in our example above?

This one coming right up.

Look at the dogs on the diagonal of the list of lines we've created. In our picture we have a diagonal that goes **B**rown, **B**rown, **W**hite, **W**hite, **B**rown, **W**hite, **W**hite, **B**rown, **W**hite, and then will keep going as we work through the list.



Now consider this line of dogs with exact opposite coloring to the diagonal.

The first dog in the diagonal is brown, the first dog in this new line is white.
The second dog in the diagonal is brown, the second dog in this new line white.
The third dog in the diagonal is white, the third dog in this is new line brown.

The dog in the 100th spot of this new line is the opposite color to the dog in the 100th position in the diagonal.

The dogs in the 703rd positions of the diagonal and of this new line are not the same color.

Every dog is opposite!

Now ask: *Where is this newly created line of dogs in our numbered list?* If we have all the lines of dogs, it has to be somewhere.

Our newly created line doesn't match the first line in the list because our line begins with a white dog, but the first line in the list begins with a brown dog. There is a mismatch in the very first spot at the very least.

Our newly created line doesn't match the second line in the list because the second dog in our line is white, but the second dog in the second line is brown. There is a mismatch in the second spot at the very least.

Our newly created line doesn't match the third line in the list because the third dog in our line is brown, but the third dog in the third line is white. There is a mismatch in the third spot at the very least.

Our newly created line doesn't match the 100^{th} line in the list because the dogs in the 100^{th} positions of each don't match at the very least.

Our newly created line doesn't match the 703^{rd} line in the list because the dogs in the 703^{rd} positions of each don't match at the very least.

And so on.

We've created a line of dogs that cannot be any line in the list we created despite believing we listed them all.

There are always going to be more lines of dogs than there are the counting numbers!

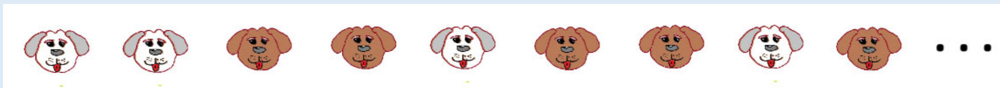
Although this is an artificial construct—"the set of all lines of dogs of two possible colors"—it does demonstrate that there are, at the very least, two different types of infinity.

Could there be more?

Comment: The argument we just presented in looking at the diagonal line of dogs in a list of lines is called **Cantor's Second Diagonal Argument**. His **First Diagonal Argument** is that of Musing 121.9 (but I actually introduced the argument earlier in that section when we first looked at a "two-dimensional array" of dogs).

MUSINGS

Musing 122.1 This line of dogs

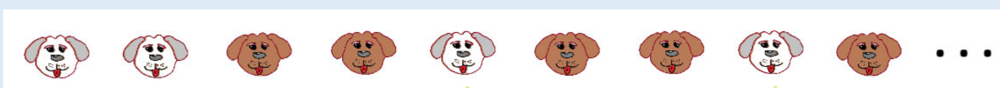


was missing from our list of lines of dogs. So why can't we just add this missing line to the list, say at the top, and make the list complete?

OPTIONAL READING: What is this new infinite set, really?

Each line of brown and white dots can be thought of as a decimal number, but not a decimal number in the $1 \leftarrow 10$ machine of Chapter 4, but in a $1 \leftarrow 2$ machine instead. (That is, as a "decimal" in binary, base-two).

For example, if we associate the digit 0 with a white dog and the digit 1 with a brown dog, then this line of dogs represents the base-two number 0.001101101 … .

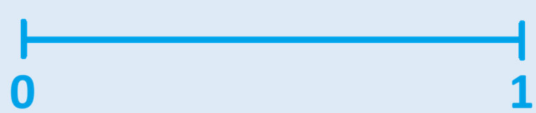


The number zero is represented a line of nothing but white dogs: 0.0000 …. The number one-half as the line of one brown dog followed white dogs thereafter: 0.10000 …. And so on.

Each number between zero and one has a binary "decimal" representation and so can be represented as a line of dogs and each line of dogs as a decimal number in binary.

There is a slight complication with some numbers having two different decimal representations. For example, we saw in Section 54, that in base ten, 0.9999 … and 1.00000 … represent the same number. A similar issue occurs in base two.

Nonetheless, Georg Cantor was able to obviate this annoyance and show that his new infinite set is the set of all numbers between 0 and 1 on the number line. That, that is, the set of all points between 0 and 1 on the number line is a set more infinite than countable infinity.

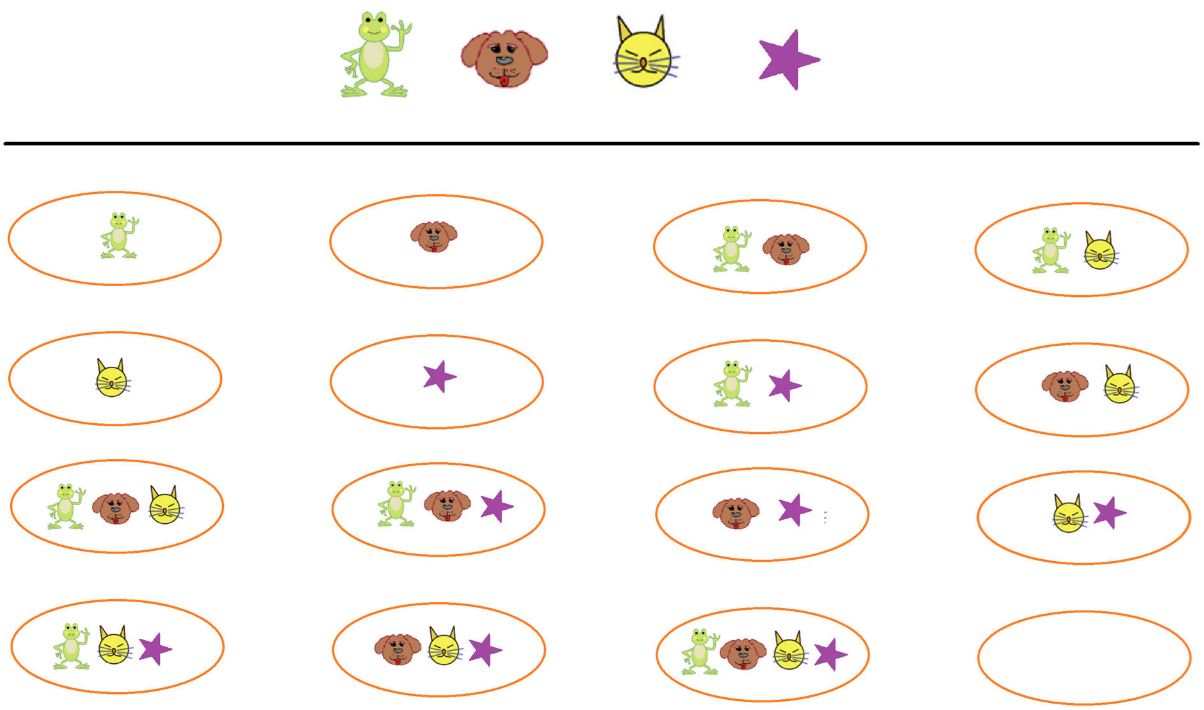


123. Infinitely Many Infinities

Here's a very boring party trick.

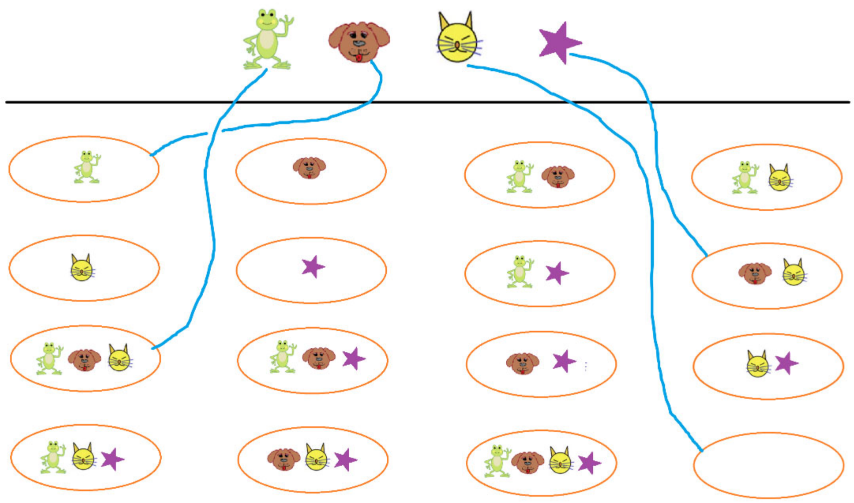
At the top of page draw four symbols or write four letters or four numbers.
I've drawn a frog, a dog, a cat, and a star.

Below them, draw all the possible sets you can make using some, all, or none of the symbols. There are 16 in all. (The bottom right set in my picture is empty. It's called the **empty set**.)



Now, without your looking, ask a friend to draw a leash from each object at the top of the page to one of the sixteen sets. Just make sure four different sets are each leashed to one of the symbols at top.

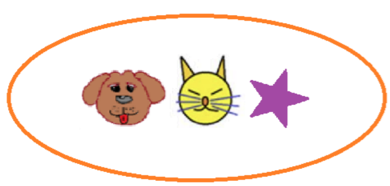
Here's an example of what your friend might draw—but you won't see the picture.



Now ask:

Is the frog in the set to which it is leashed? YES.
Is the dog in the set to which it is leashed? NO.
Is the cat in the set to which it is leashed? NO.
Is the star in the set to which it is leashed? NO.

Keep track of the NO answers. These allow you to then announce to your friend—to her absolute astonishment—a set she did not leash. In my example, the dog, cat, and start elicited NO answers and lo-and-behold the set with these three objects is not leashed.

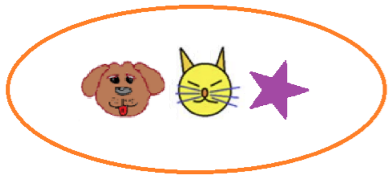


(If your friend answered YES to all four questions, then choose the empty set as the special unleashed set.)

Practice 123.1 Try playing this trick for yourself a few times to get a feel for it. Can you explain for yourself why the set coming from all the NO answers in what you draw each time is sure to be unleashed?

Practice 123.2 Try playing the trick with just three symbols at the top of the page. (How many sets can you make with some, all, or none of three symbols?) Doe the trick still work? How about with five symbols at the top of the page?

Here's why, in our example, we can be sure that the set DOG, CAT, STAR can't be leashed.



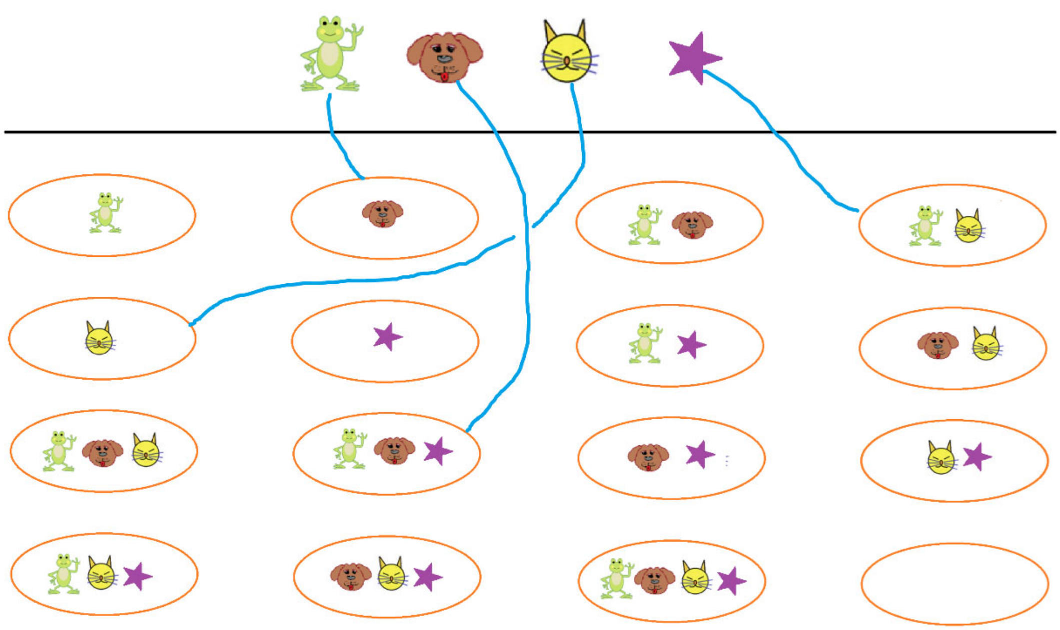
This set can't be leashed to the dog because your friend answered NO to the question "Is the dog in the set to which it is leashed?"

This set can't be leashed to the cat because your friend answered NO to the question "Is the cat in the set to which it is leashed?"

This set can't be leashed to the star because your friend answered NO to the question "Is the star in the set to which it is leashed?"

And this set cannot be leashed to the frog because your friend answered YES to the question "Is the frog in the set to which is leashed?"

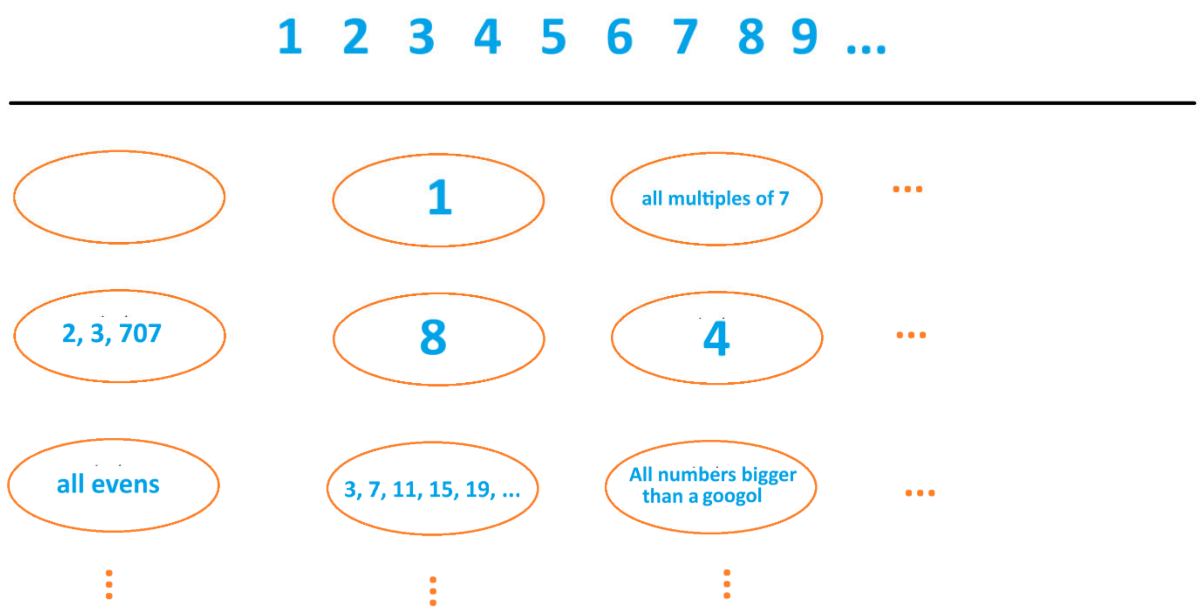
Practice 123.3 Think through this example. What is the "NO set" that must be unleashed? Explain for yourself again why this set cannot be leashed to any of the four objects at the top of the page.



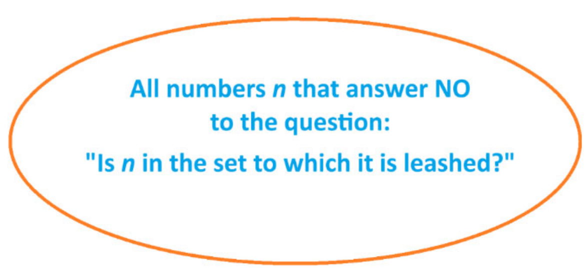
This party trick shows that the set of sets you can make from a collection of objects is sure to be bigger than the original set of objects. Whatever leashing pattern you try to draw, there will be objects that remain unleashed.

Now for the mind-bendy part. This will remain true even if the set of objects at the top of page is an infinite set!

For example, try imagining playing this party game with all the counting numbers written at the top of the page. Below them are all the sets you can make using the counting numbers.



There is no leashing pattern between the counting numbers at the top of the page and the set of all sets of counting numbers. The “NO set,” at the very least, must be unleashed.



Practice 123.4 What can’t this set be leashed to the number 7? Why can’t this set be leashed to the number 4,509,037?

We have:

The set of all sets of counting numbers is a bigger infinite set than just the counting numbers.

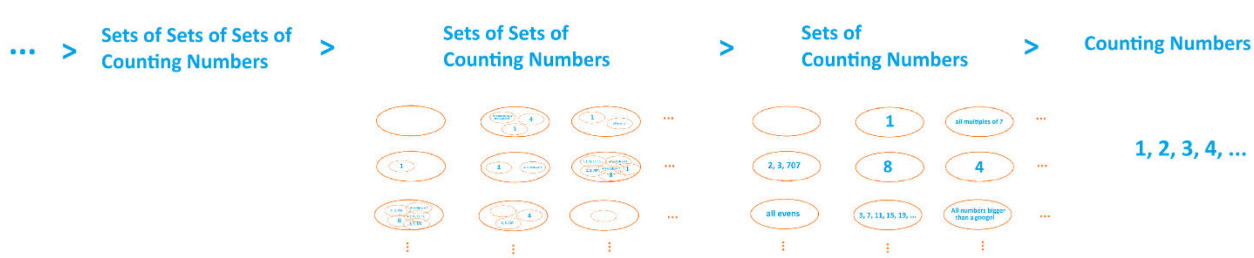
But let's go crazy!

Let's now play the party trick with the set of all sets of the counting numbers at the top of the page and all the sets of sets of counting numbers at the bottom.

Again, there can be no leashing patten that connects each set of counting numbers with each and every set of sets of counting numbers.

The set of all sets of sets of counting numbers is a bigger infinite set than the set of sets of counting numbers.

And we can keep going building more and more infinite sets, each more infinite than the previous one!



This is totally mind-boggling and nigh-on impossible to wrap one's mind around. But we have established that there many different sizes of infinite sets. "There are infinitely many infinities!"

Of course, one now wonders if there could be even more examples of different infinite sets, ones with sizes "between" the ladder of infinite sets we've just created.

The answer to that is that no one definitively knows!

Mathematicians have proven that the underpinnings of mathematics are not in danger if you choose to believe that there are more types of infinity to be found, nor are they in danger if you choose to believe there aren't. (Mathematicians phrase this as "The Continuum Hypothesis is independent of the axioms of set theory.") But that doesn't answer the questions: matters could still go either way.

Maybe you can think of an infinite set that is "more infinite" as the counting numbers, but "less infinite" than the set of all sets of counting numbers and garner world fame?

Let's stop here now that our minds—well, mine at least—are reeling.

Thank you for joining me on this excursion.

MUSINGS

Musing 123.5 Might you want to learn about the **Continuum Hypothesis** on the internet?