ARITHMETIC, ALGEBRA, and
Radical Comprehension of Math

A Refreshingly Joyous, Human, and Accessible approach to
Arithmetic and Algebra
for all those who may have experienced it otherwise

CHAPTERS 1, 2, 3, and 4

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PART 1

Arithmetic: The Gateway to All

Photo: Erick Mathew, Tanzania

Algebra is the practice of avoiding the tedium of doing arithmetic problems one instance at a time, to take a step back and see a general structure to what makes arithmetic work the way it does, and so open one's mind to more than the one view of what arithmetic, and mathematics, can be.

Part 1

Part 2

0: It All Starts with a Dot

I remember as a young lad, at around age 12 or so, wondering whether it might be possible to communicate with aliens (assuming they exist). What would it take to do so?

I was quite rational in my thinking and applied a systematic line of reasoning to consider the matter.

Communication, I first noted, requires each being involved to have a sense of “self” and a sense of “other.” If the entity with which I am trying to talk has no sense of anything but itself, all will be pointless. The fact is that I can only hope to communicate with a being who is aware of communication.

Such a being, possessing a sense of self and of other, I then reasoned is likely to have a sense of “nothing” and “something.” Maybe the something could be a physical object, or a waft of smell, or a pulse of sound or light. The “nothing” would be the absence of the something.

Figure 0: Nothing

Figure 1: Something

I next reasoned that—and this might have been a bit of a leap in thinking—if a being was aware of one thing, it might be aware of more than one thing; specifically two things, and three things, and so on. The being and I might share awareness of the *counting numbers*: 1, 2, 3, 4, … .

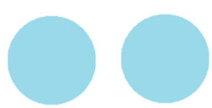


Figure 2: Two things

Figure 3: Three things

I decided that my best bet in communicating with an alien is to assume we each know how to count things and thus to communicate via the counting numbers. So, I decided we should send “blips” of sound or light out into space, in patterns of different counts to somehow say, “Hello! I am here.”

But what counts of these blips should we send? What pattern of counting numbers would be interpreted as deliberate and “intelligent” and undeniably as coming from someone trying to communicate?

I decided we should send blips that match the first few prime numbers—**2** blips, pause, **3** blips, pause, **5** blips, pause, **7** blips, pause, **11** blips, pause, **13** blips, pause, **17** blips, pause, **19** blips, pause, **23** blips, pause, **29** blips, pause, **31** blips pause, **37** blips, pause, **41** blips, pause, **43** blips, pause, **47** blips, pause, **53** blips, pause, **59** blips, pause, **61** blips, pause, **67** blips. (Maybe that's enough?)

Such a sequence of counts would be undeniably intelligent as I knew that these numbers are special and fundamental and not at all random. They take some mathematical sophistication to recognize. Sending a list of prime numbers, I thus thought, is likely to be interpreted as deliberate. (We'll learn about the prime numbers in this book.)

And this idea of mine, I later realized, is not a bad one: several science fiction writers have come up with the same proposal.

But looking back, it was clear as a young lad I was a enamored with mathematics. I could sense power to it. I could sense universality to it. I could sense that mathematics transcended my humanness—and I found that thrilling, and inspiring, and somehow comforting.

My young mind could start to see a marvelous journey to be had by simply contemplating "nothing" and "something," and then multiple copies of that something. (At least a form of mathematics that my human brain could comprehend.)

My first "something" was a dot.

It all starts with a dot.

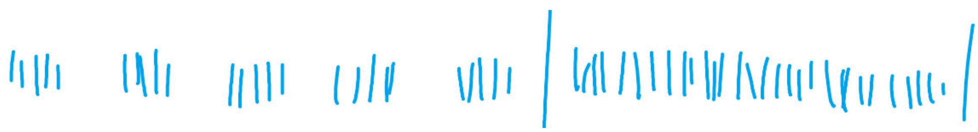
Chapter 1

The Counting Numbers and the Basis of Arithmetic

1. Humankind's First Mathematics

The first mathematical activity humankind ever conducted might have simply been counting. There is historical evidence for this claim.

In 1937, archeologists in Czechoslovakia at Dolní Věstonice in Morovia uncovered the radius bone of a wolf on which 57 notches were carved. The first 25 notches appear in groups of five and are then followed by a long notch, 30 ungrouped short notches, and a final long notch. (Or if you turn the bone the other way, maybe it's 30 ungrouped short notches followed by five groups of 5 short notches with long notches in-between?)



This bone dates back some 30,000 years and strongly suggests that pre-historic humans were counting. (Counting what? Deer? Mammoths? Full moons?)

Notched bones have also been found at Border Cave in South Africa too, dating back to possibly 44,000 BCE, suggesting that counting has been happening for a very long time indeed.

And counting certainly seems innate to our human thinking. Small children delight in the act of counting. For instance, they will count stairs going up. They will count the same stairs going down. (And if they get different answers for these two counts they might or might not think something of it.)

Be it for humankind, or for young humans, or for a young lad wondering about communicating with aliens, it does indeed seem that counting is a natural (human) start to mathematics. And we'll officially start this journey too with the positive whole numbers that count things.

1 2 3 4 5 6 7 8 9 10 11 12 ...

People sometimes call the set of counting numbers the set of **natural numbers**, probably because they are so natural to us. A funny "blackboard-script" capital N is used to denote them as a collection: $\mathbb{N}$.

No human or group of humans or a multitude of humans can write down *all* the counting numbers. The list of them simply does not stop. If you think you've written the biggest counting number there is, you are mistaken: simply add 7 or 17 or 7003 to it and you'll have a bigger counting number still.

We humans always have to “cheat” when listing the counting numbers. We do that by writing “dot dot dot” after a start to the list to mean “and keep on going forever.”

But on the other end of matters, namely, at the start of the list, there is a tricky question to be asked.

Is 1 the first counting number? Or is there another number just before it?

Humans have mused over the meaning of zero, 0 for millennia, wondering if it deserves to be called a counting number not. Does zero count something?

Think about it. If I say that there are zero sparkly purple giraffes in the room I am sitting in right now as I type this very sentence (and it is true, there are no sparkly purple giraffes here with me), do you think it is because I actually *counted* zero sparkly purple giraffes, or did I not count and just *observe* a lack of sparkly purple giraffes? That is, does one **count** zero or does one just **observe** zero?

To do this day, people—mathematicians even—choose not to give a definitive answer to this question and live with both options. Sometimes people say that zero should be included in the list of counting numbers and sometimes they choose to exclude it. There really is no standard convention on this. In fact, the same mathematician might one time write a paper in which she does regard zero as a counting number for the purposes of that work, and, later, write a separate paper, in which she states she won’t. It’s just a matter of what is appropriate for the problem being explored. Some problems naturally allow for zero. Others don’t.

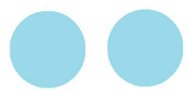
Whether or not you choose to include zero in the list of counting numbers really does not matter. You just have to state to your readers in any work you do whether or not you are including zero in your list of counting numbers.

0 1 2 3 4 5 6 7 8 9 10 11 12 …

In this book, we'll follow what their name suggests and use the counting numbers for counting things. Specifically, we'll count dots. And to be clear, we'll consider zero to be a number too.

For instance, "5" shall represent five dots

and "2" shall represent two dots



and "12,876,290,980,771,006,629,932,183" shall represent a dreadfully large count of dots.

We'll end this section with a picture of zero dots.

Here it is:

MUSINGS

Musing 1.1 In 1938, Milton Sirotta, nine-year old nephew of American mathematician Edward Kasner, coined the term *googol* for the number given by 1 followed by one hundred zeros.

$$1\ 000000000000000000000000000000000000000000$$
$$0000000000000000000000000000000000000000000000000000000000$$

At the same time, he coined the name *googolplex* for the number given by 1 followed by a googol of zeros.

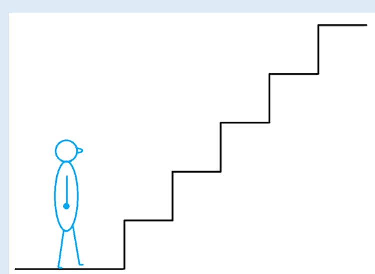
Can you describe a number bigger than a googolplex? Can you describe another number bigger still?

Musing 1.2 Many words in the English language are obvious associated with numbers. For example, a *monologue* is a speech by a single person and to speak in *unison* is to speak in one voice. (The prefix mono comes from the Greek *mónos* meaning "alone" or "single", and the prefix uni from the Latin *unis* for "one.) To have a *dilemma* is to be caught between two choices and a *bicycle* is a vehicle with two wheels. (*Bi* comes from the Latin for "two" and *di* the Greek for "two.")

a) Can you think of two English words using prefixes associated with each of the numerals three through ten?
b) September is the ninth month of the year, yet the prefix *sept* comes from the Latin for "seven." October is the tenth month of the year, yet the prefix *oct* comes from the Latin for "eight." November and December are the eleventh and twelfth months of the year, despite *nov* and *dec* being derived from the Latin words for "nine" and "ten."

   Can you find out why these months of the year are off their counts by two?

Musing 1.3 Image standing at the base of a large set of stairs.



There is 1 way to take one step up and one step down. We'll denote this as UD.

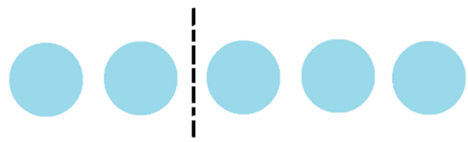
There are 2 ways to take two steps up and two steps down in some order, namely, $UUDD$ and $UDUD$. (Why isn't $UDDU$ an option?)

a) There are 5 ways to take three steps up and three steps down in some order. List them all.
b) There are 14 ways to take four steps up and four steps down in some order. Can you find them all?
c) Care to list all 42 ways to take five steps up and five steps down in some order? (The answer can be no!)

2. Addition

What does "$2+3$" mean in terms of dots?

It seems natural to regard this as "two dots placed together with three dots." And if I draw a picture of such a thing, I see five dots.



I followed a "please read left-to-right" bias in this picture, placing all my dots in a row with two dots on the left followed by three on the right. But there is no need to read this picture left to right. If I look right to left, instead, I see $3+2$. (And this is, of course, still is the same five dots.)

This is philosophically deep!

I realize now, for instance, that $4+7$ must give the same answer as $7+4$, without ever having to say, or even think, "eleven." Simply imagine placing four dots and seven dots next to each other in a row and looking at that picture left-to-right and then right-to-left. The count of dots in the picture does not change even if your perspective looking at it does.

I can even "see" why computing $176203982761 + 87799998699$ and $87799998699 + 176203982761$ must give the same answer without doing a lick of actual arithmetic. (I don't want to do such arithmetic!)

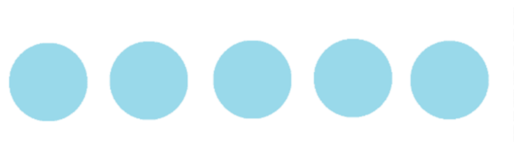
It seems we've stumbled upon a fundamental truth about the counting numbers.

$a+b$ and $b+a$ are sure to have the same value,
no matter which counting numbers a and b represent.

What about zero? Does this "truth" hold for number zero as well?

Let's explore.

Here's a picture of $5 + 0$ (or is it of $0 + 5$?)



Five dots followed by no dots is just, well, five dots.

$$5 + 0 = 5$$

Also, no dots followed by five dots is five dots.

$$0 + 5 = 5$$

It seems that not only does our first fundamental truth seem to hold if one (or both?) of the counting numbers is zero, but we've also stumbled upon a second truth.

$a + 0 = a$ and $0 + a = a$ no matter which counting number a represents.

It's fun to imagine a picture of $0 + 0$. (Can you?)

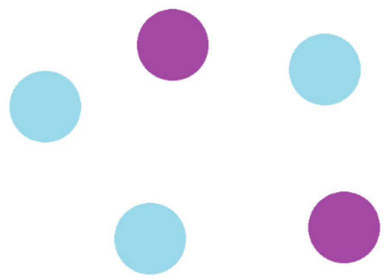
And one of $0 + 0 + 0$.

And one of

$$0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0$$
$$+0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0$$
$$+0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0$$
$$+0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0.$$

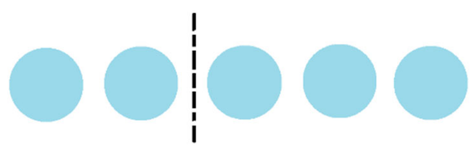
An Additional Thought (Ha!)

There is another way to interpret $2 + 3$ in terms of dots. Simply look at a set of dots with different characteristics among them: say two are purple and three are blue.



Then the answer to "$2 + 3$" is the result of recounting the dots choosing to ignore differences.

We did precisely this with our first picture of $2 + 3$: we had two left dots and three right dots and then chose to ignore leftness and rightness.



We can say that 7 apples and 9 oranges make for 16 pieces of fruit by choosing to ignore fruit details.

Perhaps this is leading us to a curious, and somewhat philosophical, idea of what addition actually is?

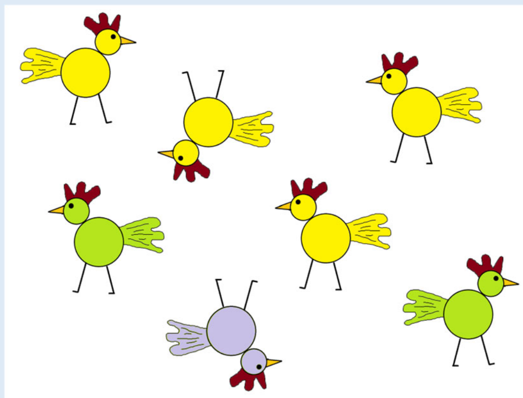
Addition is the result of recounting a set of objects after choosing a set of differences to ignore.

I don't think many people think of addition this way.

(I thank my colleague Joe Norman for opening my eyes to this!)

MUSINGS

Musing 2.1 Here is a picture of some chickens.



a) Can you interpret this as a picture for computing 2 + 5? In what way?
b) Can you interpret this as a picture for computing 1 + 2 + 4? In what way?
c) Can you interpret this as a picture for computing 4 + 3? In what way?

Musing 2.2 When we write a number such as 523 we are aware of the importance of "place." The 5 here denotes five hundreds (and not five tens nor five thousands), the 2 two tens, and the 3 three units. This spares us the need to invent new symbols beyond the ten with which we are familiar: 0,1,2,3,4,5,6,7,8, and 9. This positional notation also helps us calculate sums of numbers.

The Romans, on the other hand, used different symbols for units, tens, hundreds, thousands, as well as for five, fifty, and five hundred.

I = one
V = five
X = ten
L = fifty
C = one hundred
D = five hundred
M = one thousand

These numerals can still be seen on clock faces, monuments, and during the credits of television shows and movies.

In the system of Roman numerals, the number 523, for instance, was written DXXIII. Positional notation did not come into play (except for the convention that symbols were listed in decreasing order of size).

During the Medieval period in Europe, it became popular to use a subtractive principle to denote numbers, such as 4, 9, and 90.

4 = IV
9 = IX
90 = XC

Medieval scholars set the following rule for this subtractive principle.

> *One can only subtract a single I from a single V or a single X;*
> *or subtract a single X from a single L or a single C;*
> *or subtract a single C from a single D or a single M.*

So, writing VL for 45 would not be allowed. Nor would writing IIX for 8 or XCC for 190, for instance. (Well, perhaps this third one can be interpreted as "XC + C." But to avoid the seeming transgression of subtracting X from CC, scholars would write CXC for 190.)

With this subtraction principle the position of a symbol was made important, but the meaning of "place" was still different from what we mean it to be today.

a) Three movies were made in the years 1978, 1983, and 1999. How do their dates appear in the movie credits using Roman numerals?

b) To appreciate our place-value system for writing numbers, try computing the following sum without mentally converting the numbers you see to our decimal system. Can you do it?

```
    XVI
+    IX
+  XXIV
+    XL
-------
=
```

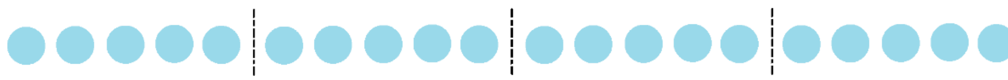
c) Look at a clock face with Roman numerals. What do you notice about the number four? Is the same true for the number nine?

d) How did Romans represent extremely large numbers, numbers in the hundreds of thousands and the millions? Find out.

3. Repeated Additions

Last section we had fun adding together multiple copies of zero. The result is always zero. Let's now add together multiple copies of a non-zero counting number, say 5.

Here are four copies of 5placed together.



In the world of counting numbers, people use the word **multiplication** for repeated addition.

The standard shorthand for $5+5+5+5$ is 4×5, using the multiplication symbol $\times$ to denote the repeated addition.

The term 4×5 is read as "four groups of five" in the U.S.. It is read as "four lots of five" in Australia, or perhaps as "four copies of five." But watch out, folk in Europe interpret "4×5" as meaning something different: they say it is "the number 4, five times," and so read it as $4+4+4+4+4$.

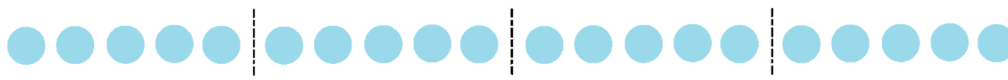
Luckily $5+5+5+5$ and $4+4+4+4+4$ have the same value, so folk on all continents are thinking "20" in the end. (But was that luck?)

We'll follow the U.S. language and thinking in these notes. In which case

$5+5+5+5$ is four groups of five: 4×5,

$4+4+4+4+4$ is five groups of four: 5×4.

And at first glance, these two quantities are philosophically different.



$5+5+5+5$

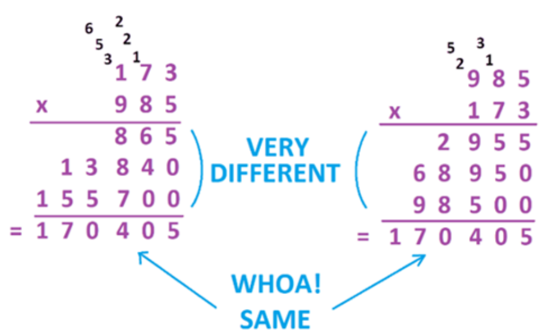
$4+4+4+4+4$

Is it remarkable coincidence that both pictures have twenty dots?

Pause. This really is remarkable!

For example, is there any reason to believe that a picture of 173 groups of 985 dots should contain the same count of dots as a picture of 985 groups of 173 dots? (Do you have the patience to draw out each of these pictures to check? Please say you don't!)

If you compute the products 173×985 and 985×173 via the long multiplication algorithm taught in school, it is not at all obvious that one is going to obtain the same final number in each: the computations look so very different in their middles! It is quite a shock to see the common answer 170,405 appear.



What magic is this?

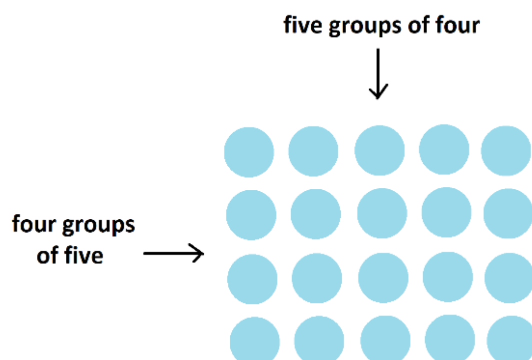
Take this in!

Work out 87×43 and then 43×87 using the school algorithm. The final answer of 3741 will appear for each, but the middles of the computations are quite different.

Is it obvious to you that the school algorithm is certain to give the same answer if you switch around the two numbers you multiply?

This is the allure and delight of mathematics. As one thinks about and plays with mathematics, little mysteries and surprising "coincidences" start to arise, and one begins to suspect there is something deep and hidden lurking behind the scenes. And then, out-of-the-blue, a flash of brilliant insight suddenly makes everything stunningly clear. All hidden machinations are revealed.

The flash of insight needed to reveal the workings of repeated addition is this: instead of drawing groups of a repeated quantity in a single row, stack those quantities instead to make a rectangle of dots.



Look at the figure from the left, focusing on the rows and you see four copies of five dots: 4×5.

Now look from above to focus on the columns to see five copies of four dots: 5×4.

It's the same collection of dots just viewed from two different perspectives. It simply must be the case, then, that 4×5 and 5×4 represent the same count of dots. (No mention of the number 20 needed!)

In the same way, a 173-by-985 rectangle of dots looked at two different ways would reveal that 173×985 and 985×173 simply must give the same count of dots. This means that the school multiplication algorithm simply can't change its answer if we switch the order of the numbers we multiply.

We seem to have stumbled upon another fundamental truth about the counting numbers.

> $a \times b$ and $b \times a$ are sure to have the same value,
> no matter which counting numbers a and b represent.

The Role of Zero with respect to Multiplication

What's "five groups of no dots"?

Well … that's no dots and no dots and no dots and no dots and another set of no dots! We have 5×0 is $0 + 0 + 0 + 0 + 0$and this is 0. Five groups of nothing is nothing.

We can argue this way that 17×0 and 62×0 and $70986798766519273 \times 0$ should all be zero as well.

What about 0×5? Is that zero as well? If I have no groups of five dots, does that mean I have no dots at all? (Check out **Musing 3.2**. Terell is worried about this.)

We just said at the top of the page that we can switch the order we multiply numbers and make no change to the answer. If we think that should apply to the number 0 as well, then we'd have to say that 0×5 has the same value as 5×0. We said that 5×0 equals 0. This means that 0×5 should be 0 as well.

We have

> $a \times 0$ and $0 \times a$ each have the value 0,
> no matter which counting number a represents

Did invoking the idea that "we can switch the order we multiply numbers" feel okay to you? Or maybe it already felt obvious to you that "0 groups of 5" has to be zero?

Zero is a tricky number. It can cause all sorts of brain-hurty troubles. Consider this question:

What is 0×0?

Here's an argument suggesting that 0×0 should equal zero.

We just set the rule that $a \times 0$ equals zero for all numbers a.
So, it works for 0 too.
We have $0 \times 0 = 0$.

Here's an argument suggesting that 0×0 can't be zero.

> *In the "real world,"* 0×0 *reads as "zero groups of nothing."*
> *How much is that?*
>
> *Well, if I have no nothing, it must be because I have something.*
> 0×0 *thus cannot be nothing.*
> *It better be something!*

The number 0 really has befuddled humankind for millennia—not just on the philosophical matter of whether or not it deserves to be considered a counting number in its own right, but also with regard to understanding on what doing arithmetic with zero actually means.

It really can hurt one's brain!

Where mathematicians have landed on this.

Seventh-century Indian mathematician and astronomer Brahmagupta was the first to lay out rules for working with the number zero and led the world to show that the mathematics is logically consistent if we do indeed assume that our four properties of numbers are valid, even if we include the number zero.

If a and b are counting numbers, including possibly being zero, then we have that

$a + b$ and $b + a$ are sure to have the same value,

$a + 0$ and $0 + a$ both have the value a,

$a \times b$ and $b \times a$ are sure to have the same value,

$a \times 0$ and $0 \times a$ both have value 0.

The logical consequence of the fourth property is that 0×0 must be 0.

Mathematicians have decided to follow the mathematics. They don't feel that all quantities and actions must always have real-world interpretations. And this is surprising to many people who experience only school mathematics.

Mathematics is exceptionally good for describing and making sense of real-world scenarios, but real-world scenarios are not good at "explaining" all mathematics.

Mathematics is bigger than the real world!

One more thing. (That's a little joke too as you will see.)

What's one group of five? Clearly five!

$$1 \times 5 = 5$$

What do five groups of one make? Clearly five!

$$5 \times 1 = 5$$

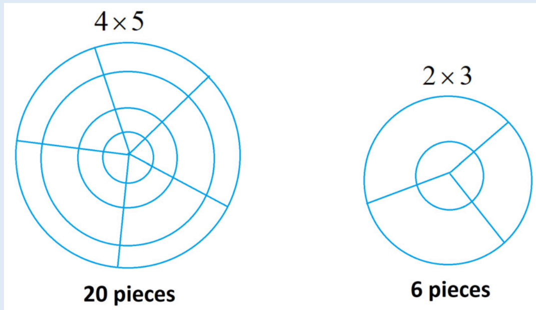
In the same way we can argue that $17 \times 1 = 17$ and $1 \times 299 = 299$ and $30012 \times 1 = 30012$.

It seems we have

$1 \times a = a$ and $a \times 1 = a$ no matter which non-zero counting number a represents (including zero).

MUSINGS

Musing 3.1 Here is an unusual (and inefficient!) way to compute the product of two **positive** counting numbers. To compute 4×5, say, first draw 4 concentric circles and then draw 5 radii for those circles. The number of pieces you get is $4 \times 5 = 20$.



In the same way, 2 concentric circles and 3 radii give $2 \times 3 = 6$ pieces.

a) Draw a picture for 3×4 and verify one does indeed see 12 pieces.

b) Can you see in your mind's eye that a picture for 20×1 must have 20 pieces?

c) Can you see in your mind's eye that a picture for 1×20 must also have 20 pieces?

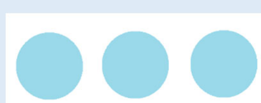
d) We drew the picture for 4×5 and saw 20 pieces. Draw the picture for 5×4 and verify that it gives 20 pieces too.

It is not obvious to me why computing $a \times b$ and $b \times a$ this weird way should give the same counts of pieces for all possible non-zero counting numbers a and b. I am curious, is it obvious to you?

For that matter, is it all obvious to you why drawing circles and radii this way and counting pieces should precisely match the ordinary multiplication of the two numbers? This is weird!

Any thoughts?

Musing 3.2 Terell is a bit worried about saying that 0×5 should be 0. He says, "Draw a picture of three dots and ask how many groups of five you see?" He reasons that you could legitimately answer that you see no groups of five in a picture of three. So, maybe, 0×5 is 3?

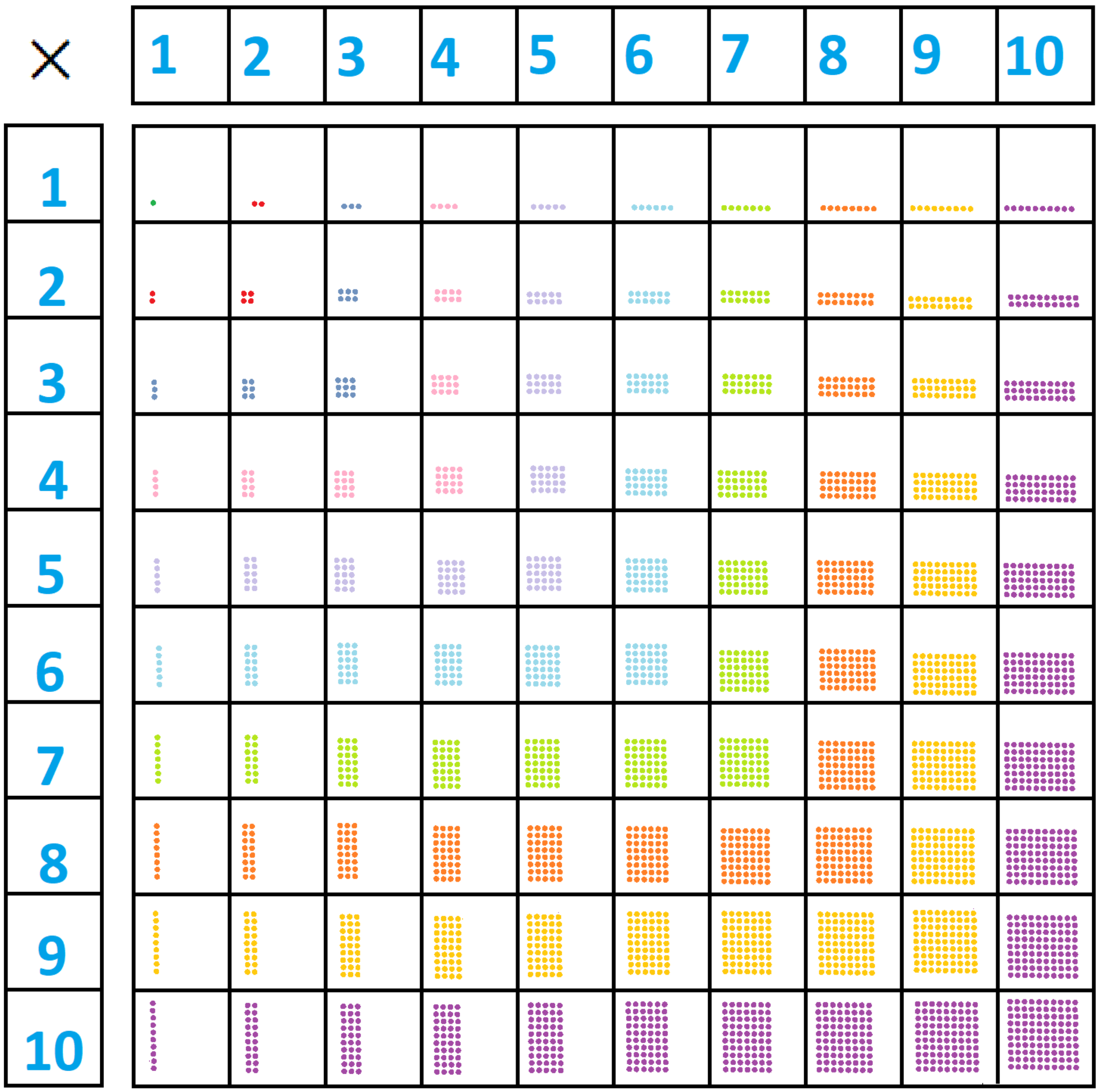


Terell has just made my brain hurt. What do you think? Or is your brain hurting too?

4. The Repeated Addition Table

Most people call this a “multiplication table,” except each entry here is a rectangular array of dots representing the multiplication fact appropriate for its cell. For example, in the third row, seventh column of the table we have a 3-by-7 rectangle of 21 dots.

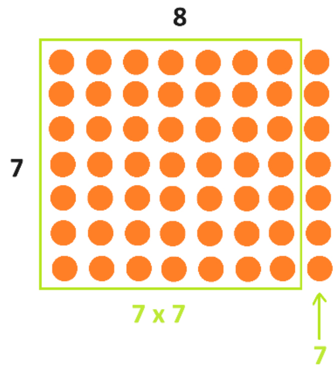
(It has become a societal convention to always mention rows first and columns second.)



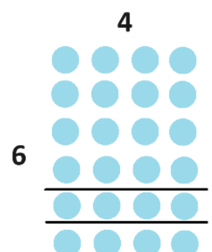
Having this imagery for multiplication answers in your mind can help you figure out new multiplication facts from known ones.

For example, if you happen to remember that 7×7 is 49, then you can figure out the answer to 7×8 somewhat readily: imagine the rectangle picture of 7×8 and identify a picture of 7×7 within it.

We then see that $7 \times 8 = 49 + 7 = 56$.

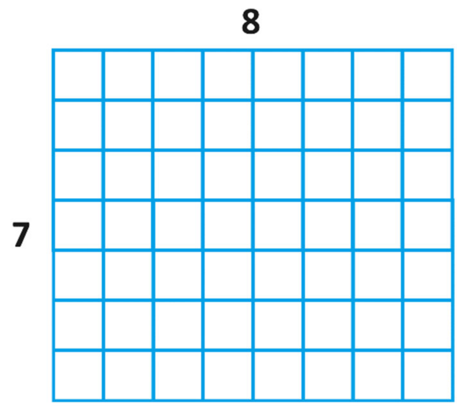


In the same way, 6×4 is $16 + 4 + 4 = 24$.



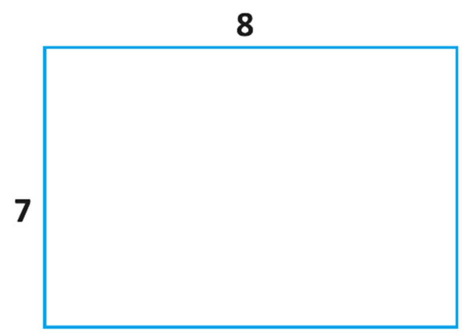
Actually, drawing dots gets tiresome pretty quickly. It's easier to draw "unit squares" (squares of area one).

For example, here's a picture of 7×8 again, but with seven rows of eight unit squares per row.
We have $7 \times 8 = 56$ squares of area one, and so this rectangle has area $7 \times 8 = 56$ square units.

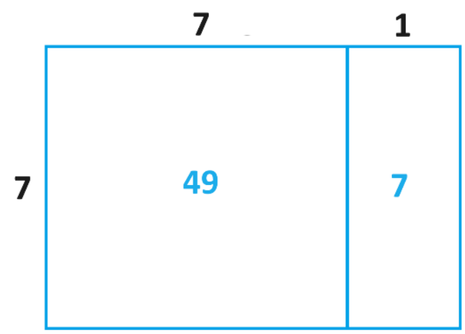


Actually … This picture is tedious to draw too!

We can just draw a rectangle and label one side as length 7 and the other side as length 8. The area of the rectangle is $7 \times 8 = 56$ and we can just imagine 56 square units.



Now we can readily see how to chop up the rectangle into two pieces and to deduce that it's area if given by 49 (which is 7×7) plus 7 (which is 7×1).



$$7 \times 8 = \quad 7 \times 7 \quad + \quad 1 \times 7 \quad = \quad 49 + 7 \quad = 56$$

In 1998, then Labour Schools Minister for the U.K. Stephen Byers was asked on the fly during an interview, "What is 7×8?" In a flustered moment he responded "54" and become a bit of a laughingstock for the nation.

It would have been lovely if he had the presence of mind to answer along these lines:

> "Ooh! Seven times eight is the hard one. Let me think.
> Well, I know that seven times seven is 49. So, adding another 7 to this gives 7×8.
>
> The answer is 56."

It would have been just brilliant for a nation to see a demonstration of beautiful mathematical thinking.

> **Practice 4.1:** Draw a picture to show that the value of 32×16 equals the sum of these four multiplication pieces: 30×10 and 2×10 and 30×6 and 2×6.

MUSINGS

Musing 4.2 The numbers from computing $1 \times 1 = \mathbf{1}$, $2\times 2 = \mathbf{4}$, $3\times 3 = \mathbf{9}$, $4\times 4 = \mathbf{16}$, and so on are called the **square numbers**. Can you see why? (Look at the multiplication table.)

Musing 4.3 The entry for 3×7 in the multiplication table is a rectangle of dots with three rows and seven columns. Describe the entry for 7×3.
In general, how would you describe the relationship between the entries for $a \times b$ and $b \times a$?

Musing 4.4 How many dots are there along the first row of the table? Along the second row? Along the third, fourth, and tenth rows?

How many dots are there altogether in the table?

Musing 4.5 What's common about all the entries of the same color?
How many dots are there of each color?

MECHANICS PRACTICE

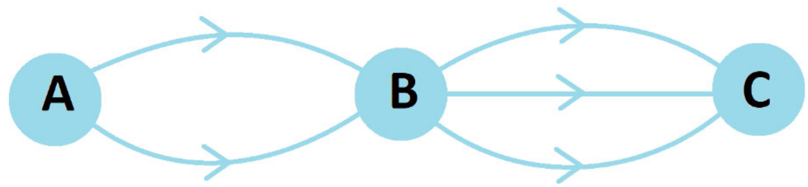
Practice 4.6 Match each quantity on the left with its matching quantity on the right. (Try to imagine rectangles here.)

8 x 9	9 + 3 + 3
7 x 6	25 +5
6 x 8	64 + 8
3 x 5	36 + 6
5 x 6	36 + 6 + 6

5. Repeated Addition in the "Real World"

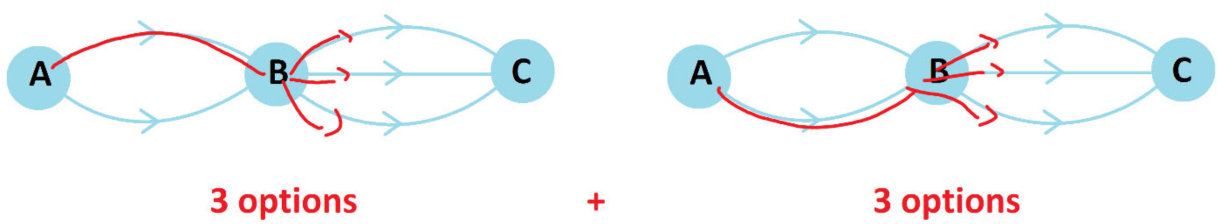
Suppose there are essentially only two different routes for driving from city A to city B, and essentially three different routes for driving from city B to city C.

How many different options do I have for driving from city A to city C?



> **Question:** Some people look at the picture without thinking and say that the answer to this question is 5. Do you see why the number 5 might first come to mind?

If I take the top route from A to B, then I am presented with 3 options on how to proceed next. And if I take the bottom route from A to B, I am again presented with 3 options for how to continue. I thus have $3 + 3 = 6$ possible ways to travel from A to C.



If, instead, there were 5 choices of route from city A to city B, and still 3 choices of route from city B to city C, then each option I choose for the first leg of my journey offers 3 choices on how to follow that choice. I'd thus have $3 + 3 + 3 + 3 + 3$, that is, 5 groups of 3 options. This is repeated addition, and so, in this revised scenario, there 5×3 routes in total from city A to city C.

> **Question:** Draw a picture of this scenario and convince yourself that there are indeed $3 + 3 + 3 + 3 + 3 = 15$ routes from city A to city C.

Now, let me tell you something about my wardrobe.

I own just 5 shirts, all different, and just 4 different pairs of trousers.
How many different shirt-trouser outfits could you see me in?

Again, this is a repeated addition challenge. For each choice of shirt, I have 4 choices of trousers to go with it. Thus, I have a total of $4 + 4 + 4 + 4 + 4 = 5 \times 4$ choices for my outfit.

Of course, this count would change if I gave you some further information that restricts my options. (For example, I will never wear my chartreuse trousers with my acid-green shirt.) But without knowledge of such restrictions, counting options is a matter of thinking through repeated addition.

We have

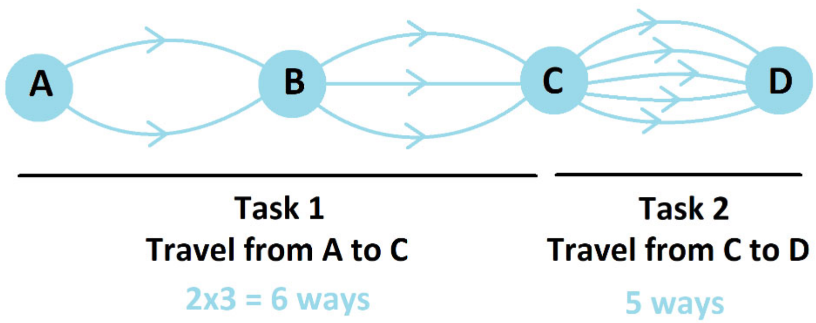
> **The Multiplication Principle**
> If there are a ways to complete a first task and b ways to complete a second task, and assuming that a choice made for one task in no way influences the choice made for the other, then there are
>
> $$a \times b$$
>
> ways to complete both tasks together.

For example, if there are 8 different movies I could watch tonight and 3 different snacks I could eat while watching them, then there are $8 \times 3 = 24$ different movie/snack combinations for me to consider.

If there 4 answers to a select from for a first question in an exam and 4 to select from for a second question, then there are $4 \times 4 = 16$ different ways I could answer those two questions. (Hopefully, I choose the answers that are correct for both questions.)

In the travel example above there are 6 ways to move from city A to city C.

If there are also 5 routes from city C to a new city D, then we can travel from A to B to C to D in $6 \times 5 = 30$ different ways. (This is really the product $2 \times 3 \times 5$.)



Question: On a menu, there are 10 choices for a starter, 12 choices for a main meal, and 5 choices for dessert. How would you explain to a friend why that provides $10 \times 12 \times 5 = 600$ options for a three-course meal?

We're seeing how to use the multiplication principle multiple times to handle counting multiple tasks!

The Full Multiplication Principle

If there are a ways to complete a first task and b ways to complete a second task, up to z ways to complete a final task, and assuming that a choice made for any one task in no way influences the choices made for any other task, then there are

$$a \times b \times \cdots \times z$$

ways to complete all the tasks together.

MUSINGS

Musing 5.1 I own 5 different shirts and 4 different trousers. If there are no restrictions on which shirt I might wear with each pair of trousers, then you could see me in 20 different shirt/trouser outfits.

a) I also own 3 different pairs of shoes. How many different shirt/trouser/shoes outfits could I wear?
b) I own 1 hat, which I might or might not wear. With the hat option, how many different shirt/trouser/shoes/hat-no-hat outfits could you see me in?

(Assume in these questions that there are no restrictions on my choices in putting together an outfit.)

Musing 5.2

a) If I were to roll a die and flip a coin, how many different outcomes are there for me to possibly see?

b) If I were to roll a red die and a blue die, how many different outcomes are there for me to possibly see?

Here's an annoying question.

c) If I were to roll two identical white dice, how many different outcomes are there for me to possibly see?

When Amit considered this third question, he said that this is just the same problem as part b) and thus has the same answer. "After all, why should the color of each die matter?" he responded.

Beatrice, on the other hand, wasn't so sure. She was worried that because one can no longer tell the dice apart, the outcomes you see might be interpreted differently and the answer to the question thus might change.

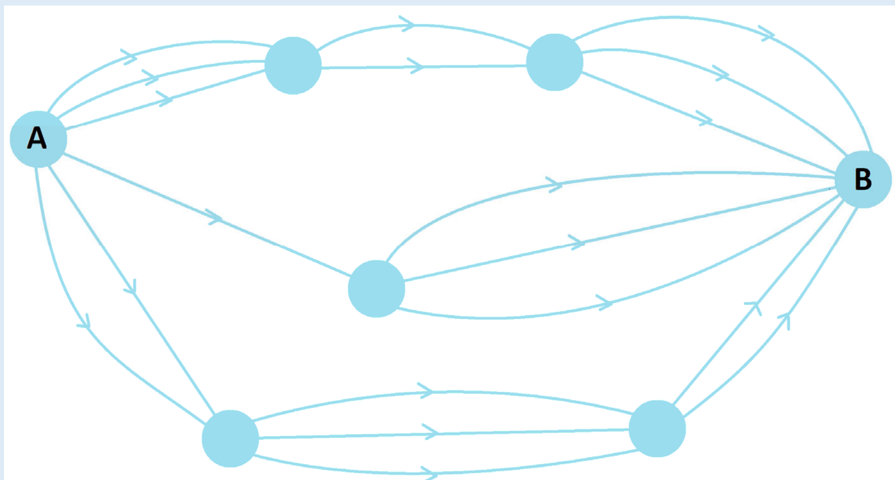
Chi thought about this and asked: "What if one white die is rolled first and the other is rolled second? Then we could tell the rolls apart and maybe the answer is the same as for part b)?"

Debjyoti said that his brain hurts and he doesn't know what to think.

Here, finally, is my question to you:

Is your brain now hurting too?

Musing 5.3 Can you see that there are 33 different ways to make your way from city A to city B in this complicated road map?



MECHANICS PRACTICE

Practice 5.4 Make up a "choice" problem whose answer is $6 \times 2 \times 3 \times 2$.

6. Ordering Additions

Try this!

ACTIVITY

A NOT-EXCITING GAME OF SOLITAIRE

Write the numbers 1, 2, 3, 4, 5, and 6 on a page.

A "move" in this game of solitaire consists of erasing two numbers and replacing them with their sum.

For example, if you cross out 3 and 5 you will then write 8 on the page and have the numbers 1, 2, 4, 6, and 8 to work with. If you next cross out 1 and 8, you will replace them with 9 and be left with 2, 4, 6, and 9. And so on.

Each move has you erasing two numbers and writing one number, so the count of numbers on the page steadily decreases. The goal of this game is to end up with the single number **21** on the page.

Do try it. Can you win?

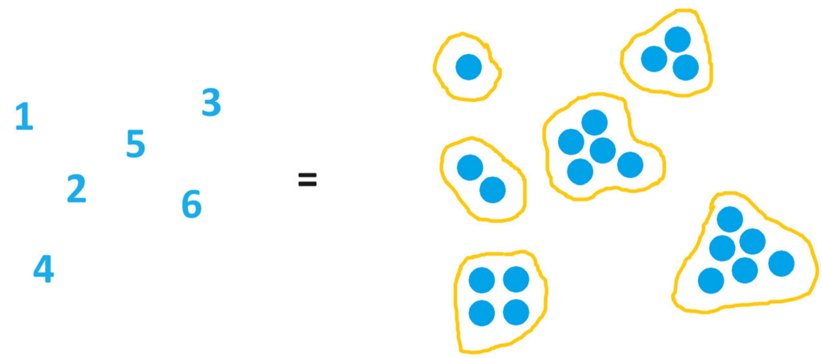
It really is worth playing the game.

I bet you can get 21 when you try it.
I bet you can get 21 again playing a second time but making different choices along the way.

Next challenge: Play the game yet again and try to **not** get the answer 21.

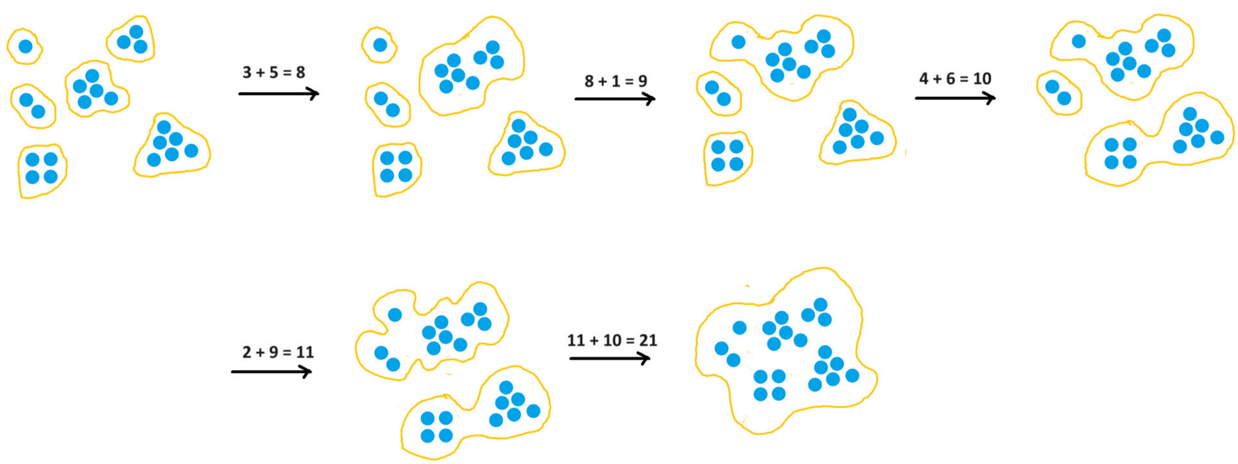
As you probably suspect, this game is rigged: you are sure to get a final single number of 21 each and every time you play no matter what choices you make along the way.

Replacing the numbers with groups of objects makes it clear why this is the case. I'll draw dots.



The act of erasing two numbers and replacing them by their sum simply combines dots in two separate groups to make one group.

Here's play of the game starting by combining 3 and 5, and then combining 1 and 8, and going from there.



Without looking at the details you can see that all we are doing is slowly combining the dots initially in separate groups into one big group. The count of dots never changes as we play this game. Thus, every game ends with the one same final state: all the dots at the start of the game combined into one big group.

As there were 21 dots to begin with, this game is destined to end with the number 21.

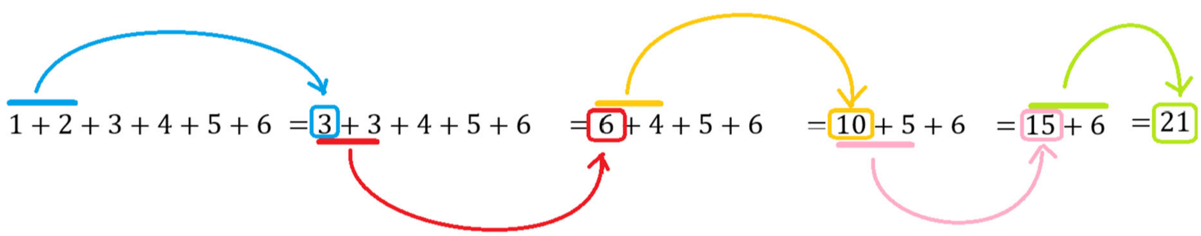
Practice 6.1: This game of solitaire is played starting with the numbers 4, 8, 8, 10, and 20 written on a page. What final single number will remain on the page at the end of the game?

Do you think you could adequately explain why this is so to another person?

In these notes so far, we've only ever added two things at a time. (Well, I did ask you to add together an absurd number of zeros in section 2). But we know from our school days we can add together any number of numbers we like. Let's consider

$$1 + 2 + 3 + 4 + 5 + 6$$

School teaches us to compute a string of additions like this by only ever adding two numbers at a time, starting at the left and working to the right.



But we've just seen from the solitaire game that we add pairs of numbers in any order we like, and we are sure to obtain the same final answer.

In $1 + 2 + 3 + 4 + 5 + 6$, we can add the 4 and 6 together first if we like and turn the sum into $1 + 2 + 3 + 5 + 10$. And now we could add the 2 and 5 and make it $1 + 7 + 3 + 10$, and so on. (Adding 3 and 7 next gets us to $1 + 10 + 10$ and the final answer of 21 is now apparent.)

We also learned in section 2 that when we add 4 and 6 we can think $4 + 6$ or can think $6 + 4$, it doesn't matter.

So, in all possible interpretations of "order does not matter," we have the following powerful realization.

> In any string of counting numbers added together, $a + b + c + d + e + \ldots + y + z$, it does not matter in which order one chooses to perform the additions. The same answer will always result.

This gives us the means to sometimes be clever when presented with a long sum.

For example, in the sum $31 + 7 + 84 + 3 + 9 + 16$, I can see:

> 31 and 9 together make 40,
> 7 and 3 make 10, and
> 84 and 16 make 100.

So, this sum can be computed as $40 + 10 + 100$, which is 150.
This is much better than working left to right!

Breaking a number down into a sum of two numbers can be helpful too. For example, seeing 47 as $3 + 44$ makes $97 + 47$ manageable.

$$97 + 47 = 97 + \underline{3 + 44} = 100 + 44 = 144$$

Practice 6.2: Can you see $16 + 92 + 4 + 39$ as the same as $20 + 100 + 31$?

MUSINGS

Musing 6.3 Can you see that each of these sums has value one hundred?

a) $3 + 50 + 47$
b) $46 + 18 + 4 + 15 + 2 + 15$
c) $17 + 17 + 17 + 17 + 17 + 2 + 2 + 2 + 2 + 2 + 1 + 1 + 1 + 1 + 1$
d) $48 + 3 + 49$

Musing 6.4

a) Can you see that

$$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 + 11 + 12 + 13 + 14 + 15 + 16 + 17 + 18 + 19 + 20$$

is ten copies of 21, and so has value 210?

b) What is the value of $1 + 3 + 5 + 7 + 9 + 11 + 13 + 15 + 17 + 19$?

c) There is a story of a famous eighteenth-century mathematician Carl Friedrich Gauss who, as a schoolboy, dumbfounded his teacher by finding the sum of all the numbers $1, 2, 3, 4$ and so on up to 100 in a matter of seconds. Can you see that the sum of these numbers equals 50 copies of 101? (This makes the sum equal to 5050.)

(Care to look up the story on the internet? You'll find several different versions of it leaving one to wonder how true the story might be in the first place!)

d) You decide to play the solitaire game described in this section with the numbers $1, 2, 3, 4$ up to 100 written on a chalk board. You keep erasing two numbers and replacing them with their sum until a single number remains on the board. What number will that be?

MECHANICS PRACTICE

Practice 6.5 Compute each of these sums in a way that feels efficient to you.

a) $19 + 18 + 16 + 2 + 4 + 1$
b) $46 + 294$
c) $998 + 875$
d) $199 + 199 + 199 + 199 + 199 + 7$
e) 19 + 18 + 18 + 19 + 17 + 9
f) $37 + 72 + 11$

OPTIONAL ADDENDUM

You may have heard the terms **commutative property** of addition and **associative property** of addition. These are not important to know, but if you are curious, they are both to do with the notion that "order does not matter" when computing addition problems.

The **commutative property** refers the fact that we can change the order of two numbers we are adding. We have that $a + b = b + a$ no matter which counting numbers a and b represent. We saw this in section 2.

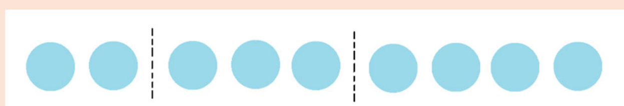
4 +6 = 6 +4

12 +10 = 10 +12

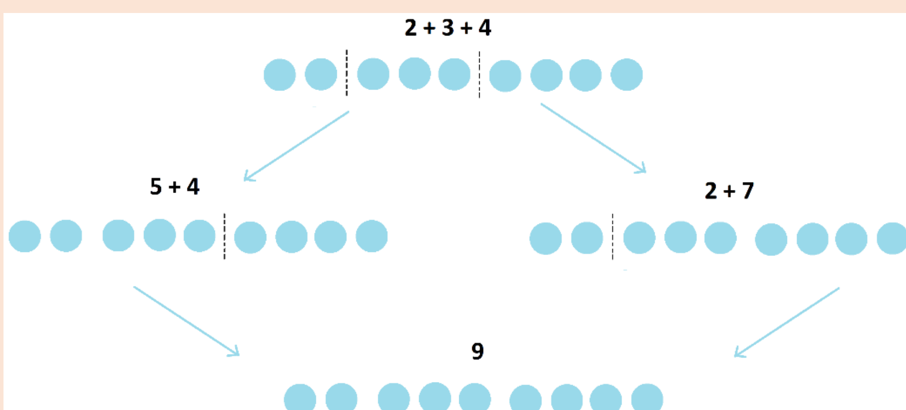
8 +92 = 92 +8

1656 + 207 = 207 +1656

The **associative property** refers to the fact that we can change the order we choose to conduct the two summations in a sum of three numbers. To illustrate what I mean, here's a picture of $2 + 3 + 4$.



We can compute the left addition first and the right addition second, if we like, or the right addition first and the left addition second. They both lead to the same result in the end (as we know they must from our solitaire game).

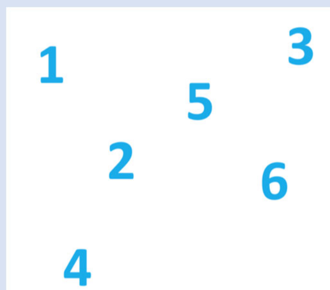


7. Ordering Multiplications

Here's another game of solitaire.

ACTIVITY

Again, write the numbers 1, 2, 3, 4, 5, and 6 on a page.



This time, instead of erasing two numbers and replacing them with their sum, replace them with their **product**, the two numbers multiplied together.

For example, if you cross out 3 and 5 you will then write 15 on the page and have the numbers 1, 2, 4, 6, and 15 to work with. If you next cross out 1 and 15, you will replace them with 15, their product (got that?) and be left with 2, 4, 6, and 15. And so on.

Each move has you erasing two numbers and writing one number, so the count of numbers on the page steadily decreases.

With which single number is this game sure to end?
(How do you know it is going to be the same number each and every time you play?)

It seems the game ends with the number 720 no matter how you choose to play.

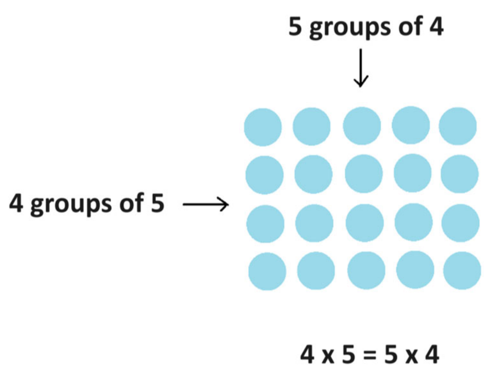
This is the value of product $1 \times 2 \times 3 \times 4 \times 5 \times 6$ computed from left to right, two numbers at a time, just as school teaches you how to evaluate a string of numbers multiplied together.

And again, the solitaire game seems to be leading us to say:

> In any string of counting numbers multiplied together, $a \times b \times c \times d \times e \times \ldots \times y \times z$ it does not matter in which order one chooses to do the individual multiplications. The same answer will always result.

Can we justify this?

We certainly have that the order in which you multiply two counting numbers together does not matter: $a \times b$ and $b \times a$ are sure to have the same value no matter which numbers a and b represent. We saw this in section 3.



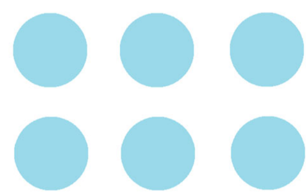
What about three counting numbers multiplied together? Can we explain why "order doesn't matter" when computing $2 \times 3 \times 4$ for instance?

Remember, school world has us compute $2 \times 3 \times 4$ from left to right. We look at 2×3 first to get 6, and then compute 6×4 next to get 24.

$$\overline{2 \times 3} \times 4 = \boxed{6} \times 4 = 24$$

How do we draw a picture of this?

Start, we can draw 2×3 as two groups of three dots drawn in a rectangle.

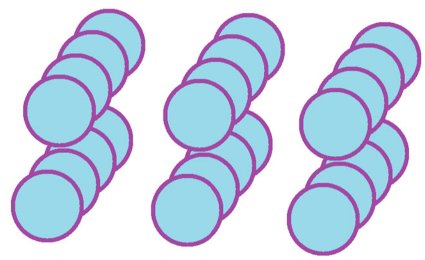


2 x 3

And now we need to draw $2 \times 3 \times 4$, which is "2×3 groups of 4," whatever that means.

Question: Think about this before turning the page. It took me a little while to figure out what one could draw here. I am curious if we come up with the same approach.

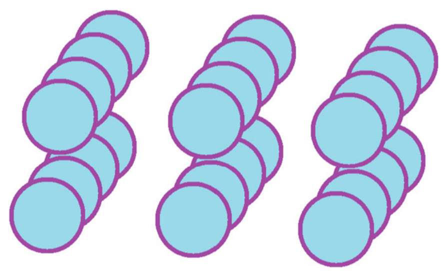
I went to the third dimension!



2 x 3 groups of 4

Here's a three-dimensional picture of 24 dots arranged in a box-like arrangement. The front face is picture of 2×3 and each dot in that picture extends 4 dots deep into the page.

But when I look at my picture, I also can't help seeing two horizontal layers of dots, and that each layer is a picture of 3×4. (Can you see that too?) The picture is also one of 2 copies of 3×4.



2 groups of 3 x 4

This picture explains why we can compute $2 \times 3 \times 4$ by focusing on 2×3 first to get 6×4 or by focusing on 3×4 first to get 2×12. Both are these 24 dots viewed two different ways.

And we can go further.

I can interpret the front face of dots as 3×2 (by looking at columns). The whole three-dimensional picture is thus also "3×2 groups of 4," which is $3 \times 2 \times 4$.

Or, I can look at the right face of dots and see "2×4." And each of those dots is the end of a row of 3 dots. So, the picture also looks like "2×4 groups of 3," which is $2 \times 4 \times 3$.

Or, I can see 3 copies of that vertical 2×4 face of dots, which is $3 \times 2 \times 4$ but with computing the second multiplication first.

Or, I can see the picture as 4×3 groups of 2 dots, which $4 \times 3 \times 2$. Or as 2 copies of 4×3, which is $2 \times 4 \times 3$ but with the second multiplication computed first.

Question: Come up with yet another way to interpret the three-dimensional picture.

We can keep playing this confusing game of switching perspectives on this single picture and explaining why we are not changing anything. All these values are the same

$$\begin{aligned} 2 \times 3 \times 4 &= 3 \times 2 \times 4 = 3 \times 4 \times 2 \\ &= 2 \times 4 \times 3 = 4 \times 2 \times 3 = 4 \times 3 \times 2 \end{aligned}$$

Even if with compute the second multiplication first and the first one second (which goes against the grain of multiplying strictly left to right).

We're seeing, for sure, that the order in which you compute $2 \times 3 \times 4$ —be it the order of the numbers or the order of which of the two multiplications you compute first—just doesn't matter. And, of course, there is nothing special about the numbers 2, 3, and 4 here: we can imagine a three-dimensional picture of any size.

This is great!

By looking at a two-dimensional figure we've explained why order doesn't matter for computing a product of two numbers, $a \times b$.

And by looking at a three-dimensional picture we've explained why order doesn't matter for computing a product of three numbers, $a \times b \times c$.

But I am nervous. For a product of four numbers do we need to draw a four-dimensional picture? (I don't know what that would mean!)

Let's hold off on justifying matters for products of four or more numbers for right now. (We'll come back to it later as there's got to be a better way than going to the fourth dimension!)

But if we do trust that order does not matter for multiplication, then we can sometimes use that idea to our advantage.

For example, here's a nice way to compute 35×14. Think of 35 as 5×7 and 14 as 2×7.

Then

$$35 \times 14 = 5 \times 7 \times \ 2 \times \ 7$$

And can you see that this is 10×49 to give 490?

Practice 7.1: Compute each of these products in a similar manner.

a) 15×12
b) 28×25
c) 68846×50

MUSINGS

Musing 7.2 Katya wondered about the two ways to interpret $2 \times 3 \times 4$ and thought she could go back to the idea of repeated addition. She thought about matters and then wrote

3×4 is $4 + 4 + 4$

2 copies of 3×4 is

$$\underline{\mathbf{4+4+4}} + \underline{\mathbf{4+4+4}}$$

(using an underline to make each copy clear)

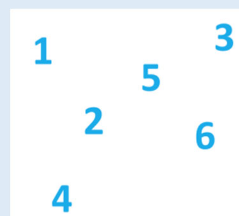
whereas "2×3 copies of 4" is

$$\begin{array}{ccc} \mathbf{4} & \mathbf{4} & \mathbf{4} \\ \mathbf{4} & \mathbf{4} & \mathbf{4} \end{array}$$

But then she wasn't sure if what she was writing was helpful, or even meaningful.

What do you think?

Musing 7.3 OPTIONAL



Here's a third variation of the solitaire game that combines addition and multiplication. Again, start with the numbers 1 through 6 on a page.

Erase two numbers and replace them with their **sum** and their **product** **added together**.

For example, if you cross out 3 and 5 you will then write 23 on the page. (This is 8, their sum, and 15, their product, added together.) If you then cross out 2 and 4 you will write 14 on the page. (This is 6 and 8 added together.)

a) Do you see the same final number each and every time you play the game?

b) What final number do you have if you start with just 1 through 5 instead? Just 1 through 4? 1 through 3? Just 1 and 2? Is there a pattern to these final numbers?

Hard Very Optional Challenge: Going back to starting with the numbers 1 through 6, can you explain *why* the same final number appears each time you play the game, no matter the choices you make along the way?

MECHANICS PRACTICE

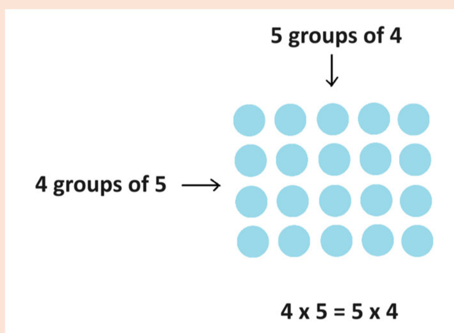
Practice 7.4 Compute each of these products in a way that feels efficient to you.

a) 25×36
b) 5×216
c) 5×846044288
d) $15 \times 6 \times 15 \times 6$
e) $72 \times 125 \times 35 \times 84 \times 55 \times 0 \times 25 \times 15 \times 8$

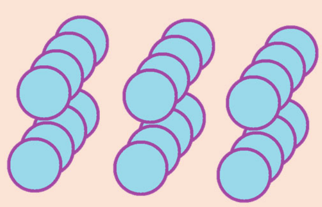
OPTIONAL ADDENDUM

You may have also heard the terms **commutative property** of multiplication and **associative property** of multiplication. If you are curious, they are the official names of the two properties we discussed and explained in this section.

The **commutative property** refers the fact that we can change the order of two numbers we are multiplying. We have that $a \times b = b \times a$ no matter which counting numbers a and b represent.



The **associative property** refers to the fact that we can change the order we choose to conduct the two multiplications in a product of three numbers.



2 x 3 groups of 4

2 groups of 3 x 4

8. The Vinculum

Going back to addition, consider again $2+3+4$.

If I were a fussy sort, I might insist that you think this through by adding 2 and 3 together first to get 5 and then add 4. Or I might insist that you compute $3+4$ first to think 7, and then compute$2+7$.

The question is

> How might I communicate to you the order I insist you conduct the two additions?

Back in 1484, French mathematician Nicolas Chuquet wrote a manuscript in which he used a horizontal bar to denote an intended order to operations. Its use became the rage among a good number of European mathematicians for the centuries that followed.

For example, mathematicians would write

$$2+\overline{3+4}$$

if they intended the reader to think $2+7$ to get to 9, or

$$\overline{2+3}+4$$

if they want the line of thought to be $5+4=9$.

The horizontal bar was—and still is—called a **vinculum**, from the Latin word for “bond” or “tie,” as it suggests which terms of an expressions are “tied together” and to be handled first. Some mathematicians liked to write their vincula on top of an expression (like I do) and others preferred to write them as underlines (like Chuquet preferred).

If I had to choose a favorite mathematical symbol, I would choose the vinculum. I just think it is neat! For example, here is a mighty complicated expression loaded with nested vincula. But despite its complexity, it is clear how I am meant to think my way through it.

$$\overline{2+\overline{4+\overline{2+\overline{\overline{8+\overline{17+\overline{\overline{1+9}+10}}}+7}}}}+1$$

(Can you see—literally see!—that one is to think 10, then 20, then 37, then 45, then 52, then 54, then 58, then 60, and then, finally, 61?)

There are some natural conventions people have settled on for working through vincula.

1. *If a mathematical expression has a vinculum placed on it, compute what is under the vinculum first.*

For example, $5 \times \overline{3+7}$ is 50. (Can you see this?)

2. *If there are nested vincula, work with the innermost vinculum first and proceed from there.*

For example, $10 + \overline{\overline{4 \times \overline{3+2}} \times 3}$ is 70. (Check this.)

3. *If there are two or more "equally nested" vincula, work them out in any order you like (left to right, or right to left, or simultaneously).*

For example, $\overline{2+3} \times \overline{4+1}$ has two "equally deep" vincula. This is to be computed as 5×5, giving 25.

Example: Evaluate $5 + \overline{\overline{\overline{4 \times 4} \times 2} + \overline{3 \times 2}}$

Answer: Looking at the innermost vinculum first we get

$$5 + \overline{\overline{16 \times 2} + \overline{3 \times 2}}$$

Next we have two equally-nested vinculums

$$5 + \overline{32 + 6}$$

And now we see $5 + 38$ to give 43.

Practice 8.1: Can you see that that value of $5 + \overline{\overline{\overline{\overline{5 + 6} \times 4} - 4 \times 2}}$ is ninety?

Reducing the Number of Vincula

To cut down on the abundance of vincula in an expression, mathematicians have settled on another convention.

> Assume every multiplication sign comes with its own hidden vinculum above the two numbers being multiplied.

For instance, $2 + 3 \times 4$ is to be understood as $2 + \overline{3 \times 4}$, and so equals 14.

And $7 \times 5 + 3 \times 1$ is to be understood as $\overline{7 \times 5} + \overline{3 \times 1}$, which equals 38.

A tricker example is

$$2 \times \overline{3 + 4} + 5 \times 4.$$

This is to be unraveled as

$$2 \times 7 + 5 \times 4,$$

which is

$$\overline{2 \times 7} + \overline{5 \times 4},$$

giving $14 + 20$, which is 34.

You may have been taught an **order of operations** rule in school which says something like *Do multiplications before doing additions.* So, in $2 + 3 \times 4$, one is to compute $3 \times 4 = 12$ first and then compute $2 + 12 = 14$. This is just our rule for vincula with the (hidden) vinculum over the product: *Always do vincula first!*

Getting a bit ahead on matters …

People don't use the vinculum anymore—except in three places.

In 1631, English mathematician Thomas Harriot suggested it might be a good idea to attach a vinculum to this symbol √ used for square roots. (This symbol is called a **radix**, by the way.) For example, an expression like

$$\surd 9 + 16$$

is ambiguous. Is this the square root of 9 (which is 3) with 16 later added to give the answer 19? Or is this the square root of the entire quantity $9 + 16$, which is 25, to give the answer 5?

The vinculum clarifies matters.

$$\sqrt{9} + 16 = 3 + 16 = 19$$

$$\sqrt{9 + 16} = \sqrt{25} = 5$$

Most people today do not realize that $\sqrt{}$ is two separate symbols combined.

In 1647 Italian mathematician Bonaventure Cavalieri used the vinculum in his geometry book. If A and B are the names of two points in space, he suggested using the notation $\overline{AB}$ for the line segment that "ties" them together. This is now standard notation in geometry books.

Today, people in some countries (the US included) use a vinculum to denote a group of digits that repeat in an infinitely long decimal. For example, we have

$$\frac{1}{7} = 0.\overline{142857} = 0.142857142857142857142857\ldots$$

$$\frac{1}{3} = 0.\overline{3} = 0.3333333\ldots$$

Folk in other parts of the world might write $0.\dot{1}4285\dot{7}$ or $0.(142857)$ or $0.\overparen{142857}$, for instance, for the repeating decimal representation of $\frac{1}{7}$.)

MUSINGS

Musing 8.2 There are **2** ways to place a vinculum in the expression $1+2+3$.

$$\overline{1+2}+3 \qquad\qquad 1+\overline{2+3}$$

There are **5** ways to place vincula in the expression $1+2+3+4$ so that one is only adding two quantities at a time.

$$\overline{\overline{1+2}+3}+4 \qquad\qquad \overline{1+\overline{2+3}}+4$$
$$1+\overline{2+\overline{3+4}} \qquad\qquad 1+\overline{\overline{2+3}+4}$$
$$\overline{1+2}+\overline{3+4}$$

Care to list the 14 ways to list vincula in the expression $1+2+3+4+5$?

Care to list the 42ways to list vincula in the expression $1+2+3+4+5+6$?

Some people like to say there is **1** way to place vincula in the sum $1+2$, namely, to not do anything and just leave it as it is as there is no need for one.

We are developing a list of “vinculum numbers”: $1, 2, 5, 14, 42$.

Do you care to guess what the next vinculum number after 42might be?
(Remember, the answer can always be: “No. I do not care to guess.”)

9. Parentheses/Brackets

Use of the vinculum remained popular all through the fifteen- and sixteen-hundreds and well into the seventeen-hundreds, which seems to befuddle some mathematical historians. After all, around the year 1440, Johannes Gutenberg invented the printing press allowing for the first time the mass production of books.

Printing lines of text and mathematical symbols was straightforward. But inserting horizontal bars between lines of text was awkward and hard to do. Why stay with a notational system that was so difficult to print, especially since other symbols for grouping terms were being proffered at the time?

One alternative was to use **parentheses** (many people in the world call them **brackets**) to group terms. Instead of writing $\overline{2+3}+4$ and $2+\overline{3+4}$, we could write $(2+3)+4$ and $2+(3+4)$. And instead of writing

$$\overline{2+\overline{4+\overline{2+\overline{\overline{8+\overline{17+\overline{\overline{1+9}+10}}}+7}}}}+1,$$

which is awfully hard to print by lining up letter and symbol tiles on the printing press table, one could write instead

$$\left(2+\left(4+\left(2+\left(\left(8+\left(17+\left(\left(1+9\right)+10\right)\right)\right)+7\right)\right)\right)\right)+1.$$

Although this is a bit harder to unravel (but, really, who in their right mind would be writing so many nested parentheses in the first place?), it is certainly straightforward to print with a press.

In the mid-1700s, the prominent and prolific Swiss mathematician Leonhard Euler often used parentheses for grouping. It seems he helped accustom European mathematicians to their use and they remain the preferred grouping symbol to this day.

Translating our Vincula Conventions to Parentheses/Brackets Conventions

Here are our vincula conventions rewritten in terms of parentheses.

1. *If a mathematical expression has a set of parentheses in it, compute what is in the parentheses first.*

For example, $5 \times (3 + 7)$ is 50.

2. *If there are nested parentheses, work with the innermost parentheses first and proceed from there.*

For example, $10 + \big((4 \times (3 + 2)) \times 3\big)$ is 70.

3. *If there are two or more "equally nested" parentheses, work them out in any order you like (left to right, or right to left, or simultaneously).*

For example, $(2 + 3) \times (4 + 1)$ has two "equally deep" parentheses. This is to be computed as 5×5, giving 25.

Example: Evaluate $5 + \Big(\big((4 + 4) \times 2\big) + (3 \times 2)\Big)$

Answer: Looking at the innermost parenthensis first we get

$$5 + \big((8 \times 2) + (3 \times 2)\big)$$

Next, we have two equally-nested parentheses

$$5 + (16 + 6)$$

And now we see $5 + 22$ to give 27.

To cut down on the abundance of parentheses in an expression, mathematicians have settled on another convention.

Assume every multiplication sign comes with its own set of parentheses immediately around the two numbers being multiplied. (They've simply been made invisible.)

For instance, $2 + 3 \times 4$ is to be understood as $2 + (3 \times 4)$, and so equals 14. And $7 \times 5 + 3 \times 1$ is to be understood as $(7 \times 5) + (3 \times 1)$, which equals 38.

A tricker example is

$2 \times (3 + 4) + 5 \times 4.$

This is to be unraveled as $2 \times 7 + 5 \times 4$,which is$(2 \times 7) + (5 \times 4)$, giving $14 + 20$, which is 34.

You may have been taught an **order of operations** rule in school which says something like:

Do multiplications before doing additions.

So, in $2+3\times4$, one is to compute $3\times4=12$ first and then compute $2+12=14$.

This is just our rule for parentheses with the (hidden) parentheses around the product:

Always do what's inside parentheses first.

Question: Do these last two pages feel like *déjà vu?*

Notations for Multiplication

As I am sure you are aware, the letter x has quite the favored status in algebra class. (We'll get to algebra.) But that letter looks awfully similar to the traditional multiplication symbol $\times$.

To avoid possible confusion, mathematicians tend to use a raised dot · to denote multiplication.

For example, $2 \cdot 3$ means 2×3, and $42 \cdot 17$ means 42×17.

And sometimes they will not write a multiplication symbol at all—just placing the two quantities to be multiplied next to each other if no possible confusion could result.

For example, instead of writing $2 \cdot (3+7)$ mathematicians will drop the dot, and write

$$2(3+7)$$

If a represents a number, instead of writing $3 \cdot a$, mathematicians will write

$$3a$$

Practice 9.1: Mathematicians would never drop the dot in $23 \cdot 17$. Can you see why?

This leads to odd looking expressions every now and then. For example, in computing

$$(3+7)(4+9)$$

one might write $(10)(13)$ and wonder why there are parentheses around the single numbers. Nonetheless, one recognizes this as $10 \cdot 3$ to get the answer 30.

In computing $5(83+17)$ one might find oneself writing $5(100)$, which is to be recognized as $5 \cdot 100$.

To prepare students for this, some elementary school curricula explicitly state that $a(b)$ and $(a)(b)$ and $(a)b$ are each alternative notations for $a \times b$. I personally suspect that this must seem weird and confusing to young'uns.

MUSINGS

Musing 9.2 Try writing these two expressions with vincula instead of parentheses. Show the hidden vincula as well.

a) $\big((4+2\cdot 3)+(3+7)\big)+(1+2)$
b) $1+\Big(1+\big(1+(1+1\cdot 1)\big)\Big)$

Try writing these two expressions in terms of parentheses. Let's keep the hidden parentheses hidden this time.

c) $\overline{2\cdot 3+4}\cdot\overline{5+6}$
d) $8\cdot 8+8\cdot\overline{\overline{8\cdot\overline{8+8}}\cdot 8}$

MECHANICS PRACTICE

Musing 9.3 Evaluate the following expressions in the order indicated via the parentheses and the multiplications.

a) $3+2\cdot 11$
b) $4\cdot 3+3\cdot 5+2$
c) $6(2+3)$
d) $(1+1)\cdot 3\cdot 8$
e) $2+(6+4(1+5)(3+2)\cdot 7+6+3\cdot 3)+10\cdot 4$

10. A String of Sums; A String of Products

We argued in section 6 that

> *In any string of sums of counting numbers $a + b + c + d + e + \ldots + y + z$ it does not matter in which order one chooses to perform the additions. The same answer will always result.*

This means that it does not matter how you might choose to place parentheses in a string of sums such as

$$2 + 3 + 4 + 5 + 6,$$

the final result will always be the same.

We have that $\left(2 + \left((3 + 4) + 5\right)\right) + 6$ gives the same answer as $2 + \left((3 + 4) + (5 + 6)\right)$, which gives the same answer as $2 + \left(\left((3 + 4) + 5\right) + 6\right)$, and so on. For this reason:

> **People never bother to put parentheses in a string of sums.**

We also wondered in section 7 if the following is true.

> *In any string of products of counting numbers $a \times b \times c \times d \times e \times \ldots \times y \times z$ it does not matter in which order one chooses to do the individual products. The same answer will always result.*

Often people just assume this is true as well. In which case, there is no need for grouping terms with parentheses.

> **People never bother to put parentheses in a string of products.**

For example, in computing $2 \times 3 \times 4 \times 5 \times 6$ it does not matter in which order one chooses to conduct the products. There is no need for parentheses.

Back in section 7, we drew pictures to justify why “order does not matter” when computing a product of two or three numbers:

A two-dimensional picture explains why, philosophically, 4×5 and 5×4, for example, have the same answer. And a three-dimensional picture explains why $(2 \times 3) \times 4$ and $2 \times (3 \times 4)$, for example, must be the same too.

We have, in general, that

$(a \times b) \times c = a \times (b \times c)$ for any three of counting numbers a, b, and c.

We were worried back in the section we would have to somehow draw four-dimensional pictures to justify why “order does not matter” when computing a product of four numbers.

Let’s attend to this now without going to the fourth dimension!

WARNING: *The remainder of this section is optional reading and is not for the faint-hearted!*

Feel free to just accept, if you like, that it is possible to justify our intuition that “order doesn’t matter” for any string of numbers multiplied together, no matter the how many numbers are in that string.

There are five ways to group a string of four numbers multiplied together.

$$\big((a \times b) \times c\big) \times d$$
$$\big(a \times (b \times c)\big) \times d$$
$$a \times \big(b \times (c \times d)\big)$$
$$a \times \big((b \times c) \times d\big)$$
$$(a \times b) \times (c \times d)$$

Our job is to explain why all five presentations of $a \times b \times c \times d$ must have the same value.
Here goes!

The first and second items in the list give the same answer because we have that $(a \times b) \times c = a \times (b \times c)$. (This is what we know about a product of three numbers.)

The third and fourth items in the list give the same answer because we have that $(b \times c) \times d = b \times (c \times d)$. (Again, what we know about a product of three numbers.)

The second and fourth items in the list give the same answer because we have that $(a \times M) \times d = a \times (M \times d)$ where M just happens to be $(b \times c)$. (Sneaky!)

So, this means the first four items on the list are sure to give the same answer.

The third and fifth items give the same answer too, as $a \times (b \times W) = (a \times b) \times W$ where W just happens to be $(c \times d)$.

This means that all five possibilities do indeed give the same answer!

Our belief about products of three terms led us to believe the same for products of four terms.

In the same way, one can show that all the ways of interpreting a product of five terms must give the same answer (by seeing all the possible products as examples of what we just showed is true about products of four terms), as do all the ways to interpret a product of six terms (by seeing what all the possible products as instances of what we will have just showed true about products of five terms), and one of seven terms, and so on.

MUSINGS

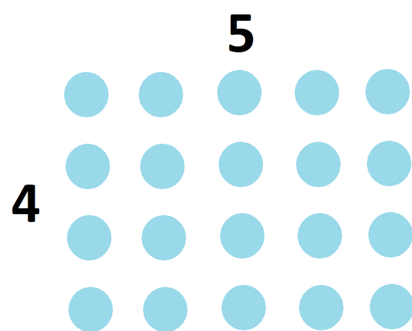
Musing 10.1 Show that there are $\mathbf{14}$ ways to compute $a \times b \times c \times d \times e$ so that one is conducting only a product of two terms at a time.

(Actually, don't bother with this question. It is really a repeat of Musing 8.2. Do you see why? Also, this section showed that all the ways to compute this product lead to the same answer, so who cares about parentheses anyway in this context?)

11. Chopping up Rectangles

Let's revisit an idea from section 4.

Here's a depiction of 4×5.



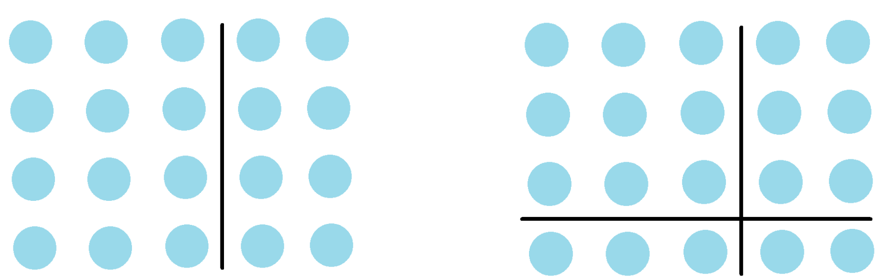
If we chop up the rectangle, we see we could also interpret the figure as

$$4 \times 3 + 4 \times 2$$

or as

$$3 \times 3 + 1 \times 3 + 3 \times 2 + 1 \times 2$$

or as many other combinations of products.



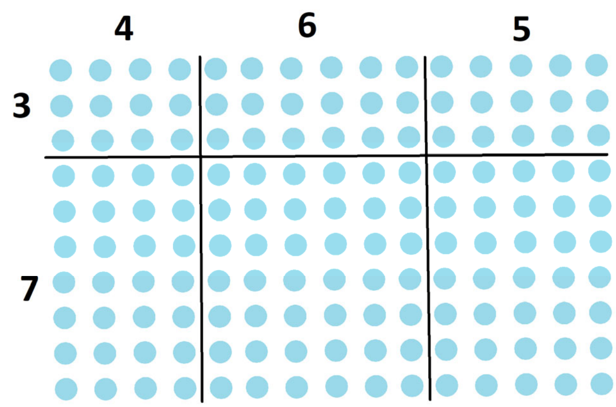
Notice: We're making use of the hidden parentheses/vinculi that come with multiplication signs.

Each sum of products is, of course, 20.

$$4 \times 3 + 4 \times 2 = 12 + 8 = 20$$

$$3 \times 3 + 1 \times 3 + 3 \times 2 + 1 \times 2 = 9 + 3 + 6 + 2 = 20$$

Consider this next figure.



It can be interpreted as

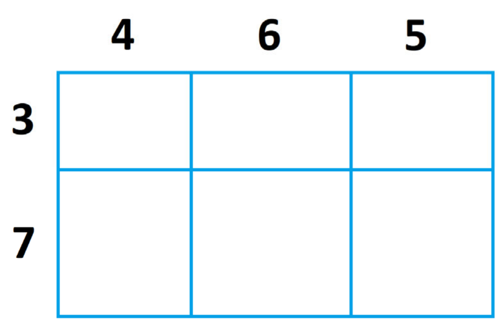
$$(3+7)\times(4+6+5)$$

which is 10×15, showing that there are 150 dots in this picture.
Or as sum of six products

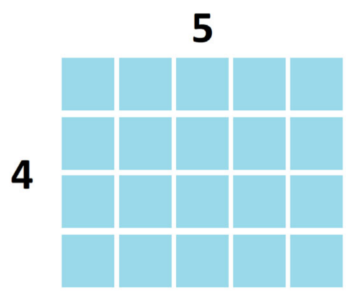
$$3\times 4+3\times 6+3\times 5+7\times 4+7\times 6+7\times 5$$

which corresponds to $12+18+15+28+42+53$, adding to 150.

Rather than keep drawing rectangles of dots, let's just draw rectangles, viewing each rectangular region is an array of dots. In this figure we imagine $3 \times 4 = 12$ dots in the top left corner of the figure region, and $7 \times 6 = 42$ dots in the middle bottom region of the figure, and so on.



As mentioned in section 4, people call these rectangle pictures examples of the **area model** of multiplication. If we replace each dot with s unit square tiles, a 4-by-5 array has area $4 \times 5 = 20$ square units.



The previous picture shows a 10-by-15 array, of area 150 square units, divided into six pieces, one of area $3 \times 4 = 12$ square units, and one of area $7 \times 6 = 42$ square units, and so on. Whether we count dots, or count square units and imagine areas, our arithmetic is the same.

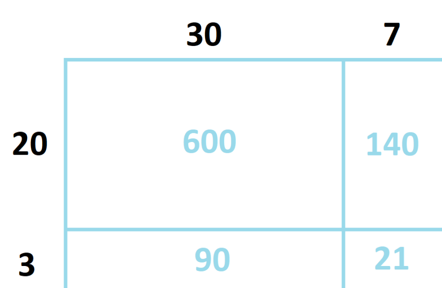
This visual representation of multiplication help us with multi-digit multiplication.

Example: Compute 23×37.

Answer 1: Ask Alexa or Siri or some virtual friend.

Answer 2: The question is asking "How many dots are there in a 23-by-37 array of dots?" or, equivalently, "What is the area of a 23-by-37 rectangle?"

The numbers are awkward. But let's simplify matters by chopping up the rectangle into regions that involve friendlier numbers. Let's think of 23 as $20 + 3$ and 37 as $30 + 7$. This divides the rectangle into four regions whose areas are easier to compute.



We see that the area of the rectangle (or, the total number of dots in the array if you are thinking dots) is $600 + 140 + 90 + 21 = 600 + (130 + 100) + 21 = 851$ square units.

That is, $23 \times 37 = 851$.

(I could almost do this computation in my head by visualizing the rectangle.)

Example: Compute 371×42.

Answer 1: Use your smartphone.

Answer 2: This picture does the trick.

$371 \times 42 = 12000 + 2800 + 40 + 600 + 140 + 2 = 15582.$

It doesn't matter that our rectangles are not drawn to scale. We just need to make sure the information presented on each diagram is correct.

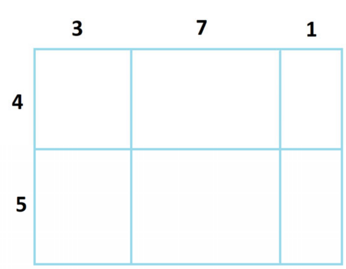
Practice 11.1: Does it matter on which sides of the rectangle you place the two numbers in the product you are computing? (Were you expecting a slightly different picture for 371×42?)

Example: Compute $(4+5)(3+7+1)$.

Answer 1: This is just $9 \times 11 = 99$.

Answer 2: In terms of chopping up rectangles, we also see the answer 99—with a lot more work along the way! We are summing the areas of six individual pieces to get there.

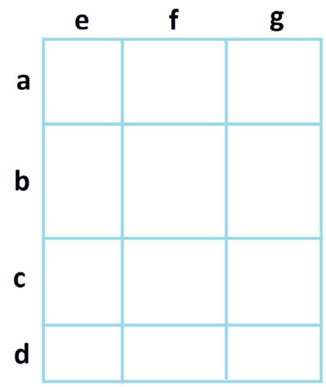
$$\begin{aligned}(4+5)(3+7+1) &= 4 \times 3 + 5 \times 3 + 4 \times 7 + 5 \times 7 + 4 \times 1 + 5 \times 1 \\ &= 12 + 15 + 28 + 35 + 4 + 5 \\ &= 99.\end{aligned}$$



Let's make matters a tad abstract

Example: If $a, b, c, , ..., g$ are numbers, what is $(a+b+c+d)(e+f+g)$?

Answer: Geometrically, it is a rectangle chopped into twelve pieces.



Those pieces are

$$(a+b+c+d)(e+f+g) = ae + af + ag + be + bf + bg + ce + cf + cg + de + df + dg$$

Examine the sum of twelve terms we see in the previous example. Each term in the sum matches a piece of the rectangle and it comes from multiplying one number displayed along the left side of the diagram with one number displayed along the top.

That is, to **expand** $(a+b+c+d)(e+f+g)$, we must select one term from the first set of parentheses, one term from the second, multiply them together, and add the results. We need to make sure to attend to all possible combinations.

$$(a+b+c+d)(e+f+g)$$
$$= ae + af + ag + be + bf + bg + ce + cf + cg + de + df + dg$$

As there are 4 ways to choose an entry in $(a+b+c+d)$ (task 1) and 3 ways to choose an entry in $(e+f+g)$ (task 2), there are indeed a total of $4 \times 3 = 12$ products to write in the sum.

Practice 11.2: Computing $(7+3)(4+5)$ corresponds to dividing a rectangle into four pieces. One piece is 7×4, another is 3×4, and so on. Can you see this? (Draw a picture.)

Can you also see that the products 7×3 and 4×4 do **not** correspond to valid pieces?

Practice 11.3: Computing 100×100 as

$$(50+20+20+7+3)(20+20+20+20+19+1)$$

corresponds to dividing a rectangle (a square, actually) into $5 \times 6 = 30$ pieces.

There are four pieces of size 7×20.
There is just one piece of size 50×19.

How many pieces of size $20 \times 20$are there?

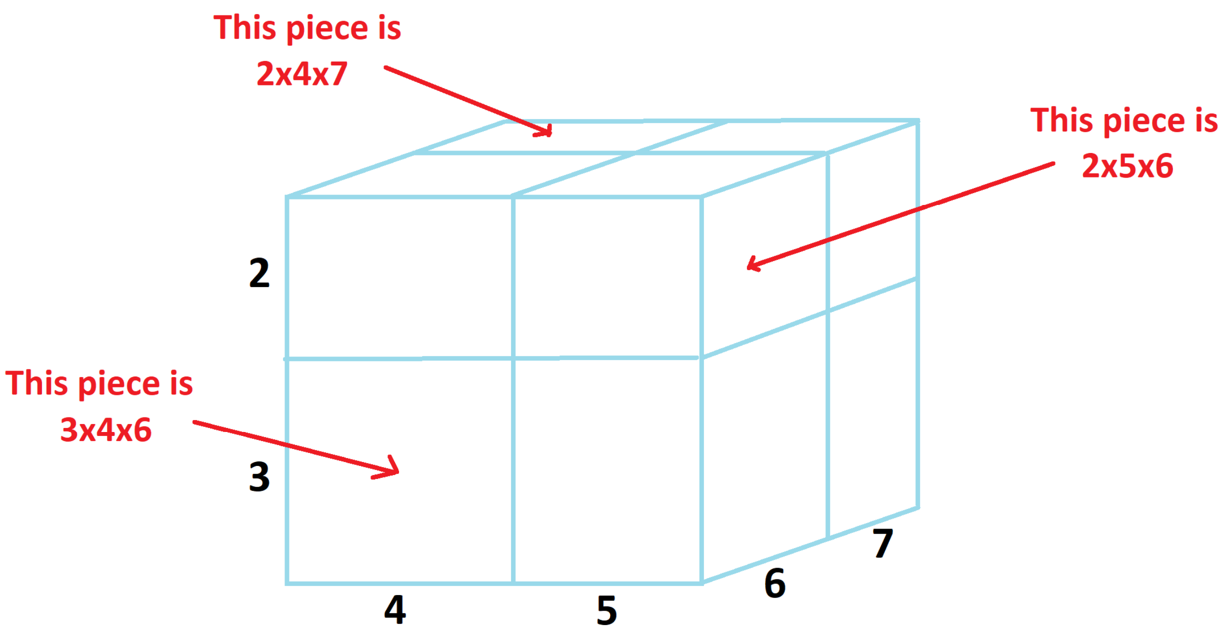
Let's have some fun.

What does $(2+3)(4+5)(6+7)$correspond to geometrically?

This is just the product $5 \times 9 \times 13 = 585$ written in a complicated way, but what is the picture to go with this product? Are we chopping up rectangle?

Think about this before turning the page.

We have that $(2+3)(4+5)(6+7)$ corresponds to chopping a three-dimensional rectangular box into eight pieces.



Here are the eight pieces.

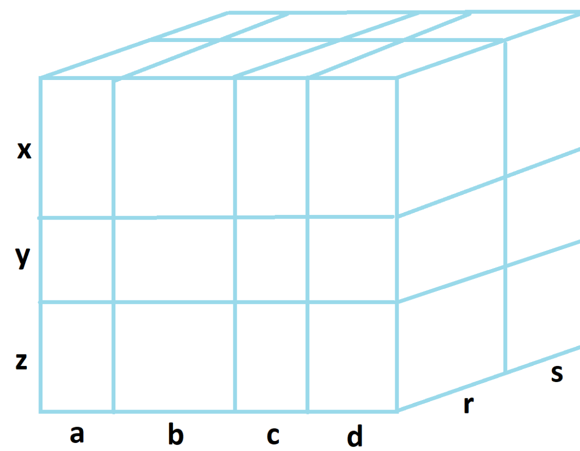
$$(2+3)(4+5)(6+7) = 2\times 4\times 6 + 2\times 4\times 7 + 2\times 5\times 6 + 2\times 5\times 7 \\ + 3\times 4\times 6 + 3\times 4\times 7 + 3\times 5\times 6 + 3\times 5\times 7$$

And this sum does indeed add to 585 if you have the patience to check.

$$5\cdot 9\cdot 13 = 48 + 56 + 60 + 70 + 72 + 84 + 90 + 105 = 585$$

Again, we are selecting one term from each set of parentheses, multiplying them together, and adding the results, making sure to attend to all possible combinations.

Example: Expanding $(x+y+z)(a+b+c+d)(r+s)$ corresponds to chopping a three-dimensional box into $3 \times 4 \times 2 = 24$ pieces.
One of the pieces is xar and another is ycs, and so on.



Example: Imagine expanding $(x+y)(x+a+b)(a+c+p)$.

a) How many pieces would there be?
b) Would xac be one of those pieces? How about cay? xcp? xax? xyc?

Answers:

a) There would be 18 terms in the final sum.
b) Yes, xac would appear.

cay appears as yac.
xcp does not appear.
xax appears as xxa
xycappears as yxc.

Example: If you were to expand

$$(a+b+c+d+e)(w+x)(a+b+x+t+r)(e+f)$$

how many terms would there be? (Is there a geometric interpretation for this scenario?)

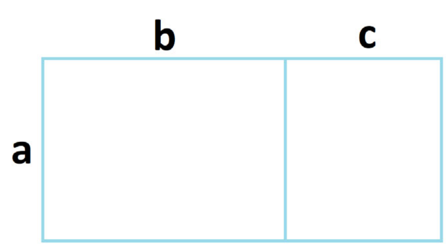
Answer: It seems we're in four-dimensions now!
But the same arithmetic will be at play.

There will be $5 \times 2 \times 5 \times 2 = 100$ terms and the resulting expansion would begin $awae + bwae + \cdots$.

If we trust that the mechanics of two-dimensional rectangles and three-dimensional rectangular boxes continues to hold in all situations (even if I don't know that the fourth dimension is) it seems we have another natural belief about arithmetic.

> To compute the product of sums of terms, select one term from each set of parentheses, multiply them, and then sum all the results. Make sure to attend to all combinations.
>
> For example,
>
> $$(a + b + c)(x + y + w + z) = ax + bx + cx + ay + by + \cdots$$
>
> $$(x + 2)(a + b) = xa + xb + 2a + 2b$$
>
> $$(x + y)(p + q) = xp + yp + xq + yq$$

Many textbooks focus on just one example of "chopping up a rectangle."



If we take the expression $a(b + c)$ and place parentheses around the single term to write

$$(a)(b + c)$$

then the process of "selecting one term from each set of parentheses" has us selecting the term a from the first set of parentheses each and every time and so the sum $ab + ac$ results, just as the diagram suggests.

Practice 11.4: Draw a picture of $(a + b)c = ac + bc$.

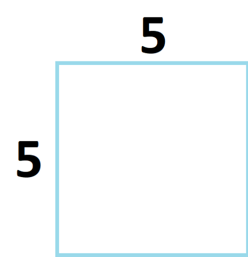
Have you noticed …?

When a number is multiplied by itself, we say that the number is **squared**. (This came up in Musing 4.1.)

We use a superscript two to denote this. For example,

$5^2 = 5 \times 5 =$ "five squared."

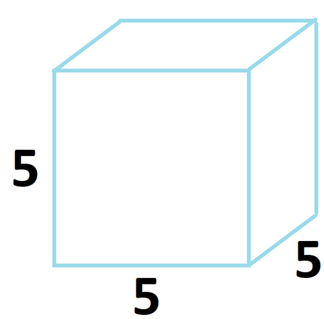
This choice of wording makes sense as the area of a five-by-five square is computed as 5×5.



We use the superscript three to denote a three-fold multiplication of the same number. We call that the number **cubed**.

$5^3 = 5 \times 5 \times 5 =$ "five cubed."

This choice of wording makes sense as the volume of a five-by-five-by-five cube is computed as $5 \times 5 \times 5$.



There is a reason why we humans don't have special words for $5^4 = 5 \times 5 \times 5 \times 5$ and $5^5 = 5 \times 5 \times 5 \times 5 \times 5$, and so on. We just can't envision matters beyond the third dimension!

MECHANICS PRACTICE

Musing 11.5 One can compute $(2+3)(7+4)$ two ways:

Short way: $(2+3)(7+4) = 5 \cdot 11 = 55$
Long way: $(2+3)(7+4) = 2 \cdot 7 + 2 \cdot 4 + 3 \cdot 7 + 3 \cdot 4 = 28 + 8 + 21 + 12 = 55$

Compute each of the following both the short way and the long way.

a) $(3+4)(5+1)$
b) $(2+3+5)(2+8)(1+9)$

Musing 11.6

a) Expand $(a+x+b)(x+y)$.
b) If one were to expand $(x+y+z+w+t+r)(a+b+c+d+e+f+g+h)$ how many terms would there be in the resulting sum?

Musing 11.7

I can compute 13×26 by imagining drawing (or actually drawing, that is okay too!) a rectangle that looks at this product as $(10+3)(20+6)$. I can then see that the answer is

$$200 + 60 + 60 + 18 = 320 + 18 = 338.$$

Compute each of the following products the same way.
Use technology to check that your answers are correct.

a) 23×14
b) 106×21
c) 213×31

Musing 11.8

If one were to expand $(p+q+2)(p+2)(a+q+p+3)(x+q+3)$, which of the following would be a term you would see in the sum of 72 terms that result?

a) $3p^3$ b) $6p^2$ c) $3q^3$ d) 18 e) xq^2p

12. Fun with Long Multiplication

We saw that a natural way to compute a multi-digit multiplication problem such as 23×12 is to use the area model.

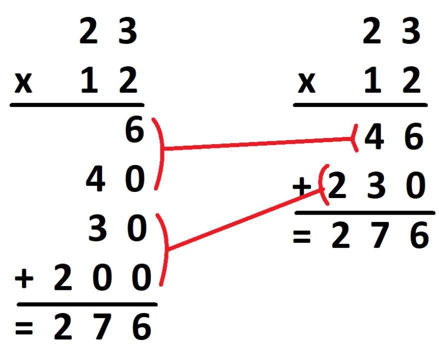
	20	3
10	200	30
2	40	6

```
   2 3
x  1 2
     6
   4 0
   3 0
+ 2 0 0
= 2 7 6
```

Some school students today are taught to compute long multiplication via a method of "partial products," which is just the area model in disguise. (It would be easier if students were shown and then allowed to draw chopped up rectangles.)

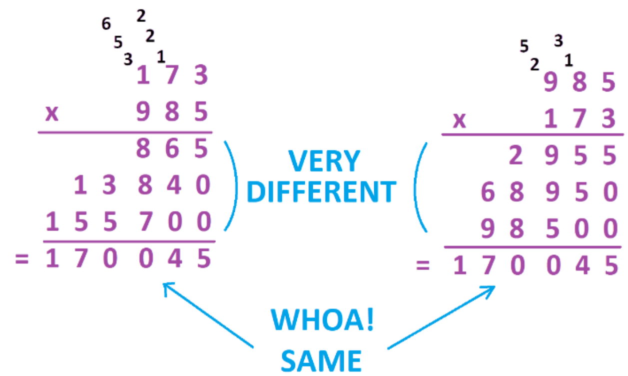
> **Question**: Do you see how the computation on the right in the figure above is indeed just the area model?

During the 1500s and the centuries that followed, paper and ink were precious. The "partial product" algorithm was compactified to save ink. It is the algorithm most students are still taught, even though ink is no longer precious. (In fact, asking Siri for the answer uses no ink whatsoever!)



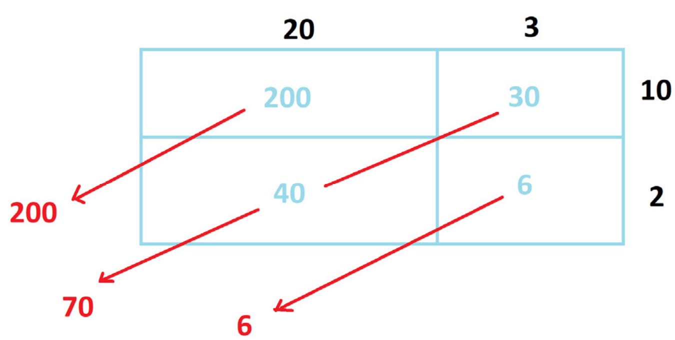
> **Question:** Were you taught to compute 23×12, say, this compactified way?

Question: In section 3 we looked at 173×985 versus 985×173.



Can you see now that if we were to compute each of these products via the "partial products" method (that is, via the area model), that there is actually nothing mysterious going on here? The nine pieces that come from a $100 + 70 + 3$ by $900 + 80 + 5$ rectangle are just being "compactified" two different ways.

One nice thing about drawing rectangles when computing long multiplications is that the units, tens, hundreds, and so on, all line up nicely along diagonals. (It helps to write numbers on the right rather than the left to see this.)



One can have fun with this as the two musings for this section show.

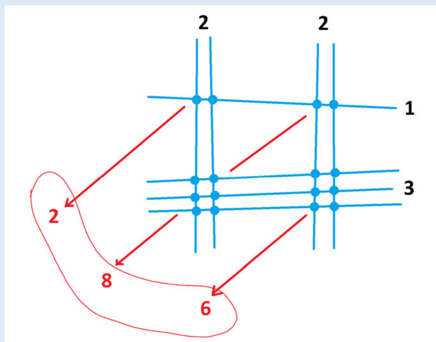
Warmup: Draw an area model picture for 218×43 and add up the diagonals of the picture. Do you see the answer of 9374 appearing?

MUSINGS

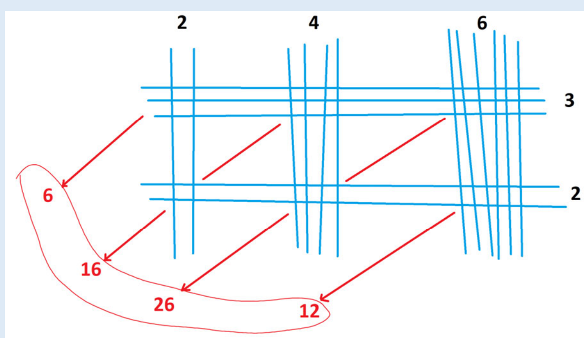
Musing 12.1 Here's a mighty strange way to conduct long multiplications.

To compute 22×13, for instance, start by drawing two sets of vertical lines, the left set containing 2 lines and the right set also containing two lines. (These match the digits of the number 22.) Also draw two sets of horizontal lines, the upper set with just 1 line and the lower set 3 lines (to match the digits of the number 13).

There are four clusters of intersection points. Count the number of intersection points in each cluster and add the counts diagonally as shown. The answer 286 appears.



There is one caveat as illustrated by the computation of 246×32.



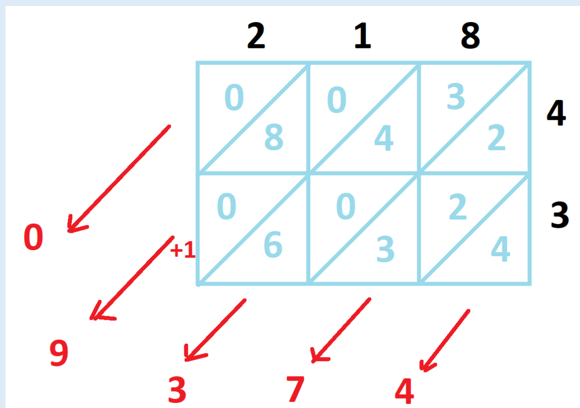
The answer of 6 thousands, 16 hundreds, 26 tens, and 12 ones, that is, $6000 + 1600 + 260 + 12 = 7872$, appears. One might have to "carry" some digits to read off the answer.

a) Compute 131×222 via this method.
b) Compute 54×1332 via this method.
c) How best should one compute 102×3001 (which equals $306{,}102$) via this method?
d) Why does this line method work?

Musing 12.2 During the 1500s in England, students were taught to compute long multiplication following the *galley method*, also called the *lattice method*. (Today, it is also referred to as the *Elizabethan method*.)

To compute 218×43, say, draw a two-by-three grid of squares. Write the digits of the first number above the columns of the grid and the digits of the second number to the right of the rows as shown.

Divide each cell of the grid with a diagonal line and write the product of the column digit and the row digit of each cell as a two-digit answer in that cell, but with its two digits split across the diagonal of that cell. (If the product is just a one-digit answer, write a 0 for the first digit of the "two-digit answer.")



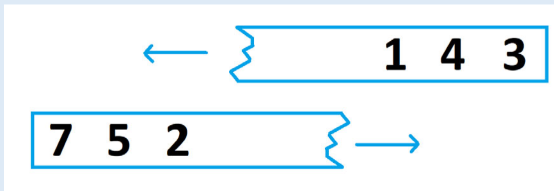
Add the entries in each diagonal, "carrying" any digits over to the next diagonal, if necessary, and read off the final answer. (Okay, this uses a lot of ink!)

In our example, we get $0 \mid 8 \mid 13 \mid 7 \mid 4$, but a carry of a "1" makes this 09374. We see that $218 \times 43 = 9374$.

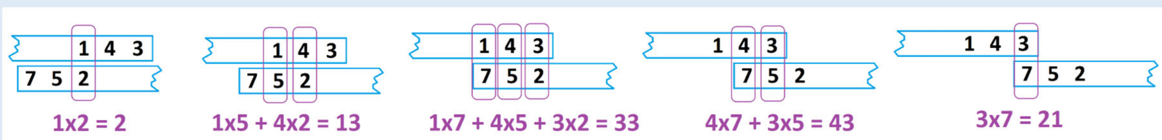
a) Compute 5763×345 via the galley method to get the answer 1988235.
b) Explain why the galley method is really the area model in disguise. (What is the specific function of the diagonal lines?)

Musing 12.3 DEFINITELY OPTIONAL!
Here's a mighty unusual way to compute long multiplication. Let's illustrate it by computing 341×752 to obtain the answer 256432.

Start by writing each number on a strip of paper, but write the first number backwards (and the second number forwards).

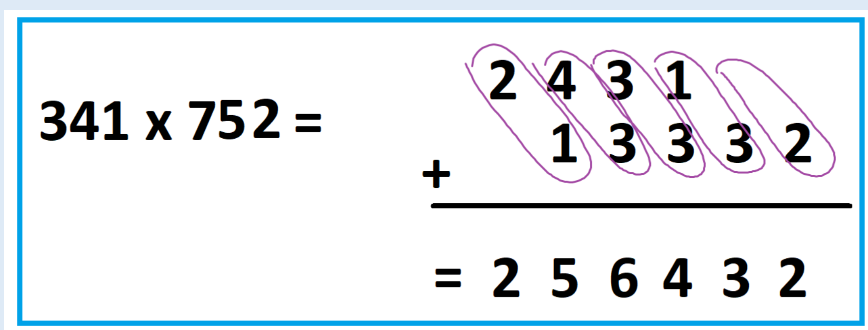


Starting with the first strip to the right and the second strip to the left, slide the first strip leftwards and the second strip rightwards until two digits align vertically. Record their product.



Keep sliding the strips recording the sums of the products the digits that align as you go along.

Write the answers you obtained in diagonally, as shown, and sum the columns. The answer 256432 appears!

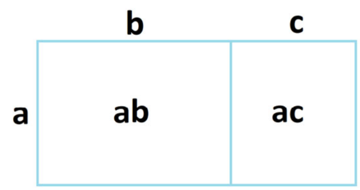


a) Compute 231×121 via the traditional school method and then again with this paper-strip method. Do you see similarity between the two methods?
b) Calculate 341×752 via the traditional school method. Do you still see similarity that approach and the paper-strip approach?
c) Can you explain, in general, why the paper-strip approach works?

13. Some Factoring

We've seen how to chop up rectangles to play with multiplication problems in clever ways.

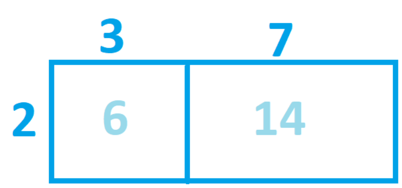
School books tend to focus on just one type of rectangle-chopping, namely, rewriting $a(b+c)$ as $ab+ac$.



And they also focus on applying this arithmetic fact backwards, namely, to recognize a quantity of the form $ab+ac$ as $a(b+c)$.

People call this backward act **factoring**. (Folk of British descent call it **factorising**.)

For example, for $6+14$, we can "factor out a 2" by recognizing this quantity as $2\times3+2\times7$, and so rewrite it as $2(3+7)$. (Why one would want to do this is not at all clear!)



In the same way we can see

$$3a+6b=3(a+2b)$$

by "pulling out a common factor of 3," and that

$$10p+5pq=5p(2+q)$$

by "pulling out a common factor of $5p$."

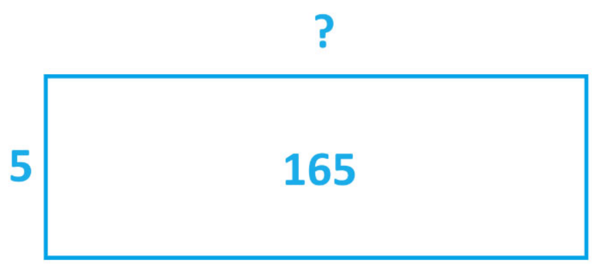
Division

Some elementary school curricula have students compute division problems essentially this way.

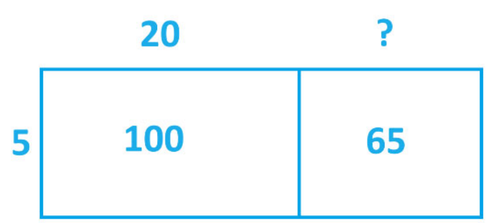
Students are often told that division is "multiplication backwards," so it seems plausible that playing with the area model backwards will help. And it does!

Consider problem $165 \div 5$. Here we are being told that two numbers multiple together to give the answer 165 and that one on the numbers is 5. Our challenge is to find the second number.

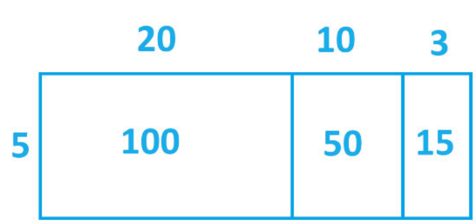
Here's a picture of the situation.



Students are advised to build up the total area of the rectangle with multiples of five they know. For example, we can get to 100 units of area by using $5 \times 20 = 100$. That leaves us with 65 units to contend with.



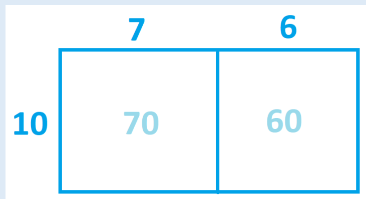
We can now use $5 \times 10 = 50$ to leave 15 units of area, which, we recognize as 3×5. (Great!)



We see $165 \div 5 = 33$.

MUSINGS

Musing 13.1 If desired, one could factor $70 + 60$ by writing it as $10(7 + 6)$.



Factor each of the following expressions in a similar way. (Drawing rectangles helps.)

a) $50 + 60 + 90$
b) $33 + 99$
c) $5x + 5y$
d) $awq + pbw$
e) $z^2 + 3z + xz$

Musing 13.2 The previous question 'factored out a 10' from $70 + 60$ to write it as $10(7 + 6)$.

a) What expression would you write if you "factored out a 1" instead?
b) **Getting Ahead of Ourselves**… What expression would you obtain if you factored out a $\frac{1}{2}$?

MECHANICS PRACTICE

Musing 13.3 Compute each of these division problems via the area method. Feel free to check your answers with a calculator.

a) $1491 \div 7$
b) $555 \div 15$
c) $516 \div 4$
d) $299 \div 13$
e) $2001 \div 23$

Musing 13.4 Does the area model for division show remainders? Can you see that $875 \div 6$ is 145 with a remainder of 5? Try it!

14. Summarizing the Rules of Arithmetic

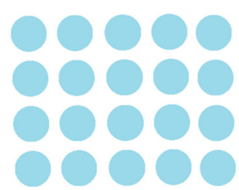
Our mathematical journey began with a study of the counting numbers and their arithmetic properties.

We discovered an operation on the counting numbers called **addition** that creates from any two counting numbers a and b a new number which we write as $a + b$.

We did this by drawing two counts of dots in a row, left and right, and then recounting the entire row.

We also created an operation called **multiplication** that produces from any two counting numbers a and b a new number which we write as $a \times b$.

We drew rectangular arrays of dot and thought of such an array as an organized picture of repeated groups. (We thus thought of multiplication as repeated addition.)



By viewing our pictures in different ways, we discovered various "rules" of arithmetic that seem natural and right for the counting numbers.

Here's a list of all the rules together in one spot.

Addition

Rule 1: We can change the order in which we add any two counting numbers and not change the final result.
That is, for any two counting numbers a and b we have that $\boldsymbol{a+b=b+a}$**.**

Rule 2: Adding zero to a counting number does not change the value of the counting number.
That is, for any counting number a we have that $\boldsymbol{a+0=a}$ and $\boldsymbol{0+a=a}$.

Rule 3: In any string of counting numbers added together

$$a+b+c+d+e+\ldots+y+z$$

it does not matter in which order one chooses to perform the additions. The same answer will always result.

(Rule 3 encompasses Rule 1.)

Multiplication

Rule 4: We can change the order in which we multiply any two counting numbers and not change the final result.
That is, for any two counting numbers a and b we have $\boldsymbol{ab=ba}$.

Rule 5: Multiplying a counting number by one does not change the value of the counting number.
That is, for each counting number a we have that $\boldsymbol{1\times a=a}$ and $\boldsymbol{a\times 1=a}$.

Rule 6: Multiplying a counting number by zero gives a result of zero.
That is, for each counting number a we have that $\boldsymbol{0\times a=0}$ and $\boldsymbol{a\times 0=0}$.

(We had to fuss a little bit to make full sense of this Rule 6.)

Rule 7: In any string of counting numbers multiplied together

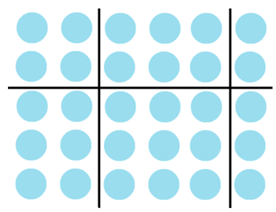
$$a \cdot b \cdot c \cdot d \cdot \cdots \cdot y \cdot z$$

it does not matter in which order one chooses to perform the products. The same answer will always result.

(Rule 7 encompasses Rule 4, and we had to go to quite some fuss to properly explain it.)

Addition and Multiplication Together

Rule 8: "We can chop up rectangles from multiplication and add up the pieces."



These eight rules give the entire the scoop on how basic arithmetic works!

Question: Can you remember the gist of how we got to each rule? (For example, rule 1 comes from reading a row of dots from left to right versus from right to left.)

Of course, feel free to look back at the earlier sections if you want o remind yourself.

FORMAL JARGON

Some school curricula insist that students know and use some formal language to describe each of these eight principles. If you are interested, here it is. (If you are not, skip this page!)

Rule 1 says that **addition is commutative**.
(What is the etymology of the strange word "commutative"? Google it!)

Rule 2 says that 0 is acting as an **additive identity**.

The aspect of Rule 3 that refers changing the order you conduct additions is described as "**addition is associative**." Mathematics books usually focus just on "$(a+b)+c=a+(b+c)$."

Rule 4 says that **multiplication is commutative**.

Rule 5 says that 1 is acting as a **multiplicative identity**.

Rule 6 doesn't have an official name in the mathematics community. Some school textbook authors call it **the zero property**.

The aspect of Rule 7 that refers changing the order you conduct multiplications is described as "**multiplication is associative.**" Mathematics books usually focus just on "$(ab)c=a(bc)$."

Rule 8 is called the **distributive property**. Mathematics books usually focus just on "$a(b+c)=ab+ac$."

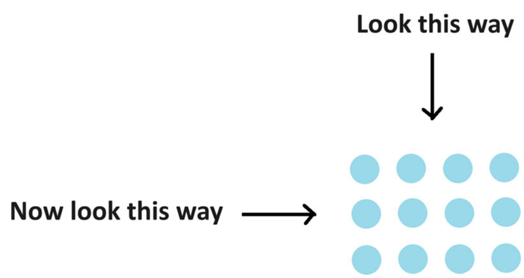
Chapter 2

Playing with the Counting Numbers

15. The Power of a Picture

Now that we've got the counting numbers and their basic arithmetic sorted out, let's engage in some sophisticated mathematical play and mathematical thinking.

We've already seen how changing one's perspective of a picture can lead to mathematical insight. The goal of this section is to do more of the same.



$$4 \times 3 = 3 \times 4$$

Let's start with this picture of 25 dots.

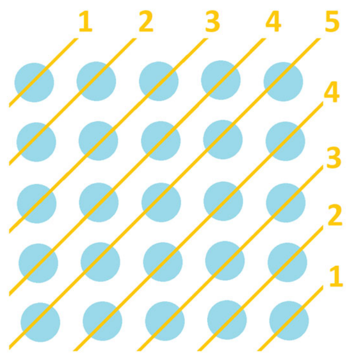
The dots are arranged as five groups of five via rows and simultaneously as five groups of five via columns. It's a picture of 5×5 two ways. That symmetry is kinda cool in-and-of itself. But can you look at this picture another way and see within it the sum of all the counting numbers from 1 up to 5 and back down again?

$$1 + 2 + 3 + 4 + 5 + 4 + 3 + 2 + 1$$

Mull on this before turning the page.

Look at the figure diagonally. That up-and-down sum then comes into view.

As all 25 dots are covered by the diagonals, we can say that the value of that sum just has to be $5 \times 5 = 25$. We don't have to do a lick of arithmetic!



But Check!
Do the arithmetic to verify that $1 + 2 + 3 + 4 + 5 + 4 + 3 + 2 + 1$ does indeed equal 25.

Practice 15.1: Draw a picture of 4 dots that illustrates the sum $1 + 2 + 1$ via diagonals. Draw a picture of 9 dots that illustrates the sum $1 + 2 + 3 + 2 + 1$ via diagonals.

We're on to a general idea.

A picture of a 10-by-10 square of dots looked at diagonally would show the sum of numbers from 1 up to 10 and back down again. As there are $10 \times 10 = 100$ dots in a ten-by-ten square of dots, this sum must have value 100.

$$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 + 9 + 8 + 7 + 6 + 5 + 4 + 3 + 2 + 1 = 10 \times 10 = 100$$

The sum of all the numbers from 1 up to 1000 and back down again appears as the diagonals of a 1000-by-1000 square of dots. There are $1000 \times 1000 = 1{,}000{,}000$, a million, dots in such a picture. We must have

$$1 + 2 + 3 + \cdots + 1000 + \cdots + 3 + 2 = 1 = 1000 \times 1000 = \text{a million.}$$

And we're doing this without a lick of arithmetic! (Well, we did have to know that a thousand times a thousand is a million.)

Practice 15.2: What is the sum of all the numbers from one up to a million and back down again?

Imagine how long it would take to conduct these ludicrously large sums on a calculator. The mind's eye is mightier than the machine here!

If it is fun to note, here's a way to express what we just discovered as a general result. (Don't memorize this as important. This is all just for mathematical amusement.)

> The sum of the numbers from 1 up to a counting number N and back down again equals $N^2 = N \times N$.
>
> $$1 + 2 + 3 + \cdots + N + \cdots + 3 + 2 + 1 = N^2$$

Let's keep going!

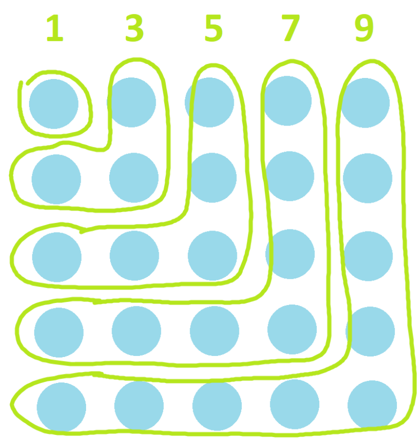
Going back to our five-by-five square of dots, can you look at the picture yet a different way and see the sum of the first five odd numbers $1 + 3 + 5 + 7 + 9$? (We haven't officially discussed even and odd numbers yet, but we shall in the next section. We're just a smidge ahead of ourselves here. I hope that is okay.)

$$1 + 3 + 5 + 7 + 9$$

Again, mull for a while before turning the page.

One can certainly randomly circle groups of 1, 3, 5, 7, and 9 dots in the square array of dots. But we want an approach that generalizes beyond just playing with this one picture.

The key this time is to look at boomerangs. (I am Australian, by the way.)



As the boomerangs I've drawn cover each and every dot, it must be that the sum $1+3+5+7+9$ has value $5 \times 5 = 25$, the number of dots in the picture.

Check It! Confirm that $1+3+5+7+9$ does indeed equal 25.

Practice 15.3: Draw a picture of 4 dots that illustrates the sum $1+3$ via boomerangs. Draw a picture of 9 dots that illustrates the sum $1+3+5$ via boomerangs.

In the same way, the sum of the first eight odd numbers comes from boomerangs covering an 8-by-8 square of dots.

$$1+3+5+7+9+11+13+15 = 8 \times 8 = 64$$

And the sum of the first one-hundred odd numbers must be $100 \times 100 = 10{,}000$. Wow!

$$1+3+5+7+\cdots+197+199 = 100 \times 100 = 10000$$

In general:

The sum of the first N odd numbers is $N^2 = N \times N$.

Now, having summed odd numbers, it is natural to ask:

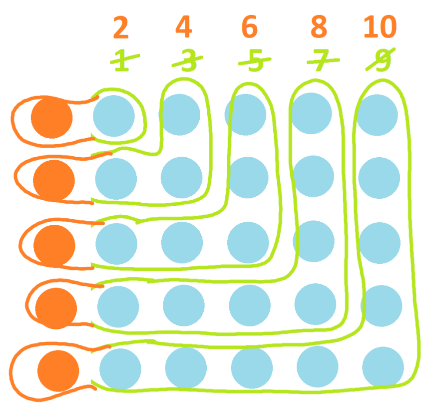
What is the sum of the first N even numbers?

Let's play with what we first learned: the sum of the first five odd numbers is twenty-five. What if we bumped each odd number up by one to get to the even numbers?

1 + 3 + 5 + 7 + 9 = 5 x 5
+1 +1 +1 +1 +1 +5

2 + 4 + 6 + 8 + 10 = 5 x 5 + 5

This is the same as adding an extra dot to each boomerang.



$$2 + 4 + 6 + 8 + 10$$

The picture now looks like a five-by-five square of dots plus five more dots. That makes for $5 \times 5 + 5 = 30$ dots. (We can also regard the picture as a 5-by-6 array of dots and, again, 5×6 equals 30).

$$2 + 4 + 6 + 8 + 10 \quad = 5 \times 5 + 5 \quad = 30$$

Practice 15.4: Draw a picture of $2 \times 2 + 2 = 6$ dots that illustrates the sum $2 + 4$ via boomerangs.

Draw a picture of $3 \times 3 + 3 = 12$ dots that illustrates the sum $2 + 4 + 6$ via boomerangs.

Practice 15.5: What sum of even numbers comes from a picture of $7 \times 7 + 7$?

We're seeing structure here.

The sum of the first six even numbers must equal $6 \times 6 + 6 = 42$.

$$2 + 4 + 6 + 8 + 10 + 12 = 6^2 + 6$$

The sum of the first ten even numbers must equal $10 \times 10 + 10 = 110$.

$$2 + 4 + 6 + 8 + 10 + 12 + 14 + 16 + 18 + 20 = 10^2 + 10$$

The sum of the first one-hundred even numbers must equal $100 \times 100 + 100 = 10{,}100$.

$$2 + 4 + 6 + \cdots + 200 = 100^2 + 100$$

In general:

The sum of the first N even numbers is $N^2 + N$.

Again, nothing to memorize or to keep in your head here. We're just playing.
But we are seeing just how powerful playing with simple pictures can be.

To finish things off for right now, I need to point out something that is a little odd. (Ha!)

We found a quick way to add together the first few odd numbers.

For example, the sum of the first five odd numbers is $1 + 3 + 5 + 7 + 9 = 5 \times 5 = 25$.

We found a quick way to add together the first few even numbers.

For example, the sum of the first five even numbers is $2 + 4 + 6 + 8 + 10 = 5 \times 5 + 5 = 30$.

But we don't yet have a quick way to add together just the first few counting numbers directly.

For example, what is $1 + 2 + 3 + 4 + 5$? What is $1 + 2 + 3 + \cdots + 100$?

Practice 15.6: Here is the sum of the first five even numbers.

$$2 + 4 + 6 + 8 + 10 = 30.$$

Can you see a way to use this sum to figure out the value of $1 + 2 + 3 + 4 + 5$?

Actually, let me answer that question right now. Just halve everything we have in $2 + 4 + 6 + 8 + 10 = 30$. That will give us $1 + 2 + 3 + 4 + 5 = 15$, just like that!

Practice 15.7: The sum of the first twelve even numbers is $12 \times 12 + 12$, which equals 156.

$$2 + 4 + 6 + 8 + 10 + 12 + 14 + 16 + 18 + 20 + 22 + 24 = 156.$$

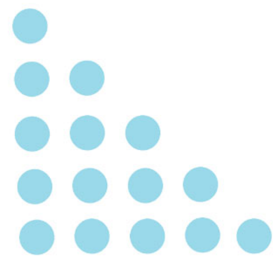
What is the value of the sum $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 + 11 + 12$?

In general, since the sum of the first N even numbers is $N^2 + N$, the sum of the first N numbers is "half of $N^2 + N$."

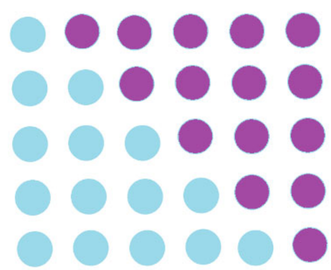
The sum of the fist N even numbers is $N^2 + N$.
The sum of the first N numbers is half of this.

Practice 15.8: Here's another approach to seeing the sum of the first few counting numbers.

I'll start with this picture of 1 dot and 2 dots and 3 dots and 4 dots and 5 dots.
It's a picture of $1 + 2 + 3 + 4 = 5$. (Do you see this via rows?)



Here's two copies of $1 + 2 + 3 + 4 + 5$ together.



a) Can you see that this new picture is telling us that $1 + 2 + 3 + 4 + 5$ is half of $5 \times 6 = 30$?
b) Draw a picture to show that $1 + 2 + 3$ is half of 3×4.
c) Draw a picture to show that $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10$ is half of 10×11. (Or, can you imagine one at least?)

Is what we are seeing here the same as the result on the previous page?

d) Are "half of 5×6" and "half of $5^2 + 5$" the same?
e) Are "half of 3×4" and "half of $3^2 + 3$" the same?
f) Are "half of 10×11" and "half of $10^2 + 10$" the same?
g) Are "half of $N \times (N + 1)$" and "half of $N^2 + N$" the same?

MUSINGS

Musing 15.9

a) On the first day of April, Joi adopts one puppy. On the second day, she adopts two puppies, three on the third day of April, and so on, all the way up to adopting 30 puppies on the 30^{th} day of the month. How many puppies in total did Joi adopt that month?

b) Alberto adopted one kitten on the first day of the year, two on the second day of the year, and so on, up to adopting 365 kittens on the 365^{th} day of the year. How many kittens did he adopt in total that year?

c) If each year on your birthday you were given a birthday cake with as many candles on it as your new age, how many birthday candles would you have blown out in your life?

Musing 15.10

e) What is the sum of the first million odd numbers?
f) What is the sum the first million even numbers?
g) What is the sum of the first million (counting) numbers?
h) **Challenge:** $1{,}000{,}000$ is the millionth counting number. What is the millionth even number?
i) **Challenge:** $1{,}000{,}000$ is the millionth counting number. What is the millionth odd number?

Musing 15.11

a) What is the value of $1 + 3 + 5 + 7 + \cdots + 97 + 99$?
b) What is the value of $2 + 4 + 6 + 8 + \cdots + 498 + 500$?
c) What is the value of $73 + 74 + 75 + 76 + \cdots + 98 + 99 + 100$?
d) What is the value of $512 + 514 + 516 + \cdots + 798 + 800$?

Musing 15.12 Here's another way to compute the sum of the first five counting numbers.

Write the sum twice, in two rows, but with the first sum forwards and the second sum backwards.

1 + 2 + 3 + 4 + 5 =	answer
5 + 4 + 3 + 2 + 1 =	answer
6 + 6 + 6 + 6 + 6 =	2 x answer

Adding by columns shows that five copies of 6 equals twice the answer. That is, we see 5×6 equals double the answer. This means that $1 + 2 + 3 + 4 + 5$ equals "half of 5×6" (as we learned before).

a) Draw a diagram like the one above that shows that the sum of first one-hundred counting numbers must be half of 100×101.

b) Use this approach to evaluate $4 + 7 + 10 + 13 + 16 + 19 + 22 + 25 + 28 + 31 + 34 + 37 + 40 + 43 + 46$.

c) **ABSOLUTELY OPTIONAL:** (But everything in this book is optional!)

Can you make sense of the following statement?

The sum of N evenly-spaced counting numbers equals half of N times the sum of the first and last numbers in the sum.

(For instance, $4 + 7 + 10 + 13 + 16 + 19 + 22 + 25 + 28 + 31 + 34 + 37 + 40 + 43 + 46$ is a sum of fifteen evenly-spaced counting numbers.)

MECHANICS PRACTICE

As mentioned, nothing in this section is "important" in, and of, itself.
But we have already played with some of the ideas of this section back in section 6 where we noted that "order does not" matter when computing a long list of additions. That idea is certainly important and helpful.

Practice 15.13 a) Can you see that the sum of all the numbers from 1 up to 8 and back down again,

$$\mathbf{1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 7 + 6 + 5 + 4 + 3 + 2 + 1}$$

is the same as $8 + 8 + 8 + 8 + 8 + 8 + 8 + 8$? (That's eight copies of 8 added together and so the answer is $8\times 8 = 64$.)

b) Can you see that the sum of the first eight odd numbers,

$$\mathbf{1 + 3 + 5 + 7 + 9 + 11 + 13 + 15}$$

is the same as $16 + 16 + 16 + 16$? (Is that answer the same as 8×8, eight copies of 8 added together?)

c) Can you see that the sum of the first eight even numbers,

$$\mathbf{2+4 + 6 + 8 + 10 + 12 + 14 + 16}$$

is the same as $18 + 18 + 18 + 18$? (Is that answer the same as eight copies of 9 added together, which would be 8×9?)

d) Can you see that the sum of the first eight counting numbers,

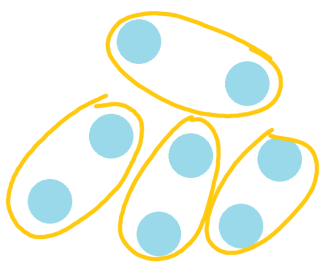
$$\mathbf{1 + 2 + 3 + 4 + 5 + 6 + 7 + 8}$$

is the same as $9 + 9 + 9 + 9$? (This is half of $9 + 9 + 9 + 9 + 9 + 9 + 9 + 9 = 8 \times 9$.)

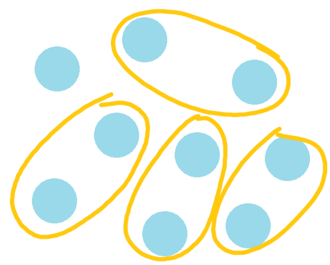
16. Even and Odd Counting Numbers

We played with sums of even and odd numbers in the last section, which was presumptuous of me: I assumed we all knew what even and odd numbers are. Let's talk about them formally now.

> A count of dots is said to be **even** if a set of that many dots can be grouped into pairs without any dots left remaining. (For example, 8 is even as we can group eight dots into pairs of two.)
>
> A count of dots is said to be **odd** if attempting this feat with that many dots always leaves one dot remaining. (For example, 9 is odd.)

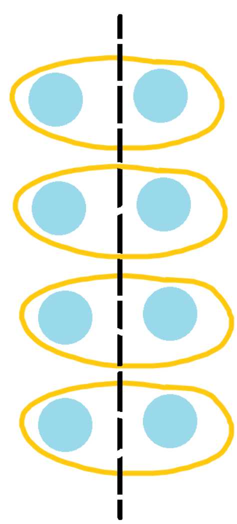


8 is even



9 is odd

Some people prefer to say that a counting number is *even* if we can split a group of that many dots into two equal-sized piles, and *odd* if this cannot be done. (We are assuming here that dots themselves cannot be split.)



8 is still even

Do you see that this is an equivalent definition of "eveneness"?

If we can split a group of dots into two piles of equal size, then move one pile to the left and the other pile to the right. Now we can make pairs by selecting left-and-right dots in turn.

And backwards, if we have a collection of pairs with no dot left over, then we can just split each pair, moving one dot in the pair to the left and the other to the right. This will give you two equal-sized piles in the end.

Practice 16.1: Here's an alternative way to envision matters.

You have two feet.

Imagine you have a huge box of socks. You start putting the socks on, one at a time, first on your left foot, then your right foot, then your left foot, then your right foot, and so on. (You'll have a lot of socks on each of your feet!)

a) Suppose you find at the end that you have the same number of socks on each foot. What does that say about the count of socks that were in the box?

b) If, instead, the count of socks in the box were odd, what would you notice about the count of socks on each of your feet? [Which foot has the greater number of socks?]

Zero

The number zero is always philosophically troublesome!

Is zero even or odd? Neither or both?

If you have zero dots, you can certainly split that collection into two piles of equal size, namely, two empty piles! Zero fits the second definition of evenness.

You could also argue that it readily fits the first definition of eveneness too: From a set of zero dots, one can form zero pairs of dots and, indeed, there are no dots left over.

For this reason(s), zero is considered to be an even number.

Practice 16.2: Is the number 1 even or odd? Convince me that your answer to this is correct.

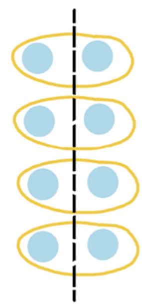
Practice 16.3:
a) A collection of dots was grouped into seven pairs with no dot left over. How many dots were there? Is that an even or an odd number?

b) Another collection of dots was grouped into 100 pairs with one left over. How many dots were there? Is that an even or an odd number?

c) A third collection of dots was split into two piles of 13 dots each and one dot was left over. Was the original number of dots even or odd?

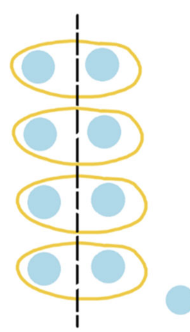
Continuing on …

Every even count of dots can be split into two groups of equal size: just split the pairs in half like we saw before. For example, 8 matches two groups of four: $8 = 2 \times 4$.



Every even number is two times a counting number.

An odd count of dots leaves an extra dot over if we split into two groups of the same size. For example, 9 is one more than two groups of four: $9 = 2 \times 4 + 1$.



Every odd number is one more than two times a counting number.

Some people prefer to use these statements as the definitions for numbers being even or odd. It leads to a more technical-looking approach, but the mathematics is really saying the same thing.

*A counting number N is **even** if we can write $N = 2a$ for some counting number a.*
*A counting number N is **odd** if we can write $N = 2a + 1$ for some counting number a.*

We can be definitive too now about the evenness and oddness of numbers.

Is 0 even or odd?

It is even because 0 equals two times a counting number, namely, it equals 2×0.

Is 1 even or odd?

The number 1 is odd because it equals $2 \times 0 + 1$. (It's one more than double a counting number.)

Is 14 even or odd?

It is even because $14 = 2 \times 7$.

Is 29 even or odd?

It is odd because $29 = 2 \times 14 + 1$.

And so on.

Practice 16.4: Write each of the following numbers either as "two times a counting number" or as "two times a counting number plus one."

50 51 100 121 A MILLION

Practice 16.5:
a) If a counting number is even, explain why the next counting number sure to be odd.
b) If a counting number is odd, explain why the next counting number sure to be even.

Adding and Subtracting Even and Odd Numbers

The sum of two even numbers is even. We can see why.

> *If one set of dots grouped into pairs is combined with a second set of dots grouped into pairs, then the resulting conglomeration of dots is a set of pairs with no dot left over.*

The sum of an even number and an odd number is odd.

> *If one set of dots grouped into pairs is combined with a second set of dots grouped into pairs but with one extra dot, the resulting conglomeration of dots is a set of pairs with one extra dot.*

The sum of an odd number and an even number is odd.

> *We can argue just as for the previous example. (But if you want to be particularly "math sophisto," cite the fact that* $a+b=b+a$ *for all counting numbers. Then you can say that this case is really no different from the previous case.)*

Your turn …
Practice 16.6: Explain why the sum of an odd number and an odd number is even.

To summarize, we have this chart.

EVEN + EVEN = EVEN
EVEN + ODD = ODD
ODD + EVEN = ODD
ODD + ODD = EVEN

Practice 16.7: We haven't yet talked about "taking away" (aka, subtraction), but can you convince yourself of these claims too? (We'll examine this properly in Chapter 3.)

EVEN - EVEN = EVEN
EVEN - ODD = ODD
ODD - EVEN = ODD
ODD - ODD = EVEN

Practice 16.8: Will the value of a sum of the following form be even or odd?

EVEN + EVEN + ODD + EVEN + ODD + ODD + ODD + EVEN + ODD

Practice 16.9: Optional Challenge Exercise not for Everybody's Taste
Here's another explanation as to why the sum of two even numbers is even.

Suppose we have two even numbers N and M.

Then we can write $N = 2a$ and $M = 2b$ for some counting numbers a and b.

The sum of these two numbers is then

$$N + M = 2a + 2b.$$

By "factoring out a 2" this can be rewritten as

$$2a + 2b = 2(a + b).$$

Thus $N + M = 2(a + b)$. This is also twice a counting number, and so $N + M$ fits the definition of also being even.

a) Can you follow the explanation presented? Do you like it? (I personally prefer imagining pairs of dots being combined. But some people like this approach as it is "more mathematical," whatever that means to them.)

b) Are you game for writing a similar explanation as to why the sum of an even number and an odd number is odd?

c) How about writing out an explanation like this for the sum of two odd numbers being even?

Going back to however you like to think of matters, can you convince yourself of the following general result? (Warning: It takes a moment or two to figure out first what exactly each sentence is saying!)

Any sum of counting numbers that involves an even number of odd numbers will be even. Any sum of counting numbers that involves an odd number of odd numbers will be odd.

A sharing moment:

Let me reveal my brain here. It might help.

On the previous page I asked whether a sum of the form

EVEN + EVEN + ODD + EVEN + ODD + ODD + ODD + EVEN + ODD

will give an even or odd answer. I personally look at this and think:

> pairs + pairs + (pairs and **1 extra dot**) + pairs + (pairs and **1 extra dot**) + (pairs and **1 extra dot**) + (pairs and **1 extra dot**) + pairs + (pairs and **1 extra dot**)

That's a whole lot of pairs and 5 extra dots, which can make two extra pairs with 1 dot remaining.

The result is loads of pairs of dots and 1 extra dot. That is, the result is odd.

> **Question:** Can you indeed "see" that any sum involving an odd number of odd numbers will give an odd-numbered answer? And that a sum with an even number of odd numbers will be even?

I think the results on the previous page are astonishingly powerful and astounding!

Let me give some examples to show why I think this.

Example: If you dare to secretly tear twenty random pages out of this book and destroy them, I will then react by telling you that the sum of the missing page numbers is even—and I know I will be right about that without even looking at my mutilated tome!

Why? Each page of a book has an odd page number on one side and an even page number on the other. The sum of the forty missing page numbers is sure to be a sum of twenty even numbers and twenty odd numbers. There is an even number of odd numbers in this sum, and so the sum of the missing page numbers is sure to be even!

Example: Using only pennies (1 cent coins), nickels (5 cent coins), and quarters (25 cent coins), it is impossible to make change for a dollar using precisely fifteen coins.

Why? Fifteen odd values of cents must add to odd value. (An odd number of odd numbers added together.) One dollar--100 cents--on the other hand, is an even number of cents. This task cannot be done because of this mismatch.

Example: The number 699 is the sum of six consecutive counting numbers.
It's $114 + 115 + 116 + 117 + 118 + 119$.

Now write $25{,}608$ as a sum of six consecutive counting numbers too.

Don't bother trying!
In any string of six consecutive counting numbers there will be 3 odd numbers and 3 even numbers. A sum with an odd number of odd numbers in it is sure to be odd. As, $25{,}608$ is an even number, the task ask for is impossible!

Alright. These three examples might now be that exciting. But consider these next four hands-on activities.

ACTIVITY

HANDSHAKE CHALLENGE

It is impossible for an odd number of people in a room to each take part in an odd number of handshakes.

In a room with friends, count how many people are there. If the count is odd, including yourself, great! If it is even, then explain in a moment that you are just going to be a facilitator for the activity and won't take part. (Just have your odd count of friends play.)

Either way, explain the activity:

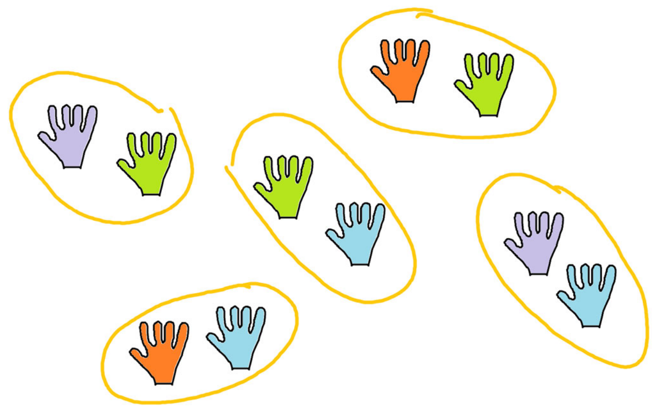
> *Let's all get to know each other by greeting each other and shaking hands. You must greet at least one person, and you may greet the same person more than once. But everyone is to take part in an odd number of handshakes in total. (So, if you shake the same person's hand more than once, you must keep track of those individual counts too.)*
>
> *Once you have taken part in a count of handshakes, any odd number of them you like, feel free to fold your arms to say: "I'm done!". (But know, it is impolite to refuse shaking a hand if someone offers a hand to you.)*
>
> *Okay, let's try it!*

It is impossible for you and your friends to complete this task. Lots of frivolity (usually) happens when people actually attempt this doomed task.

Here's the reason why the task is doomed.

Imagine doing the following:

Each time a handshake occurs, draw a picture of the two right hands that took part in the shake.



At the end of the activity—assuming the task is possible—you will have an **even number of hands drawn on your page**. (The hands are coming in pairs, after all.)

Now, show the picture to all the participants. Each participant will recognize their own hand an odd number of times if each participant really did take part in an odd number of shakes.

As there are an odd number of participants, this means that your picture is composed of an odd number of odd counts of hands. But that makes for an **odd number of hands drawn on your page** in total, not an even total.

We have a mismatch. Someone must have miscounted as an odd number of odd counts cannot add to an even result.

ACTIVITY

CUP-TURNING CHALLENGE

Thirteen cups are placed upside-down on a tabletop.

A "move" consists of choosing two cups and turning them each over. (Cups that are upside down will be turned upright and cups that happen to be upright will be turned back upside down.) Individual cups can be turned over multiple times during the play of this game.

Your job: Turning two cups over at a time, reach a state in which every cup is upright.

[Rather than using cups, try this puzzle using playing cards.]

Here's the reason why this task too is doomed.

In order for a cup to end up upright, it must be turned over an odd number of times in total. (Think about this.) So, with 13 cups, we need to perform an odd numbers of turns thirteen times. Thirteen odd counts add to an odd number of turns in total.

But by being forced to conduct two cup turns at a time, we will never be able to reach an odd number of turns: 0 turns, 2 turns, 4 turns, 6 turns, and so on.

This task described is impossible to complete.

Question: Let's keep playing!

a) Is it possible to solve the cup-turning challenge starting with 12 cups instead of 13? Starting with an even number of cups, can the challenge always be solved?

b) Is it possible to invert 14 cups turning FOUR over simultaneously at a time?

c) Is it possible to invert 15 cups turning FOUR over simultaneously at a time?

Extra Challenge:

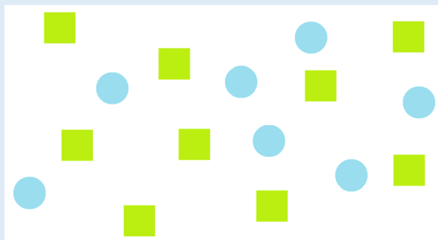
You have some upside-down cups in a row. Now a "move" consists of putting your finger on one cup, keeping it fixed in place, and turning over each of the remaining cups. Your goal is to, after some number of moves, end up with all cups upright.

For which starting counts of cups can this version of the cup-turning puzzle be solved?

ACTIVITY

CIRCLES AND SQUARES GAME

Here is a diagram of some circles and squares.



Edith and Amit will play the following game. They will each take turns choosing two symbols, erasing them, and replacing them with a single symbol determined by the following rule.

ERASE TWO IDENTICAL SHAPES → DRAW A SQUARE
ERASE TWO DIFFERENT SHAPES → DRAW A CIRCLE

The count of symbols on the page decreases by one at each turn and so eventually just one symbol will remain of the page.

If that final symbol is a circle, Edith wins.
If that final symbol is a square, Amit wins.

Play this game a few times. Who wins? Play it several more times. Who keeps winning?

Can you explain what is going on?

Here's why Edith is sure to win the game, even if Amit takes all the turns.

Focus on the count of circles on the page. We are starting with 7 of them.

A move either erases two circles completely or leaves the count of circles the same. (Again, just focus on the circles.) Thus, the count of circles on the page will eventually decrease to 5 and then to 3 and then to 1.

At the end of the game, there is one symbol left. It must be that one circle and Edith wins.

Question: Is the count of squares on the page at all relevant?

Question: Explain why Edith is always sure to win if game starts with any odd number of circles on the page.

Question: If the initial count of circles on the page is instead even, is Amit sure to win?

ACTIVITY

GNOMES, HATS, AND AN EVIL VILLAIN.

Ten gnomes are about to play a game of life and death with an evil villain. They are told that they will be asked to stand in a line, each facing the back of the next, and that hats will be placed on their heads. Each hat will be either black or red, but no indication of how many hats there will be of each color is given. No gnome will be able to see the color of his own hat, but each gnome will be able to see the colors of all the hats in front of him.

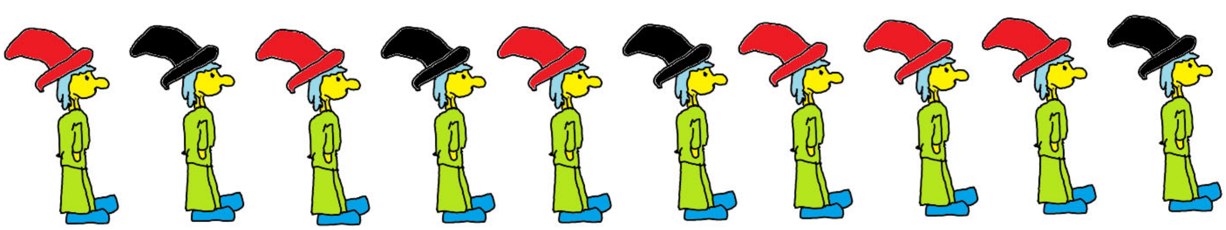
Starting with the gnome at the back of the line (the one who can see nine hats in front of him), and working along the line, the evil villain will ask each gnome in turn to guess the color of his own hat. If a gnome is correct, he will live. If not, he will meet his demise.

Gnomes will not be allowed to speak during the play of this game except for a single word, either red or black when asked, but they will be able to hear the answers – and the subsequent sighs of relief or the screams of horror – of the gnomes behind them. No other information in any shape or form will be transmitted from gnome to gnome.

What scheme could the gnomes agree on before the play of this game that would ensure the survival of a maximum of their number?

Comment 1: The very first gnome is an unfortunate position: No scheme could ever assure him of his survival. But is there a scheme that would absolutely allow for the survival of, say, the next gnome in line? How about every second gnome? Is there a way to ensure the survival of *more* than half the gnomes?

Comment 2: When you do find this special strategy, it is fun to practice it with a group of nine friends. Use playing cards to hold up high, facing backwards along the line, instead of hats.



Surprisingly, it is possible for the team to **ensure the survival of nine out of ten gnomes** (along with a 50% chance of the survival of the first gnome).

Here's how:

> *The gnomes agree on the following strategy:*
>
> *The brave gnome at the front will say "black" if he sees an even number of black hats among the nine hats in front of him, "red" otherwise.*
>
> *Each gnome among the nine will thus know whether there is an even or odd number of black hats among them. Given the number of black hats they see in front of them, and the number of times they hear the word "black" behind them, they can each correctly deduce the color of their own hat.*

In the picture on the previous page, the leftmost gnome will start and will say BLACK as he sees 4 black hats ahead of him, and even count of them. Unfortunately, he himself won't survive (his hat is red), but he has ensured the survival of the remaining nine gnomes.

The next gnome in the line, second from left, sees 3 black hats in front of him. He was told, however, that there are an even number black hats among the nine gnomes, so he deduces that his hat must be black. He says BLACK and survives.

The next gnome in line, third from the left, sees 3 black hats in front of him. He just heard that the hat behind him was black (that accounts for 4 black hats now) and he heard right at the start that there are an even number of black hats among the nine gnomes. The number 4 is even, so he deduces that his hat is red. He says RED and survives.

> **Question**: The next gnome in line, fourth from the left, sees just 2 back hats in front of him. How will he reason to deduce that his hat is black?
>
> Keep going! Explain the reasoning of each gnome as you work down the line.

This strategy works for any number of gnomes in a line.
If 1,000 gnomes play this game, for example, they can be sure of the survival of 999 of them.

That's astounding! And it's just the power of playing with even and odd numbers.

MUSINGS

Musing 16.10

a) Is the sum of seventeen even numbers and thirty-one odd numbers even or odd?
b) Is the sum of the first seven hundred odd numbers even or odd?

Musing 16.11
Is it possible to arrange one-hundred-and-fifty marbles, one of weight 1 gram, one of weight 2 grams, one of weight 3 grams, and so on up to one of weight 150 grams, into two piles of equal total weight?

Musing 16.12
Is 94315 the answer to a sum of one hundred consecutive counting numbers? If so, which sum of counting numbers?

Musing 16.13 In the world of counting numbers, multiplication is repeated addition. Explain then each of the following four claims.

EVEN x EVEN = EVEN
EVEN x ODD = EVEN
ODD x EVEN = EVEN
ODD x ODD = ODD

Musing 16.14 Explain why, if n^2 is even, then n cannot be odd (and so, must be even as well).

Musing 16.15 Explain why the following astounding claim must be true.

> **Right at this very instant, the count of people—living or deceased—who have taken part in an odd number of handshakes is even.**

Musing 16.16
The country of Oddonia only uses 3-cent and 5-cent coins.

a) Is it possible to make change for a dollar (one-hundred cents) using precisely 31 coins in Oddonia?

It is possible to buy something for 1 cent in Oddonia: give the shop clerk two 5-cent coins and receive three 3-cent coins in change, for example.

b) Is it possible to buy an item priced at 2 cents in Oddonia? 4 cents? 7 cents?
Is there a price that cannot be paid in Oddonia?

Musing 16.17 Three counting numbers are chosen at random.

a) Must there be two numbers among them whose sum is even? Explain.
b) Must there be two numbers among them whose sum is odd? Explain.

Musing 16.18 There are 25 people at a meeting. When they greeted each other, could it be that ten of them took part in exactly 4 handshakes, eleven of them in exactly 5 handshakes, and four of them in exactly 6 handshakes? Explain.

Musing 16.19
A **magic square** is a square array of counting numbers with the property that all numbers in the same row, or the same column, or along either of the two diagonals sum to the same value. For example, the picture shows the classic example of a three-by-three magic square with "magic sum" 15. (This magic square happens to use the whole numbers 1 through 9, but a magic square is allowed to repeat values.)

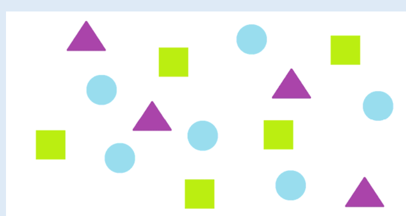
2	9	4
7	5	3
6	1	8

In the classic example, **four** of the numbers are even.

It is possible to create a three-by-three magic square with all **nine** entries even: just put the number 2 in all nine cells, for example. (The magic sum will be 6.) And putting the number 1 in each and every cell creates a magic square with **zero** even entries. (The magic sum here is 3.)

a) Explain why it is impossible to construct a three-by-three magic square with just **one** even entry.
b) Explain why it is impossible to construct a three-by-three magic square with **two** even entries.
c) Can you create a three-by-three magic square with exactly **three** even entries?

Musing 16.20 Here's another game of solitaire. Start with the following diagram of squares, circles, and triangles.



A "move" consists of erasing any two figures of different shape and drawing in their stead a single figure of the third shape. (For example, after erasing a square and a triangle, one is to draw a circle.) Keep going until no more valid moves are possible.

a) Explain why this game is sure to end with nothing but squares left on the page.
b) Why is a game that starts with 107 squares, 225 circles, and 306 triangles on the page sure to end with nothing but triangles?

<u>Hint</u>: The count of each type of shape changes by one with each move. The game will end when the count is zero (an even number) for two shapes.

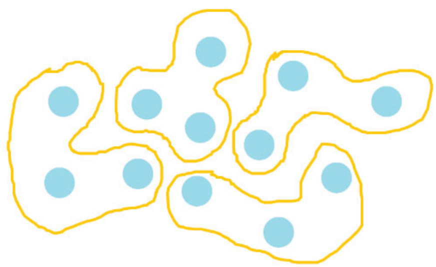
17: Division

We started the last section by saying that a number is **even** if a set of that many dots can be grouped into pairs—sets of 2—without any dots remaining. (And it is **odd** if this feat cannot be done.)

We can extend this idea to beyond focusing on pairs of dots.

> A number is **divisible by 3** (or "can be divided by 3") if a count of that many dots can be grouped into sets of 3 with no dots remaining.

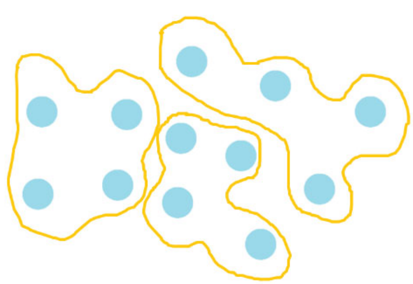
For example, a count of 12 dots can be divided into four sets of 3, and so 12 is divisible by 3.



12 is divisible by 3

A number is **divisible by 4** (or "can be divided by 4") if a count of that many dots can be grouped into sets of 4 with no dots remaining.

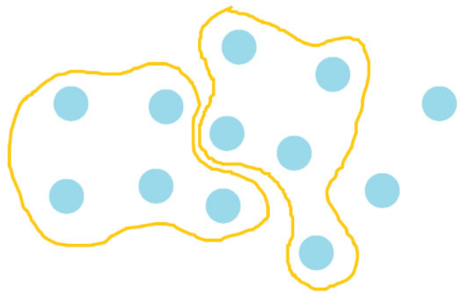
For example, a count of 12 dots can also be divided into three sets of 4, and so 12 is divisible by 4.



12 is also divisible by 4

And, in general, for any counting number N we say:

A number is **divisible by N** ("can be divided by N") if a count of that many dots can be grouped into sets of N dots with no dots remaining.



12 is not divisible by 5

In 1659, Swiss mathematician Johann Rahn used the symbol ÷ (called an **obelus**) in this context of divisibility. He would write

$$12 \div 3 = 4$$

to indicate that 12 objects, when divided into sets of 3, yields 4 such sets. (Though, saying this backwards feels a little more natural: "There are 4 groups 3 among 12 objects.")

Similarly, writing

$$12 \div 4 = 3$$

indicates that dividing set of 12 objects into sets of 4 yields 3 such sets.
("There are 3 groups 4 in a picture of 12 objects.")

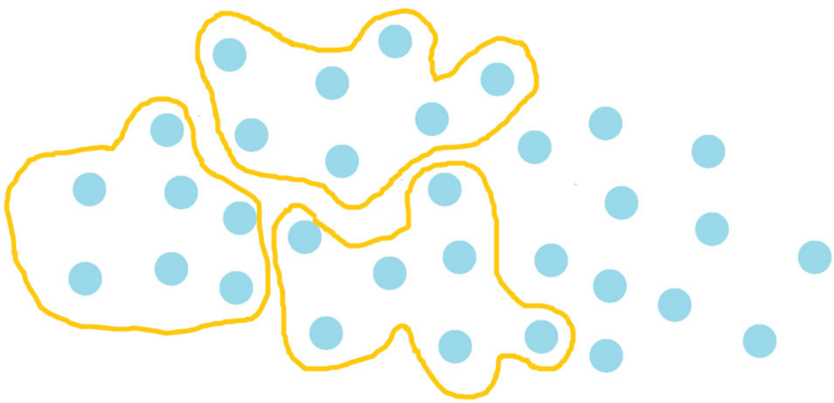
In general, a computation of the form $a \div b$ seeks to know how many groups of b objects one can find within a set of a objects, hopefully without any objects left over. If a set of objects remains unaccounted for, then we say we have a **remainder**.

For example, the picture at the top of this page shows that:

12 leaves a remainder of 2 upon division by 5.

People usually prefer to make the remainder in any division problem is as small as possible. For instance, this picture shows that

> 32 leaves a remainder of 11 upon division by 7.



This statement is absolutely correct, but it might not be seen as efficient as one can identify yet another group of 7 among the remaining dots. Stating

> 32 leaves a remainder of 4 upon division by 7

is another correct statement and is probably the preferred one.

Practice 17.1: Quentin was feeling cheeky and wrote:

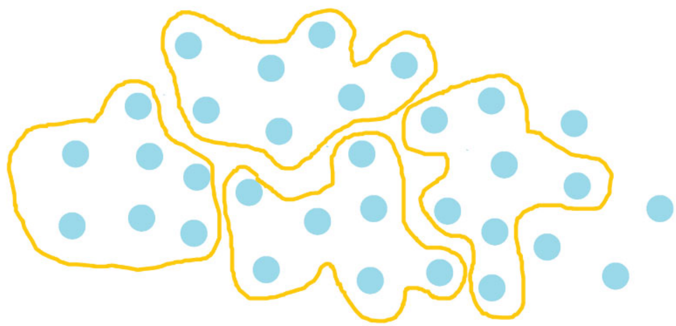
> 32 *leaves a remainder of* 32 *upon division by* 7.

Is his statement correct? Is it likely helpful?

Practice 17.2: Explain why each number can be said to leave a remainder of either 0, 1, 2, 3, 4, 5, or 6 upon division by 7.

What does a remainder of 0 indicate about the number?

Practice 17.3: This picture shows that $32 = 4 \times 7 + 4$ (four groups of 7 and four remaining dots).



How would you translate the picture at the top of the previous page into a mathematics statement like this?

Three Different Problems with the Same Answer

Consider these three questions. They each have the answer 4, but the real question is why, philosophically, should they have the same answer?

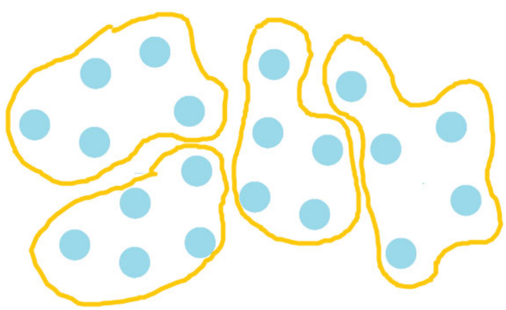
Problem 1: How many groups of 5 can be found within a collection of 20 objects? That is, what is the answer to the division problem $20 \div 5$?

Problem 2: Which number fills in this blank to this multiplication statement?

$$\blacksquare \times 5 = 20$$

Problem 3: I have 20 pies to share equally among 5 students. How many pies will each student receive?

This may seem somewhat cryptic, but do you see that this one picture illustrates the answer to each of the three problems?



With regard to Problem 1: The picture shows 20 dots divided into groups of 5. We count four such groups of 5 and thus conclude

$$20 \div 5 = 4$$

With regard to Problem 2: The picture shows four groups of 5 dots making 20 dots. It's a picture of the multiplication statement

$$4 \times 5 = 20$$

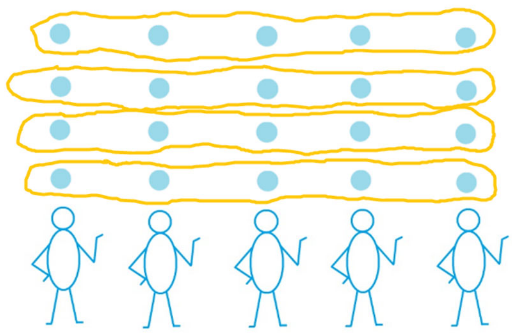
With regard to Problem 3: The picture seems unrelated to sharing pie.

But imagine being asked to distribute 20 pies equally among 5 students.

One could be methodical and hand out one pie to the first student, one pie to the second student, one to the third student, one to the fourth student, one to the fifth student, a second pie to the first student, a second pie to the second student, and so on. But it is hard to tell what final result such a (tedious) approach will yield.

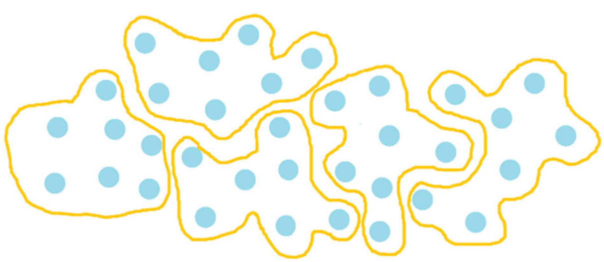
Instead, we can start by organizing pies. Group the pies into sets of 5 and then hand each student one pie from each group of 5.

We can illustrate this organized approach by stretching each group of 5 in our original picture across the five students.



As each students gets one pie per group and there are four groups of 5, each student receives 4 pies.

Example: Here's a picture.



a) A la problem 1, what division statement is it showing?
b) A la problem 2, what multiplication statement is it showing?
c) A la problem 3, what pie-sharing statement is it showing?

Answer:

a) We're seeing 35 dots divided into groups of 7 and there are 5 such groups.
It's a picture of the division statement $35 \div 7 = 5$.

b) We're seeing 5 groups of 7 dots making 35 dots.
It's a picture of the multiplication statement $5 \times 7 = 35$.

c) We're seeing how to share 35 pies equally among 7 students.
We're organizing the pies into groups of seven first and will hand each student a pie from each group. As there are five groups, each student will receive 5 pies.

Practice 17.4: For each of these three statements, find the number that fills in the blank.

1) $91 \div 7 = \blacksquare$

2) $\blacksquare \times 7 = 91$

3) *Sharing* 91 *avocados equally among* 7 *bonobos yields* $\blacksquare$ *avocados per bonobo.*

Problem 1 asked as to complete a division problem: how many groups of a given size can we find among a given collection of objects? Let's be specific and call this thinking **division by groups**.

Problem 2 asks about multiplication. But it gave us the answer and asked us about part of the question. Let's call such a challenge a task in **reverse multiplication**.

Problem 3 asks us about equally distributing a collection of objects. Let's call such a problem a **sharing** challenge.

And we have seen that all three problems, the three modes of thinking they represent, are equivalent!

We have three ways to interpret the answer to a division task $N \div a$.

1. It's the number of groups of size a we will find among N objects.

2. It the number that completes this multiplication statement: $\blacksquare \times \mathrm{a} = \mathrm{N}$.

3. If N objects are shared equally among a participants, then it is the number of objects each participant receives.

Schoolbooks call all three of interpretations "division."

So, $20 \div 5$, for instance could regarded as a task of counting groups of size five in a picture of twenty objects (**division by groups**), or as a task of seeking the number that multiplies by 5 to give the answer 20 (**division by reverse multiplication**), or as a task of determining what results when twenty objects are shared equally among five participants (**division by sharing**).

We've seen that all three modes of thinking are equivalent and so there is no harm done in bouncing back and forth between these three modes and in calling all of them **division**.

Practice 17.5: Make up three word problems that could appear in an elementary school textbook that has students think about $42 \div 6$ via each of these frameworks.

a) division by groups.
b) division by reverse multiplication.
c) division by sharing.

ABSOLUTELY OPTIONAL COMMENT

Folk who work in the field of mathematics education use tricky terms for the ways we can interpret a division statement.

They call "division by groups" **quotative division**.
They call "division by sharing" **partitive division**.
(There doesn't seem to be a special term for thinking of division as reverse multiplication.)

Can you research the etymology of these terms?

Practice 17.6 Knowing that $1035 \div 45 = 23$, what then is $1035 \div 23$?
What's the rationale for your answer?

The "reverse multiplication" approach has us compute $48 \div 6$, say, by asking: *What times* 6 *gives* 48*?* The answer is 8. (This is the way most people go about thinking through a basic division question.)

This thinking also provides a way to check answers to division problems.

We see:

$30 \div 6 = 5$ is correct because 5×6 is 30.

$45 \div 5 = 9$ is correct because 9×5 is 45.

$80 \div 6 = 10$ is **not** correct because 10×6 is not 80.

Why Can't You Divide by Zero?

Let's end off this section with an age-old question.

What exactly goes wrong if you try to divide by zero?

Let's explore $5 \div 0$. I think it equals 3.

Following our reverse multiplication check …

$5 \div 0 = 3$ is not correct because 3×0 is not 5.

Okay, I've changed my mind. I think $5 \div 0 = 7$.

$5 \div 0$ is not 7 because 7×0 is not 5.

I can keep guessing and keep seeing I am wrong.

$5 \div 0$ is not 43 because 43×0 is not 5.

$5 \div 0$ is not 679 because 679×0 is not 5.

$5 \div 0$ is not $1{,}000{,}002$ because $1{,}000{,}002 \times 0$ is not 5.

We can see that there is no number N we can dream of to make the statement $5 \div 0 = N$ correct. This is because $N \times 0$ is sure to be zero—not 5—no matter the value of N. (This was a rule of arithmetic we noted back in section 14.)

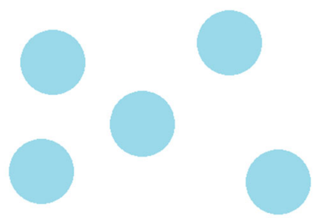
Math is telling us that there is no answer to be had in dividing 5 by 0.
(Mathematicians say that the expression $5 \div 0$ is **undefined**. There is no possible value for it.)

Practice 17.7: Hang on! Might $5 \div 0$ be equal to 0?

Practice 17.8: Convince me that $123 \div 0$ is also undefined.

It's fun to contemplate the "real-world" interpretations of division by zero.

Practice 17.9: **$5 \div 0$ via Division by Groups**
Here is a picture of five dots. How many groups of zero you see in the picture? Is that a meaningful question?



Practice 17.10: **$5 \div 0$ via Division by Sharing**
I have 5 pies to share equally among zero students. If I conduct this sharing task, how many pies will each student receive?

Does that sound like a meaningful question to you?

Many schoolbooks say that dividing by zero is impossible because it has no meaning when you think of division by groups or division as sharing.

But it is the use of multiplication that gives the true mathematical explanation as to why we must reject division by zero.

Except there is a hiccup!

What's the value of $0 \div 0$?

Playing with reverse multiplication, we see that $0 \div 0$ seems to have value 36.

$0 \div 0 = 36$ could be correct because $36 \times 0 = 0$.

Question: If you look at a picture of zero dots, do you see 36 groups of zero?
If you share no pies equally among no students, will each student get 36 pies?

Both these questions (and answers) feel like nonsense to me!

But multiplication also suggest that $0 \div 0$ could have value 7 or value 777 or value 7056452089!

$0 \div 0 = 7$ could be correct because $7 \times 0 = 0$.

$0 \div 0 = 777$ could be correct because $777 \times 0 = 0$.

$0 \div 0 = 7056452089$ could be correct because $7056452089 \times 0 = 0$.

The trouble with $0 \div 0$ is that **every** value passes our multiplication check.
Too many numbers are possible values for it!

(If you want the fancy language, mathematicians sometimes say that value of $0 \div 0$ is **indeterminate**.)

For $5 \div 0$, there is no number that is a possible value for it.
And $0 \div 0$ has the opposite problem: there are too many numbers that are possible values for it!

We're seeing that dividing by zero is deeply problematic both intuitively—when we are thinking of division as groups and when thinking of division as sharing—and mathematically via multiplication in reverse.

So, people say: **Just don't do it! Don't divide by zero!**

The mathematics and our intuition are aligned on this matter.

Practice 17.11: Can the number 0 be divided by 5? If so, what is the value of $0 \div 5$?

(Try exploring this matter by thinking about division by groups and division by sharing, and then see if your answer is supported by making use of reverse multiplication.)

MUSINGS

Musing 17.12 Every few months a math challenge makes the rounds on social media asking for the value of this expression:

$$8 \div 2(2+2)$$

a) Some people argue vehemently that it has value 1. Can you see how those folk might come to that value?

b) Some people argue vehemently that is has value 16. Can you see how these folk might come to that value?

The conventions of arithmetic dictate that if an expression includes parentheses, we are to compute the value of the expression inside the parentheses first. So, we are certainly being asked to evaluate

$$8 \div 2 \times 4$$

The trouble is that we have no clear convention on how to handle a division sign and a multiplication together in an expression—hence the debate and the confusion!

But we can avoid the confusion by use of more parentheses.

For example, writing $8 \div (2(2+2))$ ensures that the expression will be evaluated to give 1 as the answer.

c) Insert parentheses into the original expression to ensure all who look at it will obtain the value 16.

The original expression is an example of intentionally ambiguous writing. No respectable math author would write this. (Just saying!)

d) Here are some English sentences that are deliberately ambiguous.

"I painted the room with the lights off."
"I saw a man with binoculars."
"She spoke to her friend with an accent."

For fun, try coming up with some more deliberately ambiguous sentences.

Musing 17.13 Give three different division problems all with the answer "6 with a remainder of 2."

Musing 17.14 A number leaves a remainder of 5 upon division by 7. What remainder will double the number leave when divided by 7?

MECHANICS PRACTICE

Practice 17.15 Use multiplication to determine which of these division statements are correct.

a) $103 \div 103 = 1$ b) $1000 \div 125 = 10$ c) $999 \div 1 = 999$

d) $0 \div 5 = 0$ e) $1 \div 3 = 3$ f) $0 \div 1 = 1$

Practice 17.16 In this question a is a counting number different from zero.

a) What is the value of $a \div a$?
b) What is the value of $a \div 1$?

How do you know each of your answers are correct?

Practice 17.17 Fill in the appropriate value for each box.

$7 \times \square = 56$ $56 \div \square = 8$

$7 \times 8 = \square$ $56 \div 7 = \square$

$\square \times 8 = 56$ $\square \div 7 = 8$

$56 \div \square = 7$

(This is not a very exciting problem, I know. But is illustrating that the single math fact "$7 \times 8 = 56$," for instance, can be enquired about in many different forms.)

18. Factors, Prime Numbers, and Composite Numbers

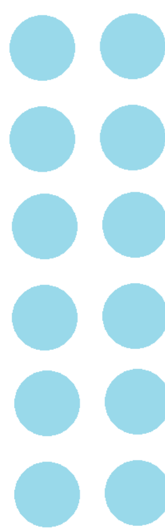
Let's go back to playing with dots directly and ask:

> How many different rectangles can one make with 12 dots?

Here's one way, a 2-by-6 rectangle. (Remember, it has become the convention to mention the count of rows first and the count of columns second.)

2×6

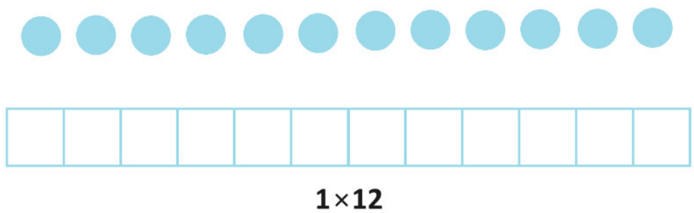
If you turn the picture 90-degrees you get a second rectangle for free.



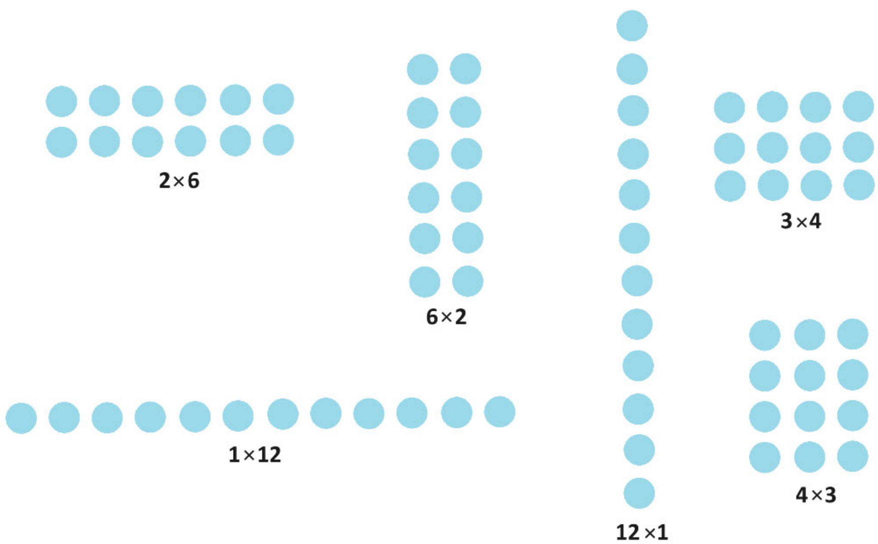
6×2

Practice 18.1: How many different rectangles in total can you make with 12 dots?

There is some debate as to whether or not a single row of 12 dots or a single column of 12 dots really counts as a "rectangle." But if we were playing this game with 12 unit squares instead of dots, then most everyone agrees a picture representing 1×12 is a rectangle.



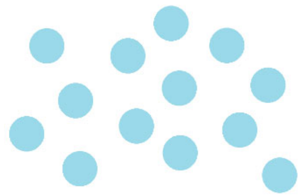
So, let's include these degenerate cases too. In which case, there are a total of six rectangles you can make with 12 dots.



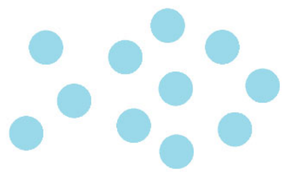
The numbers that arise as the widths and lengths of rectangles – 1, 2, 3, 4, 6, and 12 – are called the **factors** of the number 12. The factors 1 and the number 12 itself are sometimes called the **improper factors** of 12 as the rectangles associated with them might be deemed that way. The remaining factors, 2 3, 4, and 6, are the **proper factors** of 12.

Practice 18.2: List all eight factors of 150.

Practice 18.3: How many different rectangles can one make with 13 dots?



Practice 18.4: How many different rectangles can one make with 11 dots?



Some counts of dots are just resistant to making rectangles. The numbers 13 and 11 each only have two factors, the two improper ones, and you only can make two (improper) rectangles with them.

Practice 18.5: How many different rectangles can you make with just 3 dots?

Practice 18.6: How many different rectangles can you make with just 2 dots?

Practice 18.7: How many different rectangles can you make with just 1 dot?

The numbers 3 and 2 are also resistant to making rectangles—they each have only two factors—and the number 1 is especially resistant: it has only one (improper) factor.

Practice 18.8: Would you regard the picture of a single dot a rectangle? Maybe it's a picture of a one-by-one square?

It is common to regard squares to be examples of rectangles too.

Practice 18.9: How many different rectangles can you make with 9 dots?

Here's the proper language and formal mathematics behind these ideas.

A counting number a is a **factor** of the counting number N if $N = a \times b$ for some counting number b.
(That is, a is the width of some rectangle composed of N dots. b is the rectangle length.)

A factor a of N is a **proper factor** if it is different from 1 and different from the number N itself.
(That is, a is a side-length of a "meaningful" rectangle composed of N dots.)

A number N is called **composite** if has more than two factors.
(That is, you can make at least one meaningful rectangle with N dots. That is, N has at least one proper factor beyond the two improper factors.)

A number N is called **prime** if it has exactly two factors.
(That is, N is a "resistant" number that allows you to make two (not just one) improper rectangles with N dots. So, N is a number different from 1 that has only its two improper factors 1 and N as factors.)

Question 18.10: Explain why the number 0 is composite.

Question 18.11: Give an example of a composite number with exactly one proper factor.

Question 18.12: Does the number 1 fit the definition of being composite? Does it fit the definition of being prime?

The Number 1

The number 1 is neither prime nor composite: it doesn't have more than two factors and it doesn't have exactly two factors. It has one factor, namely, itself.

And yep, there is only one "rectangle" you can make with one dot.

One dot trying to be a rectangle.

During the 1800s, mathematicians decided to keep 1 out of the list of prime numbers. They were studying how prime numbers work in arithmetic, in particular, how they multiply together to create other numbers, and they found it irritating to have to keep track of any 1s that might be part of the product.

For example, 6 is the product of two prime numbers—exactly two!

$$6 = 2 \times 3$$

But if we were to consider 1 to be a prime number, then there infinitely many (irritating) ways to write 6 as a product of prime numbers.

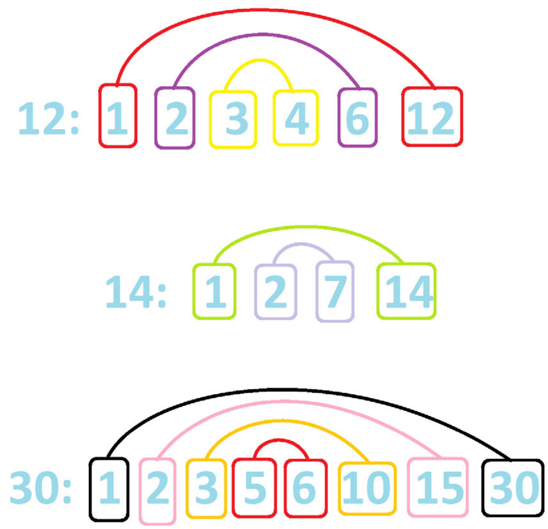
$$6 = 2 \times 3 = 1 \times 2 \times 3 = 1 \times 1 \times 2 \times 3 = \cdots = 1 \times 1 \times 1 \times 1 \times 1 \times 1 \times 1 \times 1 \times 1 \times 2 \times 3 = \cdots$$

For ease, mathematicians decided to deny the number 1 the status of being prime.
It is the only counting number that is neither composite nor prime.

Practice 18.13: If $32 = a \times b$ for two counting numbers and a is a proper factor of 32, explain why b must be a proper factor of 32 as well.

Factor Pairs

It looks like the factors of numbers naturally come in pairs: rectangle lengths and widths. Each pair consists of two numbers that multiply together to the given number.



Practice 18.14: Draw a diagram of the factor pairs for 19.

Since the factors of a number seem to come in pairs, we might conclude that each counting number has an even number of factors. Alas, this is not so.

Practice 18.15:

a) List all the factors of the number of 36 and attempt to identify its factor pairs. What's up?

b) Draw all the rectangles one can make with 16 dots.
(Remember squares are considered to be rectangles too.)

What are the factors of 16? What are the factor pairs? Why does this number have an odd number of factors?

c) Does the number 100 have an even or odd number of factors? How do you know?

d) Why does every square number have an odd number of factors?

e) If a number has an odd number of factors, why must it be a square number?

We have the following observation.

> Every square number has an odd number of factors.
> All other non-zero counting numbers have an even number of factors.

Practice 18.16: Does 0 have an even or odd number of factors? (Is it even possible to answer this question?)

We can have some fun with this.

ACTIVITY

THE CLASSIC LOCKER PROBLEM

A corridor in a school has a line of 25 lockers on its left side. Initially all the locker doors are closed.

Twenty-five students decide to conduct the following experiment:

Student number 1 will walk down the corridor and open every locker door.

Student number 2 will walk down the corridor and close every second locker door. (Locker numbers 2, 4, 6, and so on.)

Student number 3 will walk down the corridor and change the state of every third locker door (lockers 3, 6, 9, …). She will close if it is open and open it if it is closed.

Student number 4 will change the state of every fourth locker door.

Student number 5 the state of every fifth locker door, and so on, all the way to student number 25 who changes the state of every 25th locker door (namely, just the last one).

At the end of this process, which locker doors are left open?

a) Try this experiment using a line of 25 playing cards, all initially face down. What do you notice?
b) Which students touched locker number 12 during this experiment?
c) Which students touched locker number 16 during this experiment?
d) Can you explain the results of this experiment before reading on?

Explaining the Locker Experiment

To get a feel for what is going on, ask: *Which students touched locker 12?*

Answer: Students 1, 2, 3, 4, 6, and 12 each touched locker 12, and only these students. (Do you see this?) This locker door was touched six times—once for each of these factors of 12—and so was returned to a closed state by the end of the experiment.

Which students touched locker 16?

Answer: Students 1, 2, 4, 8, and 16 each touched locker 16, and only these students. Door 16 was thus touched an odd number of times and so was left open at the end of the experiment.

We're now seeing that locker number N is touched only by students whose number is a factor of N. And if N has an even number of factors, its door will be left closed. If N has an odd number of factors, its door will be left open.

Only the square numbers have an odd number of factors. Thus doors 1, 4, 9, 16, and 25 are left open by the end of this experiment.

Practice 18.17: Going Further Here are some more issues to think about if you like.

a) Does it matter in which order the students walk down the corridor? Will the final outcome be the same if students decided to walk down the corridor in a random order?

b) Suppose I want locker number 1 open and all the rest closed. I don't want to send all the students down the corridor (this will leave lockers 4, 9, 16, and 25 open as well), so let's send down only a subset of students. Which students should we send down?

Primes: The Atoms of Arithmetic

People sometimes refer to the prime numbers as the "atoms" of arithmetic since every counting number is built from the prime numbers in the following sense.

Every counting number different from 1 is either a prime number or can be written as a product of prime numbers.

For example:

23 is a prime number. There is nothing more to say about it here.

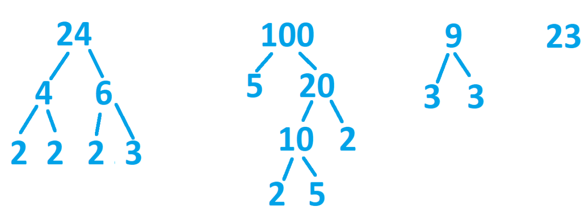
26 is a composite number. It factors as 2×13 and 2 and 13 are each prime.

30 is a composite number. It factors as 3×10, for example, and 10 factors as 2×5.
Put these observations together and we see we can write 30 as $3 \times 2 \times 5$.
That's a product of three prime numbers.

32 can be written as 4×8, which is equal to $2 \times 2 \times 8$, which equals $2 \times 2 \times 2 \times 4$, which equals $2 \times 2 \times 2 \times 2 \times 2$. Thus 32 is a product of five prime numbers, which each happen to be 2.

In short, if a number is not already a prime number, write it as a product of two proper factors.
If each of these factors is prime, then we're done. If not, factor the factors into proper factors and keep doing this until you cannot factor further. At this point, you'll have a nothing but prime factors of the original number, and they all multiply together to make that number.

Question: Did you draw "factor trees" when you were in school? (The factor tree of a prime number is quite short!)



$$24 = 2 \times 2 \times 2 \times 3$$

$$100 = 2 \times 5 \times 2 \times 5$$

$$9 = 3 \times 3$$

$$23 = 23$$

Practice 18.18: Write the number 1000 as a product of primes.

Again, we see why we don't want to regard 1 to be a prime number.

We have

$$24 = 2 \times 2 \times 2 \times 3$$

Allowing "1" into our considerations means we could also write

$$24 = 1 \times 2 \times 2 \times 2 \times 3$$

$$24 = 1 \times 1 \times 2 \times 2 \times 2 \times 3$$

$$24 = 1 \times 1 \times 1 \times 2 \times 2 \times 2 \times 3$$

$$24 = 1 \times 1 \times 1 \times 1 \times 2 \times 2 \times 2 \times 3$$

and so on.

We wouldn't know when to stop!

Practice 18.19: Which prime numbers are even? List them all.
Convince me that your list is complete.

The Prime Numbers are Tricksters

The list of prime numbers begins

$$\mathbf{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, \ldots}$$

and it was proved 2300 years ago by Greek scholar Euclid that this list goes on forever. (See Musing 18.26 if you are curious.)

> **Practice 18.20:** Continue the list until you get to the first prime number bigger than 100.

In this list, we see some pairs of consecutive odd numbers that are both prime.

3 and 5	5 and 7	11 and 13	17 and 19
29 and 31	41 and 43	59 and 61	

Such pairs of prime numbers are called **twin primes** and, surprisingly, no one on the planet currently knows if the list of examples of twin primes goes on forever, or if the list of twin primes eventually halts. (This is a famous unsolved problem in mathematics.)

> **Practice 18.21:** According to the internet, what is the largest example of a pair of twin primes currently known? (Can you make sense of the answer offered?)

It is known that the list of **triplet primes**—three consecutive odd numbers, each prime—stops. The list starts with

3 and 5 and 7

and, well, stops there! There is no other example of a triplet of primes.

> **Hard Challenge 18.22:** Can you explain why this is the case?
> (A good place to start is to see if you can explain why, for three consecutive odd numbers, one of those numbers must be divisible by 3. Then ask: Which prime number(s) are divisible by 3?)

There are many examples of false patterns that occur with the prime numbers.

Here's a famous one.

Example: In the mid-1700s, Swiss mathematician Leonard Euler noticed that

$0 \times 1 + 41 = 41$ is prime
$1 \times 2 + 41 = 43$ is prime
$2 \times 3 + 41 = 47$ is prime
$3 \times 4 + 41 = 53$ is prime
$4 \times 5 + 41 = 61$ is prime
…
$10 \times 11 + 41 = 151$ is prime
…
$27 \times 28 + 41 = 497$ is prime
…

(Use Google or Siri or some such to check these.)

Does this pattern persist? That is, if we kept going with the pattern, do we obtain a prime number value each-and-every time?

As you can guess, the answer is NO!

Amazingly, this pattern consistently gives prime number values for $0 \times 1 + 41$ and $1 \times 2 + 41$ all the way through to $39 \times 40 + 41$, but not for $40 \times 41 + 41 = 1681$. There luck runs out.

Here's why:

"Factor out a 41*" from* $40 \times 41 + 41$ *to write the expression as*

$$(40 + 1) \times 41.$$

This is $41 \times 41 = 1681$*, a square number, not a prime number.*

After this incredible run of prime number values, this pattern becomes quite hit and miss (more miss, than hit) as to when it gives prime numbers again.

Practice 18.23: Is $41 \times 42 + 41$ a prime number?

MUSINGS

Musing 18.24 What is the largest known prime number today?
(Can you make sense of the answer presented on the internet?)

Musing 18.25 Take the number 11, which is prime, and keep adding multiples of 30 to it. This produces a list of numbers that begins

$$11, 41, 71, 101, 131, \ldots.$$

Each number mentioned so far is a prime number. (Check this with Alexa if you like.)

If you keep going, does this list produce only prime numbers?

Musing 18.26 Greek scholars some 2300 years ago called a number **perfect** if the sum of all its factors, excluding the number itself, sum to the number. For example, the number 6 is perfect. Its factors (excluding the number 6 itself) are 1, 2, and 3 and we have

$$1 + 2 + 3 = 6.$$

The next perfect number is 28. We have

$$1 + 2 + 4 + 7 + 14 = 28.$$

a) Show that the number 496 is also perfect.

b) Use the internet to find the next few perfect numbers.

c) No one on this planet knows how many examples of perfect numbers should exist—finitely many? Infinitely many? Who knows? (And if they know, why aren't they sharing?)

   According to the internet, how many examples of perfect numbers are currently known?

   No one knows if there is an example of a perfect number that is odd.
   (This is another famous unsolved problem in mathematics!)

Musing 18.27 Some 2300 years ago, Greek scholar Euclid established that the list of primes cannot stop. (That is, the list must go on forever and so there are infinitely many prime numbers.)

Your job in this question—if you choose to keep reading it—is to answer this question at the end of it:

> *On a scale of 1 to 5—where 1 means "not at all, not even one tiny-eensy bit" and 5 means "utterly and completely" –to what degree did you understand what you just read?*

Okay, here's the thing to read.

> To prove that the list of primes cannot stop, Euclid argued this way.
>
> If you think you know all the primes there are to know, then I suggest you multiply them all together, and then add one to that product.
>
> > *My Example:*
> > *I am going to pretend I know the prime numbers 2, 5, and 11, but no more!*
> >
> > *Euclid wants me to consider* $2 \times 5 \times 11 + 1$*, which equals* 111*.*
>
> The number you now have is one more than a multiple of each of your prime numbers. It is not divisible by any of your prime numbers, and so has none as your prime numbers as a factor.
>
> > *My Example: True!*
> > $2 \times 5 \times 11 + 1$ *is one more than a multiple of* 2*, and* 2 *is not a factor of* 111*.*
> > $2 \times 5 \times 11 + 1$ *is one more than a multiple of* 5*, and* 5 *is not a factor of* 111*.*
> > $2 \times 5 \times 11 + 1$ *is one more than a multiple of* 11*, and* 11 *is not a factor of* 111*.*
>
> When we write this number as a product of primes (draw a factor tree for it), the product will use only primes that are factors. None of your primes are factors, so you must be using new primes that weren't on your first list!
>
> > *My Example:*
> > *Heavens! All right.*
> >
> > 111 *factors as* 3×37*.And yes! I've just discovered two new primes to add to my list.*
> >
> > *Okay, I now know the prime numbers* 2*,* 3*,* 5*,* 11*, and* 37*, but no more!*
>
> Keep repeating this process to discover more and more new primes to add to your list. Multiply together all the primes you now know, add 1 to the answer, and then draw a factor tree for that final result to discover new primes.

My Example:
Okay, now I am going to consider $2 \times 3 \times 5 \times 11 \times 37 + 1 = 12211$.

The factor tree for this number is quick: 12211 *is already prime according to my computer. Ah! It's new prime number to add to my list.*

Okay … I am aways going to be able to find new primes to add to any list I have.

This process will keep producing more and more and more primes.

No finite list of primes will thus ever be complete!

Here's the question:

On a scale of 1 to 5—where 1 means "not at all, not even one tiny-eensy bit" and 5 means "utterly and completely" –to what degree did you understand what you just read?

MECHANICS PRACTICE

Practice 18.28
The number of degrees in a circle is 360. This number is highly divisible.

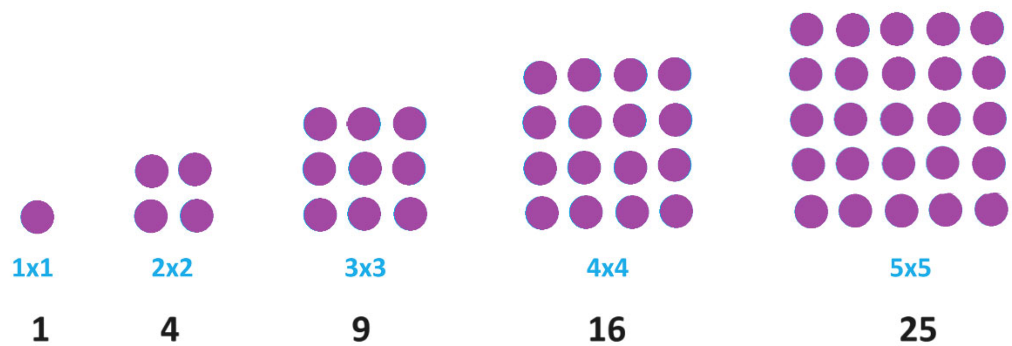
a) List all its factors of the number 360. (It has a total of twenty-four of them.)
b) Is the number 361 prime? If not, list all its factors.
c) Is the number 359 prime? If not, list all its factors.

[It's okay to use google or Siri or some 21^{st}-century tool to ask if a number is prime.]

Musing 18.29 Write the number 360 as a product of prime numbers.

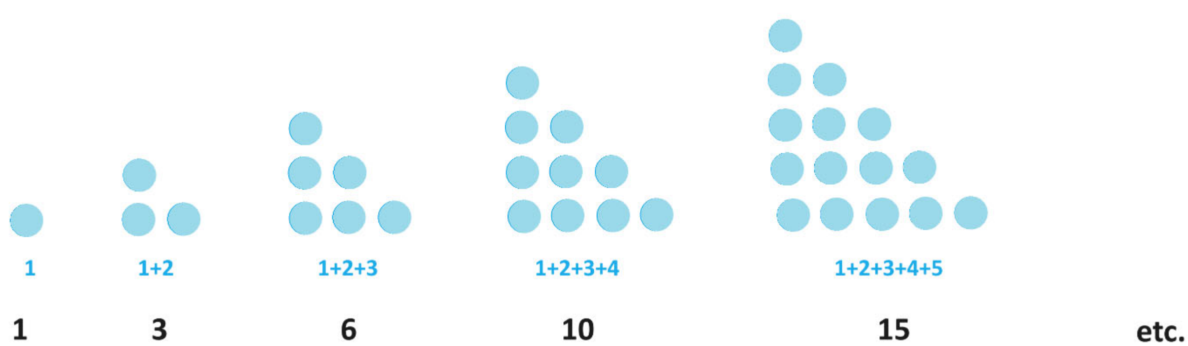
19. Figurate Numbers

We started this chapter with a picture of a square array of dots. The counts of dots that arise in square arrangements of dots are called the **square numbers**.



The fifth square number is $5^2 = 5 \times 5 = 25$.
The seventh square number is $7^2 = 7 \times 7 = 49$.
The thousandth square number is $1000^2 = 1{,}000{,}000$, a million.
In general, the Nth square number in the list is $N^2 = N \times N$.

We also saw in Practice 15.8 an example of a **triangular number** made from a row of one dot, then a row of two dots, then a row of three dots, and so on. Here are the first few triangular numbers.



The fifth triangular number is $1 + 2 + 3 + 4 + 5 = 15$.
The seventh triangular number is $1 + 2 + 3 + 4 + 5 + 6 + 7 = 28$.
And in section 15 we kept going and found a general formula for the Nth triangular number. (Feel free to look up the formula for the sum of the first N counting numbers.)

Greek scholars of some 2,500 years ago enjoyed thinking about square and triangular numbers, and other "figurate numbers" that had a natural geometry associated with them.

They discovered stunning patterns and interactions within and between these numbers. I thought it would be fun to end this chapter with some of that delightful play.

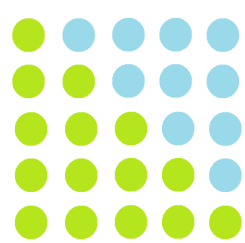
Triangular Numbers: 1 3 6 10 15 21 28 36 45 55 ...
Square Numbers: 1 4 9 16 25 36 49 64 81 100 ...

INTERACTION 1

The sum of any two consecutive triangular numbers is sure to be a square number.

1 + 3 = 4
3 + 6 = 9
6 + 10 = 16
10 + 15 = 25
15 + 21 = 36
etc.

A single picture reveals why this is so. Here we see that the 4th and 5th triangular arrays lock together to make a square array, and the same will be case for any two consecutive triangular arrays.



Question: Does this picture indeed explain matters for you? Do you see two consecutive triangular arrays in the picture?

Do we need infinitely many pictures like this, or does this one picture show how, in general, any two consecutive triangular arrays of dots will combine to make a square array of dots?

Triangular Numbers: 1 3 6 10 15 21 28 36 45 55 ...

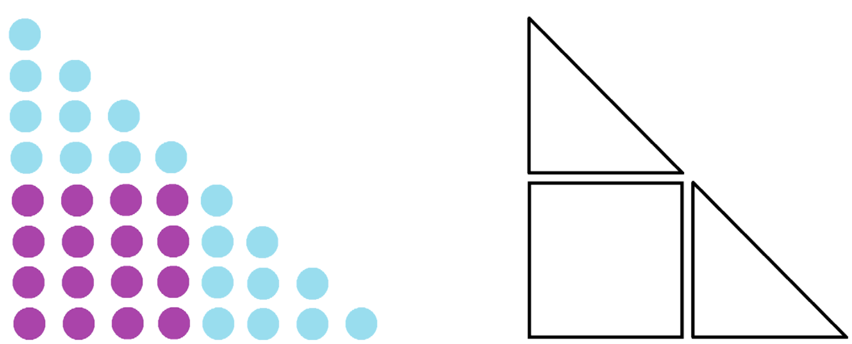
Square Numbers: 1 4 9 16 25 36 49 64 81 100 ...

INTERACTION 2

Take any triangular number and double it. Then add the matching square number to the result. The final answer is sure to be another triangular number.

2 x 1 + 1 = 3
2 x 3 + 4 = 10
2 x 6 + 9 = 21
2 x 10 + 16 = 36
2 x 15 + 25 = 55
etc.

Here's an explanation (maybe).



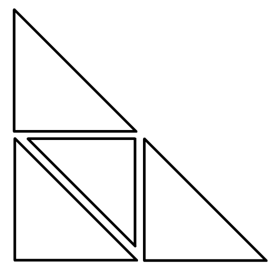
Question: Does this figure provide an explanation for you?

Question: What do you think of the following statement?

> *Twice the Nth triangular number plus the Nth square number gives the $2N$th triangular number.*

(The answer can be: "I don't get it and I don't like it. Don't ask me a question like this again.")

Practice 19.1 What does the following picture suggest about the triangular numbers? (Here we have three triangles the same size and one slightly smaller triangle.)



Triangular Numbers:	1	3	6	10	15	21	28	36	45	55	...
Square Numbers:	1	4	9	16	25	36	49	64	81	100	...

INTERACTION 3

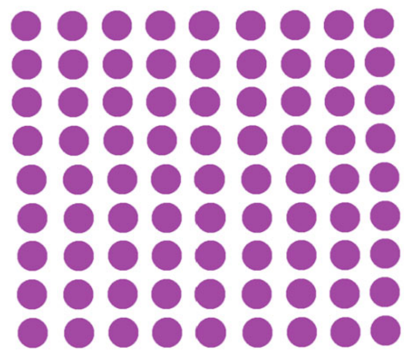
Take any triangular number. Multiply it by eight and then add one to the result.
This is sure to produce an odd square number.

$$8 \times 1 + 1 = 9$$
$$8 \times 3 + 1 = 25$$
$$8 \times 6 + 1 = 49$$
$$8 \times 10 + 1 = 81$$
$$8 \times 15 + 1 = 121$$

etc.

Let's look at one specific example: $8 \times 10 + 1 = 81$.

Here's the nine-by-nine square array of dots that represents the square number 81.



Our job is to see this picture as eight copies of the fourth triangular array of ten dots, plus one extra dot.

Practice 19.2: There is meant to be one special dot in that nine-by-nine array. If you had to make a guess, which dot in that array might be singled out as special?

Can you now draw in eight copies of the small triangular array around that special dot? Can you come up with a design that clearly generalizes beyond this specific example and would work for any odd square array of dots?

Question: What do you think of the following statement?

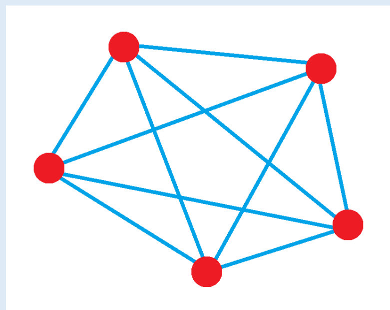
Eight times the Nth triangular number plus one equals the $(2N + 1)$th square number.

MUSINGS

Musing 19.3

a) What is the one-hundredth square number?
b) What is the one-hundredth triangular number?

Musing 19.4 In the picture, five dots were drawn in a circle and each dot was connected to each and every other dot with a line segment. A total of 10 line segments were drawn.



a) Draw six dots on a circle and the line segments that connect each and every pair of dots. How many line segments in total did you draw?
b) How many line segments does one draw starting with three dots? With four dots? With two dots? With seven dots?
c) Make a prediction as to how many line segments you would draw if you started with one hundred dots on a circle.
d) Can you explain what you are noticing?

Musing 19.5

The number 1 is both a square number and a triangular number, but that is not very exciting.

The number 36 is also both a square number and a triangular number. (We have $36 = 6 \times 6$ and $36 = 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8$.) It's a **squangular number**!

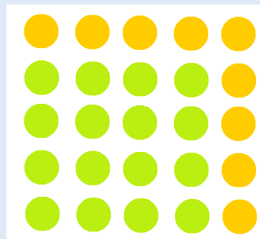
The next squangular number is 1225.

a) Which square number is 1225?
b) Can you figure out which triangular number 1225 is? (This, technically, is a YES/NO question.)

(If you are curious, the list of squangular numbers is infinitely long and it begins $1, 36, 1225, 41616, 1413721, 48024900, 1631423881, \ldots$.)

Musing 19.6

This picture shows that the fourth and fifth square numbers differ by 9. (Do you see that?)



a) What do the 10^{th} and 11^{th} square numbers differ by?
b) Explain why two consecutive square numbers are sure to differ by an odd number.
c) Which two consecutive square numbers differ by 101?
d) Which two consecutive triangular numbers differ by 101?

Musing 19.7

Triangular Numbers: 1 3 6 10 15 21 28 36 45 55 ...

Square Numbers: 1 4 9 16 25 36 49 64 81 100 ...

The following table suggests that any three consecutive triangular numbers have the following property:

Six copies of a triangular number plus one copy of the triangular number before it and one copy of the triangular number after it gives a square number.

$$1 + 6 \times 3 + 6 = 25$$
$$3 + 6 \times 6 + 10 = 49$$
$$6 + 6 \times 10 + 15 = 81$$
$$10 + 6 \times 15 + 21 = 121$$

etc.

a) Draw a nine-by-nine square array of 81 dots. Can you identify within it one triangular array of 6 dots, six triangular arrays of 10 dots, and one triangular array of 15 dots?
b) Do you think the pattern suggested in the table is always true?

Musing 19.8 The Prime Numbers are Tricksters!

Consider the list of triangular numbers 1, 3, 6, 10, 15, 21, 28, 36, 45, 55, … .

Starting with 6, do the following:

Add 1 to the triangular number if it is even.
Subtract 2 from the triangular number if it is odd.

Is the result always a prime number? It seems so at first!

$6 + 1 = 7$ is prime
$10 + 1 = 11$ is prime
$15 - 2 = 13$ is prime
$21 - 2 = 19$ is prime
$28 + 1 = 29$ is prime
$36 + 1 = 37$ is prime
$45 - 2 = 43$ is prime
$55 - 2 = 53$ is prime

a) What's the fifteenth triangular number? Does this observation hold for it?
a) What's the eighteenth triangular number? Does this observation hold for it?

Musing 19.9
In Musing 18.25 we were introduced to the perfect numbers: 6, 28, and 496. (Over fifty examples of perfect numbers are currently known and, curiously, each example is an even number.)

6 is also a triangular number. It is the third one, and 3 is a prime number.

28 is also a triangular number. It is the seventh one, and 7 is a prime number.

496 is also a triangular number. It is also in a prime-numbered place on the list of triangular numbers.

Can you figure out which triangular number 496 is?

Swiss mathematician Leonard Euler proved in the mid-1700s that every even perfect number is sure to be a triangular number and in a prime-numbered position in the list of triangular numbers.

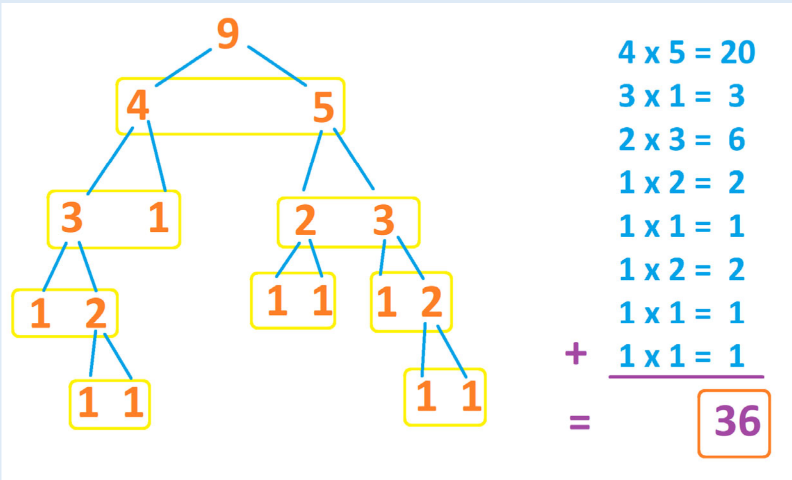
ACTIVITY

Musing 19.10 PILE SPLITTING

Sichun splits a pile of 9 buttons into two piles: a pile of 4 and a pile of 5.
He writes on a piece of paper: $4 \times 5 = 20$.

He then splits the pile of 4 buttons into two piles: a pile of 3 and a "pile" of 1.
He writes on a piece of paper: $3 \times 1 = 3$.

He keeps doing this with all the piles he sees—splitting each pile into two and recording the product of the pile sizes on his paper. He does this until he has nothing but single buttons, that is, until he has nine "piles" of 1.



Sichun then adds the results of all the products he wrote down. In the example shown, he gets the "magic sum" of 36.

a) Play the game for yourself. Start with nine objects and keep splitting piles into two, recording products along the way. Make different choices than Sichun did for splitting piles.
What magic sum do you get?

Play the game a second time–but predict first what magic sum you think you will get. Do you?

b) Play the game again, but this time start with 8 buttons. What magic sum do you get this time?
c) Play the game yet again, starting with 2 buttons, with 3 buttons, with 4 buttons, and so on, up to 7 buttons. Any comments?

OPTIONAL HARD CHALLENGE: Can you explain what is going on?

Chapter 3

The Integers

20. New Numbers: The Opposites of the Counting Numbers

In chapter 1, we made sense of addition and multiplication, but skipped over subtraction. Why? Because I don't believe subtraction exists! To me, subtraction is just addition again—but it is the addition of a new type of number.

Let me explain what I mean by telling a story that is blatantly not true.

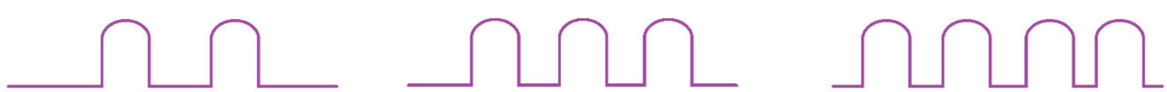
> When I was a young child, I spent my entire days playing in a sandbox at the back of my yard. (Not true.)
>
> And being a very quiet and contemplative child (not true), I would start each day leveling out the sand in my box to make a perfectly flat horizontal surface. This appealed to my tranquil sensibilities and calmed my mind. I even gave this special level state a name. I called it **zero**, **0**.

0

> And I spent many an hour admiring my zero state. (Still not true.)
>
> But then, one day, I had a flash of insight. I realized I could reach behind where I was sitting, grab a handful of sand, and make a pile. And I called the one pile, **1**.

1

> And then I realized I could do grab even more sand and make two piles, which I called **2**, and three piles, **3**, and so forth.

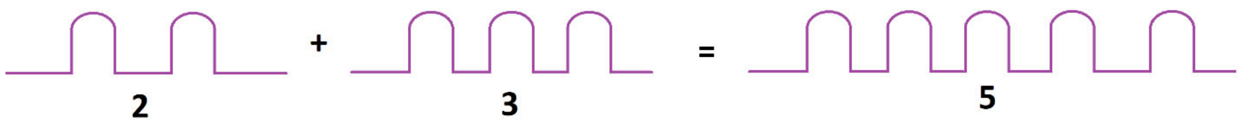


2 **3** **4** **etc.**

> I had even more hours of mathematical fun creating and admiring more and more piles of sand.
>
> I had discovered the **counting numbers**.

And, by lining up piles, I also discovered **addition** with the counting numbers.

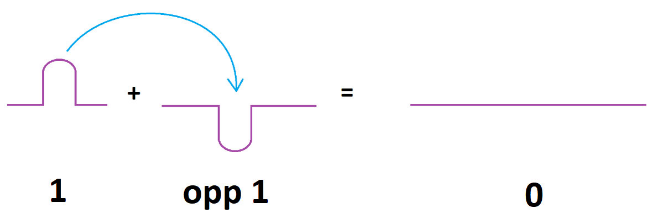
For example, I saw that two piles plus three piles equals five using piles.



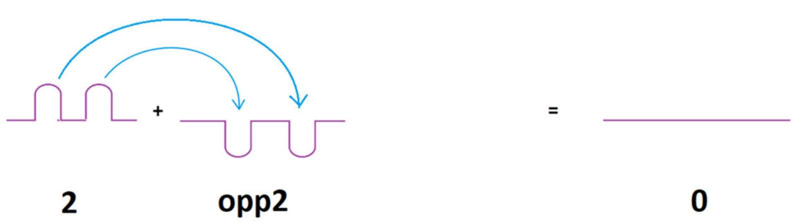
Question: Can you imagine a picture of $7 + 9 + 13 + 26$?

But then, one day I had the most astounding flash of insight of all! Instead of using a handful of sand to make a pile, I realized I *take away* a handful of sand and make the opposite of a pile, namely, a hole!

I wrote **opp 1** for a hole because I realized that one hole truly is the opposite of 1 pile: a pile and a hole together cancel each other out to return to the zero state. Whoa!

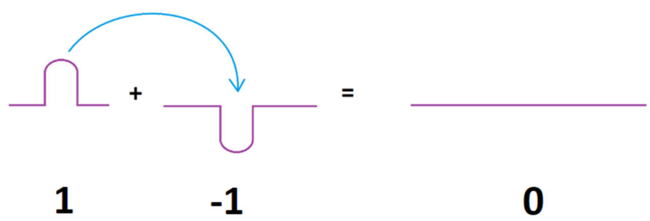


I also wrote **opp 2** for the two holes, **opp 3** for three holes, and so on.



Question: Can you see in your mind's eye that $7 + \mathbf{opp}\ 7$ is zero? That is, can you see how the sand of seven piles perfectly fills seven holes?
(I guess, in this untrue story we are assuming that all piles and all holes are the same size.)

Later in school I was taught draw a little dash for “opposite” and write -1 for the opposite of a one. I was also taught to call this **negative one**. This felt strange and unnatural to me, but so be it.



I was also taught to write -2 for two holes and to call that **negative two**. And so on.

But even though I would say “negative 5” with my words, my brain was still secretly thinking the opposite of five, **opp 5**, and I was imagining five holes.

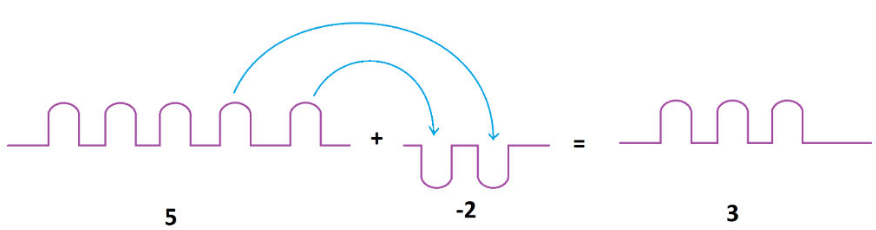
And strange matters went even further.

School also wanted me to do this thing called **subtraction**. But I never bought into it.

For example, my schoolmates would read

$$5 - 2$$

as **five take away two** and they would talk about removing two objects from a set of five objects. I, on the other hand, thought of this as $5 + \textbf{opp}\ 2$, five piles and the addition of two holes. I could see that makes for three piles.



Of course, knew that my two holes, in effect, “took away” two piles just as my colleagues were doing, but I knew my supposedly unusual way of thinking had an advantage!

Question: Can you see, by drawing a picture of piles and holes, that **10 plus the opposite of 4** (which is, $10 + -4$) is the same as **10 take away 4** $(10 - 4)$?

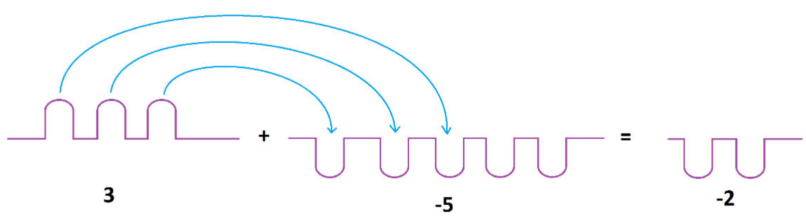
I asked my colleagues:

What's three take away five, $3 - 5$*?*

They all said:

That's impossible!
You can't take away five things if you only have three to begin with.
There is no answer to $3 - 5$*.*

But I saw that $3 - 5$ does have an answer. It's three piles (3) and five holes (-5), which makes two holes (-2).



Practice 20.1: Many people say that $2 - 6$ has no answer.

But think of this as $2 + -6$.

How many piles are there? How many holes are there? And when we combine them, what are we left with?

I was having fun doing arithmetic that my teacher told me I won't be learning until a few more years' time. (This part is true!)

This whimsical story makes an important point.
It shows that we can view any subtraction problem as an addition problem.

Subtraction is just the **addition** of the opposite.

For example, we have

$10 - 3$ is really $10 + \ -3$ (ten piles and three holes)

$26 - 6$ is really $26 + -6$ (twenty-six piles and six holes)

$100 - 40$ is really $100 + -40$ (one hundred piles and forty holes)

People tend not to write the opposite numbers first in an expression, but something like the following is actually fine and makes good sense.

$$-3 + 7 = 4$$

Three holes and seven piles combine to leave four piles.

In fact, it is very handy to rewrite traditional statements using the subtraction symbol as addition statements.

For example, writing $6 - 9 + 2 - 1$ as

$$6 + -9 + 2 + -1$$

allows us to see eight piles and ten holes which combine to make two holes: $6 - 9 + 2 - 1 = -2$.

Comment: I personally think it is odd and confusing that we use the same symbol "$-$" both for a verb (*subtract* or *take away*, as in "$8 - 5$") and for an adjective (*negative*, as in "-5").

I do write statements such as $6 + \ -9 \ + 2 + \ -1$, but I understand they look confusing. Many educators won't let students write such things probably because it is hard to decipher in and of itself, and messy handwriting won't help. Writing

$$6 + \textbf{opp}\ 9 + 2 + \textbf{opp}\ 1$$

is clear and fun, but it is cumbersome.

Question: What do you think of writing $\overline{6} + \underline{9} + \overline{2} + \underline{1}$ for six piles and nine holes and two piles and one hole? (No one writes this, but I am just wondering what you think.)

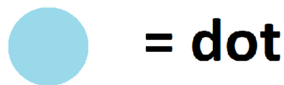
Practice 20.2:
a) A friend wrote $6 - 7 - 1$. How do you think I would rewrite this expression?
b) If I wrote $5 + -1 + 3 + -2 + -1$, what do you think my friends would write?
c) Can you imagine piles-and-holes pictures these two sums? What final value do they each give?

By the way, people call the little dash in front of a number a **negative sign**.

In many parts of the world, it is called a **minus sign**. But folk in the U.S. object to this language. They say that "minus" is a verb and so shouldn't be used as an adjective (but, as I mentioned, folk in the U.S. are still happy to use the one symbol "−" for both, nonetheless!)

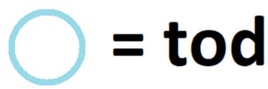
DOTS and ANTIDOTS

Piles and holes are well and good. But we began this book with a dot.



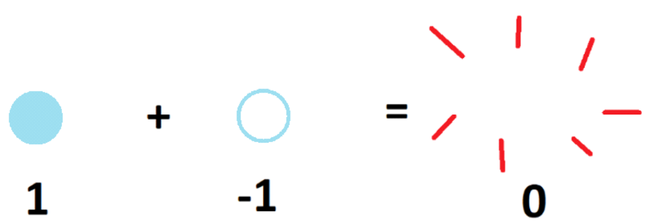
A dot perhaps looks like a pile viewed from above. So, what should we draw for a hole looked at from above? That is, what should we draw for the opposite of a dot?

> **True Story Moment:** When I ask this question to students in countries all over the world, kids always tell me the same one answer.
>
> *Draw an open circle for the opposite of a dot and call it a **tod**. That's the word dot, backwards.*



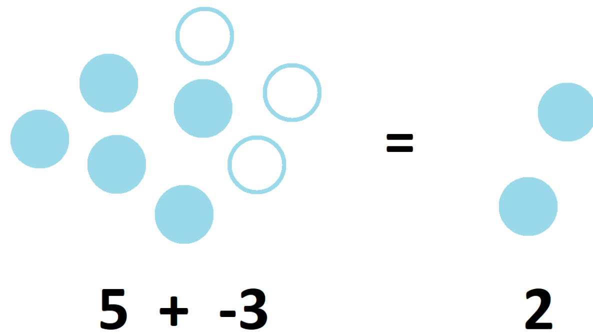
> Adults usually suggest drawing an open circle too but, less imaginatively, suggest calling it an **antidot**.
>
> I personally like the term *tod* very much, but *antidot* have the advantage of evoking a sense oppositeness, which is a help. I'll be adult and go with that name here.

Like a pile and a hole which annihilate one another when brought together, a dot and an antidot annihilate too – POOF! – when brought together and leave nothing behind.

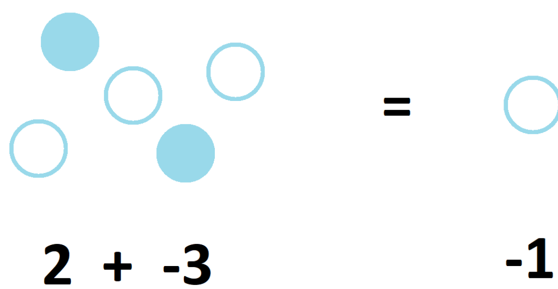


We can conduct arithmetic with dots and antidots, just like we did with piles and holes.

For example, in $5 - 3$ ("five plus the opposite of three") there are three annihilations – POOF! POOF! POOF! – leaving behind two dots.



In $2 + \ -3$ there are two annihilations—POOF! POOF!—leaving behind one antidot.



Practice 20.3 Draw, or just imagine drawing, a dots-and-antidots picture of $10 - 20$ and find the value of this quantity.

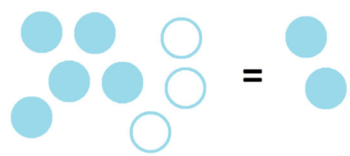
"Take Away" Again

Some people really do prefer the "take away" mindset over the "add the opposite" mindset. Here's a way to bridge both approaches—if you are interested.

Example: $5 - 3$.

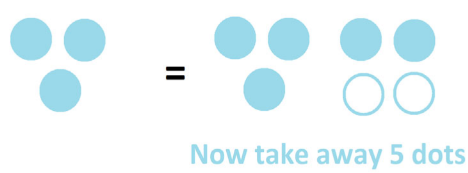
Take Away Approach:
Draw five dots and then take three away.

Adding the Opposite Approach:
Read the problem as $5 + -3$. Draw five dots and then three antidots and see the annihilations. (This, of course, has the effect of "taking three away.")



Example: $3 - 5$.

Take Away Approach:
Draw three dots. We don't have five dots to take away, so draw in an extra two dots and counteract that move by drawing two antidots to balance them out. Now take away five dots. That will leave -2.



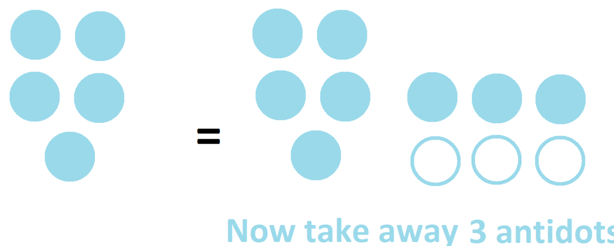
Adding the Opposite Approach:
Read the problem as $3 + -5$ and draw three dots and five antidots. Two antidots result.

Now let's get quirky.

Example: $5 - (-3)$.

Try making sense of this before turning the page.

Take Away Approach:
Think of this as "five dots take away three antidots." We don't have any antidots in a picture of five dots, so draw them in, three of them, and counteract that move by drawing three dots as well.



Now we can take away three antidots. That leaves behind 8 dots.

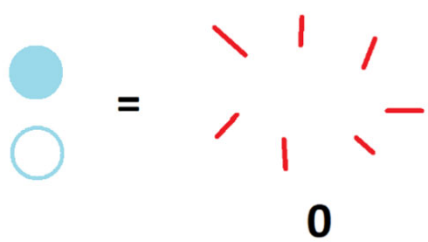
Adding the Opposite Approach:
Read the problem as $5 + -(-3)$, which is "five dots and the opposite of three antidots."

What's the opposite of three antidots? Well, that must be three actual dots!

So, we're being asked to add together five dots and three dots. The result is 8 dots.

Whoa!

This idea of drawing in pairs dots and antidots, with each pair technically being "nothing" and so not affecting the value of a problem, is sneaky and helpful! It creates items you can then take away if you are a "take away" kind of person.



MUSINGS

Musing 20.4 What do you think should be the value of -0?

Musing 20.5 What do you think should be the value of $---5$?
(Maybe think of this as $-\left(-\left(-5\right)\right)$.)

Musing 20.6 Multiple Choice:

A picture of $1-1+1-1+1-1+1-1+1$ would contain …

a) five dots and four antidots, which would combine to leave one dot (1).
b) four dots and five antidots, which would combine to leave one antidots (-1).
c) five dots and four antidots, which would combine to leave one antidots (-1).
d) four dots and five antidots, which would combine to leave one dot (1).
e) This is absurd! Who writes such a sum in the first place?

MECHANICS PRACTICE

Musing 20.7 Rewrite each expression involving subtraction as one involving only addition. For example, $5-9-1$ can be rewritten as $5+\ -9\ +\ -1$.

f) $67-33-88+102-46$.
g) $-9-9$.
h) $-9-(-9)$.

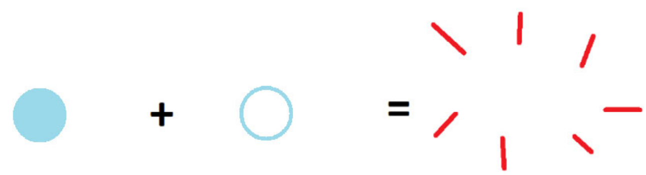
Musing 20.8 Find the value of each of these expressions.
(Practice envisioning dots and antidots, or piles and holes.)

a) $3-7$
b) $2000-1000+65$
c) $15-15$
i) $15-(-15)$
j) $-15-15$
k) $-15-(-15)$

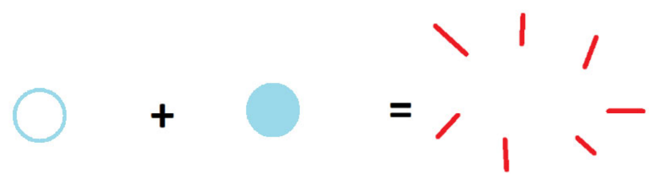
21. Distributing the Negative Sign

A dot (1) and an antidot (−1) annihilate one another.

An antidot is the opposite of a dot because it annihilates it.



And what is the opposite of an antidot? That is, what annihilates an antidot? A dot!



Since society uses a little dash for "opposite" this second statement is written as follows.

$$-(-1)=1$$

We can read this as saying: the opposite of the opposite of a dot is … a dot!

Okay then, what's this quantity?

$$- - - - - - - 1$$

(I hope it is okay for me to omit the parentheses. I don't think any confusion results.)

We have here the opposite of the opposite of the opposite of the opposite of the opposite of the opposite of the opposite of a dot. That's seven "opposites." What have we got in the end?

Well, toggling dot-antidot-dot-antidot- … seven times, starting with a dot, we will land us with an antidot.

$$- - - - - - - 1 = -1$$

We can keep playing this toggle game, starting with any number of dots.

5 **= five dots**
-5 **= five antidots**
- - 5 **= five dots**
- - - 5 **= five antidots**
- - - - 5 **= five dots**
- - - - - 5 **= five antidots**

Let me now ask some trickier questions.

Example: What is the opposite of "three **dots** and two **dots**" altogether?

Answer: That would be three **antidots** and two **antidots**.

Actually, the tricky part is to translate what we just asked, and answered, into mathematics. Parentheses help us make clear what we're making opposite.

The opposite of a group of three dots and two dots is $-(3+2)$. And we computed this as three antidots, -3, and two antidots, -2. So, we have

$$-(3+2) = -3 + -2.$$

Of course, we can go the step further and say this is -5. But let's not worry about carrying the actual arithmetic for now.

Example: What is the opposite of "three dots and two antidots" altogether?
That is, what is $-(3+-2)$?

Answer: That would be three antidots and two dots.
We have $-(3+-2) = -3+2$.

Example: What is the opposite of "a dots and b antidots and 2 dots"?
That is, what is $-(a+-b+2)$?

Answer: That would be a antidots and b dots and 2 antidots.
Thus, the expression $-(a+-b+2)$ can be rewritten as $-a+b+-2$

Practice 21.1 What is

$$-(x + -y)$$

expressing in words? How can this expression thus be rewritten?

Next question:

Example: What is the opposite of $10 - T + 7 - 3 + b$?

Answer: Think of the quantity we're given as $10 + -T + 7 + -3 + b$. Then the opposite of this is

$$-(10 + -T + 7 + -3 + b) = -10 + T + -7 + 3 + -b.$$

Since the author expressed the question using subtractions, we can rewrite our answer in their preferred style.

$$-(10 - T + 7 - 3 + b) = -10 + T - 7 + 3 - b$$

People call the game we're playing **distributing the negative sign**. But that sounds too scary for what we are actually doing: we're just identifying the opposite of everything given to us. Nothing more!

This next example is typical of textbook exercises.

Example: Please make $2 - (20 - x)$ look friendlier.

Answer: This is $2 + -(20 + -x)$, which I read as

2 dots and the opposite of "20 dots and x antidots."

Unravelling that, we get

2 dots and 20 antidots and x dots.

This is $2 + -20 + x$.

We can make this friendlier still by doing arithmetic. We have 18 antidots and x dots, so this answer is $-18 + x$.

Since the world likes subtraction, let's write this as $x + \ -18$ and use the subtraction notation. This, finally, gives the answer the world likes best: $x - 18$.

Example: Kindly make $(x - y) - (x + y)$ look friendlier.

Answer: This as $(x + -y) + \ -(x + y)$ which we read as:

x dots and y antidots and the opposite of "x dots and y dots."

That gives

x dots and y antidots and x antidots and y antidots.

Within this collection are x dots and x antidots, which would all annihilate each other. So, that just leaves y antidots and y antidots. That's, $y + y$ antidots. But remember, repeated addition is multiplication, so we have here $2y$ antidots.

Answer: $-(2y)$

Oh no! Wait! Multiplications come with invisible parentheses. We can make the parentheses invisible.

Final answer: $-2y$.

Whoa!

This example brings up an important point:

Multiplications still come with invisible parentheses, even if there is a negative sign involved.

For example, the expression

$$-2 \times 3$$

has hidden parentheses. It is $-(2 \times 3)$, which equals $-(6)$, and that is the opposite of six dots, which is six antidots, -6.

If n is the name of some number (maybe, it's 17 or maybe it's 3? I am not going to say), then $2n$ represents two times that number ($2 \times n$), and

$$-2n$$

is the opposite of two times that number. If we make the hidden parentheses visible, it's $-(2 \times n)$.

We'll learn soon that this technical fussiness doesn't actually matter: all the different ways you might correctly or incorrectly interpret -2×3 will likely lead you to the same final answer when evaluating it!

MUSINGS

Musing 21.2 What is $- - - - - - - - - - 3$?

Musing 21.3 What is $-x$ if

a) x is three dots?
b) x is three antidots?
c) x is 16 ?
d) x is -16?

MECHANICS PRACTICE

Practice 21.4
There are two ways to evaluate $3 - (3 - 7)$.
Either be evaluating the quantity inside the parentheses first

$$3 - (3 - 7) = 3 - (-4) = 3 + 4 = 7$$

or by distributing the negative sign.

$$3 - (3 - 7) = 3 + -3 + 7 = 0 + 7 = 7$$

Evaluate each of the following expressions two different ways (and get the same answer!)

a) $(15 - 2) - (13 - 2)$

b) $100 - (100 - 2)$

c) $-9003 - (2 + 1)$

Practice 21.5 In the following, x, y, and R each just represent some unspecified number.

a) Show that $x - (x - 2)$ is really just 2.

b) Show that $20 - (15 - y)$ is really $5 + y$.

c) Show that $\left(1 - \left(1 - \left(1 - (1 - R)\right)\right)\right)$ is really just R.

22. Interlude: Milk and Soda

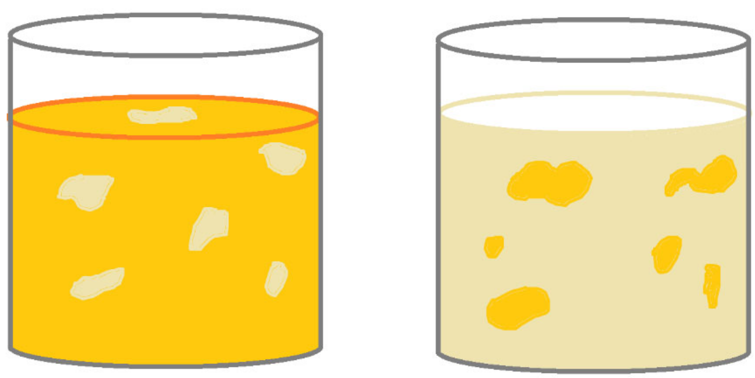
Here's a classic puzzle.

Penelope has a glass of soda and a glass of milk.

She takes a tablespoon of soda from the soda glass and haphazardly stirs it into the milk. She then takes a tablespoon of the milk/soda mixture and transfers it to the soda. Both drinks are now "contaminated."

Here's the question: *Which drink has more foreign substance?*

Is there more foreign milk in the soda than foreign soda in the milk? Or is it the other way round? Or is it impossible to say as it depends on how much or how little mixing took place?



What are your thoughts on this matter?

We can explore this puzzler by modeling it with some playing cards.

Read on!

ACTIVITY

CARD PILE MYSTERY

a) Take 10 red cards and 10 black cards from a deck of cards. Shuffle your 20 cards and arbitrarily split them into two equal piles. Count the number of red cards in the left pile and the number of black cards in the right pile. What do you notice? Repeat this activity two more times.

b) Shuffle your 20 cards and this time split them into a pile of 6 cards and a pile of 14 cards. Count the number of red cards in the small pile and count the number of black cards in the large pile. Take the (positive) difference of those two counts. Did you get 4? Repeat this exercise two more times.

c) Shuffle the 20 cards again and this time split them into a pile of 9 cards and a pile of 11 cards. Count the number of red cards in the small pile, count the number of black cards in the large pile and take the (positive) difference of this count. What did you get? Repeat two more times. What do you notice?

d) Complete the following table.

Small Pile	Large Pile	Difference count of red in small versus count of black in large
10	10	0
9	11	
8	12	
7	13	
6	14	4
5	15	
4	16	
3	17	
2	18	
1	19	
0	20	

Any patterns?

e) Suppose in a game with 5 cards in the small pile and 15 cards in the large pile, I counted three red cards in the small pile. Complete the following table giving the number of cards of each type in the remaining three piles.

	Small Pile 5	Large Pile 15
# Reds	3	
# Blacks		

What is the difference of counts of red cards in the small pile and black cards in the large pile?

f) Let's go back to the first scenario with two piles of equal size, 10 cards per pile. Let's say the number of red cards in one pile is N. Complete the table. What do you notice about the number black cards in the other pile?

	Pile 10	Pile 10
# Reds	N	
# Blacks		

Explanation:

Let me go straight to parts a) and f).

There are ten red cards in total. If N of them are in one pile, the remaining red cards, $10 - N$ of them, must be in the other pile.

	Pile 10	Pile 10
# Reds	N	10 - N
# Blacks		

But that pile only has ten cards. If there are $10 - N$ red cards in that pile, the remaining $10 - (10 - N)$ of them must be black.

	Pile 10	Pile 10
# Reds	N	10 - N
# Blacks		10 - (10 - N)

So, we have $10 - (10 - N)$ black cards in that second pile. And what is this quantity?

$$\begin{aligned} 10 - (10 - N) \quad &= 10 + -(10 + -N) \\ &= 10 + -10 + N \\ &= 0 + N \\ &= N \end{aligned}$$

Lo and behold! There are N black cards in that second pile, the same as the number of red cards in the first pile.

The number of red cards in one pile is always equal to the number of black cards in the second!

Okay. The algebra here provides proof that the number of red cards in one pile matches the number of black cards in the second pile—but the work doesn't feel intuitive and satisfying. Really, why should these two counts be the same?

An Intuitive Explanation:

Count the number of red cards in the first pile.

This tells you how many black cards you need to make the whole pile black.

And where are those missing black cards? They must be in the second pile!

The number of black cards in the second pile must match the number of red cards in the first pile.

Practice 22.1: Use algebra, or intuition, to explain what you observed in the remaining sections of the activity.

We can answer the milk and soda puzzle now in a similar intuitive manner. It must be the case that each glass must have the same volume of contaminant!

> Imagine taking out from the milk glass all the molecules that belong to the soda. This leaves a missing volume from that milk glass. Where must the missing milk be? It must be contaminating the soda—the same volume of it!

> **Practice 22.2:** Must the two glasses have equal volumes of soda and milk at the start of this puzzle? By transferring a tablespoon of liquid one way and then back, are we always guaranteed to have the same volume of contaminant in each cup?

23. The Rules of Arithmetic and Negative Numbers

In chapter 1 we made sense of the counting numbers $1, 2, 3, 4, \ldots$ including 0. These numbers are also called the **natural numbers**.

In this chapter we developed an intuitive model for the opposites of these numbers $-1, -2, -3, -4, \ldots$. (And Musing 20.4, if you thought about it, might have suggested to you that -0, the opposite of zero, is no different from 0. We'll see in a moment that mathematics itself also wants this to be so.)

To be clear, we've been led to the following belief:

For each (counting) number a, there is one other number "$-a$" such that $a + -a = 0$.

If you want to be fancy with language, but there is no need for it, people call $-a$ the **additive inverse** of a.

-5 is the additive inverse of 5 because it is the number you add to 5 to get to zero:

$$5 + -5 = 0$$

-7 is the additive inverse of 7 because it is the number you add to 7 to get to zero:

$$7 + -7 = 0$$

Example: Show that -0, the additive inverse of zero (the "opposite of 0"), is actually 0 itself!

Answer: Well, -0 is the number must we add to 0 to get the answer zero.

$$0 + ? = 0$$

What number works? Well, 0 works!

Ahh, -0 is the number 0.

We've just established that $-0 = 0$.

Example: Show that $-(-5)$, the "opposite of -5," is actually 5.

Answer: Now $-(-5)$ is the number must we add to -5 to get the answer zero.

$$-5+?=0$$

Well, 5 works! So, $-(-5)$ is 5.

We've just established that $--5=5$. This matches our intuition about piles and holes, or dots and antidots. Phew!

Practice 23.1: Explain for yourself, mathematically, why $-(-17)$ must be 17.

Example: Show that $-(2+3)$ is actually $-2+-3$.

This shows that all the work we were doing in the last section "distributing the negative sign" is mathematically valid!

Answer: Now $-(2+3)$ is the number we need to add to $2+3$ to get the answer zero. What could that number be?

$$2+3+?=0$$

Well, $-2+-3$ does the trick.

$$2+3+-2+-3=0$$

So, $-(2+3)$ is indeed $-2+-3$.

Practice 23.2: Show mathematically that $-(5-7)$ is $5+-7$.

People call this entire collection of numbers—the counting numbers and their opposites—the set of **integers** and they use the symbol $\mathbb{Z}$ to denote this set. (This symbol comes from the German word *Zahlen* for "numbers.")

So, we now have the set of natural numbers $\mathbb{N}$ and the set of integers $\mathbb{Z}$.

$$\mathbb{N} = \{1, 2, 3, 4, \dots\} \text{ or } \{0, 1, 2, 3, 4, \dots\}$$

$$\mathbb{Z} = \{\dots, -4, -3, -2, -1, 0, 1, 2, 3, 4, \dots\}$$

People also call the counting numbers $1, 2, 3, 4, \dots$ the **positive integers** and the opposite numbers -1, -2, -3, -4, … the **negative integers**. The number zero, with the property that -0 is the same as 0, is deemed neither positive nor negative.

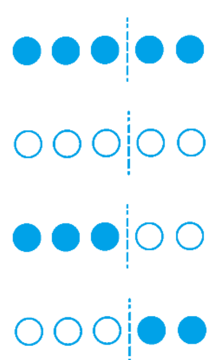
> **Comment:** There is confusion in school textbooks over the term "set of *whole numbers*." Some authors use this term to mean the set of numbers $0, 1, 2, 3, 4, \dots$. Others use it as another name for the set of integers $\{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$. Mathematicians tend not to use this term. (Or, if they do, in a context that causes no confusion.)

In section 14 we listed the rules of arithmetic that seem appropriate for how we think about addition and multiplication with the counting numbers.

These rules were motivated by drawing pictures of dots.

We can also try drawing pictures of antidots to see to what if the same rules might hold for our new opposite numbers too.

For example, we can still imagine a notion of "addition" as coming from lining up dots and/or antidots. And by reading such pictures from left to right and then from right to left, we will come to believe that we still can switch the order in which we add two numbers, even if one or both of those numbers happen to be negative integers.



3 + 2 = 2 + 3

-3 + -2 = -2 + -3

3 + -2 = -2 + 3

-3 + 2 = 2 + -3

It feels like at least one of our familiar rules of arithmetic is still valid in the world of integers.

For any two numbers a and b we have that $a + b = b + a$.

Other rules seem valid too.

For example, adding nothing to a picture of dots or to a picture of antidots changes nothing about the quantity that picture represents.

For any number a, we have that $a + 0 = a$ and $0 + a = a$.

We can even play the same not-exciting solitaire game from section 10 with negative numbers.

-2 5

0

6 -7

-3 6

Erase two numbers and replace them with their sum.
Repeat until a single number remains.

(See Musing 23.3 at the end of this section.)

Reasoning just as before, we deduce:

For any string of numbers added together

$$a + b + c + d + e + \ldots + y + z$$

it does not matter in which order one chooses to perform the additions. The same answer will always result.

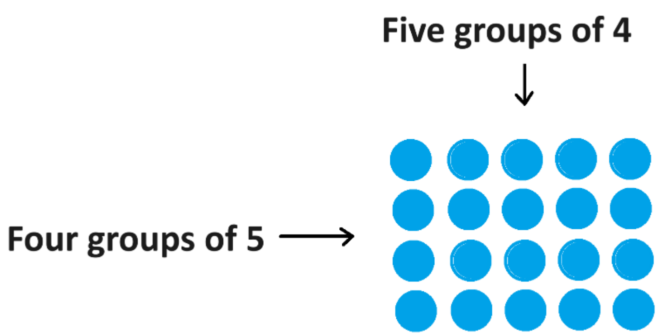
All feels fun and good when thinking about addition with positive and negative integers via dots and antidots. But thinking about multiplication with negative integers, on the other hand, forces us to take a new stance on matters.

Let me explain.

Within the system of counting numbers, multiplication is defined as *repeated addition*.

For instance, 4×5 is "four groups of 5," namely, $5 + 5 + 5 + 5$.

Drawing a rectangular array of dots to match this notion of repeated addition allowed us to deduce some properties of multiplication, that 5×4 is sure to have the same numerical value as 4×5, for instance.



But the notion of "repeated addition" is sometimes meaningless when it comes to working with negative numbers and drawing guiding pictures with dots and antidots is impossible.

Let's engage in some mind-bendiness!

Consider this question:

> What does $(-4) \times 5$ mean?
> That is, what are "negative four groups of 5"?

The issue:

> What's a "negative group"?

We know what a negative dot is (it's an antidot). But we've never talked about a "negative group" of something.

Schoolbooks typically "cheat" right at this moment and simply assert

> $(-4) \times 5$, *negative four groups of five, is just the opposite of … four groups of* 5.

That is, they assert that $(-4) \times 5$ is the same as $-(4 \times 5)$.

Or they might assert that

> $(-4) \times 5$, *negative four groups of five, is the same four groups of negative five.*

That is, they assert that $(-4) \times 5$ is the same as $4 \times (-5)$.

The extra confusing thing is that it turns out mathematics will come to tell us that $(-4) \times 5$ and $-(4 \times 5)$ and $4 \times (-5)$ are all sure to have the same value. So, the schoolbooks are not suggesting incorrect mathematics and so they can get away with this.

But they are sidestepping the question of what $(-4) \times 5$ itself actually means.

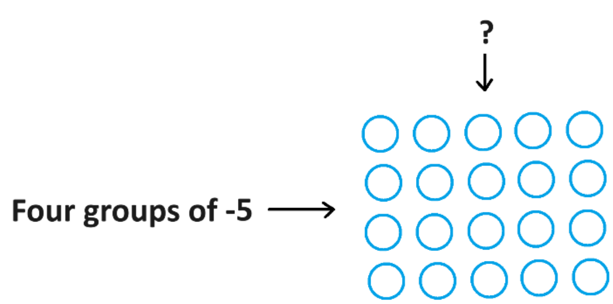
Question: Most people do not understand the issue being pointed out here because they have been told over and over again: "This is how it is in math. It just is. It just is. It just is." The chance to step back and be confused and to question claims made about mathematics has been denied to most students.

So, how are you doing processing the previous page?

Do you see that $-(4 \times 5)$—the opposite of "four groups of five"—means something different from $4 \times (-5)$—four groups of negative five—even though you can imagine them each to give a picture of twenty antidots?

Do you see that there is no *a priori* reason to think that $(-4) \times 5$ should be the same as either one of these?

Practice 23.3: Let's lean into the quantity $4 \times (-5)$ a little bit. It makes sense in our "repeated addition" thinking to see it as "four groups of -5" and represent it as a picture four groups of five antidots in a rectangular array.



What quantity do you see if you look at this picture via columns?

We, and schoolbooks, can make intuitive sense of 4×5 and of $4 \times (-5)$ with dots and antidots. They are four groups of five dots and four groups of five antidots, respectively.

Making concrete sense of $(-4) \times 5$, however, is problematic and leads to some schoolbook shenanigans.

(And we haven't even touched on something like $(-4) \times (-5)$ yet! *Why is negative times negative positive* is an age-old question students—and adults—have been crying out for decades and decades.)

Here's the truth about multiplication with negative numbers—and I know you can handle the truth.

> Mathematics is bigger and bolder than the real world. It is therefore bigger and bolder than all schoolbook attempts to make every part of it concrete and real. Mathematics certainly incorporates real-world models and is immensely powerful in helping describe them. But mathematics sits at a higher plane to them.
>
> The concept of multiplication—for numbers beyond just the counting numbers—is one of those higher-plane mathematical concepts.
>
> We choose to believe that there is a general operation on numbers, called **multiplication**, which behaves the same way as we defined multiplication for just the counting numbers.
>
> We recognize that we can sometimes give "real-world" meaning to the notion of multiplication, when it is appropriate to do so, and are not at all phased when we are in a context where we cannot.

In other words, we have chosen to believe that there is a notion of "multiplication" that can be applied to all numbers. We make no attempt to assert what it means and what it is in concrete terms. Rather, we define it by how it behaves.

We understand what multiplication is when applied to just the counting numbers and how it behaves for those numbers. We now extend that notion of multiplication to all numbers by insisting its observed behavior continues.

The mathematical operation multiplication is defined by how it behaves, not by what it is.

This is the mathematician's take on matters, and it is a powerful take. It's also very hard for people who have undergone years and years of school training to shift to it.

To be upfront:

Mathematicians have no trouble saying that they don't actually know what $(-4) \times 5$ means in a concrete sense. (They even think it is folly to always insist on "real world" interpretations.)

But they do know how to do mathematics with this quantity and get meaningful and practical results from it, nonetheless.

So, here's the honest way to move from the world of counting numbers to the world of integers (the counting numbers and their opposites).

We work with the same eight rules we know so well from playing with the counting numbers but now use these rules to define the behavior of our two basic operations: addition and multiplication. And to get us beyond just the counting numbers, we add a ninth rule that shows how the new "opposite numbers" should work.

Everything is spelled out on the next two pages.

There is nothing to memorize!

Your job is to just have a quick read-through what comes next and observe how these nine rules are simply bringing together the intuition we've already developed about arithmetic and how it works.

The Integers

There is a set of numbers, called the **integers**, $\mathbb{Z}$, which contains the counting numbers.

For these numbers there are notions of "addition" and "multiplication." Each is a way to combine two numbers to produce a new number. These two operations …

a) match the expected addition and multiplication we know for the counting numbers when they are applied to just the counting numbers

b) continue to follow the behavior of addition and multiplication we know for the counting numbers when applied to numbers that are not themselves counting numbers.

This behavior is outlined in the following nine rules.

Addition

Rule 1: We can change the order in which we add any two numbers and not change the final result. That is, for any two numbers a and b we have that $a + b = b + a$.

Rule 2: Adding zero to a number does not change the value of the number.
That is, for any counting number a we have that $a + 0 = a$ and $0 + a = a$.

Rule 3: In any string of numbers added together

$$a + b + c + d + e + \ldots + y + z$$

it does not matter in which order one chooses to perform the additions. The same answer will always result.

Multiplication

Rule 4: We can change the order in which we multiply any two numbers and not change the final result. That is, for any two numbers a and b we have $ab = ba$.

Rule 5: Multiplying a number by 1 does not change the value of the number.
That is, for each number a we have that $1 \times \mathrm{a} = a$ and $\mathrm{a} \times 1 = a$.

Rule 6: Multiplying a number by 0 gives a result of zero.
That is, for each number a we have that $0 \times a = 0$ and $a \times 0 = 0$.

Rule 7: In any string of numbers multiplied together

$$a \cdot b \cdot c \cdot d \cdot \cdots \cdot y \cdot z$$

it does not matter in which order one chooses to perform the products. The same answer will always result.

Addition and Multiplication Together

Rule 8: "We can chop up rectangles from multiplication and add up the pieces."

"Opposite Numbers"

Rule 9: For each number a there is one other number, denoted $-a$, such that $a + -a = 0$.

Again, we add Rule 9 to make sure we are getting the "opposite numbers" to create all of the integers.

MUSINGS

Musing 23.4 On a scale of 1 to 5, how perturbed are you by the idea that mathematicians are so brutally honest to admit that they do not know what multiplication **is** for numbers beyond the counting numbers, but are comfortable to simply define multiplication as an operation on numbers that behaves a certain way?

> **1** = "What a cop out! Mathematicians are wimps!"
> **5** = "Brutal honesty! At last, someone is admitting the truth about the nature of mathematics. This stance is refreshing, and I can handle it."

Musing 23.5
My mathematical colleagues, who like logical austerity, will object to my phrasing of Rule 9:

> For each number a there is **one** other number, denoted $-a$, such that $a + -a = 0$.

They say that pointing out that there is only **one** other number that deserves to be called $-a$ is unnecessary; logic dictates that there can't be more than one! So, simply say "there is a number …".

For example, here's an argument that shows that there can only be one number that behaves like -5.

> *Suppose there is another number, let's call is b, that behaves like -5.*
> *That is, when you add it to 5 you get zero:* $5 + b = 0$.
>
> *Consider the sum* $b + 5 + -5$.
>
> *We can add this in any order we like. (Rule 3.)*
> *Adding together b and 5 first gives us*
>
> $$b + 5 + -5 = 0 + -5 = -5$$
>
> *On the other hand, adding together 5 and -5 first gives*
>
> $$b + 5 + -5 = b + 0 = b$$
>
> *These answers must match as they are both just $b + 5 + -5$, after all.*
> *So, b must -5.*

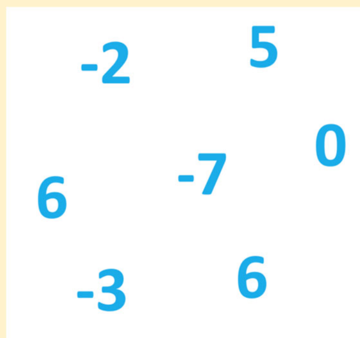
On a scale of 1 to 5, to what extent did you follow the logic here of my colleagues?

> **1** = "What on Earth are we talking about here? This is all gobbly gook."
> **5** = "Wowza! I get what they are saying."

MECHANICS PRACTICE

Musing 23.6
Let's play a not-exciting game of solitaire again, a la the game presented in section 6. But this time, let's play it with some negative numbers thrown in for fun!

Here are some numbers drawn on a page.



Recall that a "move" in this game consists of erasing two numbers and replacing them with their sum.

For example, if you cross out 5 and -7, you then write -2 on the page in their stead.

You keep repeating such moves until a single number remains on the page.

a) In this particular game, what will that final number be no matter the choice of moves you make along the way? Can you explain why this will be the case? (Perhaps draw pictures of groups of dots and of antidots.)

b) What number should you add to the starting board above so that the final number that remains on the pages is sure to be -10?

24. Making Sense of Rule 8: "Chopping Up Rectangles"

I quietly slipped over Rule 8 in the previous section without any comment or fuss, as though it is all fine and not worthy of any extra comment.

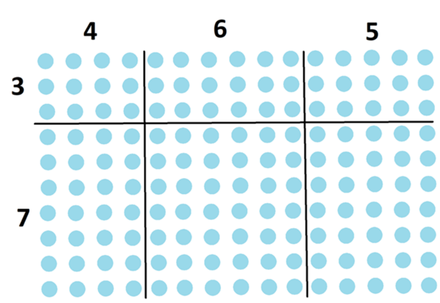
> **Rule 8: "**We can chop up rectangles from multiplication and add up the pieces."

But that was wrong of me.

We really need to talk about what this rule means now that we have negative numbers in our minds.

Playing with just counting numbers in chapter 1 allowed us to represent any multiplication problem we encountered with a picture, a rectangle of dots. And we could chop up these rectangles in any way we liked without a care in the world. The number of dots in each of the small rectangular pieces we created were sure to sum to the total number of dots in the picture to begin with.

For example, here's a picture representing 10×15 broken up into six pieces: 3×4 and 7×4 and 3×6, and so on.



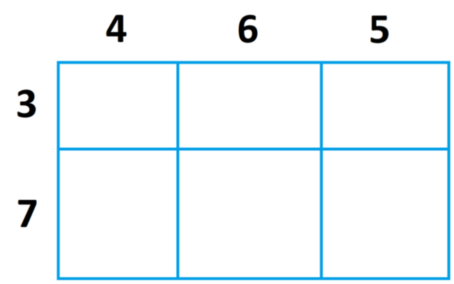
But, as we saw in the previous section, is not at all clear what dot or antidot pictures we could draw for some multiplication statements involving negative numbers. (We couldn't even give a meaningful interpretation of $(-4) \times 5$, for instance, in the last section, let alone draw a picture to represent it!)

So, what then is Rule 8 now saying in the context of negative numbers?

In order to move forward, we need to let go of dot and antidot thinking. Instead, we need to shift to thinking about the <u>behavior</u> of numbers as represented by the statement “we can chop up rectangles” (since, after all, we are now defining multiplication by how it behaves, not by “what it is”).

And we kinda did that already in Chapter 1 by drawing just rectangles, not rectangles of dots, and chopping up the pictures of those rectangles.

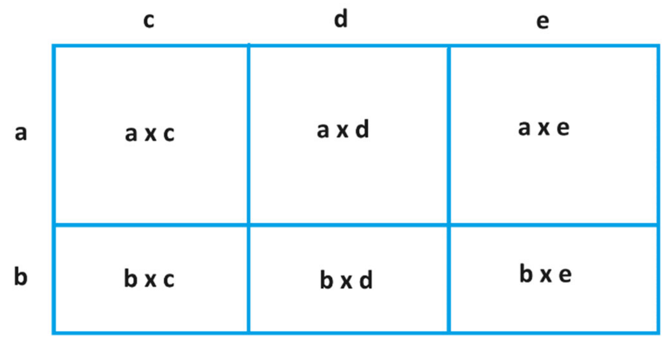
For example, this picture illustrates 10×15 and says that it matches the sum of six smaller products: 3×4 and 7×4 and 3×6 and so on. It’s a picture that shows how arithmetic is working rather than showing you literal counts of dots.



Practice 24.1: What is the value of $(3+7)\times(4+6+5)$?
What is the value of $3\times4+7\times4+3\times6+7\times6+3\times5+7\times5$?
(They should be the same!)

This is how we need to represent Rule 8. Not by attempting to draw literal pictures of dots and antidots in rectangular arrays, but by drawing general rectangles to illustrate the behavior of the arithmetic.

For example, this picture is showing that we can think of a product $(a + b) \times (c + d + e)$ as the same as the sum of six smaller products (just as was the case for $(3 + 7) \times (4 + 6 + 5)$).



But now we are allowing some or all of our numbers to be negative.

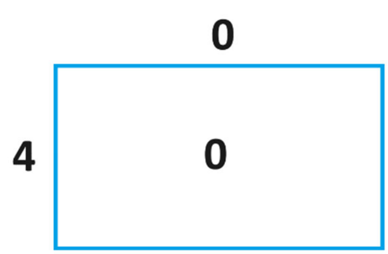
And this will look confusing to some people, “How can a rectangle have a negative side length or a negative area?” they might cry out.

Again, we can’t read the pictures literally: just as they are not representing pictures of actual dots and antidots, and they are also not representing pictures from geometry class of actual rectangles with real-world measurements for side lengths and areas.

The rectangle pictures illustrate how arithmetic is working—nothing more.

(But this, I know, can be confusing because, if all the numbers shown in a picture happen to be positive, then the picture does perfectly match a rectangle that could have been drawn in geometry class!)

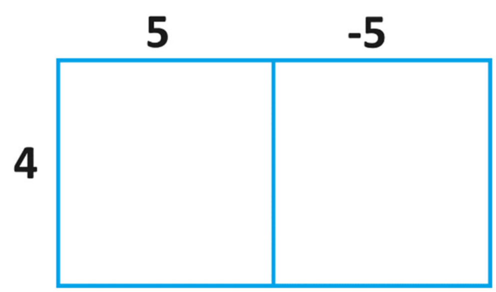
For example, here’s a picture representing 4×0 with the answer 0.



One doesn’t draw side lengths of zero and areas of zero in a geometry class. But this picture is still illustrating for us that $4 \times 0 = 0$ and so is a helpful picture.

It is particularly helpful when we change how we represent that top side length of zero. Let's think of zero as $5 + \ -5$ and chop up the rectangle accordingly.

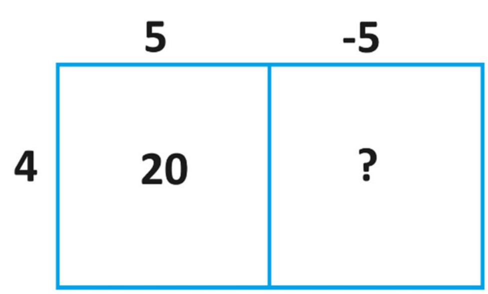
The total "area" of this rectangle is still zero.



But we now see two pieces: a left piece corresponding to 4×5 and a right piece to $4 \times (-5)$.

We know the value of the left piece: it's just multiplication with the counting numbers and we are assuming that hasn't changed. So $4 \times 5 = 20$.

It remains to work out the value of the right piece



Whatever its value, we know that 20 and it have to add to zero.

$$20 + ? = 0$$

Ahh! That right piece has to be -20.

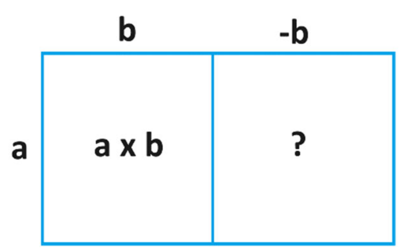
So, rule 8 has just forced us to conclude that $4 \times (-5)$ has value -20.

And this is lovely. It matches our intuitive sense that $4 \times (-5)$ might be interpreted as "four copies of -5," which is $-5 + -5 + -5 + -5$. And, yes, that's -20.

Math and intuition are aligned … again!

> **Practice 24.2** Draw a rectangle that represents the statement $8 \times 0 = 0$, but make the top side length $7 + -7$ instead of 0.
>
> Use your picture to then explain why the value of $8 \times (-7)$ must be -56.

If you want to be fancy and use letter names for numbers and not specific numbers (like 4 and 5 or 7 and 8), here's a picture of $a \times 0$, which has value 0, but with the top side length presented as $b + -b$, instead of 0.



The value of the question mark must be $-(a \times b)$, the number we add to $a \times b$ to get zero. But it is also $a \times (-b)$ from looking at the picture.

We've discovered:

> For any two numbers a and b, we have $a \times (-b) = -(a \times b)$.

For example, $4 \times (-5) = -(4 \times 5)$, and working out what is inside the parentheses, this is -20.

Also, $7 \times (-8) = -(7 \times 8)$, and working out what is inside the parentheses, this is -56.

People say "we can just pull out a negative sign from a product."

Here's a next example.

In the last section, we couldn't give meaning to the quantity $(-4) \times 5$ (although we have all been trained from our school days to say that is has value -20).

We are now ready to see what value math assigns to $(-4) \times 5$.

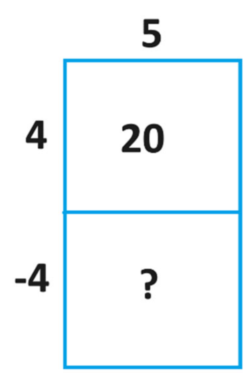
In fact, we can get to it two ways!

Approach 1:
By Rule 4, we can change the order in which we multiply two numbers.
So, $(-4) \times 5$ has the same value as $5 \times (-4)$.

By "pulling out the negative sign" as we just learned, this is $-(5 \times 4)$, which is -20. (Work out what is inside the parentheses.)

Approach 2: We know that 0×5 is zero.

Draw a picture to represent this, but with 0 presented as $4 + \ -4$.



One piece of the rectangle has value $4 \times 5 = 20$.
The second piece is given by $(-4) \times 5$, whose value we seek.

But the two pieces combined have value 0. It must be that $(-4) \times 5 = -20$.

Practice 24.3 Draw a picture of $0 \times b = 0$ to show that for any two numbers a and b, we have $(-a) \times b = -(a \times b)$.

So,

$4 \times (-5)$ has value -20,

and

$(-4) \times 5$ has value -20,

and

$-(4 \times 5)$ has value -20 too. (Work out what is inside the parentheses first.)

And this is why schoolbooks can get away with being unclear in their attempts to make sense of a product like $(-4) \times 5$ and just assert "It's obviously the same as $4 \times (-5)$" or "It's obviously the same as $-(4 \times 5)$." Our mathematics has shown that all three products have the same value, so no false claims are being made (except for the "obviousness" of it).

You can see too why schoolbooks avoid explaining the mathematics of all this. Look at the amount of hard thinking we had to go through to get this point!

In general, for any two numbers, a and b, we have established that

$$a \times (-b)$$

$$(-a) \times b$$

$$-(a \times b)$$

are all sure to have the same value.

We can just "pull out negative signs" from products in any way we like. Doing so won't affect the value of the product.

Practice 24.4 Explain why $(-1) \times 8$ is -8.

For extra fun, try to justify this more than one way, perhaps by drawing a picture of 0×8 but presenting 0 as $1 + -1$; or perhaps by "pulling out a negative sign" from $(-1) \times 8$ and see where that leads you; or perhaps by adding $8 + (-1) \times 8$ and factoring (section 13) to see if you get zero.

Where We Are At

We feel we have a concrete understanding of addition and multiplication ("repeated addition") for the counting numbers. This was the work of Chapter 1.

And now we've extended addition and multiplication to a wider class of numbers, namely, to the counting numbers and their opposites—the integers.

We've had to let go of always trying to interpret what addition and multiplication concretely mean ("What is $(-4) \times 5$, really?") and have moved instead to defining addition and multiplication by how they behave rather than what they are.

The nine rules of arithmetic we listed are to define the rules of behavior of arithmetic.
And these nine rules really do capture all the arithmetic we were taught in school.

We saw in this section and the last that we can:

"**distribute negative signs**" ($-(4-5) = -4 + -5$, for instance)

"**pull out negative signs**" ($(-4) \times 5 = -(4 \times 5)$ and that's -20, for instance)

multiply by negative one to make a number negative ($-(1) \times 8 = -8$, for instance)

and more.

On the next page, in one place, is a much quicker and cleaner summary of everything we need to know about the arithmetic of the integers, that is, the counting numbers, zero, and their negatives.

There is nothing to memorize or do here!

This list is just to say that everything we've been taught to do in arithmetic is sound.

Numbers come with two operations—addition and multiplication—which behave as we expect when applied to just the counting numbers and, more generally, behave as follows:

Rule 1: For any two numbers a and b we have $a + b = b + a$.

Rule 2: For any number a we have $a + 0 = a$ and $0 + a = a$.

Rule 3: In a string of additions, it does not matter in which order one conducts individual additions.

Rule 4: For any two numbers a and b we have $a \times b = b \times a$

Rule 5: For any number a we have $a \times 1 = a$ and $1 \times a = a$.

Rule 6: In a string of multiplications, it does not matter in which order one conducts individual multiplications.

Rule 7: For any number a we have $a \times 0 = 0$ and $0 \times a = 0$.

Rule 8: "We can chop up rectangles from multiplication and add up the pieces."

Rule 9: For each number a , there is one other number "$-a$" such that $a + -a = 0$.

Some Logical Consequences: For any two numbers a and b

i) $-0 = 0$
("The opposite of zero is zero")

ii) $- - a = a$
("The opposite of the opposite is back to the original")

iii) $-(a + b) = -a + -b$
(We can "distribute a negative sign")

iv) $(-a) \times b$ and $a \times (-b)$ and $-(a \times b)$ all have the same value
(We can "pull out a negative sign")

v) $(-1) \times a = -a$
("Multiplying by -1 gives you the opposite number")

25. Why Negative times Negative is Positive

We're all set to address this age-old question of why multiplying together two negative numbers should give a positive answer.

It is the only "case" remaining for us to consider.

Positive times Positive

Example: What is 2×3?

This is just the product of two counting numbers, and multiplication is still just "repeated addition" in this context. Thus, $2 \times 3 = 3 + 3 = 6$, which is positive six.

The product of two positive integers is positive.

Positive times Negative

Example: What is $2 \times (-3)$?

We no longer have the product of two counting numbers, so we (technically) cannot rely on repeated addition.

But we did learn that we can "pull out a negative sign" from a product. Thus

$$2 \times (-3) = -(2 \times 3)$$

Working out what is inside the parentheses, we get

$$2 \times (-3) = -6$$

(And this answer does happen to match "repeated addition" thinking!)

The product of a positive integer and a negative integer is negative.

Negative times Positive

Example: What is $(-2) \times 3$?

This example of "meatier." Now the even intuition of "repeated addition" fails here.

But we know we can again "pull out a negative sign." Thus

$$(-2) \times 3 = -(2 \times 3)$$

Working out what is inside the parentheses, we get

$$(-2) \times 3 = -6$$

The product of a negative integer and a positive integer is negative.

And now to the challenge case …

Negative times Negative

Example: What is $(-2) \times (-3)$?

Why should the answer to this product be <u>positive</u> six?

Let's establish why this must be so two different ways.

Explanation 1: Pull out a negative sign

Let's pull out a negative sign once.

$$(-2) \times (-3) = -(2 \times (-3))$$

And let's now work out what is inside the parentheses: $2 \times (-3) = -6$, as we saw.

So,

$$(-2) \times (-3) = -(-6)$$

We showed that the opposite of the opposite of a number is the original number: $- - 6$ is 6. What we have now is that

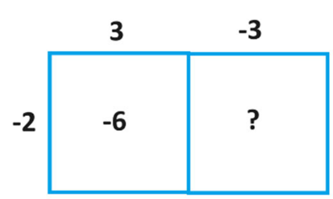
$$(-2) \times (-3) = - - 6 = 6$$

The product of a negative integer and a negative integer is positive.

Explanation 2: Chop up a rectangle

We know that $(-2) \times 0 = 0$ by Rule 7.

Let's draw a picture of this, but with zero presented as $3 + -3$.
This gives a rectangle divided into two pieces.



The left piece is $(-2) \times 3$, which we've already seen has value -6.
The right piece is $(-2) \times (-3)$, which is the product we are wondering about.

But we do know that the two pieces have values that sum to 0.

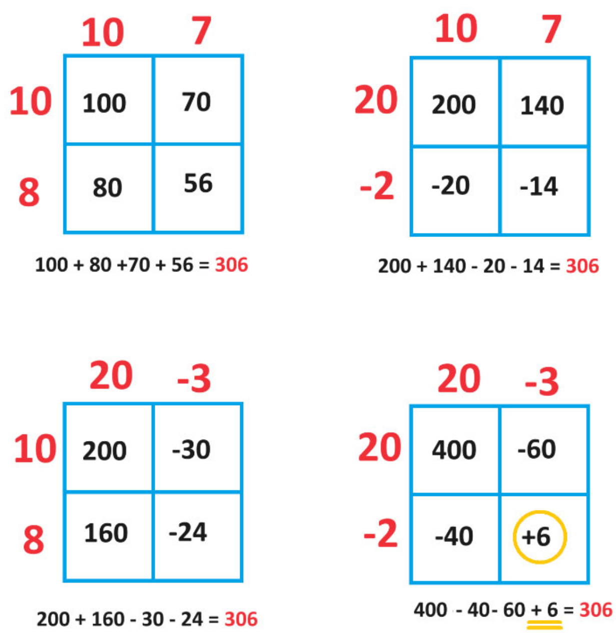
It must be that $(-2) \times (-3) = 6$.

The product of a negative integer and a negative integer is positive.

In summary, we have established what we were taught (told?) in school:

positive times positive is positive
positive times negative is negative
negative times positive is negative
negative times negative is positive

For fun, here are four different ways to work our 17×18, making use of each of these four facts. You can see in the fourth picture that math is telling us that we really do need $(-2) \times (-3)$ to be positive six.



Practice 25.1 Draw four different pictures like these that each compute 14×15. See that mathematics really does want $(-6) \times (-5)$ to be positive thirty.

Practice 25.2 How might you reason why $(-11) \times (-12)$ has to be 132, a positive answer?

Practice 25.3

a) What is the value of $(-1) \times (-1)$?

b) What is the value of $(-1) \times (-1) \times (-1)$? How do you know?

c) What is the value of $(-1) \times (-1) \times (-1) \times (-1) \times (-1) \times (-1) \times (-1) \times (-1) \times (-1)$? How do you know?

Let's practice some ideas from Chapter 1, but with negative numbers involved.

Example: Work out $(4-6)(10-3)$ two different ways.

Answer:

By working out what is inside the parentheses first we get $(-2) \times (7)$, which is -14.

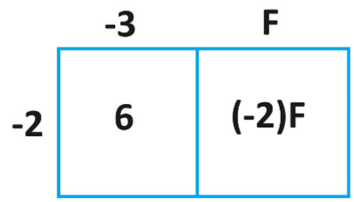
Alternatively, thinking of this as coming from chopping up a rectangle,

	10	-3
4	**40**	**-12**
-6	**-60**	**18**

we get $40 + 18 - 60 - 12$, which is -14.

Example: Make $(-2)(-3+F)$ look a bit friendlier.

Answer: Draw, or imagine, a rectangle to see that $(-2)(-3+F)$ is $6+(-2)F$.



Now, $(-2)F$ is really $(-2)\times F$.

Pulling out a negative sign, this is $-(2\times F)$, which most people write as $-2F$. (Do you remember parentheses are usually kept hidden for multiplications?)

So, we have that $(-2)(-3+F)$ is the same as

$$6+\ -2F$$

And we can rewrite this in terms of subtraction

$$6-2F$$

So, $(-2)(-3+F)=6-2F.$

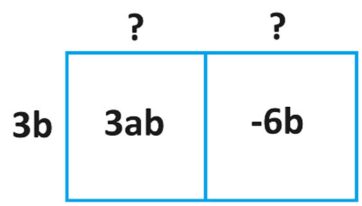
Practice 25.4 Show that $(-5-G)\cdot(-3)$ can be rewritten as $3G+15$. (Drawing picture really does help!)

Practice 25.5 Show that $2-5(2-3w)$ can be rewritten as $15w-8$.

Example: Factor $3ab - 6b$.

Answer: Think of this as $3ab + \ -6b$.

There is a common factor of "$3b$" in the terms $3ab$ and $-6b$.



The picture suggests that $3ab - 6b$ is $3b(a + \ -2)$. That is,

$$3ab - 6b = 3b(a - 2).$$

Practice 25.6 Factor $10xy - 2x^2 - x$.

Practice 25.7 Show that $3(F - 2) - 2(F - 3)$ is the same as just F.

MUSINGS

Musing 25.8 Do you recall how "negative times negative is positive" was explained to you back in school? (Was it explained?)

Musing 25.9
a) Draw rectangles to show that $(10 + a)^2$ and $10(10 + 2a) + a^2$ are sure to have the same value, no matter what number the letter a represents.

b) Eoin computed 14^2 as $10 \times 18 + 4^2$, which is $180 + 16 = 196$ and is correct, and he computed 15^2 as $10 \times 20 + 5^2$, which is 200+25 $= 225$ and is again correct. Can you explain what he was doing?

c) How might Eoin compute 17^2?

MECHANICS PRACTICE

Musing 25.10
Which of these expressions evaluates to a positive value?
Which evaluate to a negative value?
And which three of them evaluate to zero?

a) $(-2) + (-7)$
b) $4 \cdot (-3)$
c) $(-10) \times 110$
d) $-(-6)$
e) $(-5)(-4)$
f) $(-10)(-3 - 5)$
g) $7 - (10 - 3)$
h) $(4 - 8)(2 - 1)$
i) $(-143) \cdot (542) \cdot 0 \cdot (-1987)$
j) $(5 - 2) \cdot (100 - 50 - 20)$
k) $(f - a)(f - b)(f - c)(f - d)(f - e)(f - f)(f - g)(f - h)$

Musing 25.11 Make each of the following expressions look friendlier.
(Each letter is just a symbol representing some unspecified number.)

a) $3(x-4)$
b) $(-3)(4-x)$
c) $2-y-4(2y-5)$
d) $5(F-4)-4(F-5)$

Musing 25.12 Rewrite each of these expressions by factoring and see if you can match each with the expression given in square brackets.

a) $5c+cd$	$[c(5+d)]$
b) w^2-2w	$[w(w-2)]$
c) $px-py$	$[p(x-y)]$
d) $(r^2-7y+43)x-(r^2-7y+43)y$	$[(r^2-7y+43)(x-y)]$
e) $(3z-1)(4-z)-(3z-1)(3-z)$	$[3z-1]$
f) $3(f-2)-(f-2)$	$[2(f-2)]$

26. Interlude: Finger Multiplication

ACTIVITY

FINGER MULTIPLICATION

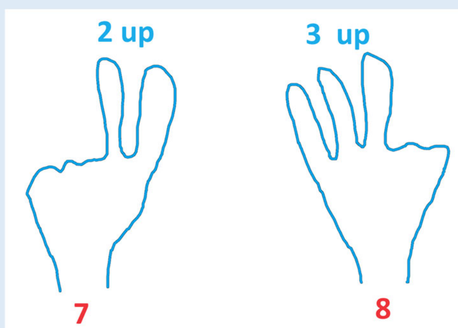
Don't memorize your multiplication table facts. Let your fingers do the work!

If you are comfortable with multiples of one, two, three, four, and five, then there is an easy way to compute the products in the six- through-ten part of the times table.

Start by encoding the two numbers you wish to multiply on your hands as follows.

A closed fist represents "five" and any finger raised on that hand adds one to that value.

For example, a hand with two fingers raised represents $5 + 2$, that is seven, and a hand with three fingers raised represents $5 + 3$, that is, eight.



To multiply together the two numbers you've encoded, follow these steps.

1. *Each raised finger is worth ten. Count ten for each finger raised you see.*
2. *Count the number of <u>unraised</u> fingers you see on each hand and multiply together those two numbers.*
3. *Add the results of steps 1 and 2. The sum in the desired product.*

For example, 7×8 is represented as two raised fingers on the left hand and three on the right hand.

There are five raised fingers in all, yielding the number $\mathbf{50}$ for step 1.
The left hand has **3** lowered fingers and the right hand **2**, and $3 \times 2 = \mathbf{6}$ for step 2.
The product we seek is then $50 + 6 = 56$, which is indeed the value of 7×8!

Magic!

In the same way, one computes 9×7 as follows:

One hand with **four** fingers raised is "nine." $(5 + 4 = 9)$.
One hand with **two** fingers raised is "seven." $(5 + 2 = 7)$.
That's six fingers up in total, making $\mathbf{60.}$

We have 1 finger down on one hand, 3 fingers down on the other, and $1 \times 3 = \mathbf{3}$.
Thus $9 \times 7 = 60 + 3 = 63$, and it does!

And one computes 6×8 as follows:

One hand with **one** finger raised is "six." $(5 + 1 = 6)$.
One hand with **three** fingers raised is "eight." $(5 + 3 = 8)$.
That's four fingers up in total, making $\mathbf{40.}$

We have 4 fingers down on one hand, 2 fingers down on the other, and $4 \times 2 = \mathbf{8}$.
Thus $6 \times 8 = 40 + 8 = 48$, and it does!

a) Practice this method by computing 9×9 and 6×7.
b) How about going to extremes? Does this method work for 5×5 and 10×10?
c) Can you explain why this strange method of multiplication is working?

Fingers and Toes

We can go to the ten- through twenty-times tables too if we use all of our twenty digits!

Here's how to work out 17×18, as an example.

On the left side of my body I have ten digits (five fingers and five toes). All ten digits down represents 10. To make this 17, raise **seven** of those digits. (Or, at least, imagine doing so.)
On the right, all ten digits down also represents 10. To make 18, say, raise **eight** of those digits.

Each raised digit is now worth 20.
We currently have a total of fifteen raised digits. That makes for $\mathbf{300}$.

Now multiply the count of digits down: 3 on the left and 2 on the right giving $3 \times 2 = \mathbf{6.}$

And, lo and behold, 17×18 does equal $300 + 6 = 306$.

d) Compute 14×18 and 16×19 this way.
e) Again, why is this bizarre method working?

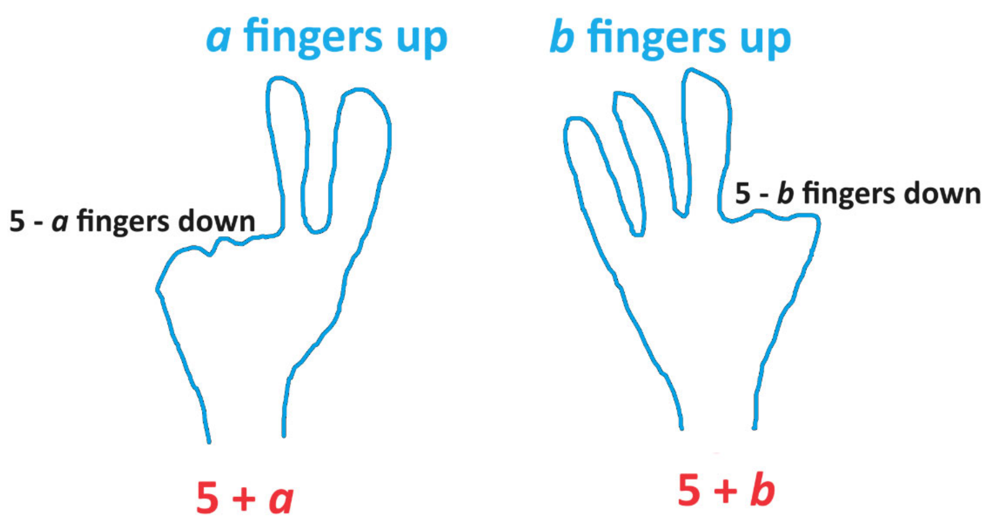
I remember being taught several tricks for recalling math facts when I was a kid, but I don't remember having anyone help me think through why they worked. (Understanding seemed much more interesting to me than memorizing the facts!)

Surprisingly, this finger trick utilizes all the math we've developed so far—including multiplying negative numbers—which seems surprising since we're just doing multiplication of counting numbers a la chapter 1.

Let's first consider just finger multiplication. (No toes.) And let's be a little abstract in our thinking.

Suppose we have a fingers up on the left hand. This would represent the number $5 + a$.
(For instance, if a is the number 2, then $5 + a$ is 7.)

And suppose we have b fingers up on the right hand. This would represent the number $5 + b$.
(For instance, if b is the number 3, then $5 + b$ is 8.)



We are trying to multiply together the numbers $5 + a$ and $5 + b$.
That is, we're computing

$$(5 + a)(5 + b)$$

(For $a = 2$ and $b = 3$, this is the product 7×8.)

But the finger tricks tells us to do this in an unusual way.

Step 1: *Regard each raised finger as "worth ten."*

We currently have $a + b$ fingers raised.
So, step 1 of the algorithm gives us the value: $\mathbf{10(a + b)}$.

Step 2: *Multiply together the counts of fingers down on each side.*

There are 5 fingers in total on our left hands. If a of them are raised up, then $5 - a$ are down.
There are 5 fingers in total on our right hands. If b of them are raised up, then $5 - b$ are down.
We need to multiply these two counts together.
So, step 2 of the algorithm gives us the value $\mathbf{(5 - a)(5 - b)}$.

Step 3: *The product we seek is these two values added together.*

We're looking for the value of the product $\mathbf{(5 + a)(5 + b)}$.

The algorithm says that it equals $\mathbf{10(a + b) + (5 - a)(5 - b)}$.

These don't look the same!

Are they?

Let's play with each of these expressions and see if they are really the same thing in disguise.

Playing with $(5 + a)(5 + b)$

Let's expand the brackets.

We see that $(5 + a)(5 + b)$ is the same as $25 + 5a + 5b + ab$.

	5	b
5	25	5b
a	5a	ab

Playing with $10(a+b)+(5-a)(5-b)$

We have two pieces here.

Expanding $10(a+b)$ gives us $10a+10b$.

	a	b
10	10a	10b

Expanding $(5+\ -a)(5+\ -b)$ gives

$$25+\ -5a\ +\ -5b\ +ab$$

	5	-b
5	25	-5b
-a	-5a	ab

Putting these two pieces together we get:

$$10a+10b+25+\ -5a\ +\ -5b\ +ab$$

But we can add this string of additions in any order we like. And I can't help noticing "$10a+\ -5a$" and "$10b\ +\ -5b$" sitting within the string.

Practice: Factor to show that $10a+\ -5a$ is the same as $5a$.
Practice: Factor to show that $10b+\ -5b$ is the same as $5b$.

So, our string of additions is really

$$5a+5b+25+ab$$

Comparing the Two:

$(5 + a)(5 + b)$ is really the same as $25 + 5a + 5b + ab$.

$10(a + b) + (5 - a)(5 - b)$ is really the same as $5a + 5b + 25 + ab$.

Are these the same?

Yes! They are each the same sum of four terms, just in a different order.

The result of the finger trick algorithm is sure to match the product we seek!

Practice 26.1
a) Show that $(10 + a)(10 + b)$ is the same as $20(a + b) + (10 - a)(10 - b)$.
b) Do you see that this observation explains the finger-and-toe version of the trick?

MUSINGS

Musing 26.2 A Martian has two hands, but with six fingers per hand.

a) Describe a finger-multiplication method the Martian can use to compute the values of products in the six- through- twelve times tables.
b) Can you provide an explanation as to why your method works?

Musing 26.3: Optional Plutonians have bodies that are not symmetrical. They have two hands, but with four fingers on one hand and six fingers on the other. Might there be a finger-multiplication method what will work for them?

27. Even and Odd Negative Integers

Let's end this chapter by playing with even and odd numbers again. We met them in Section 16.

Our definitions of *even* and *odd* actually work for negative numbers too.

An integer N, positive or negative (or zero), is said to be **even** if it equals twice another integer. That is, we can write $N = 2a$ for some integer a.

An integer N, positive or negative, is said to be **odd** if it is one more than twice another integer. That is, we can write $N = 2a + 1$ for some integer a.

For example:

-14 is even as it is double -7.

-15 is odd because is double -8, plus 1. (Think though that!)

Practice 27.1
a) If I have 14 antidots, can I split them into two groups of equal size? If so, how many dots are in each group?
b) If I have 15 antidots, what happens if I try to split them into two groups of equal size?
c) Draw a picture of two sets of eight antidots and one dot. Does this picture represent the number -15?

Practice 27.2 Write each of these integers as either double another integer or double another integer plus one.

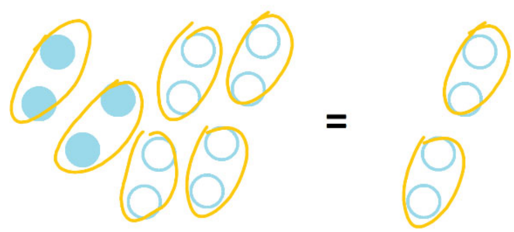
-20 -402 -5 -401 -1

In section 16 we briefly contemplated the validity of the following claims for counting numbers using a "take away" mindset.

EVEN - EVEN = EVEN
EVEN - ODD = ODD
ODD - EVEN = ODD
ODD - ODD = EVEN

We can intuitively justify these claims for integers too by thinking about pairs of dots and pairs of antidots. (Remember, subtraction is really the addition of the opposite.)

For example, this picture of $4 + -8$ is an example of "EVEN – EVEN" being EVEN.



We can certainly give an abstract mathematical proof of "EVEN – EVEN = EVEN," but it is a bit tedious. Can you make sense of this proof?

Claim: If N and M are each even integers, then $N - M$ is sure to be even too.

Proof: Since N and $M$are each even, we can write $N = 2a$ and $M = 2b$ for some integers a and b.

Then, $N - M$ is $2a - 2b$.

This is really $2a + -2b$, which we can think of as $2a + 2(-b)$.

Factoring out the two gives $2(a + -b)$.

So, $N - M$ is $2(a - b)$, which is double and integer. It is even!

Optional Practice 27.3 Prove mathematically that N is an even integer and M is an odd integer, then $N - M$ is sure to be an odd integer.

(Also, feel free to prove than if N is odd and M is even, then $N - M$ is sure to be odd; and also prove that if N and M are both odd, them $N - M$ is sure to be even.)

Here's something fun-ish.

ACTIVITY

THE PLUS/MINUS GAME

Niko and Zoe play the following game with the following seven numbers written in a row.

15	8	12	7	3	4	11

They each take turns inserting a plus sign $+$ or a minus sign $-$between two consecutive numbers until all six spaces are filled. They then compute the sum.

If the sum is odd, Niko wins.
If the sum is even, Zoe wins.

Who is sure to win?

Explanation:

Just like we did in Section 16, we can argue that any sum of integers containing an even number of odd integers is sure to be even, and any sum of integers containing an odd number of odd integers is sure to be odd.

The final sum in this game will contain four odd integers and so is sure to be even, and so Zoe is sure to win!

Practice 27.4 Design a plus/minus game Niko is sure to win.

MUSINGS

Musing 27.5

a) A grasshopper jumps along a line, moving left or right, one inch at a time. After 105 hops to the left and 106 hops to the right, in some random order, could it be that the grasshopper is back at its starting position?

b) A grasshopper again jumps randomly left and right along a line. Its first jump is 1 inch long. Its second jump is 2 inches long. Its third is 3 inches long, and so on. Could it be that after 50 jumps the grasshopper is back at its starting position?

c) Suppose instead that the grasshopper jumps two inches on its first hop, 4 inches on its second, 6 inches on its third, and so on. Could the grasshopper return to its starting position after 50 hops?

Chapter 4

Exploding Dots and the Power of Place Value

28. A Mind-Reading Trick

Let's start off this chapter with a classic piece of math magic.

ACTIVITY

A MIND-READING TRICK

In front of an audience present the following five groups of numbers. Write them on a chalk board or on a Power-Point slide or write out the groups of numbers on big cards.

GROUP A	GROUP B	GROUP C	GROUP D	GROUP E
16 20 24 28	8 12 24 28	4 12 20 28	2 10 18 26	1 9 17 25
17 21 25 29	9 13 25 29	5 13 21 29	3 11 19 27	3 11 19 27
18 22 26 30	10 14 26 30	6 14 22 30	6 14 22 30	5 13 21 29
19 23 27 31	11 15 27 31	7 15 23 31	7 15 23 31	7 15 23 31

The numbers 1 through 31 appear throughout these five groups, numbers that match the days of a month. Ask your audience members to each silently think of the day of the month they were born (a number) and look for their birthday among each of the five groups.

Now perform the mindreading trick by having the following question-and answer session with individual audience members.

"Suzzy. Is the number you are thinking of in group A?"	"Yes."
"Is the number you are thinking of in group B?"	"Yes."
"Is it in group C?"	"No."
"Is it in group D?"	"No."
"Group E?"	"Yes!"
"Ahh … you were born the 25th day of the month!"	"Wow! How did you know?"

GROUP A	GROUP B	GROUP C	GROUP D	GROUP E
16 20 24 28	8 12 24 28	4 12 20 28	2 10 18 26	1 9 17 25
17 21 25 29	9 13 25 29	5 13 21 29	3 11 19 27	3 11 19 27
18 22 26 30	10 14 26 30	6 14 22 30	6 14 22 30	5 13 21 29
19 23 27 31	11 15 27 31	7 15 23 31	7 15 23 31	7 15 23 31

Have the same question-and-response conversation with a few more audience members, noting each time which groups elicit a "yes" answer. Any given birthday number is then simply the sum of the top-left corner numbers in each group with a YES answer. For example, Suzzy answered YES YES NO NO YES. Groups A, B, and E have top left numbers 16, 8, and 1, respectively, and indeed $16 + 8 + 1 = 25$ was sure to be Suzzy's birthday.

As practice, check that if Sameer is thinking the number 13, he will answer NO YES YES NO YES and indeed $8 + 4 + 1 = 13$.

Do this as many times as your audience desires. Invite them to figure out what you are doing, and then, why what you are doing works!

Note: Rather than ask each audience member five questions, it is easier to just ask each person "In which group or groups does your birthday appear?"

Question: Examine each of the cards. What do you notice about the numbers in each group? What do you wonder about them?

Here are some things I personally notice and question.

- Group E contains all the odd numbers.
- Group A contains all the numbers 16 and above.
- The top left corner numbers of these groups, 1, 2, 4, 8, and 16, are doubling.

The doubling numbers 1, 2, 4, 8, 16 are clearly the key to this trick.

To explain their magic, let me tell you another personal story that isn't true.

MUSINGS

Musing 28.1

a) Derarcha says that her birthday appears in groups A, D, and E. On what day of the month was she born?

b) Erik says that his birthday appears in every group. On what day of the month was he born?

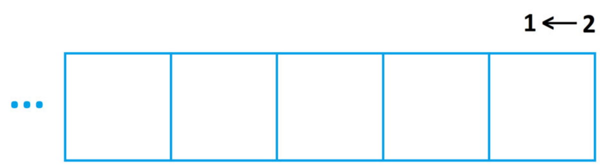
Musing 28.2 Before reading on the rest of this chapter, do you want to try to figure out why the mind-reading trick works all on your own? (This is a YES/NO question!)

29. A Story that is Not True

When I was a child, I invented a machine (not true!). It isn't a physical machine, but a machine I can work with using pencil and paper, or better yet, a board and erasable markers.

The machine is nothing more than a row of boxes that extend as far to the left as I please. For example, I could have 5 boxes heading off to the left, or 7 boxes, or 777 boxes.

And in this untrue story, I gave this machine a name. I called it a "two-one machine" both written and said in a funny backwards way. (I knew no different as a child.)



And what do you do with this machine? You put in dots, of course! I like dots.

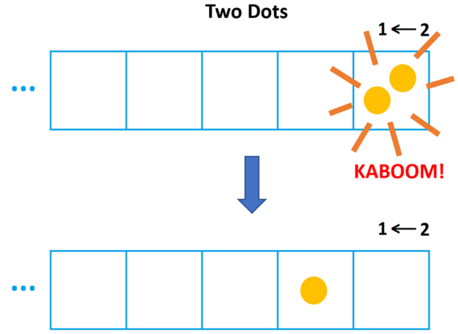
The thing note is that **dots always go into the rightmost box.**

Put in one dot, and, well, nothing happens: it stays there as one dot. Ho hum.

One Dot

But put in a second dot – always in the rightmost box – and then something exciting happens.

Whenever there are two dots in a box they explode and disappear – KAPOW! – to be replaced by one dot, one box to the left.



Question: Do you see now why I called this a "1 ← 2 machine" written in this funny way?

We see that two dots placed into the machine yields one dot followed by zero dots.

Actually, it's zero dots and zero dots and zero dots and one dot and zero dots for my picture with five boxes (this would be a much longer sentence if I happened to draw 707 boxes). But let's ignore all the zeros for empty boxes to the left and write the result of putting two dots in the machine as "**1 0**"—one dot followed by zero dots.

The 1 ← 2 machine code for the number *two* is **1 0**.

Two Dots

2: 10

Warning: The code "1 0" looks like the ordinary number ten. But when we are thinking of codes, let's read 1 0, for instance, as "one zero."

Putting in a third dot – always the rightmost box – gives the picture one dot followed by one dot.

Three dots

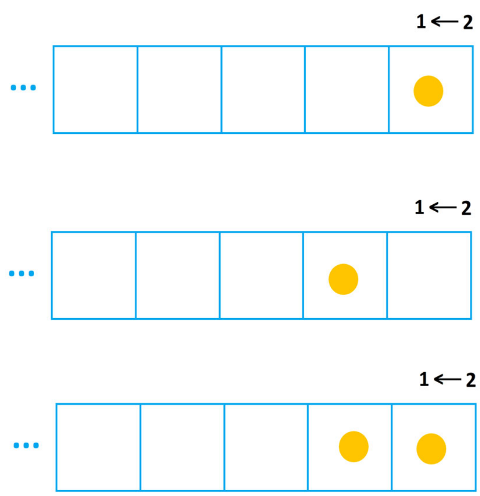
3: **11**

We're getting codes for numbers.

Just one dot placed in the machine stayed as one dot.
The 1 ← 2 machine code for the number *one* is **1**.

Two dots placed into the machine, one after the other, yielded one dot in a box followed by zero dots.
The 1 ← 2 machine code for the number *two* is **1 0**.

Adding a third dot in the machine gave the code **1 1** for *three*.



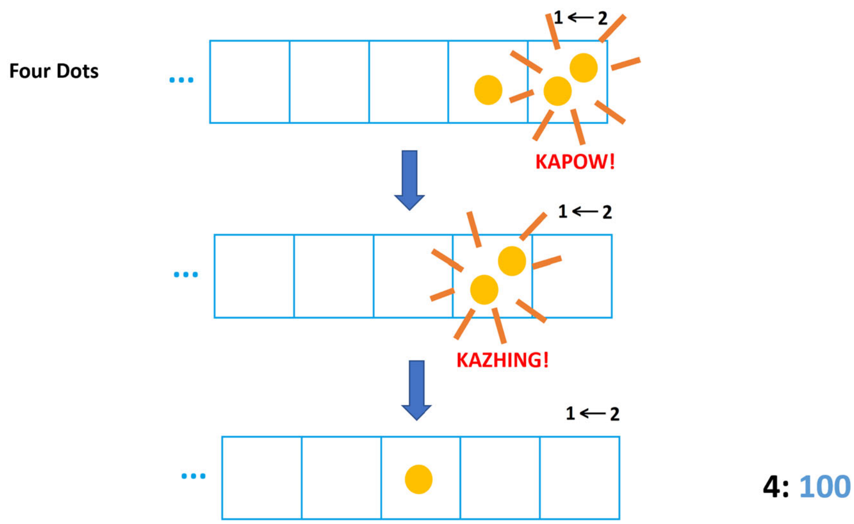
1: **1**

2: **10**

3: **11**

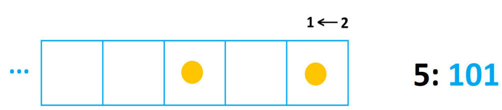
Adding a fourth dot into the machine after three dots have exploded is particularly exciting: we are in for multiple explosions!

The 1 ← 2 machine code for *four* is **1 0 0**. (Remember: Read this is "one zero zero.")

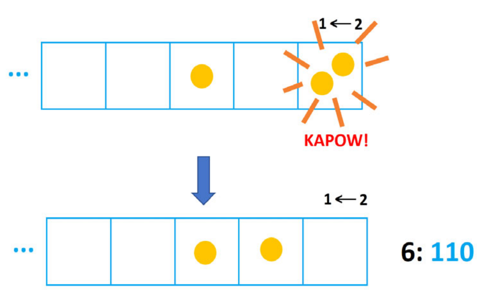


We can keep going, adding dots to the rightmost box one at a time.

The 1 ← 2 machine code for *five* is **1 0 1**.

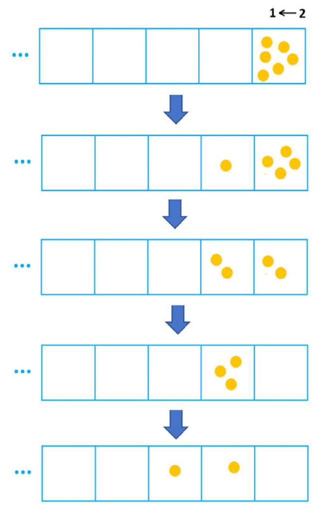


And the code for *six*? Adding one more dot to the code for five gives **1 1 0** for *six*.



We can also get this code for *six* by clearing the machine and putting all six dots in at once. Pairs of dots will explode in turn to each produce one dot, one box to their left.

Here is one possible series of explosions. (Sound effects omitted!)



Practice 29.1: Do you get the same final code of **1 1 0** if you perform explosions in a different order? (Try it!)

Practice 29.2: What is the 1 ← 2 machine code for the number *thirteen*?
(It turns out to be **1101**. Can you get that answer?)

There are hours of fun to be had playing with codes in a 1 ← 2 machine.

Practice 29.3 Here are the 1 ← 2 machine codes for the first ten numbers.
Care to work out the code for all the numbers up to twenty?
(This is technically a YES/NO question.)

1: 1	**4: 100**	**7: 111**	**10: 1010**
2: 10	**5: 101**	**8: 1000**	
3: 11	**6: 110**	**9: 1001**	

Practice 29.4 Experiment and try to find the number which has 1 ← 2 machine code **10111**.

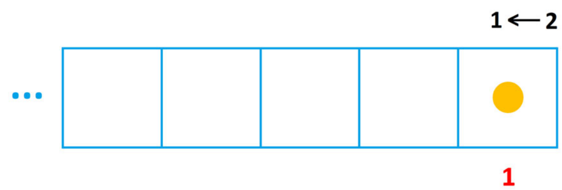
Explaining the 1 ← 2 machine

Let's figure out what's really going on with this machine and the codes it produces.

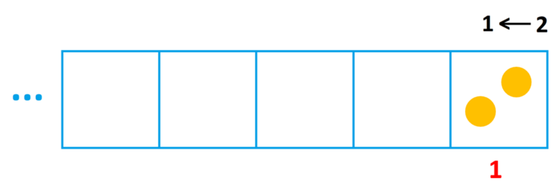
We set up matters in our 1 ← 2 machine so that …

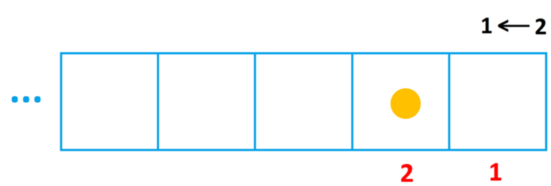
> *Whenever there are two dots in any one box, they "explode," that is, disappear, to be replaced by one dot, one place to their left.*

Also, the machine is set up so that dots in the rightmost box are always worth one.

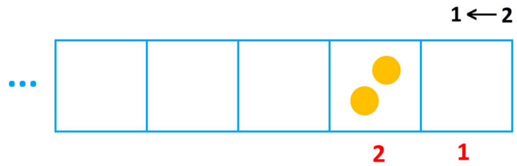


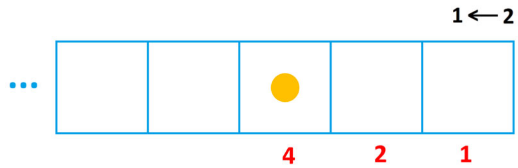
With an explosion, two dots in the rightmost box are equivalent to one dot in the next box to the left. And since each dot in the rightmost box is worth 1, each dot one place over must be worth two 1s, that is, 2.



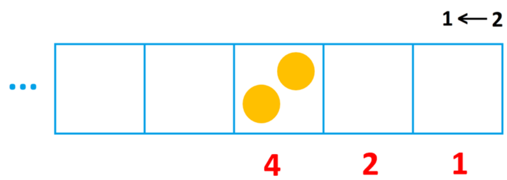


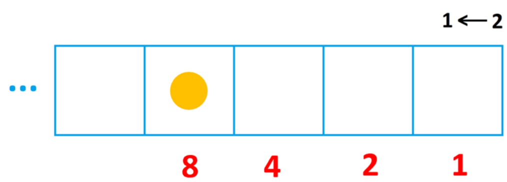
And if you happen to have two dots in this second box, they world explode to make one dot, one place to the left. That new dot is with two 2s, so it is worth 4.





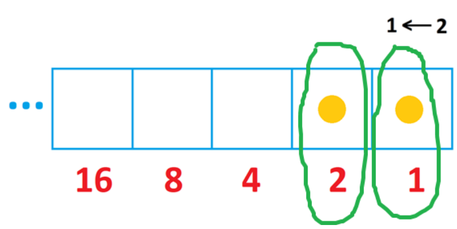
And two 4s makes 8, so a dot in the next place to the left is worth 8.



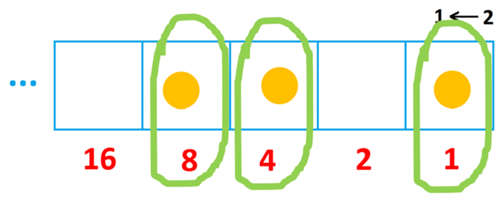


And two 8s make 16, and two 16s make 32, and two 32s make 64, and so on.

Earlier we saw that the $1 \leftarrow 2$ machine code for the number *three* is **1 1**. And one dot and one dot in the last two boxes does indeed make give a picture of dots with total value three: $2 + 1 = 3$. Neat!



If you tried Practice 29.2, you would have seen that the 1 ← 2 machine code for *thirteen* is **1 1 0 1**. Now we can see too—literally see!—that this code is correct: one 8 and one 4 and no 2s and one 1 does indeed make thirteen!



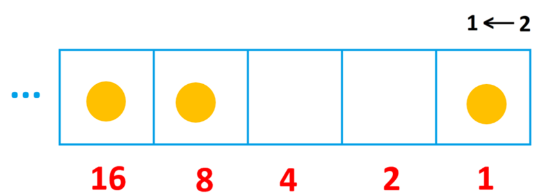
Practice 29.5 Answering question 29.4 this new way, which number which has 1 ← 2 machine code **1 0 1 1 1**?

Practice 29.6 The number seventeen equals 16 + 1. What is the 1 ← 2 machine code for *seventeen*?

Practice 29.7 What is the is the 1 ← 2 machine code for *thirty*?

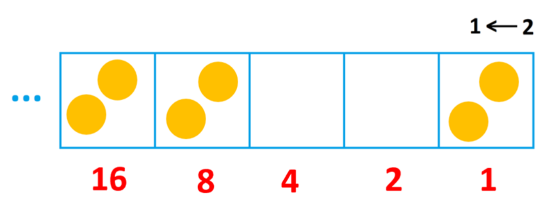
The 1 ← 2 machine shows how to write any number as a sum of the doubling numbers 1, 2, 4, 8, 16, and so on: just put in that many dots into the rightmost box, let them explode, and the code of **0**s and **1**s tells you which doubling numbers to use to make that number.

Practice 29.8 The 1 ← 2 machine code for the number *twenty-five* is **11001**.



Double twenty-five is fifty.

What code do you get if you double the number of dots in each box in the picture and the conduct the explosions. Do you get the 1 ← 2 machine code for the number *fifty*?



Practice 29.9
The largest number with a five-digit 1 ← 2 machine code is *thirty-one* with code **11111**. What is the smallest number with a five-digit code?

Some people might answer this question by noting that we could write the code for *one* as **00001** if we decided to write the **leading zeros** to the left. (For that matter, we could write the code for *zero* as **00000**.) The answer to this question could be one (or it could be zero).

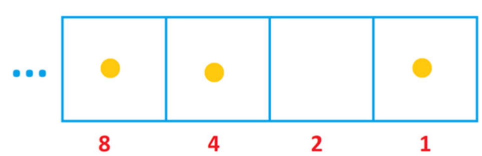
But let's do indeed follow the convention of *not writing leading zeros* in the codes for numbers. In which case, the number with the smallest five-digit code has code **10000**. That corresponds to the number *sixteen*.

Some Language

People call the 1 ← 2 machine codes for numbers the **binary** representations of numbers (with the prefix *bi-* meaning "two"). They are also called **base two** representations. One only ever uses the two symbols **0** and **1** when writing numbers in binary.

Computers are built on electrical switches that are either on, or off. So, it is very natural in computer science to encode all arithmetic in a code that uses only two symbols: say **1** for "on" and **0** for "off." Thus, base two, binary, is used at the heart of computer science.

BASE TWO



thirteen = 1101

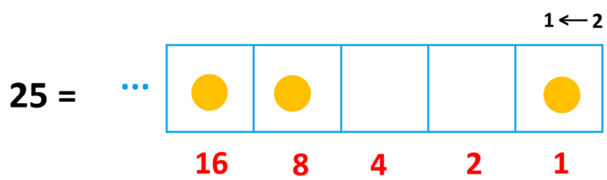
Explaining the Mind-Reading Trick

Let's end this section by explaining the mind-reading trick.

Recall that Suzzy was thinking of the number 25 and saw it groups A, B, and E.

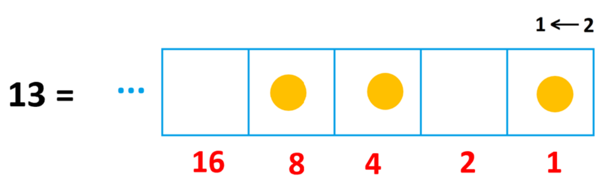
GROUP A	GROUP B	GROUP C	GROUP D	GROUP E
16 20 24 28	8 12 24 28	4 12 20 28	2 10 18 26	1 9 17 25
17 21 25 29	9 13 25 29	5 13 21 29	3 11 19 27	3 11 19 27
18 22 26 30	10 14 26 30	6 14 22 30	6 14 22 30	5 13 21 29
19 23 27 31	11 15 27 31	7 15 23 31	7 15 23 31	7 15 23 31

The $1 \leftarrow 2$ machine code for twenty-five shows us that $25 = 16 + 8 + 1$.



And in creating the cards, I made sure the number 25 appeared in the card with 16 at its top left, in the card with 8 at its top left, and in the card 1 at its top left. That is, I made sure that the number 25 appears in groups A, B, and E so that if Suzzy were to say "A, B, and E" I'd know to compute $16 + 8 + 1$ and deduce her number.

Here's the binary code for *thirteen.*



And look! I made sure 13 appears in cards B, C, and E so that I could quickly compute $8 + 4 + 1$ if those three groups are mentioned!

Practice 29.10: The binary code for *ten* is **1010**. In which cards should ten appear? Does it? (And does it appear only in those groups?)

Practice 29.11: Does it make sense that the number 31 appears in each and every card?

It is a happy coincidence that largest number we can make with the doubling numbers 1, 2, 4, 8, and 16, namely 31, matches the number of days in a (long) month of the year. Having people think of their birthdays makes this trick feel a little more mysterious and special.

Practice 29.12: Why does card A contain all the numbers 16 and higher?

Practice 29.13: Why does card E contain all the odd numbers?

Practice 29.14: The number with the biggest six-digit binary code **111111** is

$$32 + 16 + 8 + 4 + 2 + 1 = 63.$$

Create a six-card mind-reading trick where an audience member is to pick a number between 1 and 63 and, after telling you in which cards they see their chosen number, you read their mind.

What numbers do you place on each card?

MUSINGS

Musing 29.15 Could a number ever have code **100211** in a $1 \leftarrow 2$ machine, assuming we always choose to explode dots if we can?

Musing 29.16 The list of doubling numbers continues indefinitely.

1, 2 4, 8, 16, 32, 64, 128, 256, 512, 1024, ….

Since $100 = 64 + 32 + 4$, we see that the binary code for one hundred is **1100100**.

a) What is the binary code for two hundred?
b) What is the binary code for one thousand?

Musing 29.17 What is the largest number with a ten-digit binary code?

Musing 29.18 Here's a fun question.

As we noted, the codes from a $1 \leftarrow 2$ machine are called the *binary* codes with the prefix "bi" meaning two. Can you guess what each of these English words have to do with the number two?

bicycle **binoculars** **bisect** **biped**
bivalve (an oyster and a clam are examples of bivalves)

ACTIVITY

Musing 29.19

Imagine the numbers 1, 2, 4, 8, and 16 painted on your fingers as shown.

You make any number from 1 to 31 by raising some fingers and lowering others.

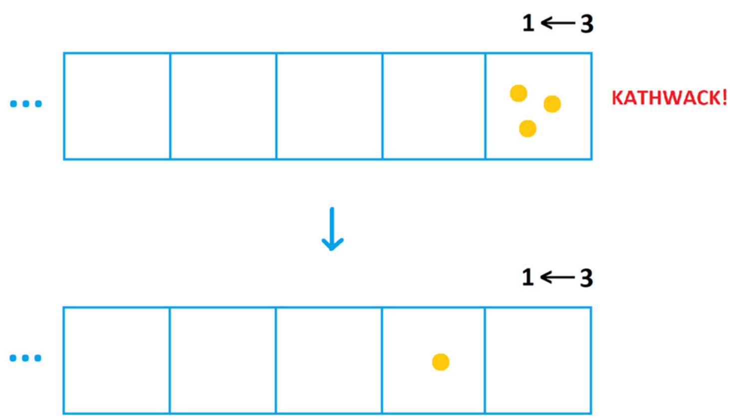
It's quite fun to run through the numbers 1, 2, 3, 4,5, ..., 30, 31 in turn on your fingers. Try it!

How high can you count in binary if you use two hands?

30. More Machines

Let me continue the story of the machines.

After playing with the $1 \leftarrow 2$ machine for a while, I suddenly had a flash of insight. Instead of playing with a $1 \leftarrow 2$ machine, I realized I could also play with a $1 \leftarrow 3$ machine. (Again, this is written and read backwards: a “three-one “machine.) Now, whenever there are three dots in a box, they explode away to be replaced with one dot, one box to the left.



Question: Work out the codes for the numbers *one*, *two*, *three*, *four*, *five*, and *six* in a $1 \leftarrow 3$ machine. Double-check that my list here is correct.

1:	**1**	**4:**	**11**
2:	**2**	**5:**	**12**
3:	**10**	**6:**	**20**

Practice 30.1 What’s the code for *thirteen* in a $1 \leftarrow 3$ machine?
(Perhaps try putting thirteen dots all at once into the rightmost box.)

Practice 30.2 Which number has1 $\leftarrow$ 3 machine code **222**?

And hours of fun are to be had playing with numbers in a $1 \leftarrow 3$ machine.

But then …
Another flash of insight!

Instead of playing with a 1 ← 3 machine, I could create a 1 ← 4 machine!

And then …
Another flash of insight!

Instead of playing with a 1 ← 4 machine, I could create a 1 ← 5 machine!

And then …
Another flash of insight!

Instead of playing with a 1 ← 5 machine, I could create a 1 ← 6 machine!

Practice 30.3: In a 1 ← 4 machine, how many dots in a box explode to make one dot, one place to the left?

Practice 30.4: What is the code for *ten* in a 1 ← 5 machine?

Practice 30.5: Multiple Choice
Would the code **30721** be stable in a 1 ← 6 machine?

a) No. In the group of seven dots in the middle, six of them would explode to make one dot, one place to the left.
b) Answer a) is correct.

And then …

I decided to go wild!

The 1 ← 10 Machine

Okay. Let's go all the way up to a 1 ← 10 machine. Crazy!

And let's put in 273 dots. Crazier!

What is the 1 ← 10 machine code for the number 273?

273:

I thought my way through this by asking a series of questions.

Are there any groups of ten that will explode? Certainly!

How many explosions will there be initially in that rightmost box? Twenty-seven.

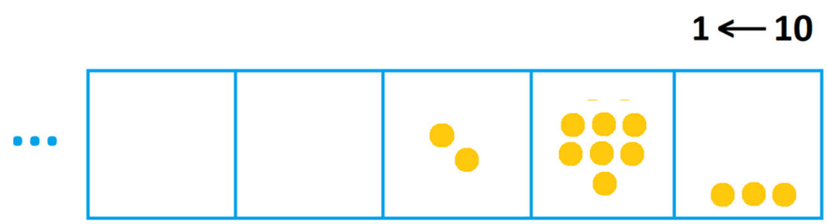
Will there be any dots left behind? Yes. Three.

Okay. So, there are twenty-seven explosions, each making one dot one place to the left, leaving three dots behind. (I hope it is okay that just write the number "27" rather than draw twenty-seven dots in a box.)

Next questions …

Will there be any more explosions? Yes. Two more.

Any dots left behind? Seven left behind.



273: 273

And look at what we have!

The 1 ← 10 machine code for two hundred seventy-three is … **273**.

Whoa!

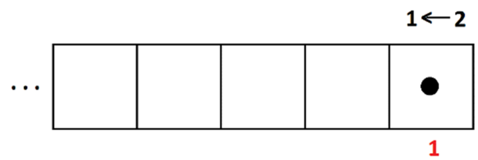
Something curious is going on!

Practice 30.6 Draw twenty-four dots in the rightmost box of a 1 ← 10 machine. Do all the explosions and see the code "24" appear for the number twenty-four.

Practice 30.7
a) Imagine three-hundred-eighty-two dots in the rightmost box of a 1 ← 10 machine. Can you think your way through all the explosions to see that code **382** will appear?
b) What do you think will be the 1 ← 10 machine code for the number one-thousand, eight-hundred, and forty-nine?

Explaining the Machines

We saw what was going on with the 1 ← 2 machines in the last section. This machine is set so that dots in the rightmost box are always worth one.



And as we keep adding dots to the machine, we follow the rule:

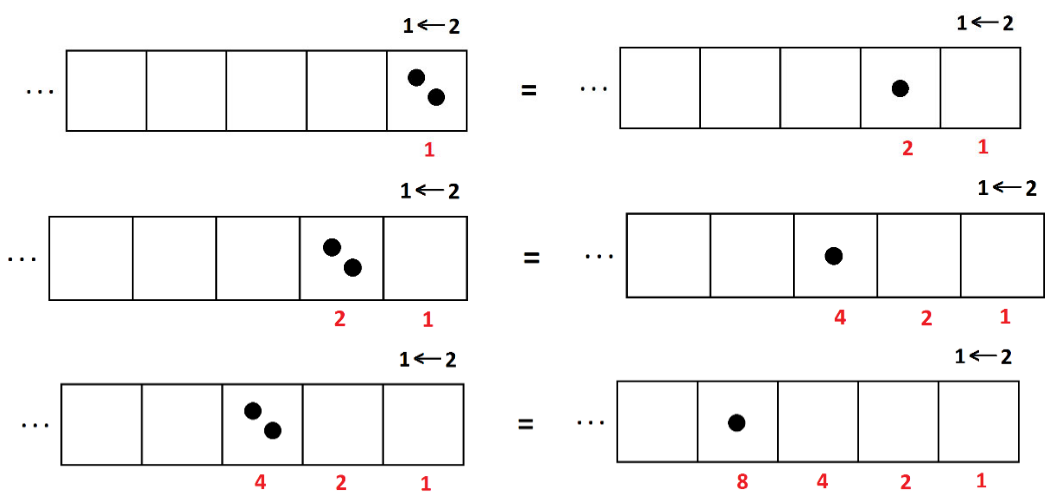
Whenever there are two dots in any one box, they "explode," that is, disappear, and are replaced by one dot, one place to their left.

This means, with an explosion, two dots in the rightmost box are equivalent to one dot in the next box to the left. And since each dot in the rightmost box is worth 1, each dot one place over must be worth two 1s, that is, 2.

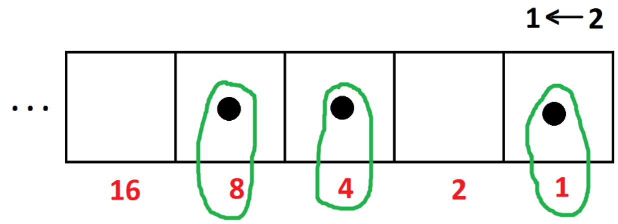
And two dots in this second box are equivalent to one dot, one place to the left. Such a dot must be worth two 2s, that is, worth 4.

And two 4s makes 8 for the value of a dot the next box over.

And two 8s make 16, and two 16s make 32, and two 32s make 64, and so on.

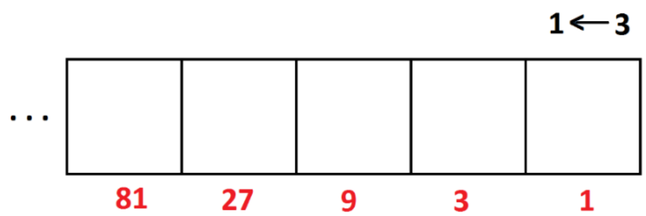


And recall, the 1 ← 2 machine code for thirteen is **1101** and we saw this is correct by looking at the values of the dots in this machine.



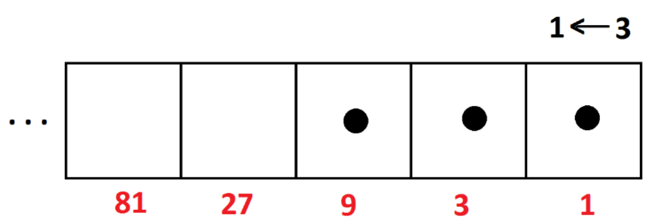
The same idea must be at play for the 1 ← 3 machine.

Here dots in the rightmost box again are each worth one, but now three dots in any one box are equivalent to one dot, one place to the left. We get the dot values in this machine by noting that three 1s is 3, and three 3s is 9, and three 9s is 27, and so on.



> **Practice 30.8** If one more box was added to the machine, what would be the value of a dot in that sixth box?

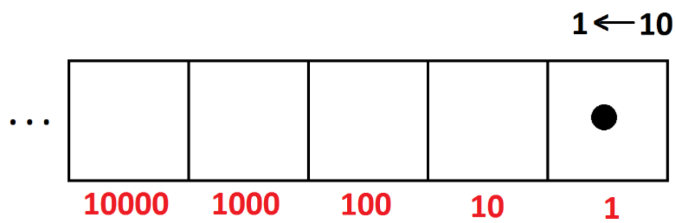
Earlier I asked for the 1 ← 3 machine code of number thirteen. It's **111**, and we see that this is correct because one 9 and one 3 and one 1 do indeed make thirteen.



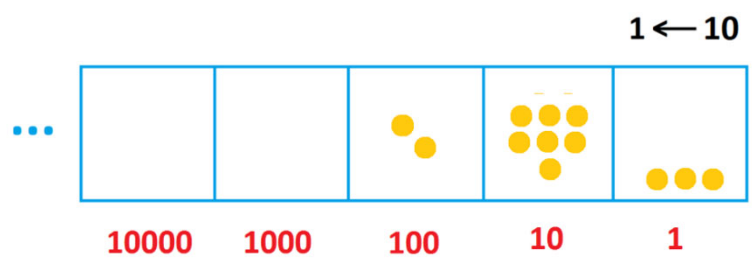
The 1 ← 3 machine codes for numbers are called **ternary** or **base three** representations of numbers. Only the three symbols **0**, **1**, and **2** are ever needed to represent numbers in this system.

> **Practice 30.9** What is the base-3 code for the number *fifteen*?

In a $1 \leftarrow 10$ machine, dots in the rightmost place are worth 1, and we have that ten ones make 10, ten tens make 100, ten one-hundreds make 1000, and so on. A $1 \leftarrow 10$ machine has 1, 10, 100, 1000, and so on, as dot values.



The code for the number 273 in a $1 \leftarrow 10$ machine is **273** and we see that this is absolutely correct because two 100s, seven 10s, and three 1s do make 273.



273: 273

In fact, we even speak the language of a $1 \leftarrow 10$ machine. In words, 273 is

273 = two hundred seventy three

We literally say, in English at least, two hundreds and seven tens (that "ty" is short for *ten*) and three.

We call the $1 \leftarrow 10$ codes for numbers the **base ten** or **decimal** representation of numbers.

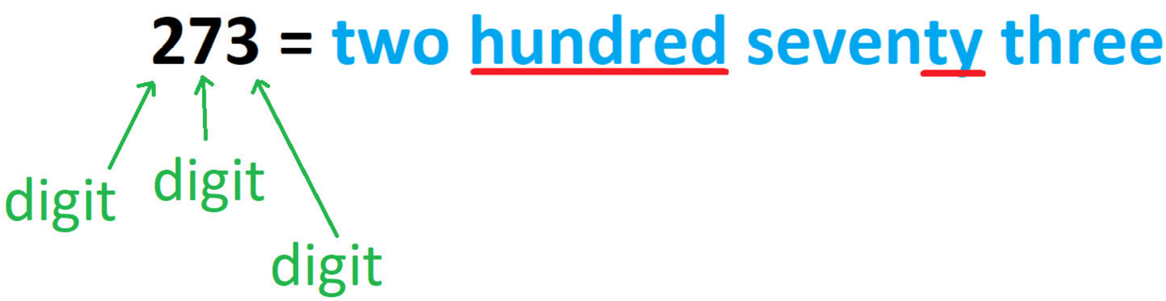
The prefix *dec*- means "ten" and we have, for example, that a *decagon* is a figure with ten sides and a *decade* is a period of ten years. Also, *December*, at one time, was the tenth month of the year. (Care to research the history of how we came up with twelve months for the year and why these months have the names they do?)

Practice 30.10 What's a *decapod*?

Number Bases in Society

We have discovered **number bases:** base two, base three, base ten, and so on.

And we have noticed that society has decided to speak the language of the base ten machine.



> **Question:** Why do you think we humans settled on 1 ← 10 machine to play with? Why do we like the number ten so much for matters of arithmetic and counting?

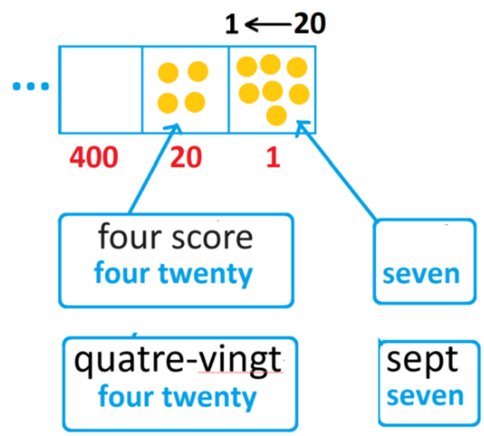
One answer could be because of our human anatomy: we are born with ten digits on our hands (thumbs and fingers—and toes too—are called *digits*) and we are thus naturally prone to think in terms of groups of ten. The connection is particularly strong when you notice we use the word **digit** for the individual symbols when we write in long numbers!

> **Practice 30.11:** Some cultures on this planet used base twenty. Why might they have chosen that number, do you think?

In fact, there are vestiges of base twenty thinking in the western European culture of today.

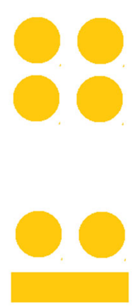
Abraham Lincoln's famous Gettysburg address begins: "Four score and seven years ago." The word *score* is an old word for "twenty" and so Lincoln was saying: "four-twenties and seven years ago." That's 87 years.

And, in French, the number 87 is said just this way too: *quatre-vingt-sept* translates, word for word, as "four twenties seven."



Ibo, a Nigerian language of the south-east region, says 87 as *ogu anon na asaa*, which, word-for-word, translates to "twenty into four and seven." It's the same idea of thinking of four groups of twenty and seven more.

The Maya of the Mayan Civilization of Mesoamerica used a base twenty system, but they wrote their numbers vertically. They used dots and bars in their number system. Can you see how the picture below is meant to be read as "four twenties and seven"?



Question: Do you know another language? How is 87 said in your language: in a base-ten way, like English, in a base-twenty way, like French, or some other way?

It seems natural for humans to develop base ten and base twenty number systems given our anatomy. But some cultures on Earth actually developed a base-12 number system instead!

This could be because there is also a very natural way to count to twelve on one hand. We have four long digits naturally broken into three segments each and a natural pointer to point at them—a thumb!

In some parts of the world, often in India and south-east Asia, it is still very common for people to count this way.

And there is still “twelveness” in our everyday life.

> How many items in a dozen? Answer: 12
> How many inches in a foot? Answer: 12
> How many hours are in a day—literally? Answer: 12!

The first clocks humans constructed were sundials, which, of course, work only during daylight hours. The ancient Egyptians divided the day into ten main hours and two extra twilight hours—early morning and early evening. With the invention of water clocks and mechanical clocks people could start measuring time during the night as well. Since the day was divided into 12 hours, they divided the night into 12 hours as well.

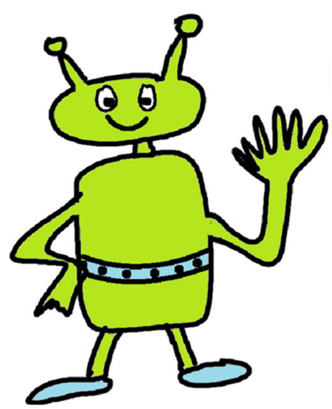
Question: The number twelve is very handy in matters of weights and measures. For instance, one might not want to purchase a full unit of some quantity, but perhaps only a half, or a third, or a quarter of that quantity. (These fractions are common in everyday operations.)

a) Eggs are sold in quantities of twelve. How many eggs is half a dozen of them? A third of a dozen? A quarter of a dozen?
b) If eggs were sold in quantities of ten (to follow base ten thinking), how many eggs is half of that quantity? A quarter of that quantity? A third of that quantity?

Question: Multiple Choice

I happen to know that Martians have six fingers on each of two hands. What base do you think they might use in their society?

A. If they focus on one hand, perhaps base six?
B. If they focus on two hands, perhaps base twelve?
C. If they focus on fingers and toes (assuming they have two feet and six toes on each foot), perhaps base twenty-four?
D. Everything stated in A, B, and C is reasonable.



The English Language is a Bit of Everything

Even though we write our numbers with ten digits—base ten—we have special words for the first twelve numbers. We apparently still think twelve-ness is important.

one
two
three
four
five
six
seven
eight
nine
ten
eleven
twelve
thirteen
fourteen
fifteen
sixteen
seventeen
eighteen
nineteen
twenty
twenty-one
twenty-two
....

base ten system

After twelve, we fall into a systematic naming pattern: thirteen, fourteen, fifteen, and so on, up to nineteen, using the special suffix "teen."

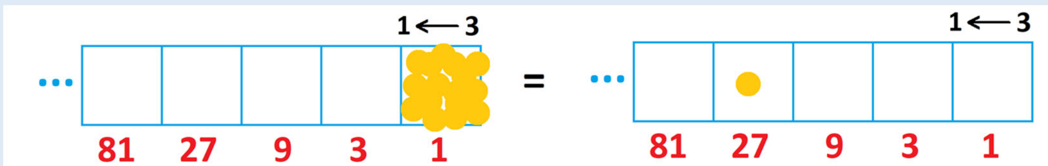
But then at twenty, we change to a different systematic pattern, which we then stay with—twenty-one, twenty-two, …, thirty-one, …, sixty-one, …, and so on. So, we think the first twenty numbers are somewhat special too, but that the numbers thereafter don't need special reference.

So, in speaking English, we follow a base-10 number system, but use special words for the first 12 numbers and a special pattern for numbers up to 20, and then change to a different pattern for saying numbers twenty and larger.

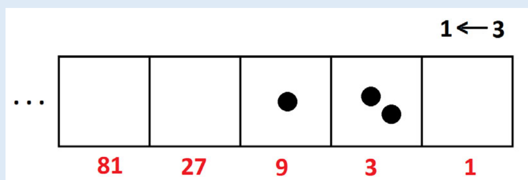
English is trying to do it all. That makes English hard and strange! (How does anyone learn English?)

MUSINGS

Musing 30.12 Place 27 dots into a 1 ← 3 all at once. Verify that after all the explosions one dot four boxes to the left remains. (This shows that a single dot in that box really is "worth" 27.)



Musing 30.13 The 1 ← 3 machine code for *fifteen* is **120**. What number has base-three code **1200**? What number has base-three code just **12**?



Musing 30.14 What is the code for the number 67,620,938 in a 1 ← 10 machine?

Musing 30.15 Write the codes for the number *thirty-one* in each of the following machines.

a) A 1 ← 10 machine.
b) A 1 ← 2 machine.
c) A 1 ← 3 machine.
d) A 1 ← 4 machine.
e) A 1 ← 5 machine.
f) A 1 ← 6 machine.
g) A 1 ← 31 machine.

Musing 30.16

In section 28 we presented a five-card mindreading number trick.
Would you like to consider another trick like that?

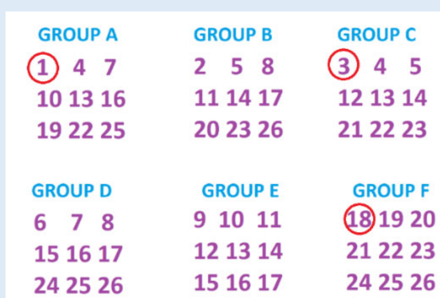
Present the following six groups of numbers to a friend and ask them to silently think of a number between 1 and 26.

GROUP A	GROUP B	GROUP C
1 4 7	2 5 8	3 4 5
10 13 16	11 14 17	12 13 14
19 22 25	20 23 26	21 22 23
GROUP D	**GROUP E**	**GROUP F**
6 7 8	9 10 11	18 19 20
15 16 17	12 13 14	21 22 23
24 25 26	15 16 17	24 25 26

As before, have the following conversation with your friend.

> "Shankar, tell me in which groups your secret number lies."
> "It's in groups A, C, and F."
> "Ahh. Your secret number is 22."

You might suspect that we are looking at the top left values in each group mentioned and adding them to find the secret number. And this is the case. Shankar's secret number 22 appears in the groups with corner numbers 1, 3, and 18 and indeed $18 + 3 + 1 = 22$.



As another example, 14 appears in groups B, C, and E with respective top left numbers 9, 3, and 2, and $9 + 3 + 2 = 14$.

The number 5 appears in groups B and C with top left numbers 3 and 2, respectively, and $3 + 2 = 5$.

The top left numbers of the cards are 1 and 2 (and 2 is two copies of 1), 3 and 6 (and 6 is two copies of 3), 9 and 18 (and 18 is two copies of 9).

What *Exploding Dots* machine might be at play here, do you think?

Write the codes for the numbers 1 through 26 in the machine you have in mind.
Can you now explain why each number appears on the card that it does and why the trick works?

31. English is Weird

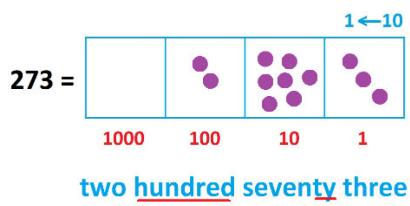
We noted in the last section that English is quirky. But it is actually down and outright weird!

For starters …

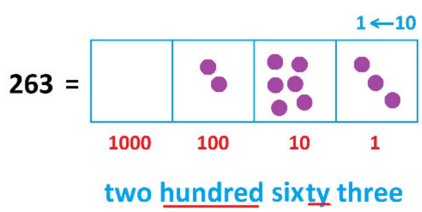
> **Question:** Have you ever noticed that the spelling of "weird" is weird?
> What happened to "i before e, except after c"?

The strangeness of English applies to how we write and say numbers too.

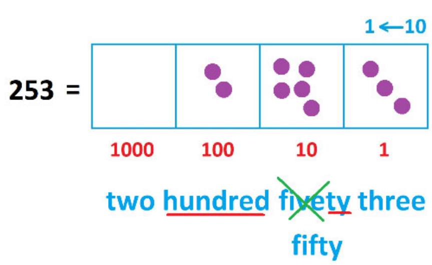
Here's the number 273 in a $1 \leftarrow 10$ machine. It is literally two hundreds, seven tens (*ty* is short for "ten" in English), and three. Nothing too strange there. (Well, "ty" is a bit strange.)



And here is 263. Nothing strange here either.

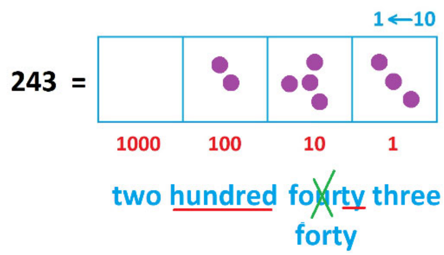


But listen to the number 253. We should say "two hundred five-ty three" but we don't. English has us say "fifty" instead of five-ty.

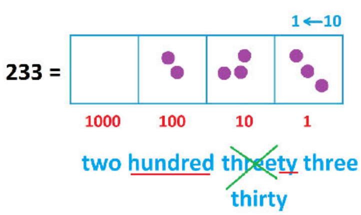


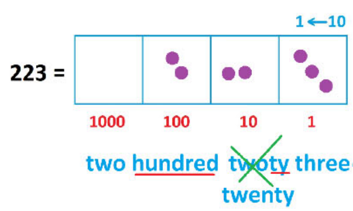
Now saying 243 out loud sounds right, but when writing it out we should write "four-ty," but English insists we write "forty" instead.

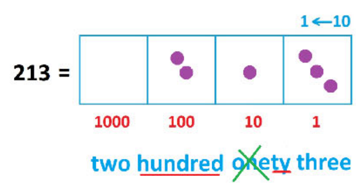
Weird!



And there are 233 and 223 and 213, each particularly weird.

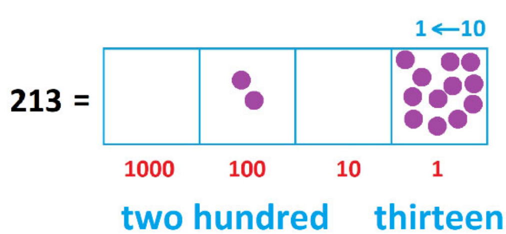




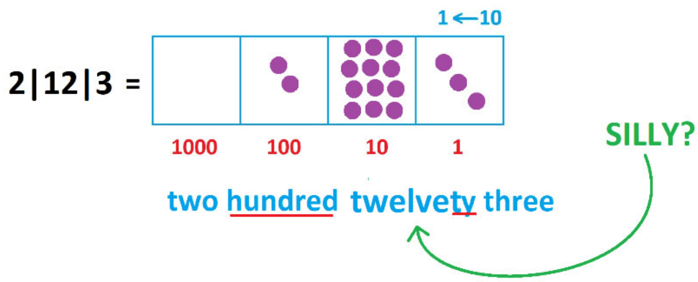


Question: Something additionally strange happens with the number 213. What do we say instead of "two hundred onety three"? Does what we say make sense?

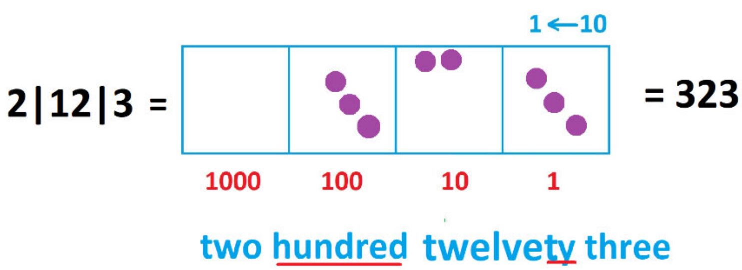
Question: When Katya thought about this question, she said that "two hundred and thirteen" is really this picture: two hundreds and an extra thirteen dots all in the ones place! What do you think? Are we allowed to have 13 dots in a single box?



Most people in society think it is silly to have a large number of dots in any single box. You would never say "two hundred and twelvety three," for example.



But, of course, we know ten dots in any box explode away to produce one dot, one place to their left. So, this number is really 323 in disguise.



Question 31.1: Draw a dots-and-boxes picture of "two-hundred eleventy three." What number is this?

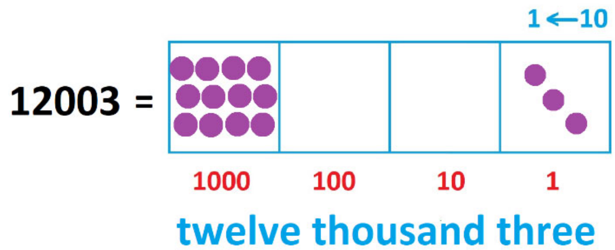
Question 31.2: Over a thousand years ago people spoke a version of English we today call "Old English." It included words equivalent to twelvety and eleventy.

What number is twelvety? What number is eleventy?
(Would saying "thirteen-ty" be just too silly?)

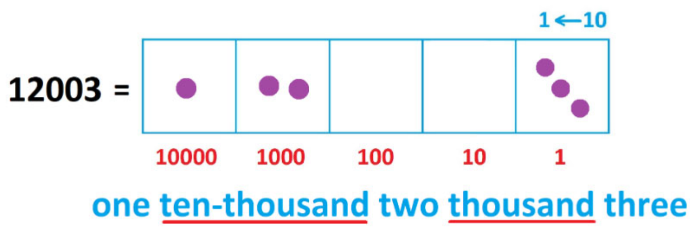
But society is inconsistent.

Question: How do you say the number 12,003? Draw a dots and boxes picture of the number.

The number 12,003 is pronounced "twelve thousand three" as though we have twelve thousands and three ones. Here we **do** allow more than nine dots in a box, it seems.



But we don't write what we say. Here's a picture of what we write:



If English and society were consistent, we would say "one ten-thousand two thousand three." But we don't!

What we're learning here is that society and the English language has all sorts of strange demands on how we write and say numbers. The math is always solid and clear. It is just society that makes different, and sometimes strange, choices about speaking the math.

So, if society has permission to be strange, I say … let's go for it! Let's be a little strange too and use the strangeness to our advantage for doing math.

Let's go through the arithmetic we've learned in school and see how ignoring society actually makes the mathematics so much simpler! We can always fix up what we do to make society happy at any time.

This is going to be fun!

Question: How is the number 12,003 pronounced in other languages?

MUSINGS

Musing 31.3 Here are four boxes of the $1 \leftarrow 10$ machine. Let's put twelve dots in each box. (We have 12 thousands, 12 hundreds, 12 tens, and 12 ones.)

12	12	12	12

a) How might you say this number out loud?
b) Did you notice as you said this out loud that saying "twelve thousand" sounds acceptable to our ear? And saying "twelve hundred" is okay too? Saying "twelvety" is odd, but saying "and twelve" sounds okay?

   That is, did you notice that 75% of what you said in part a) sounds acceptable to our ear?

c) What number is 12|12|12|12 really?

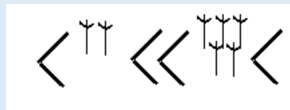
d) How do you say your answer to part c) out loud? Do you start with "One ten-thousand, three thousand, ..."?

Musing 31.4 What number is "eleven-thousand eleven-hundred eleventy eleven"?

Musing 31.5 The Babylonians of ca. 1700 B.C.E. were adept astronomers and based their representations of numbers on multiples of 60. (Is this because 60 is a number that unites base 10, base 12, and base 20 thinking? Is it because the number of days in a year is approximately six 60s? Mathematics historians are not sure.)

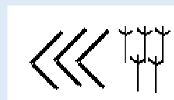
They used the symbols  and , respectively, for one and ten, and imagined they had numbers in boxes like a $1 \leftarrow 60$ machine. The values of the boxes were 1, 60, $60 \times 60 = 3600$, $60 \times 360 = 21600$m and so on.. For example,

represented $23 \times 60 + 11 = 1391$, and



represented $12 \times 360 + 25 \times 60 + 10 = 44710.$

a) Translate each of the following numbers into our base ten system.



b) Write the numbers 323 and 2016 using the Babylonian system.

c) There were problems with the Babylonian's method. Try to write the numbers 61 and 1200 and 3600 in their system. What do you notice?

Remnants of the Babylonian system still exist today. We measure time in units of 60 (sixty seconds in a minute, sixty minutes in an hour) and divide circles into 360 degrees (°). Furthermore, each degree is divided into sixty minutes (') and each minute into sixty seconds (").

EXTRA
As we saw in this question, the Babylonians had a system for representing numbers that used only two symbols and relied on the placement of symbols to represent counts of units, counts of 60, counts of 360, and so on. They developed a **place value numeral system**.

But their lack of a symbol for zero to indicate a lack of any one of these particular quantities causes confusion.

Our current base ten place-value system does use zero to indicate a lack of ones, or tens, or hundreds, and so on, in writing numbers. This system was invented and used by Indian scholars around 100 C.E. and was widely adopted throughout the middle east by the year 900 CE. Some 300 years later it was noticed and used by scholars in Europe.

d) Chinese scholars of ancient times also used a place-value numeral system based on the number ten. Did they too make use of a symbol for zero within their place-value system?

32. Addition

Let’s now focus on the base-ten place-value system, the one our society today likes best. That is, let’s look at doing arithmetic numbers written in $1 \leftarrow 10$ machine codes.

And let’s have some quirky fun with it all!

Here’s an addition problem:

Compute $251 + 124$.

Such a problem is usually set up this way.

$$\begin{array}{r} 251 \\ +\,124 \\ \hline \end{array}$$

This addition problem is easy to compute: $2 + 1$ is 3, and $5 + 2$ is 7, and $1 + 4$ is 5.
The answer 375 appears.

$$\begin{array}{r} 251 \\ +\,124 \\ \hline 375 \end{array}$$

But did you notice something curious just then?

I worked from left to right—just as I was taught to read. But this is the opposite to what most people are taught to do in a math class: always work from right to left.

Question: Does it matter? Compute the problem from right to left instead. Do you get the same answer 375?

Why are we taught to work right to left in mathematics classes?

Perhaps the issue at hand is hidden because the problem we just did is "too nice." We should do a more awkward addition problem, one like $358 + 287$.

$$\begin{array}{r} 358 \\ +\,287 \\ \hline \end{array}$$

Okay. Let's do it!
But I'll be naughty and go left to right again.

$3 + 2$ is 5 and $5 + 8$ is 13 and $8 + 7$ is 15

The answer five-hundred thirteen-ty fifteen appears. (Remember, "ty" is short for *ten*.)

$$\begin{array}{r} 358 \\ +\,287 \\ \hline 5\,|\,13\,|\,15 \end{array}$$

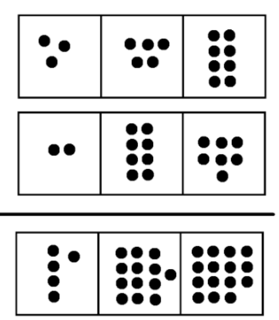
And this answer is absolutely, mathematically correct! You can see it is correct in a $1 \leftarrow 10$ machine.

Draw pictures of 358 and 287.

358

+ 287

=



5 | 13 | 15

Adding together 3 hundreds and 2 hundreds really does give 5 hundreds.
Adding together 5 tens and 8 tens really does give 13 tens.
Adding together 8 ones and 7 ones really does give 15 ones.

“Five-hundred thirteen-ty fifteen” is absolutely correct as an answer – and I even said it correctly. We really do have 5 hundreds, 13 tens, and 15 ones. There is nothing mathematically wrong with this answer. It just sounds weird.

But, as we realized in the last section, English is inconsistent about what it thinks is weird and what it thinks is not. I say, let’s use weirdness to our mathematical advantage and be weird when it makes matters natural and easy.

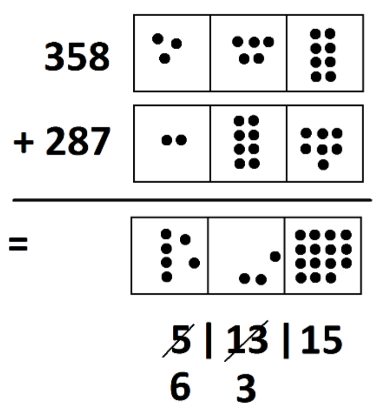
> Can we fix up this strange-sounding answer for society’s sake—not mathematics’ sake, but for society’s sake?

The answer is yes! We can do some explosions. (This is a $1 \leftarrow 10$ machine, after all.)

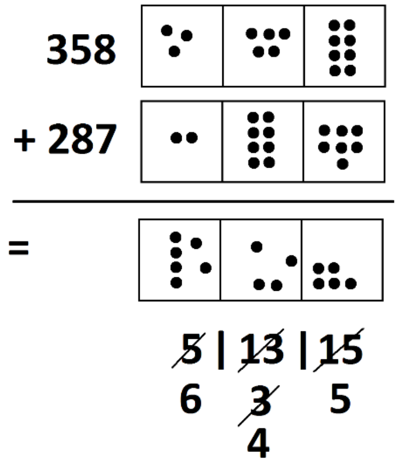
Which do you want to explode first: the 13 or the 15?

It doesn’t matter! Let’s explode from the 13.

Ten dots in the middle box explode to be replaced by one dot, one place to the left.



The answer “six hundred three-ty fifteen” now appears. This is still a lovely, mathematically correct answer. But society at large might not agree. Let’s do another explosion: ten dots in the rightmost box.



Now we see the answer "six hundred four-ty five," which is one that society understands. (Although, in English, "four-ty" is usually spelled *forty*.)

Practice 32.1 Write down the answers to the following addition problems working left to right and not worrying about what society thinks! Then, do some explosions to translate each answer into something society understands. You can check your final answers if you like by doing the calculations the traditional way or by using a calculator.

$$\begin{array}{r} 148 \\ +\ 323 \\ \hline = \end{array} \qquad \begin{array}{r} 567 \\ +\ 271 \\ \hline = \end{array} \qquad \begin{array}{r} 377 \\ +\ 188 \\ \hline = \end{array} \qquad \begin{array}{r} 582 \\ +\ 714 \\ \hline = \end{array}$$

$$\begin{array}{r} 310462872 \\ +\ 389107123 \\ \hline = \end{array} \qquad \begin{array}{r} 87263716381 \\ +\ 18778274824 \\ \hline = \end{array}$$

The Traditional Algorithm

How does this *Exploding Dots* approach to addition compare to the standard algorithm most people know?

Let's go back to the example $358 + 287$. Most people are surprised (maybe even perturbed) by the straightforward left-to-right answer $5 \mid 13 \mid 15$.

$$\begin{array}{r} 358 \\ +\,287 \\ \hline 5 \mid 13 \mid 15 \end{array}$$

This is because the traditional algorithm has us work from right to left, looking at $8 + 7$ first.
But, in the algorithm we don't write down the answer 15. Instead, we explode ten dots right away and write on paper a 5 in the answer line together with a small 1 tacked on to the middle column. People call this **carrying the one** and it – correctly – corresponds to adding an extra dot in the tens position.

$$\begin{array}{r} {}^{1} \\ 358 \\ +\,287 \\ \hline 5 \end{array}$$

Now we attend to the middle boxes. Adding gives 14 dots in the tens box ($5 + 8$ gives thirteen dots, plus the extra dot from the previous explosion).

And we perform another explosion.

$$\begin{array}{r} {}^{1}\,{}^{1} \\ 358 \\ +\,287 \\ \hline 45 \end{array}$$

On paper, one writes “4” in the tens position of the answer line, with another little “1” placed in the next column over. This matches the idea of the dots-and-boxes picture precisely.
And now we finish the problem by adding the dots in the hundreds position.

$$\begin{array}{r} {}^{1}\,{}^{1} \\ 358 \\ +\,287 \\ \hline 645 \end{array}$$

The traditional algorithm works right to left and does explosions (**carries**) as one goes along. On paper, it is swift and compact, and this might be why it has been the favored way of doing long addition for centuries.

The *Exploding Dots* approach works left to right, just as we are taught to read in English, and leaves all the explosions to the end. It is easy to understand and kind of fun.

Both approaches, of course, are good and correct. It is just a matter of taste and personal style which one you choose to do. (And feel free to come up with your own new, and correct, approach too!)

MUSINGS

Musing 32.2 When doing a long addition problem the traditional way, we work from right to left and conduct explosions as we go along. Most people use the word "carrying" instead of "exploding." For example, the problem shown has us "carry a 1" two separate times.

$$\begin{array}{r} {}^{1}\,\,{}^{1}\,\, \\ 4\;8\;7 \\ +\;1\;2\;4 \\ \hline =\;6\;1\;1 \end{array}$$

a) When adding two numbers the traditional way, why won't we ever carry a 2?

b) Give an example of adding a three-digit number and a two-digit number that results in carrying a 1 exactly one time.

c) Give an example of adding a three-digit number and a two-digit number that results in carrying a 1 exactly two times.

d) Could there be an example of adding a three-digit number and a two-digit number that results in carrying a 1 three times?

Musing 32.3 Multiple Choice
Which of the following are mathematically correct answers to the following addition problem?

$$\begin{array}{r} 5\;3\;9 \\ +\;2\;8\;9 \\ \hline = \end{array}$$

a) 7|11|18
b) 8|1|18
c) 8|2|8
d) These are all the same number, really, and all are mathematically correct. It's just that society doesn't like options a) and b).

33. Multiplication

Without regard to what society thinks what would be a good—and correct—three-second answer to this multiplication problem?

2784 x 3

Can you see that 6 | 21 | 24 | 12 is the natural answer to this? After all, if we have 2 thousands and 7 hundreds and 8 tens and 4 ones and we triple everything, we'd have 6 thousands and 21 hundreds and 24 tens and 12 ones as a result. Easy! The answer is "six thousand, twenty-one hundred, twenty-four-ty, twelve."

2	7	8	4	x 3 =	6	21	24	12

Now, how can we fix up this answer for society? With some explosions of course!

Let's do two explosions from the 24 first, say. (It doesn't matter which explosions we do when). It gives

6 | 23 | 4 | 12

Now maybe explode from the 12.

6 | 23 | 5 | 2

Now from the 23.

8 | 3 | 5 | 2

We have that $2784 \times 3 = 8352$.

Practice 33.1 Compute each of the following in the same manner.

a) 2784×2
b) 2784×4
c) 2784×5
d) 2784×10

"To Multiply by Ten, Add a Zero." Huh?

Here's a true story this time.

When I was in school, I was told a rule for multiplying by ten: *just add a zero.*

This rule made no sense to me as stated. To compute 213×10, for instance, you don't add zero.

$$\begin{array}{rr} & 2\,1\,3 \\ + & 0 \\ \hline = & \end{array} \qquad \text{HUH?}$$

Of course, I realized that people meant, "tack a zero to the end of the number."

$$213 \times 10 = 2130$$

Why does multiplying a number by ten seem to have the effect of appending a zero to the digits of the number? *Exploding Dots* thinking explains why.

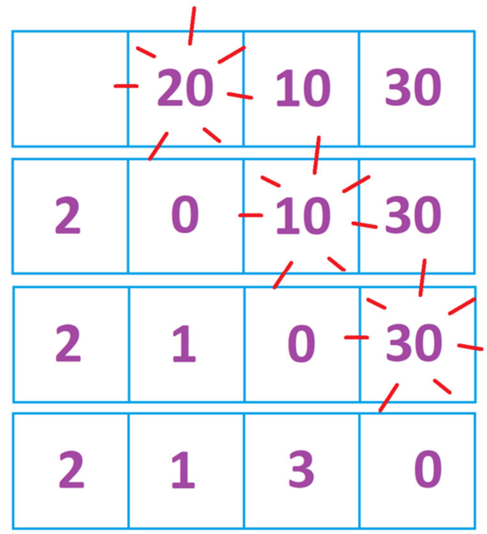
Here's the number 213 in a $1 \leftarrow 10$machine.

	2	1	3

And here is 213×10.

	20	10	30

Now let's perform the explosions, one at a time. We'll need the extra box to the left.



We have that 2 groups of ten explode in the hundreds place to give 2 dots one place to the left, and 1 group of ten explodes in the tens place to give 1 dot one place to the left, and 3 groups of ten explode in the ones place to give 3 dots one place to the left. The net effect of what we see is all digits of the original number shifting one place to the left to *reveal* zero dots in the ones place.

Indeed, the end result looks like we just tacked on a zero to the right end of 213. But really it was a whole lot of explosions that pushed each digit one place to the left to leave an empty box at the very end.

Question: How would you explain to a young student why 2222×10 is 22220?

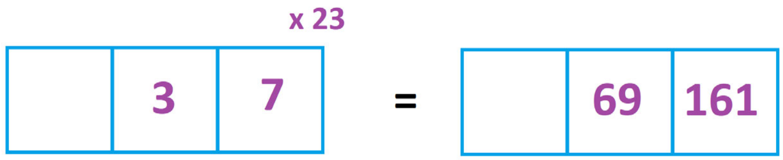
Practice 33.2
a) What must be the answer to 476×10?
b) What must be the answer to 476×100? Why?

Practice 33.3
a) What number was multiplied by 10 to give the answer 9190?
b) What must be the answer to $55740 \div 10$?
c) What must be the answer to $3310000 \div 100$? Why?

Long Multiplication

People wonder if there is a way to conduct long multiplication with dots and boxes. For example, can we compute 37×23 with *Exploding Dots*?

There is, but it is not very nice. This particular example requires you to know your multiples of 23.



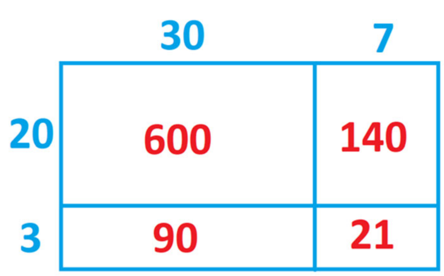
Question: After explosions, what number is 69|161? Do you see it is 851?

Question 33.4 Fill in the question marks.

$37 \times 2 = 6 \,|\, 14 = 74$ $\quad$ $37 \times 11 = 33 \,|\, 77 = ?$ $\quad$ $375 \times 11 = 33 \,|\, ? \,|\, ? = ?$

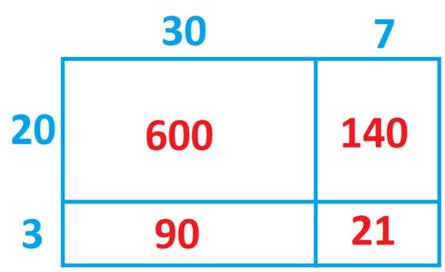
The most natural way to conduct long multiplication, as we saw in Chapter 1, is to use the area model.

Here's a picture of 37×23.



We get $37 \times 23 = 600 + 140 + 90 + 21 = 851$.

Let's be very clear about what is going on in this picture of 37×23.



We found a piece of the rectangle of area 600.

This came from computing 30×20. This is essentially the same as computing $3 \times 2 = 6$, but our answer is in the hundreds because we are really multiplying $30 = 3 \times 10$ and $20 = 2 \times 10$ and are seeing 3×2 multiplied by ten twice.

$$30 \times 20 = \mathbf{3} \times \mathbf{2} \times 10 \times 10 = 6 \times 10 \times 10$$

We found a piece of the rectangle of area 140.

This came from computing 7×20. This is essentially the same as computing $7 \times 2 = 14$, but our answer is in the tens because we are really multiplying 7 and $20 = 2 \times 10$ and are seeing 7×2 multiplied by ten once.

$$7 \times 20 = \mathbf{7} \times \mathbf{2} \times 10 = 14 \times 10$$

We found a piece of the rectangle of area 90.

This came from computing 30×3. This is essentially the same as computing $3 \times 3 = 9$, but our answer is in the tens because we are really multiplying $30 = 3 \times 10$, and 3. We see 3×3multiplied by ten once.

$$30 \times 3 = \mathbf{3} \times \mathbf{3} \times 10 = 9 \times 10$$

And we found a piece of area 21, which comes from computing 7×3.

$$\mathbf{7} \times \mathbf{3} = 21$$

In short, we're just multiplying the digits of the two numbers in turn and are keeping track of whether the answers—6, 14, 9, 21—are in the hundreds, tens, or units.

This is what the school long-multiplication algorithm does—without the visuals of the area model to show what is happening. (Section 12.)

But look at this. What do you think of this area model/dots-and-boxes/long-multiplication mash up?

		3	7
x		2	3
		9	21
+	6	14	
=	6	23	21

We see the numbers 6, 14, 9, 21 in the right spots for being hundreds, tens, or units.

We can just add up what we see and get the answer 6 | 23 | 21.

Some explosions show that this is 851 in disguise.

Whoa!

Question: There's a lot to "see" here. Is this making sense at all?

Practice 33.5 My calculator says that 3125×832 is 2600000. (That's the sensible way to do long multiplication in the 21st century!)

Can you get that answer the area/dots-and-boxes/long-multiplication mash-up way?

$$\begin{array}{r} 3\ 1\ 2\ 5 \\ \times\quad 8\ 3\ 2 \\ \hline = \qquad\qquad \end{array}$$

Just so you can compare efforts, this is what I wrote when I tried this question.

			3	1	2	5
		x		8	3	2
=			6	2	4	10
		9	3	6	15	0
	24	8	16	40	0	0
	24	17	25	48	19	10
=	2 | 6	0	0	0	0	0

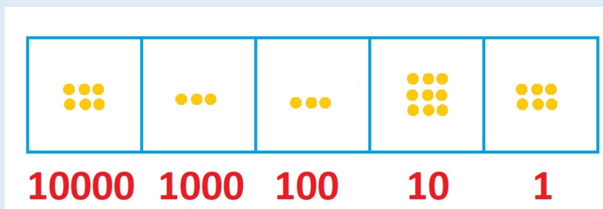
MUSINGS

Musing 33.6 Compute each of the following.

a) 26417×4
b) 26417×5
c) 26417×9.
d) 26417×10

Musing 33.7 Compute 236×34 first with a calculator (as any sensible human would do), and then with the hybrid area/dots-and-boxes approach.

Musing 33.8 Which number was multiplied by 3 to give this picture?



Musing 33.9

a) Evgenya thought of 37×11 as $37 \times (10 + 1)$, which is $37 \times 10 + 37 \times 1$, to get the value $370 + 37 = 407$.

   How might she compute 125×11?

b) What's 313131×11? What's $3131313131313131 \times 11$?

c) What's $11 \times 11$and $11 \times 11 \times 11$ and $11 \times 11 \times 11 \times 11$?

Musing 33.10 Here's a hack for multiplying a two-digit number by 11.

To compute 14×11*, say, split apart the digits* 1 *and* 4 *and write their sum* $1 + 4 = 5$ *between them.*

$$14 \times 11 = 154$$

To compute 26×11*, say, split apart the digits* 2 *and* 6 *and write their sum* $2 + 6 = 8$ *between them.*

$$26 \times 11 = 286$$

Some more examples: $54 \times 11 = 594$, $80 \times 11 = 880$, *and* $08 \times 11 = 088$. (Ooh!)

One might have to explode some dots doing this trick.

$$67 \times 11 = 6|13|7 = 737$$

$$99 \times 11 = 9|18|9 = 1089$$

a) Can you explain why this hack works?

b) What's $693 \div 11$?

c) What's $924 \div 11$?

34. Subtraction

Recall from Chapter 3 that I do not believe that subtraction exists.

> Subtraction is the addition of the opposite.

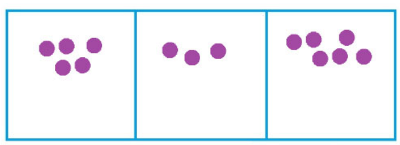
Let's play with some multidigit subtraction. Consider this subtraction problem $536 - 123$.

I think of this as being asked to compute "536 plus the opposite of 123," but schoolbooks don't and will present the problem this way and expect students to fill in the bottom line.

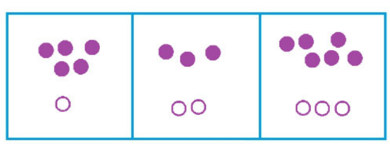
$$\begin{array}{r} 536 \\ -\ 123 \\ \hline = \end{array}$$

But let's think about this the *Exploding Dots* way.

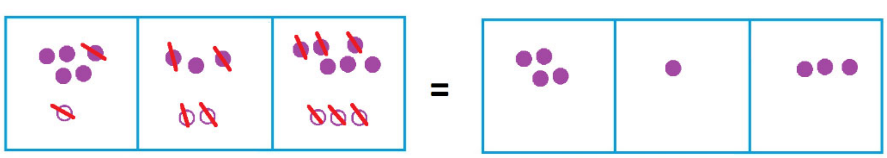
The first number, 536, looks like this in a $1 \leftarrow 10$ machine: five dots, three dots, six dots.



We must add to this the opposite of 123. That is, we're adding one anti-hundred, two anti-tens, and three anti-ones.



And now there are a lot of annihilations: POOF! and POOF POOF! and POOF POOF POOF!



The answer 413 appears.

And notice, we get this answer as though we just work left to right and saying

5 take away 1 is 4,

3 take away 2 is 1,

and

6 take away 3 is 3.

$$\begin{array}{r} 536 \\ -\ 123 \\ \hline =\ 413 \end{array}$$

Yes. Left to right again!

All right. That example was too nice. How about 512 – 347?

$$\begin{array}{r} 512 \\ -\ 347 \\ \hline = \quad\quad \end{array}$$

Going from left to right, we have

5 take away 3 is 2,

1 take away 4 is -3,

and

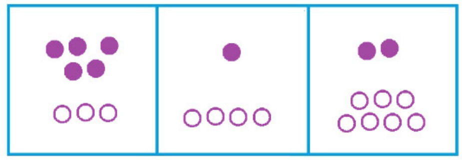
2 take away 7 is -5.

$$\begin{array}{rrrr} & 5 & 1 & 2 \\ - & 3 & 4 & 7 \\ \hline = & 2 & |-3 & |-5 \end{array}$$

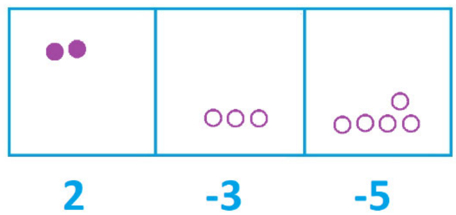
The answer is two-hundred negative-three-ty negative-five.

This answer is absolutely, mathematically correct!

Here's five hundreds, one ten and two ones together with three anti-hundreds, four anti-tens, and seven anti-ones.



And after lots of annihilations we are left with two actual hundreds, three anti-tens, and five anti-ones.



The answer really is "two-hundred negative-three-ty negative-five"!

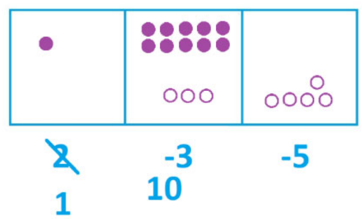
Question: But, of course, giving this answer to the subtraction problem will seem mighty weird to society. Can we fix up this mathematically correct answer to one acceptable by society?

Think about this for a moment. What could we possibly do to fix this answer?

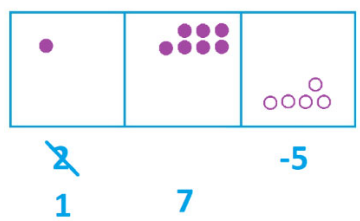
After a moment it might occur to you to unexplode dots. Any dot in a box to the left must have come from ten dots in the box just to its right, so we can just unexplode it to make ten dots.

Important Question: What sound effect should we make for unexploding?

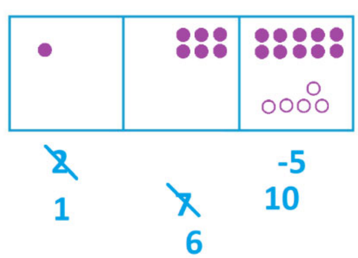
Okay. Let's unexplode one of the two dots we have in the leftmost box. Doing so gives this picture.



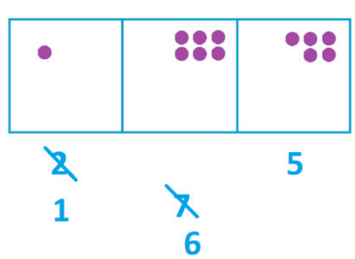
After annihilations, we see we now have the answer one-hundred seventy negative-five. Beautiful!



Let's unexplode again.



And with some more annihilations we see an answer society can understand: one-hundred sixty-five.



Practice 34.1 Show that the number $4 \mid -2 \mid 1 \mid -7$, "four-thousand, negative-two-hundred, onety, negative seven" is really the number 3803.

Practice 34.2 When Raj saw

$$\begin{array}{r} 5\ \ 1\ \ 2 \\ -\ 3\ \ 4\ \ 7 \\ \hline = 2 \mid -3 \mid -5 \end{array}$$

he wrote on his paper the following lines.

$$\begin{array}{r} 200 \\ -30 \\ -5 \end{array}$$

He then said that the answer has to be 165.

a) Can you explain what he is seeing and thinking?

b) What would Raj likely write on the page for $7109 - 3384$?

The Traditional Algorithm

How does the dots-and-boxes approach to long subtraction compare to the standard algorithm we were taught in school?

Consider again $512 - 347$.

$$\begin{array}{rccc} & 5 & 1 & 2 \\ - & 3 & 4 & 7 \\ \hline = & & & \end{array}$$

The standard algorithm has you start at the rightmost column to first consider *2 take away 7*.
This is deemed "not possible."

So, what do you do? You "borrow" a one from the left.

That is, you take a dot from the tens column and unexplode it to make ten ones. In this case, that leaves zero dots in the tens column.

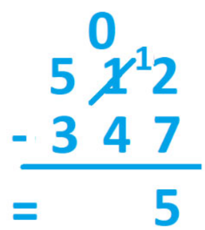
Now, we should write ten ones to go with the two in the ones column like this, but we don't.

$$\begin{array}{rccc} & & 0 & {\scriptstyle 10} \\ & 5 & \not{1} & 2 \\ - & 3 & 4 & 7 \\ \hline = & & & \end{array}$$

We do something a bit sneaky and write "12" rather than $10 + 2$, by putting a little 1 next to the digit 2 to make it look like twelve.

$$\begin{array}{rccc} & & 0 & \\ & 5 & \not{1} & {}^{1}2 \\ - & 3 & 4 & 7 \\ \hline = & & & \end{array}$$

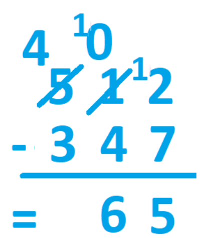
Then we say, “twelve take seven is five,” and write that answer.

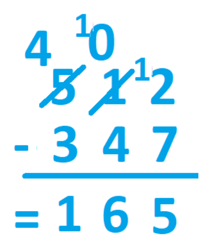


The rightmost column is complete. Shift now to the middle column.

We see “zero take away four,” which can’t be done. So, perform another unexplosion, that is, another “borrow,” to see $10 - 4$ in that column. We write the answer 6.

We then move to the last remaining column where we have $4-3$, which is 1.





This is complicated and looks mysterious at face value.

But if you draw a picture of dots and antidots side-by-side with the steps of the standard algorithm, you can see what is going on.

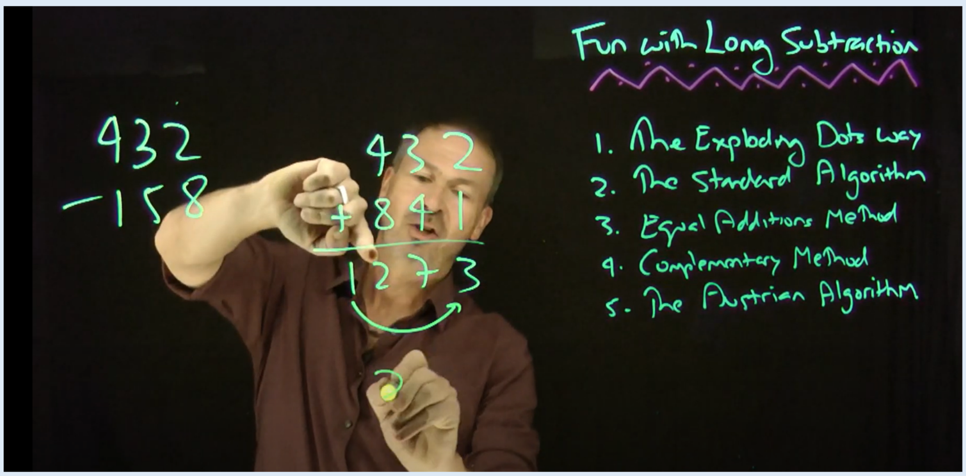
Of course, all correct approaches to mathematics are correct. It is just a matter of style as to which approach you like best for long subtraction. The traditional algorithm has you work from right to left and do all the unexplosions as you go along. The dots-and-boxes approach has you “just do it!” and conduct all the unexplosions at the end.

Both methods are fine and correct.

ACTIVITY

OTHER METHODS OF LONG SUBTRACTION

We might think our traditional algorithm for multi-digit subtraction is “standard,” but at other times and in other cultures alternative approaches were considered standard. Can you make sense of the “Equal Additions Method,” the “Complementary Method,” and the “Austrian Method” I describe in this video? Perhaps pause the video before I go on to give the explanations away.



https://youtu.be/KJb5nfGvXp0

MUSINGS

Musing 34.3 Compute each of the following arithmetic problems three ways:

i) Just use a calculator and see what the answer is. (This is the smart approach.)
ii) Use a method you were taught in school for conducting long subtraction.
iii) Try the *Exploding Dots* way, working from left to right and fixing up the answer for society's liking afterwards.

You should get the same answer via each of these approaches.

$$\begin{array}{r} 6328 \\ -\ 4469 \\ \hline = \end{array} \qquad\qquad \begin{array}{r} 78390231 \\ -\ 32495846 \\ \hline = \end{array}$$

Thinking question along the way: As you fix up your answers for society, does it seem easier to unexplode from left to right, or from right to left?

Additional question: Do you think you could become just as speedy the dots-and-boxes way as you currently are with the traditional approach? (Not that one should become speedy at this. We have calculators, after all, if our goal is to just get answers.)

Musing 34.4 Create a subtraction problem whose answer is $1 \mid -2 \mid -3 \mid -4$. (What number is this answer?)

Musing 34.5 How might you handle this subtraction problem? How would you interpret its answer?

$$\begin{array}{r} 148 \\ -\ 677 \\ \hline = \end{array}$$

Musing 34.6 Something Mind-Bendy

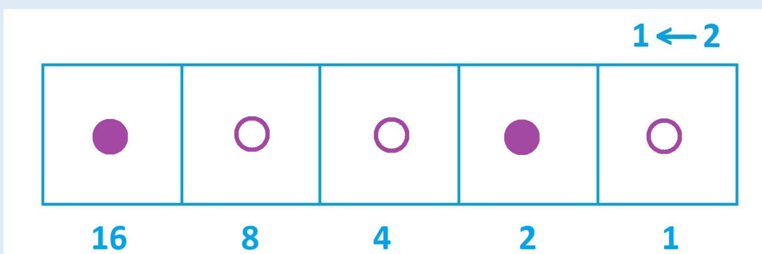
The number *thirteen* in base two is 1101. The number *six* is 110. Compute this long subtraction problem in base two. Do you get an answer that is the base two code for *seven*?
(After all, $11 - 6 = 7$ in ordinary arithmetic.)

$$\begin{array}{r} 1\,1\,0\,1 \\ -\quad 1\,1\,0 \\ \hline = \qquad\quad \end{array}$$

Musing 34. 7 WILD EXPLORATION

Here's a puzzle to explore if you like.

Let's go back to a $1 \leftarrow 2$ machine and just work with its five rightmost boxes.



And let's fill each box with either one dot or one antidot.

The picture shows the number $16 + (-8) + (-4) + 2 + (-1)$, which is 5.

a) Show the that the number 17 can be represented in the same way, with each box containing either one dot or one antidot (and with no box empty).

b) Show that the number -17 can also be represented this way.

c) Find all the numbers that can be formed by placing single dots and antidots in the five rightmost boxes of this machine. How do you know your list is complete?

35. Division

Archeologists often find artifacts from an ancient past and wonder how they were made. They don't see the process that produced the object, just the end of result of the process.

In much the same way, some people like to think of division as the reverse of multiplication. (We saw this in section 17.)

For instance, from

$$3 \times 7 = 21$$

we could focus on the answer 21 and ask about the process: *What times 7 makes 21?*

This would now be considered a division problem and we'd write $21 \div 7 = 3$ after recognizing multiplying 7 by 3 is what makes 21.

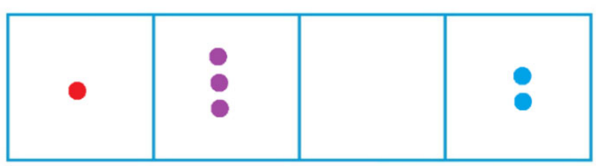
Let's revisit multiplication for a moment to then see if we can follow it backwards to get to division. We'll start with a straightforward multiplication problem, say,

$$1302 \times 3$$

(with answer 3906).

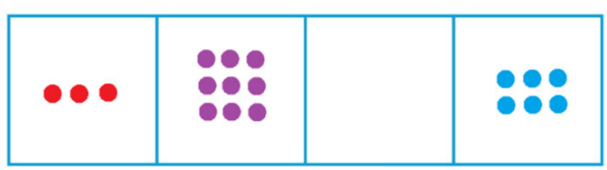
Here's what 1302 looks like in a $1 \leftarrow 10$ machine. (I've colored the dots for fun.)

1302 =



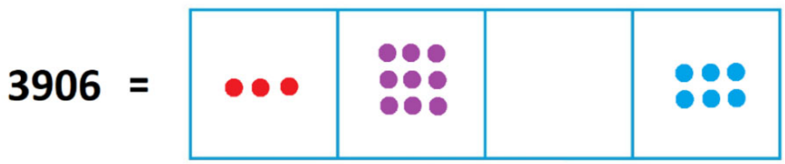
To triple this quantity, we just need to replace each dot in the picture with three dots. We see the answer 3906.

1302 x 3 =

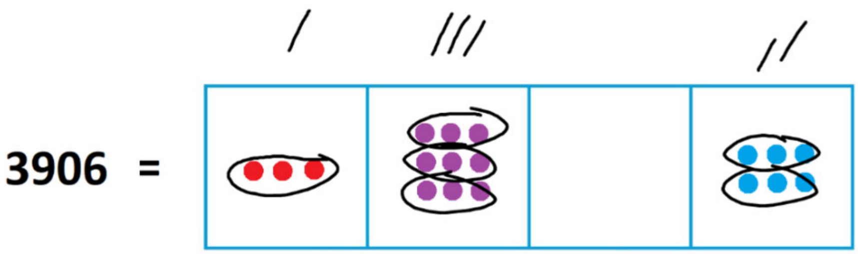


Now let's go backwards and start with the answer to ask:

Here's a picture of 3906*. What times three gives this picture?*



To answer it, you would look for triples of dots that must have come from single dots. And you'd see plenty of those.



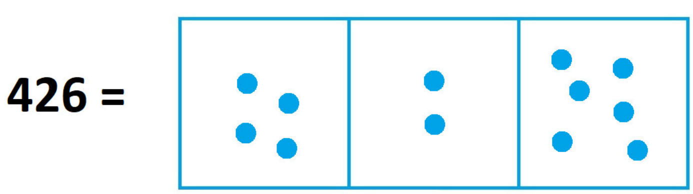
We see the picture as the result of tripling one dot at the thousands level, tripling three dots at the hundreds level, and tripling two dots at the ones level. That is, we see 3906 as the number 1302 tripled.

We have just deduced, from the picture, that $3906 \div 3 = 1302$!

So, to divide a number by three, all we need to do is to look for groups of three in the picture of the number. Each group of three corresponds to a dot that must have been tripled. We can just read off the answer to the division problem then by looking at the groups we find!

And we can do the same for any single-digit division problem.

Question: Here's a picture of 426. Can you see in the picture that $426 \div 2$ must equal 213? (What was doubled to give this picture?)



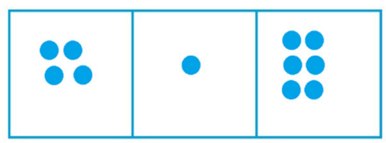
Practice 35.1: Draw a dots-and-boxes picture of 848.
Use your picture to show that $848 \div 4$ must be 212.

How would you explain what is happening to a curious friend?

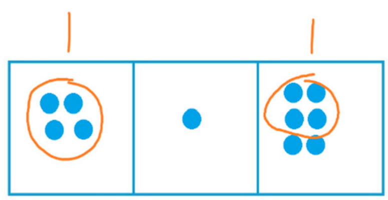
There could be a hiccup in our approach.

Let's try computing $416 \div 4$. (The answer is going to be 104.)

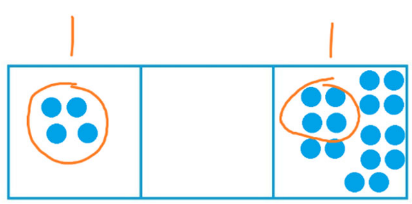
This means we are looking for groups of four (dots that got quadrupled) in a picture of 416.

416 = 

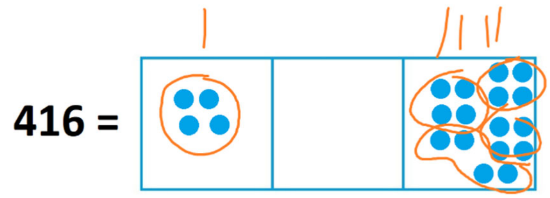
We see one group of four at the hundreds level, and one at the ones level. But no more. Hmm.

416 = 

But an unexplosion might help!

416 = 

This allows to see more groups four.



All dots are now accounted for.

We have one group of four at the hundreds level and four at the ones level. We see that $416 \div 4 = 104$. (That is, 104 was quadrupled to give 416.)

Practice 35.2 Try finding $402 \div 3$ with just a dots-and-boxes picture. How can you get to the answer 134?

Practice 35.3 Use a dots and boxes picture to show that $102 \div 2$ equals 51.

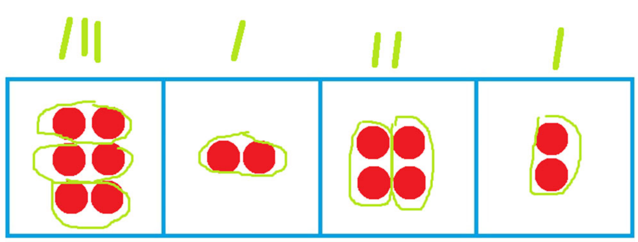
Practice 35.4

a) How does the *Exploding Dots* approach show that $10 \div 5$ equals 2? Draw the picture.

b) Can you work out $1000 \div 8$ with a dots-and-boxes picture? You might be drawing a lot of dots!

Practice 35.5 We saw that $416 \div 4 = 104$. What's $417 \div 4$? What does a dots-and-boxes picture show?

Practice 35.6 Genelle did a division problem the dots-and-boxes way and drew this picture. But she forgot what the problem was.



What division was she solving and what answer did she get for it?

Practice 35.7 Is computing $452 \div 1$ the dots-and-boxes way weird?

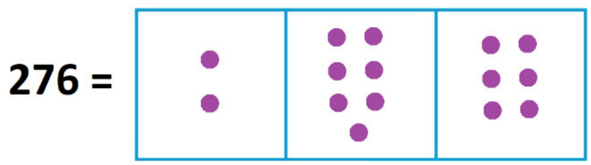
Long Division

Division by single-digit numbers is all well and good. What about division by multi-digit numbers? People usually call that **long division**.

Let's consider the problem

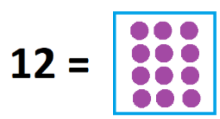
$$276 \div 12.$$

Here is a picture of 276 in a $1 \leftarrow 10$ machine.

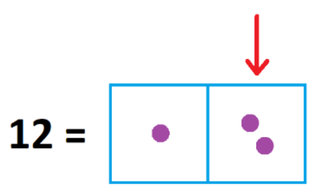


And we are looking for groups of twelve in this picture of 276. That is, we are looking for what got multiplied by twelve to create this picture of 276.

Here's what twelve looks like.



Actually, this is not quite right as there would be an explosion in our $1 \leftarrow 10$ machine. Twelve will look like one dot next to two dots.

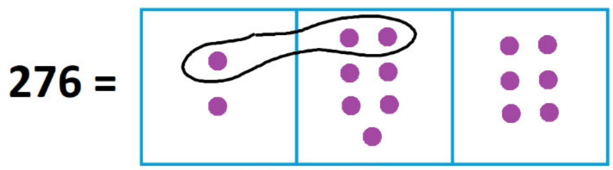


This is going to be confusing, as we don't actually see twelve dots in this picture. We'll have to remember that there really are twelve dots in the rightmost box and that an explosion caused some "spillage."

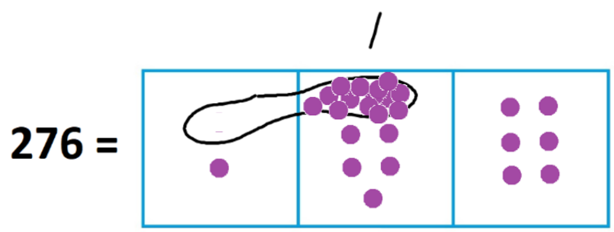
Okay. We are looking for groups of 12 in our picture of 276.

Do we see any "one-dot-next-to-two-dots" in the diagram?

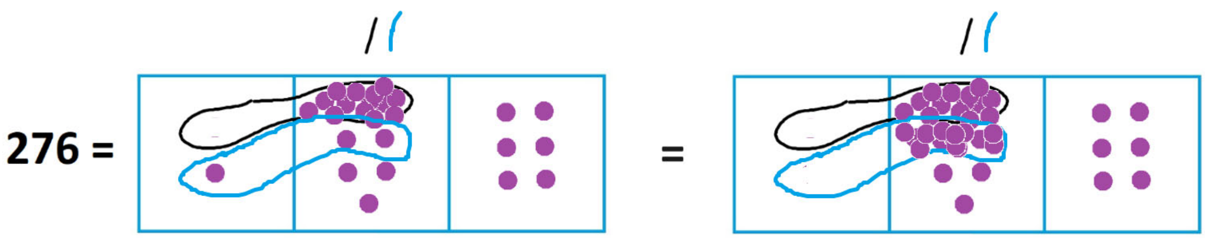
Yes. Here's one.



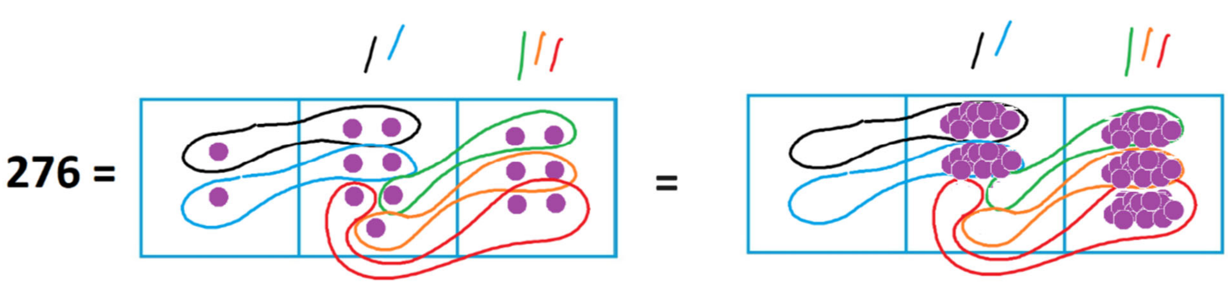
If I do the unexplosion within this loop, we really do see twelve dots at the tens level. One dot at the tens level was multiplied by 12.



There's a second group of twelve at the tens level. A second dot was multiplied by 12 at the tens level.

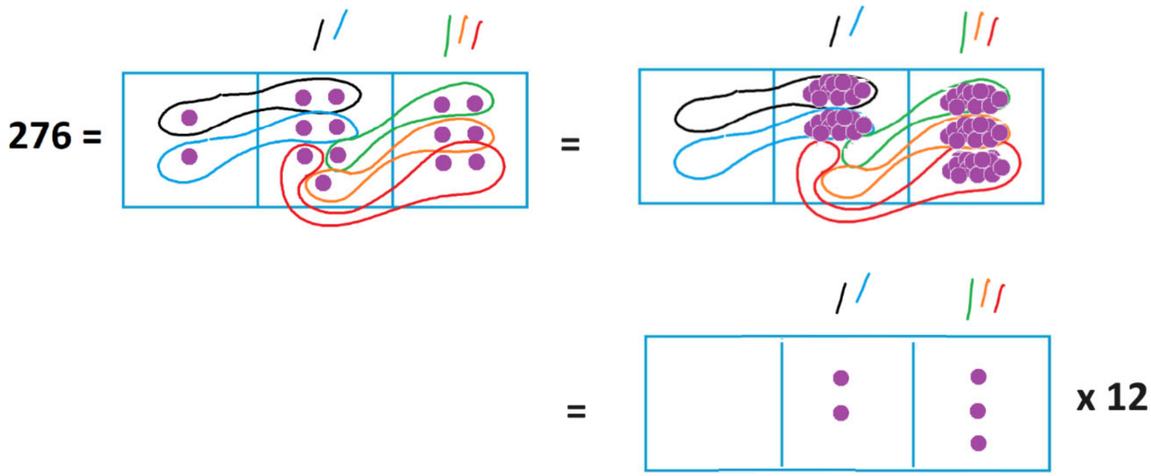


And we can see three more "one-dot-next-to-two-dots" loops, but now at the ones level.



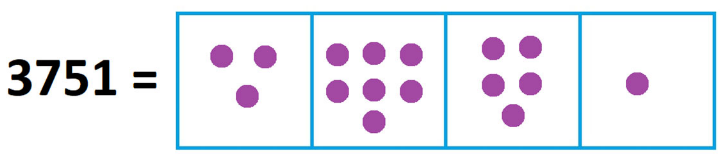
Our picture shows that 276 is really the number 23, two dots in the tens place and three dots in the ones place, with each dot replaced by a group of 12. We see that $276 = 23 \times 12$ and so

$$276 \div 12 = 23.$$

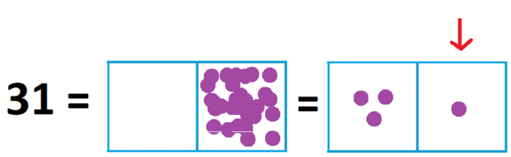


As another example, let's compute $3751 \div 31$ this dots-and-boxes way.

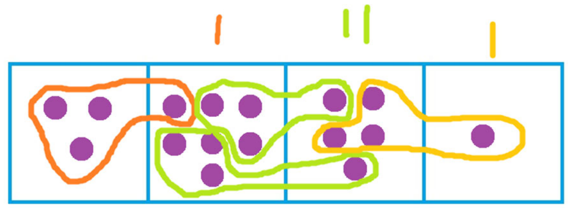
Here's a picture of 3751.



And here's what a group of 31looks like. (All 31 dots are in the rightmost box.)



Here are all the 31s I can find in the picture of 3751.

3751 = 

Question: Can you imagine the unexplosions within each loop?
Can you see the picture as one dot at the hundreds level, two dots at the tens level, and one at the ones level that each got multiplied by 31.

Can you see that it must be that 121 was multiplied by 31 to give the answer 3751?

We have:

$$3751 \div 31 = 121$$

Practice 35.8: See if you can compute

$$2783 \div 23$$

using dots and boxes. Do you see the answer 121?

How would computing $2784 \div 23$ be different?

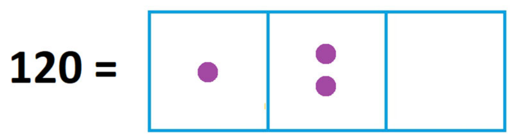
Practice 35.9:

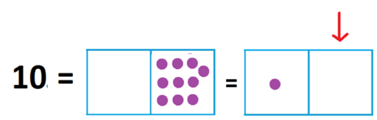
a) Compute $4473 \div 21$ with dots and boxes to get the answer 213.
b) Now compute $4473 \div 213$. Do you see the answer 21?

Practice 35.10: SOMETHING TRICKY

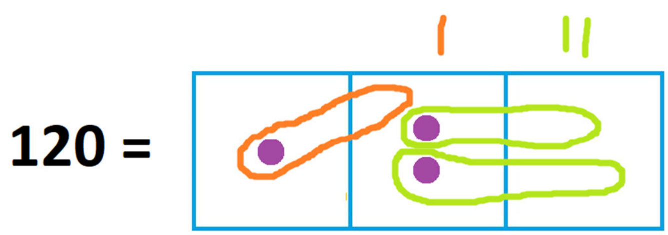
a) Ricky is wondering about 120 ÷ 10. He knows the answer is going to be 12, but he is wondering how the dots-and-boxes approach is going to show this.

He starts by drawing this.





He says he is looking for "one dot next to no dots" and then hunts for them. He thinks he's finding them.



Is he finding them? Is he seeing the answer 12 to 120 ÷ 10?
Is everything he is doing good, and fabulous, and correct?

What do you think?

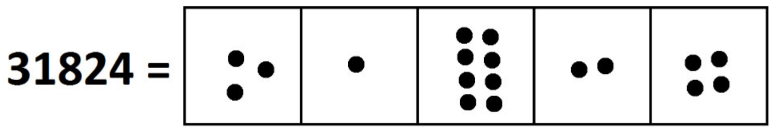
b) Can you see how a dots-and-boxes picture shows that 700 ÷ 70 equals 10?

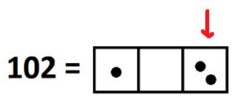
Try computing

$$31824 \div 102$$

before I do it on the next page. Notice the zero in that "102."

Here the pictures to set us up for computing $31824 \div 102$.

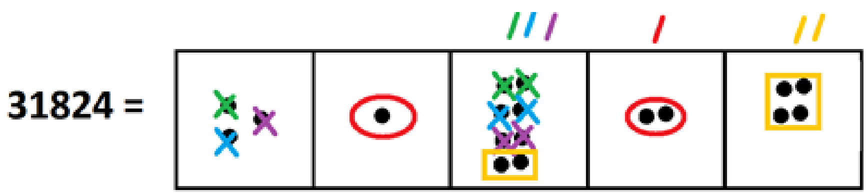




(In the picture of 102, all one-hundred-and-two dots are actually sitting in the rightmost position. But after many explosions, lots of dots "spilled" over to the left.)

We are looking for groups of "one dot–no dots–two dots" in our picture of 31824.

And we can spot a number of these groups.



Question: Can you make sense of what I am doing here? I found drawing loops to be messy, so I drew Xs and circles and boxes instead. Is that okay?

And do you also see how I circled a double group in one hit at the very end?

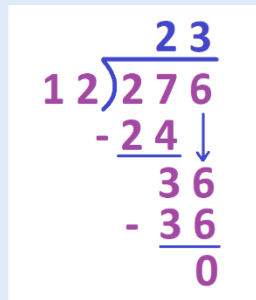
We see that $31824 \div 102$ equals 312.

Practice 35.11 If I am looking for a pattern of "three dots-no dots-one dot-two dots" when doing a long division problem the dots-and-boxes way, what number am I dividing by?

TRADTIONAL LONG DIVISION

I remember as a young lad being taught an algorithm for conducting long division.

It looked like this.



Example: Compute $276 \div 12$.

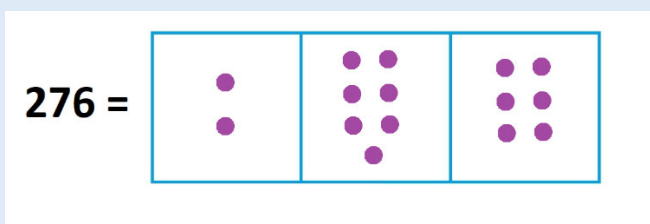
Answer: We see $276 \div 12 = 23$.

And I remember as a young lad being thoroughly perplexed by this algorithm!

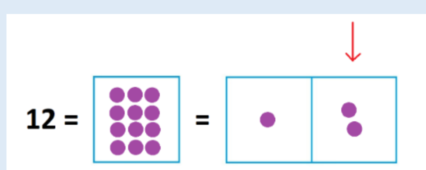
Did you learn an algorithm like this one too?
Can see for yourself what this algorithm is doing?
Can you explain why it works?

I couldn't then! But you and I can now.

The dots-and-boxes approach to computing $276 \div 12$ starts with a picture of 276 in a $1 \leftarrow 10$ machine



and a picture of 12 (which appears as one dot next to two dots, but it is really twelve dots in the rightmost box).



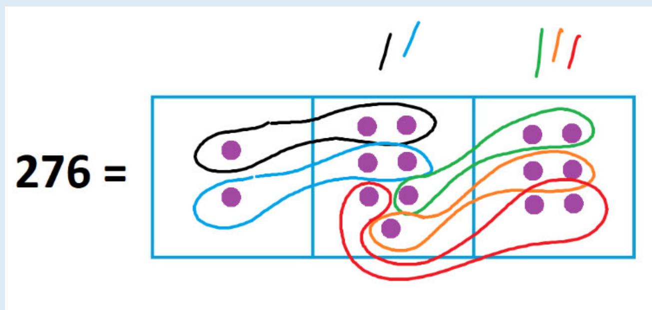
We looked for groups of 12 in the picture of 276.

We first saw two groups among the left two boxes. That is, we first found 2 groups of twelve at the tens level. This accounts for two dots in the left most box, and four of the dots in the middle box.

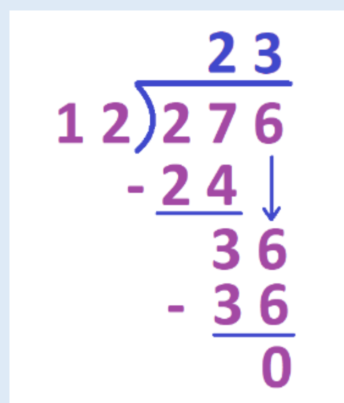
That left three dots still to consider in the middle box and the six in the rightmost box.

Then we found 3 more groups of twelve at the ones level.

This now accounts for all the dots in the picture. Zero dots are left over.



Now think through the school algorithm. Can you see now that it is actually the same process—just without the pictures?



The algorithm first identifies 2 groups of twelve at the tens level just by looking at the first and second boxes.

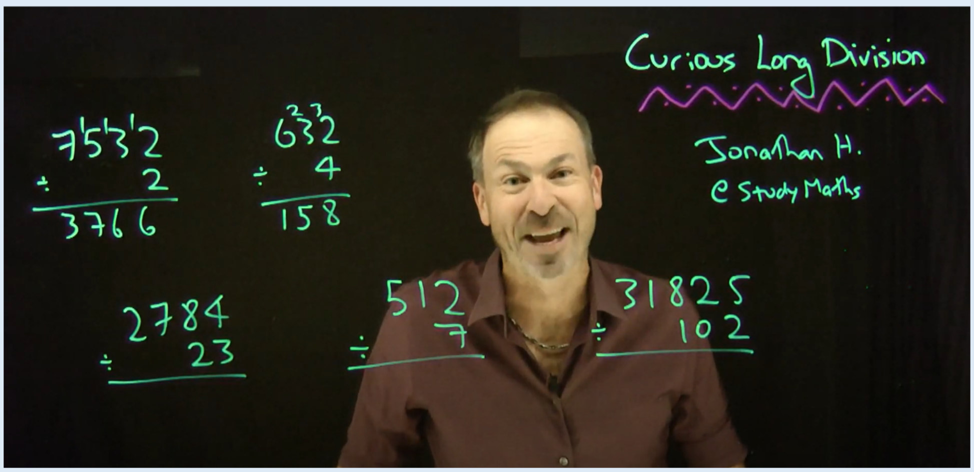
Then the subtraction simply observes that there are 3 dots in the middle box unaccounted for after doing this.

The algorithm then draws our attention to the second and third boxes, where we can next see 3 groups of twelve at the ones level with no dots are left unaccounted for.

Practice 35.12 Compute 2756 ÷ 13 the traditional way and the dots-and-boxes way at the same time, side by side. Can you see that both approaches are doing the same thing?

SPEED DIVISION

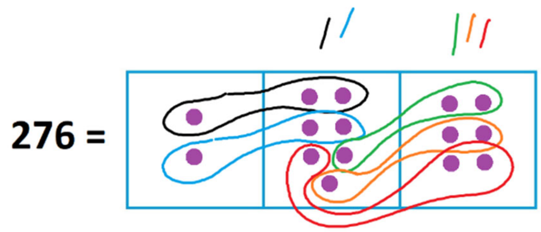
Here's another video. It shows another fun way to conduct long division! (Fun, if you are the sort of person who enjoys doing computations just with pencil and paper.)



https://youtu.be/sWrjGjM2Vjc

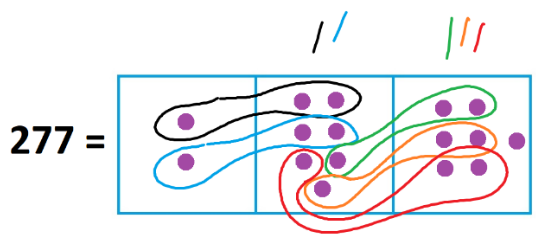
Remainders

We saw that 276 ÷ 12 equals 23



Suppose we tried to compute 277 ÷ 12 instead.
What picture would we get? How should we interpret the picture?

Well, we'd see the same picture as before except for the appearance of one extra dot, which we fail to include in a group of twelve.



This shows that 277 ÷ 12 equals 23 with a remainder of 1.

You might write this as

$$277 \div 12 = 23R1$$

or with some equivalent notation for remainders. (People use different notations for remainders in different countries.)

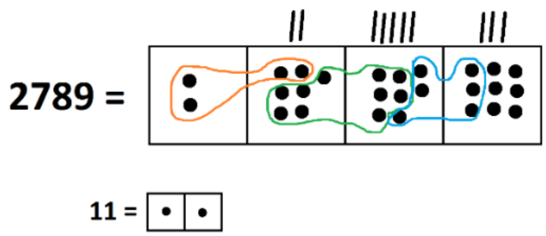
Or you might be a bit more mathematically precise and say that 277 ÷ 12 equals 23 with one more dot still to be divided by twelve.

$$277 \div 12 = 23 + \frac{1}{12}.$$

(And some people might like to think of that single dot as one-twelfth of a group of twelve. All interpretations are good!)

Practice 35.13 Compute 2789 ÷ 11 the dots-and-boxes way.

Do you get a picture like this?



How do you interpret this picture?

Practice 35.14 Use dots and boxes to show that 4366 ÷ 14 equals 311 with a remainder of 12.

Practice 35.15 Use dots and boxes to show that 5481 ÷ 131 equals 41 with a remainder of 110.

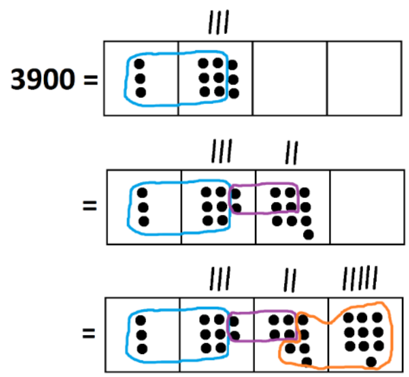
Question: Here are some tricky practice problems you might or might not want to try.

1. Compute 3900 ÷ 12.
2. Compute 46632 ÷ 201.
3. Show that 31533 ÷ 101 equals 312 with a remainder of 21.
4. Compute 2789 ÷ 11.
5. Compute 4366 ÷ 14.
6. Compute 5481 ÷ 131.
7. Compute 61230 ÷ 5.

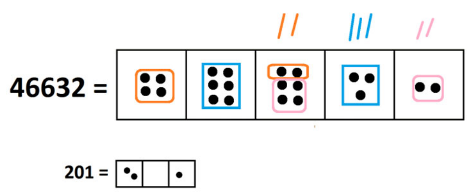
I give my answers to all of these on the next two pages.

Answers:

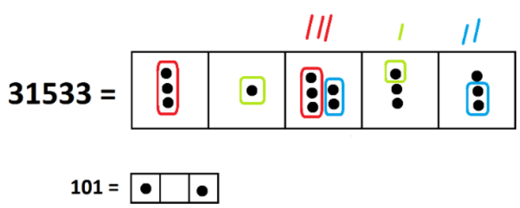
1. 3900 ÷ 12 = 325. We need some unexplosions along the way. (And can you see how I am getting efficient with my loop drawing?)



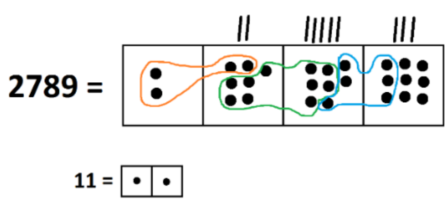
2. 46632 ÷ 201 = 232.



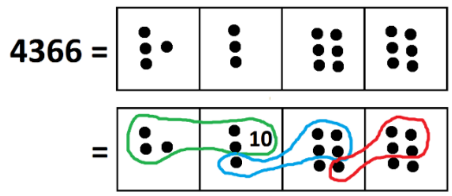
3. 31533 ÷ 101 = 312 with a remainder of 21. That is, $31533 \div 101 = 312 + \frac{21}{101}$



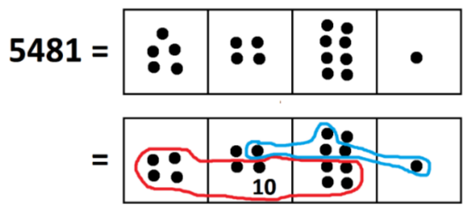
4. We have 2789 ÷ 11 = 253 with a remainder of 6. That is, $2789 \div 11 = 253 + \frac{6}{11}$.



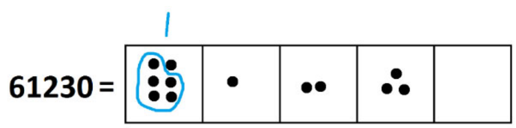
5. $4366 \div 14 = 311 + \frac{12}{14}$.



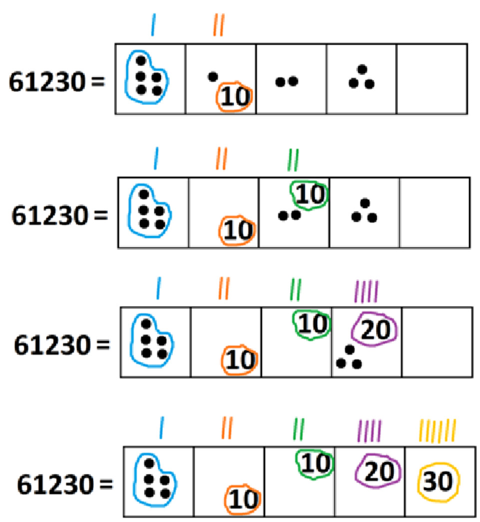
6. $5481 \div 131 = 41 + \frac{110}{131}$.



7. We certainly see one group of five right away.



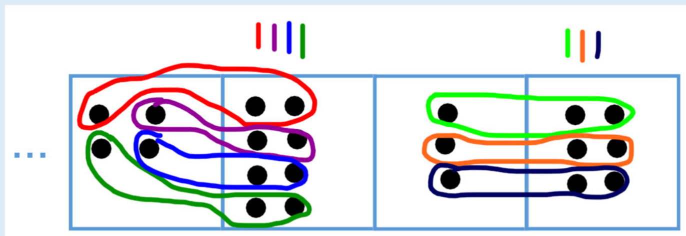
Let's perform some unexplosions. (And let's write numbers rather than draw lots of dots. Drawing dots gets tedious!)



We see $61230 \div 5 = 12246$.

MUSINGS

Musing 35.16 Ebube was working on a division problem, but lost all the details of his work. He just had the picture below.



What division problem was Ebube working on? What answer did he get for it?
(Assume Ebube is working with a $1 \leftarrow 10$ machine.)

Musing 35.17 Here are some division problems you might or might not want to try.
Just pick a few to do! There are too many of these!

a) Compute 4840 ÷ 4.

b) Compute 721 ÷ 7.

c) Compute 126 ÷ 6.

d) Compute 126 ÷ 3.

e) Compute 126 ÷ 2.

f) Compute 126 ÷ 1.

g) Compute 3641 ÷ 11.

h) Compute 3642 ÷ 11.

i) Compute 3649 ÷ 11.

j) Compute 3900 ÷ 12.

k) Compute 100 ÷ 9.

l) Compute 1000000000 ÷ 9.

Musing 35.18 It turns out that $222 \div 3$equals 74. What then is the answer to each of the following?

a) $223 \div 3$?
b) $225 \div 3$?
c) $3222 \div 3$?
d) $444 \div 3$?
e) $2220 \div 3$?

Musing 35.19 **Challenge** Ebube just informed me that his picture back in question 30.1 is actually of a $1 \leftarrow 9$ machine! Okay then. What division problem was he working on, what answer did he get, and how does all that translate to base-ten numbers?

Musing 35.20 DIVISIBILITY RULE FOR NINE

Some people know a rule for quickly determining whether or not a number is divisible by nine.

A number is divisible by 9 only if the sum of its digits is divisible by 9.

For example, 387261 is divisible by 9—apparently—since $3 + 8 + 7 + 6 + 2 + 1 = 27$ is. (And if we weren't sure about the number 27, we could test that it is divisible by 9 by noting that $2 + 7 = 9$ certainly is.)

And to check: $387261 \div 9 = 43029$ with no remainder.

Actually, this rule can be made a little stronger.

A number leaves the same remainder upon division by 9 as does the sum of its digits.

For example, 40061 is two more than a multiple of nine (it equals $4451 \times 9 + 2$) and its sum of digits, $4 + 0 + 0 + 6 + 1 = 11$ is also two more than a multiple of nine.

Also, 77 is five more than a multiple of 9, just as $7 + 7 = 14$, the sum of it digits, is.

Also, 2808 is a multiple of nine, that is, leaves a remainder of 0, and $2 + 8 + 0 + 8 = 18$ leaves a remainder of 0 too upon division by nine.

Let's see if we can explain these rules. (Follow along if you like.)

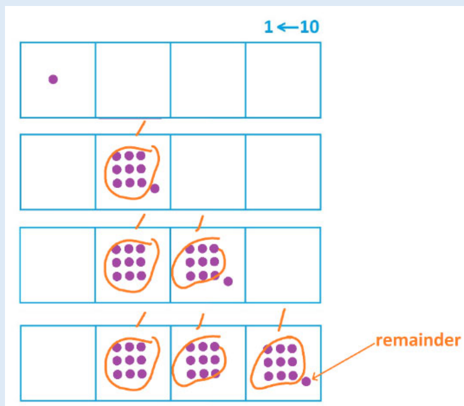
Let's look at division by 9 in a $1 \leftarrow 10$ machine.

a) Draw a dots and boxes picture to compute $210 \div 9$. Show that your work leaves a remainder of $2 + 1 = 3$ dots in the rightmost box.

b) Here is a claim.

Each dot in a $1 \leftarrow 10$machine leaves a remainder of 1 upon division by 9.

Does this picture convince you the claim is true?



Consider these two observations.

The dots-and-boxes picture of 210 uses $2 + 1 + 0 = 3$ dots.
Each dot leaves a remainder of one dot in the right most box.
Thus $210 \div 9$ must have a remainder of 3.

The dots-and-boxes picture of 2213 uses $2 + 2 + 1 + 3 = 8$ dots.
Each dot leaves a remainder of one dot in the right most box.
Thus $2213 \div 9$ must have a remainder of 8.

c) Do these observations make sense to you?
d) What are the matching observations for $2005 \div 9$ and $11111 \div 9$?
e) What is the matching observation for $473 \div 9$?

f) Is it true that each number, upon division of by nine, has the same remainder as the sum of its digits divided by nine?

g) Is it consequently also true that if the sum of the digits of a number is divisible by nine, then the number itself is divisible by nine?

36. Advanced Algebra is not that Advanced Really.

Everything we do in current society is based on the number 10.

We humans are drawn to the number ten for counting, most likely because we were born with ten digits on our hands.

But there is absolutely nothing special about the number ten for doing mathematics. We could do arithmetic in base 6 or base 12 (Martian), or base 4 or base 8 (Venution), or base 2 (computer), or in any other number machine we care to work in. The mathematics is exactly the same!

Algebra is about doing mathematics in any system whatsoever and not being locked into our humanness. (Did you read the introduction to this book? I really do believe that mathematics is universal.) Moreover, not only can we do addition and subtraction and multiplication and division in any base in any base we like, we can do arithmetic before we've even decided which base we want to work in!

The way to do that is to just use a symbol to represent a base number.

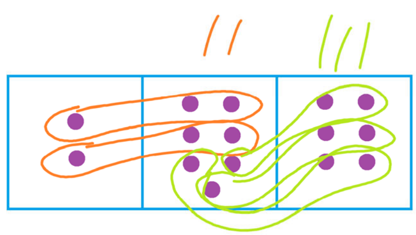
But there is one unfortunate thing. People seem to use the same symbol over and over again when they think "any old number." They use the letter x. And now that letter has been so overused for so many decades that people are now actually scared of that one letter of the alphabet in math class. So unfortunate. (I personally like to use the symbol n for "number" or maybe b for "base.")

Take a deep breath and look at this picture. It has the symbol x.

Can you see that it is showing that division in base 10, as we've been doing, and division in algebra look exactly the same?

Arithmetic

$$276 \div 12 = 23$$



Algebra

$$\left(2x^2 + 7x + 6\right) \div (x+2) = 2x + 3$$

IT'S THE SAME!

And remember, the symbol x here just means "any base you like." We can go back to being human and choose x to be 10 if we like, and this takes us back to the familiar calculation $276 \div 12 = 23$. But it is also fun to play with other possibilities as to what x could be. We could be human or Martian or Venutian or a computer, or something else entirely, with ease.

Algebra is about just having fun not being locked into our humanness.