# ARITHMETIC, ALGEBRA, and
## Radical Comprehension of Math

A Refreshingly Joyous, Human, and Accessible approach to
Arithmetic and Algebra
for all those who may have experienced it otherwise

# CHAPTERS 5, 6, 7, and 8

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# PART 1

# Arithmetic: The Gateway to All


Photo: Erick Mathew, Tanzania

*Algebra is the practice of avoiding the tedium of doing arithmetic problems one instance at a time, to take a step back and see a general structure to what makes arithmetic work the way it does, and so open one's mind to more than the one view of what arithmetic, and mathematics, can be.*

# Part 1

# Part 2

# Chapter 5

## Fractions:
## Not Getting Them is Not Your Fault

## 37. It’s Not Your Fault

Many people fear fractions. And this fear seems somewhat universal.

I have a theory as to why.

People first experience learning about fractions as youngsters at home, typically from sharing goods and desserts. Understanding portions of items—such as *halves*, *thirds*, *quarters*—is intuitive and natural.

The typical school curriculum then picks up on this intuitive start and over the course of several years next attempts to build a robust and coherent mathematical story of fractions from this start.

But the school curriculum is caught between two contradictory demands: the demand to be age appropriate and to develop “real world” meaning to the mathematics of fractions each step of the way, and to be honest about the mathematics itself, which, for fractions, very quickly steps beyond real world loyalties.

As I said in Section 23,

> *Mathematics is bigger and bolder than the real world. It is therefore bigger and bolder than all schoolbook attempts to make every part of it concrete and real. Mathematics certainly incorporates real-world models and is immensely powerful in helping describe them. But mathematics sits at a higher plane to them.*

The teaching of fractions is particularly challenged by this fact.

As a result, many elementary- and middle-school curriculums on the topic are muddled and confounding. One doesn’t usually see that confusion right away as each step of the curriculum brings in a real-world idea that makes sense for that one step. It is only when you look back and try to make sense of the story as a whole do you say: *Hang on! So, what is a fraction really? Which real-world model applies when, and why not to everything?*

To see what I mean, what comes next is an overview of how the story of fractions is often presented from grades K to 7.

Read this section, but don’t take it too seriously. The only message I hope you glean from by its end is this:

**It is not your fault!**

If you are befuddled, confused, and scared by fractions, it is not your fault for not “getting” them. (Actually, not getting them is a sign of your intelligence: you’ve picked up that something is awry.)

Here’s the school story of fractions as you may well have experienced it.

## VERY EARLY GRADES: Fractions are Parts of Things

Here's a task for you, right off the bat.

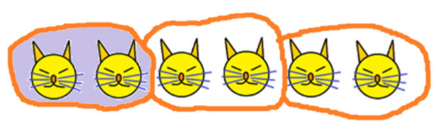
*Please circle **one third** of these six kittens.*

*Now please circle **half** of these stars.*

These tasks represent how young students often first experience fractions.

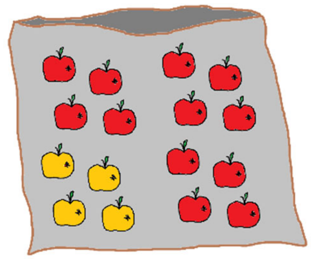
They are taught the notions of *half*, *third*, *quarter*, *fifth*, and so on, and practice dividing sets of objects into equal-sized parts. And they learn that equal-sized parts have these special names.

And you just did this (mentally, at least). You saw that the kittens naturally divide into three sets of equal size, and you selected one of those sets. That's a third of the kittens.



The stars naturally divide into two sets of equal size, and you selected one of those sets: half the stars.

Which word describes the proportion of yellow apples in this bag?



**Answer:** One quarter—also called a fourth.

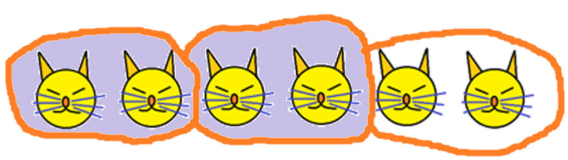
It can be hard for young students to recognize equal-sized groups, and there is genuine confusion to be had here.

For example, in this bag of apples, one can see four equal groups of size 4, two equal groups of size 8, eight equal groups of size 2, and sixteen equal groups of size 1. We can even say there is one group of 16.

There are many sets of equal-sized groups!

Students are expected to intuit which size of equal sized groups make sense for a question and respond appropriately.

Sometimes young students might be asked to go a step further. Rather than select one third of the kittens, they might be asked to select **two thirds** of the kittens.



The English language helps here.

"Two houses" is one house and another house. "Two kumquats" is one kumquat and another kumquat. And "two thirds" is one third and another third.

Similarly, "three quarters" is literally three quarters, and "seven eighths" is literally seven eighths.

Still at this early stage, students learn to write the symbols $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, … and so on for the words one half, one third, one quarter, and so forth.

They also learn to write $\frac{2}{3}$, for "two thirds" and $\frac{3}{8}$, for "three eights," for example.

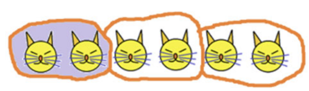
Weirdly, many curricula then seem to soon lose sight of what the English language is suggesting: $\frac{3}{8}$, for instance, really is three eighths: an eighth and an eighth and an eighth, that is, it's three copies of a basic fraction called one eighth.

### Where Does this First Story Leave Us?

It's not clear what fractions are at this point.

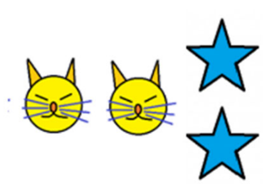
We certainly use numbers to describe them—$\frac{1}{2}$ and $\frac{1}{3}$ and the like—and we count things to find them. But they are not really numbers in-and-of themselves.

A fraction is more like a *call to action* or a thing to do: "circle a third of the kittens" or "select half the stars."



And right now, the idea of doing arithmetic with fractions makes no sense.

> **Question:** *What is a third plus a half?*
> Which of these might be a reasonable answer a young student might give to this question?



**4**

**HUH?**

All three seem like reasonable answers to me.

An older student will be expected to give the answer $\frac{5}{6}$ for a half plus a third, but this answer makes no sense whatsoever in this part of the story. (Where is that $5$ coming from? Where is that $6$ coming from? The fraction $\frac{5}{6}$ is a bizarre and nonsensical answer to this problem.)

The upshot here is that people are first typically taught intuitive and practical understanding of the notions of halves, thirds, sevenths, tenths, and so on.

But there is no reason to think that fractions are numbers in any way, and there is certainly no reason to think we should be able to do arithmetic with them.

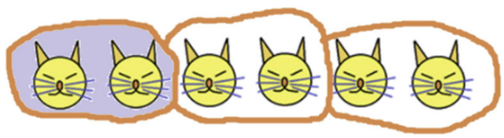
But the typical curriculum wants to change that.

And it does so by next making an edict.

## LESS EARLY GRADES: Fractions Should Come from the Same "Whole."

We just saw that a third could be two kittens and that a half could be two stars.

But as it is not possible to compare kittens and stars, the idea of combining fractions or doing arithmetic with fractions has no meaning—at least not in this context.



So, the typical curriculum next takes a stand and makes an edict.

**In a conversation about fractions, the fractions must come from the same whole.**

Let's think about what this means.

First, what type of word is "whole"?
It's used both as an adjective and a noun in everyday life.

"I spent the whole day doing math."
"On the whole, life is good."

But it is a bit weird to use it as a stand-alone noun: "They come from the same whole."

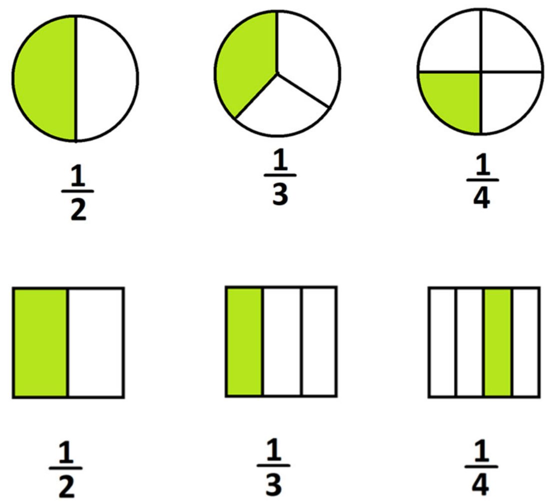
The language of the edict feels a bit strange.

It is easier to just give examples of "wholes" and slide in the use of that word with those examples.
And the stereotypical example of whole is a pie, actually, to be clear, a whole pie. (Nouns and adjectives again!)

People tend to think of pizza pie, or an apple pie, or just a generic round pie.

Drawing circles for pies and dividing up circles is actually hard! It is so much easier to draw square or rectangular pies.

But no matter what type of pie you draw and divide into equal-sized parts, the basic fractions $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, … are then seen as slices of pie, each just one of an equal-sized part.

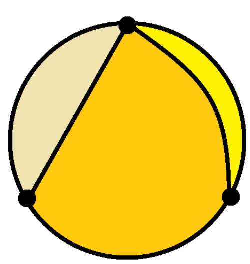



But there is confusion to be had here.

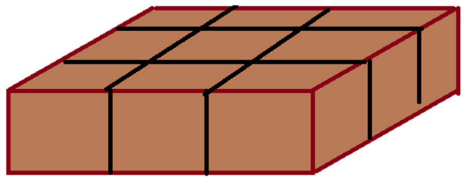
We all know in the real world that not all people deem all slices as "equal."

A slice of pizza with more anchovies on it might be more desirable to some. (To me, for instance!)

People who only like the cheese-stuffed crust and don't care one whit about the topping might agree that this pizza shown is divided into equal thirds.



If we are talking about square cake, then corner pieces have more icing on the side and might be deemed more valuable.



Young students like to think like lawyers and find loopholes and exceptions.
And they are right to do so. That is mathematical thinking!

So, in our discussion of fractions with pie as “the whole,” we have to make some points absolutely clear.

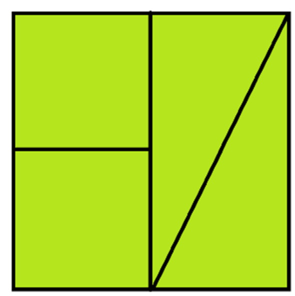
1. We’re talking about the top surface area of the pie.
2. We are assuming that the pie is perfectly uniform in its topping.
3. The perimeter of the pie (its crust) is irrelevant.

We are also assuming that no one cares about the shape of the pieces, just as long as each piece has the same area.

This is a lot for a young student to take in. (And this is especially hard if a student hasn’t yet properly learned what area is from a geometry class.)

**Question:** Here’s a square pie divided into four pieces.

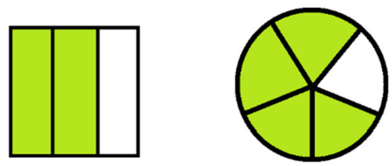
Does it look like it has been divided into four **equal** pieces according to the parameters just outlined?



Next, students are reminded that $\frac{2}{3}$ means "two copies of $\frac{1}{3}$" and $\frac{4}{5}$ means "four copies of $\frac{1}{5}$," for instance. It is understood that all the copies (slices) are coming from the same pie and all the slices are "equal."

In general,

> The fraction $\frac{a}{b}$ is interpreted as "$a$ copies of $\frac{1}{b}$," where $\frac{1}{b}$ represents one slice that comes from dividing a pie into $b$ equal slices.

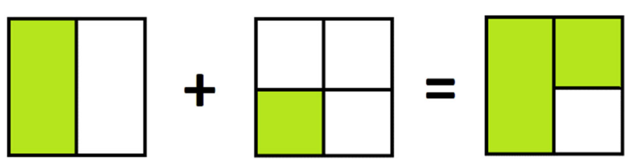


$\frac{2}{3}$ $\quad$ $\frac{4}{5}$

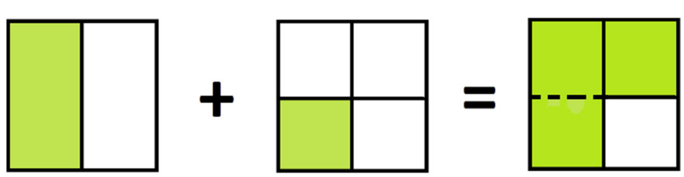
So, fractions are represented by portions of pie. But it is still not yet clear whether or not fractions are numbers in and of themselves.

But if two fractions come from the same pie, it does start to feel possible to do some legitimate arithmetic with them.

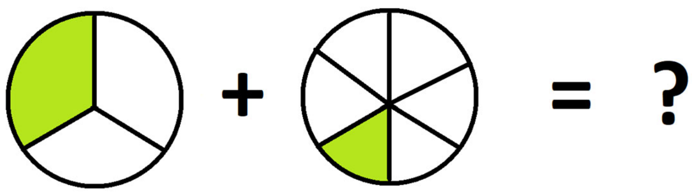
For example, $\frac{1}{2}+\frac{1}{4}$ can be interpreted as putting half of a pie and a quarter of that same pie together on one plate.



In this example, students might be encouraged to extend the horizontal line in the middle of the right picture to see that the answer is the same as $\frac{3}{4}$, three copies of a quarter of the pie.



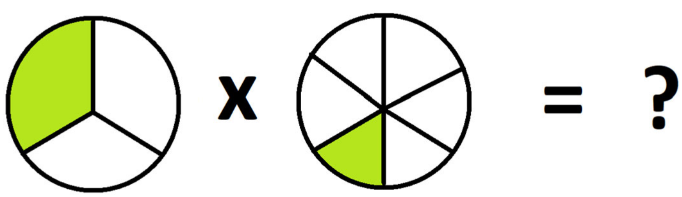
**Question:** What's $\frac{1}{3} + \frac{1}{6}$ with this type of thinking?



We're now doing arithmetic with fractions. They are starting to feel like numbers.

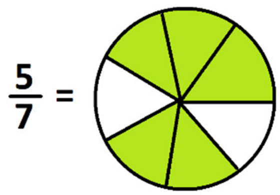
But are they numbers?

**Question:** If $\frac{1}{3}$ and $\frac{1}{6}$ truly are numbers, then we should be able to multiply them as well. Can we? Does "a third of a pie times a sixth of a pie" make sense to you? (It sounds like nonsense to me.)



Here's a little more on jargon.

The fraction $\frac{5}{7}$ means "five copies of a seventh."
It counts out five copies of a certain "type of thing," namely, a *seventh*.



The late Latin word for "a count of things" is *numerare* and the word for "a type of a thing" is *denominare* with *de-* meaning "completely" and *nominare* meaning "to name".

These led to words we use in everyday life. For example, to *enumerate* means to count out something and a *denomination* is the name of a certain class or type.

These Latin words also led to the modern names of the two numbers we use to write fractions.

For the fraction $\frac{5}{7}$ , for instance, the top number "5" is called its **numerator**. It tells us the count of things (slices) we have. The bottom number "7" is called its **denominator**. It tells us the type of thing we are counting, namely, sevenths.

While we are at it …

The word **fraction** itself comes from the late Latin word *fractionem* which means "the act of breaking into pieces." One might break one's arm and have a bone *fracture*, or one might break the law and commit an *infraction*, or one might break a pie into pieces and create examples of *fractions*.

## Where Does This Second Story Leave Us?

We have the edict that in any conversation about fractions, the fractions should come from the same "whole" (pie). This allows us to legitimately compare fractions and we can start to do what feels like some arithmetic with fractions.

But it is only a feeling. Fractions aren't numbers here – they are portions of pie – and the "addition" of fractions is just the physical act of bringing portions of pie together.

Still, students can be asked some mind-bendy questions.

**Question 1:** Betta ate $\frac{1}{6}$ of a pie and Cuthbert ate $\frac{1}{7}$ of a pie. Who ate the most pie? Select all options that could be true.

a) Betta did if we know the pies were the same size.
b) Cuthbert did if we know the pies were the same size.
c) Cuthbert could have eaten more if his pie was sufficiently larger than Betta's.
d) Perhaps neither. They could have eaten the exact same amount of pie if their pies were differently sized.

**Question 2:** A pie is divided into five pieces. Only one piece can legitimately be dubbed as $\frac{1}{5}$ of the pie. Which of the following statements must be true?

a) There is at least one piece larger than $\frac{1}{5}$ of the pie.
b) There is at least one piece smaller than $\frac{1}{5}$ of the pie.
c) Neither of these statements need be true.

To fix up the worry of whether or not fractions are actually numbers, the curriculum takes a new turn.

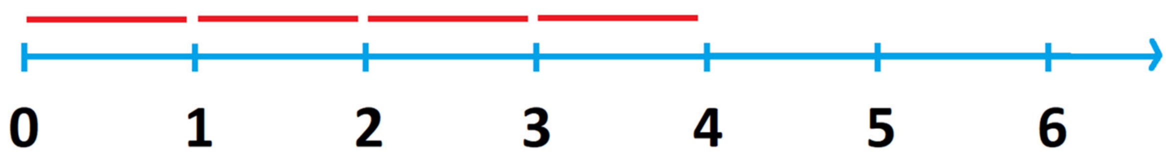
**Answers to the Two Questions:**

1. a) c) d)
2. a) b)

## SLIGHTLY LATER GRADES: Fractions are Points on a Line

School math likes to stack numbers along a **number line**.

It's a line that is built from laying sticks of length $1$ end-to-end with the left end of the line labeled $0$. The point labeled "$4$" on the line, for instance, tells us that four sticks laid together from $0$ on the left reach that point on the line.




Up to this point, fractions have been portions of a whole, with that whole typically being a pie.

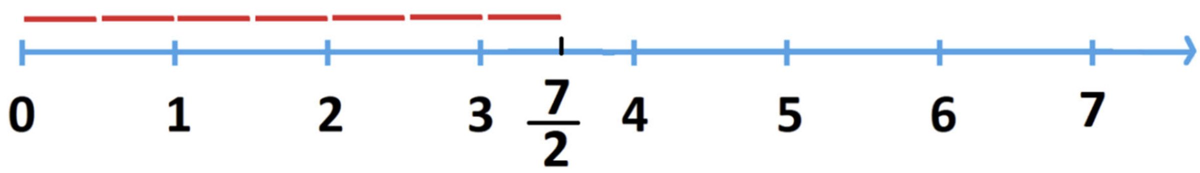
But as soon as schools introduce the number line, a switch in fraction thinking occurs.

**Make your stick of length 1 your new pie!**

You are now slicing up a stick. (You can think of the stick as a very thin, flat, linear pie. Or maybe a strand of uncooked spaghetti?)

The fraction $\frac{1}{2}$ now represents half a stick and we mark on the number line the point where that portion of the stick reaches. We label that point $\frac{1}{2}$ as well.

The fraction $\frac{7}{2}$ means "seven copies of $\frac{1}{2}$" and we mark on the line the point where seven half sticks reach.




And this number-line thinking suggests that $\frac{7}{2}$ is the same as $3+\frac{1}{2}$ (though we haven't officially talked about what addition means in this context of a number line).

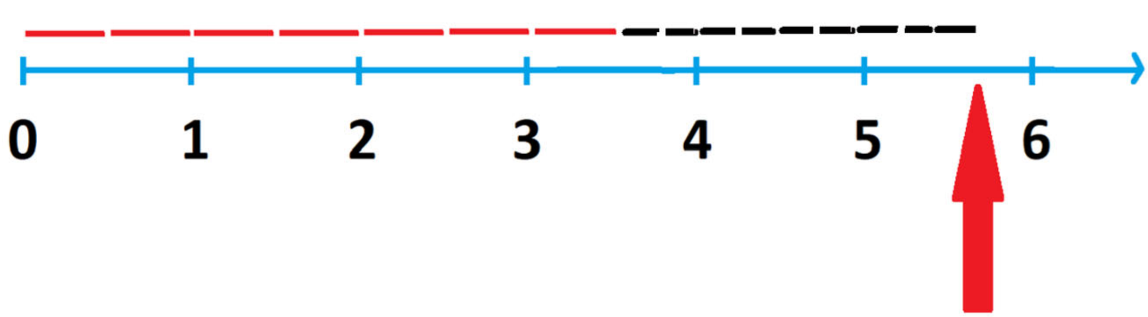
> **Aside:** Using a number line for visualizing addition (and subtraction) came quite late in the history of mathematics. English mathematician John Wallis (1616-1703) seems to have been the first person to write about a number line in this context.

Having students work with fractions on the number line could be taken as a psychological ploy.

> Well, if fractions are on the number line, they must be numbers!

Hmm.

Following Wallis' ideas, we can do arithmetic with fractions on the number line. For example, by placing down a length of $\frac{7}{2}$ and a length of $\frac{9}{4}$ side by side, it looks like we're landing at the mark $5+\frac{3}{4}$ on the number line.




We marked fractions on the number line. We're doing some arithmetic with fractions.

So, are fractions numbers?

## Where Does this Third Story Leave Us?

I am not sure!

I feel like I am being coerced to say that fractions are numbers.
But even here fractions are still portions of pie– just a long thing spaghetti-like pie.

Sure, we can label off the lengths along a number line. But the labels themselves are not the objects we're talking about. Or are they now?

This is confusing.

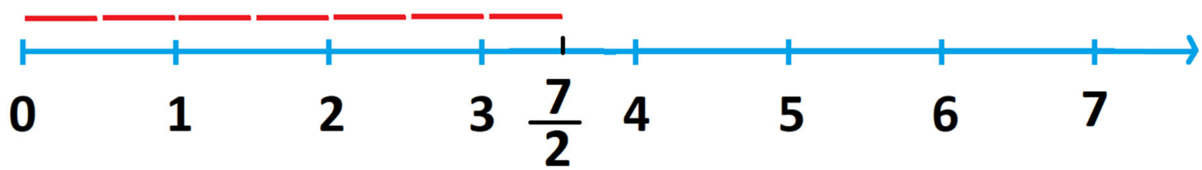
Plus ... If fractions are numbers, I still don't know what it means to multiply two fractions.
How do I multiply two labels to make a third label?

My brain hurts!

But never mind, the school curriculum pushes on.

## LATER GRADES: Fractions are Numbers

We're at the point of our curriculum story where fractions, that were once portions of pie, are now being seen almost solely as labels on the number line.




And the coercion is strong: Because fractions are on the number line, they must be numbers.

Are they?

The typical school curriculum next just turns this question into an assertion.

**Fractions are numbers. They are answers to division problems.**

For example, $\frac{7}{2}$ is no longer "seven copies of $\frac{1}{2}$," *per* se. It is now the answer to the division problem $7 \div 2$. And $\frac{1}{3}$ is the answer to the division problem $1 \div 3$.

**A fraction $\frac{a}{b}$ is the number that answers the division problem $a \div b$.**

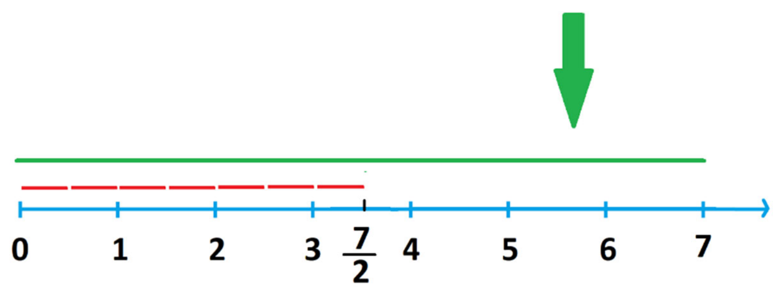
This is typically justified by thinking of division as "division by sharing" (from Section 17).

For example, a textbook might be naughty and have students ignore the number line and go back to portions-of-pie thinking (despite weening students off this thinking).

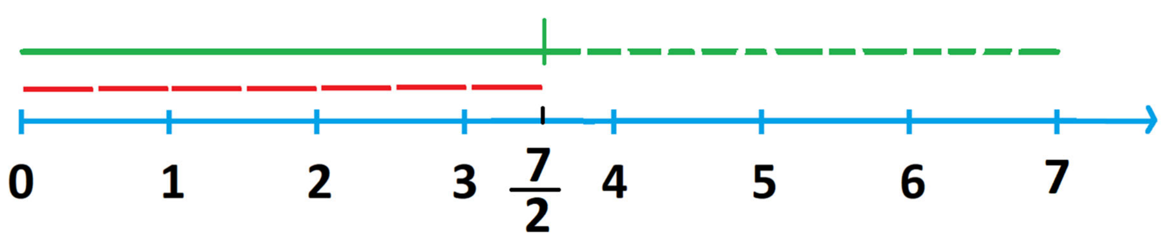
> *Interpret $1 \div 3$, say, as the result of sharing one pie equally among three students. The result is one third of pie per student. We called that $\frac{1}{3}$. So, $1 \div 3 = \frac{1}{3}$.*

Or textbooks attempt to conduct division by dividing (sharing) lengths on the number line.

For example, let's try to find the value of $7 \div 2$ by diving a length of 7 sitting on the number line into two equal parts. Can you imagine sliding the green arrow to the appropriate spot?




What is surprising—or should come across as surprising—is that the arrow lands on the spot that matches "seven copies of $\frac{1}{2}$," which is how we previously interpreted $\frac{7}{2}$.




Similarly, if we divide a line segment of length 2 into three equal parts, the length of left piece seems to match the position of the fraction $\frac{2}{3}$, which we previously understood to be two copies of one third.

But this leaves me wondering, in general:

Will the result of constructing $a \div b$ always land at the same spot as "$a$ copies of $\frac{1}{b}$"?

Thinking through this question feels weird and hard! (Plus, I am not sure I "get" the question!)

Okay. I'll just swallow that fractions are numbers and that they do match the answers to division problems.

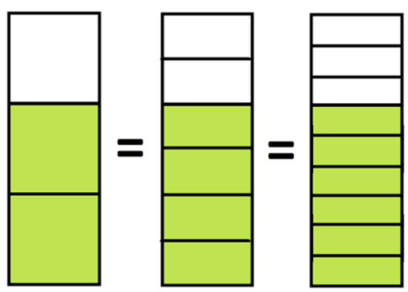
Let's just accept that somehow $\frac{a}{b}$ and $a \div b$ and "$a$ copies of $\frac{1}{b}$" all match up to be the same thing.

But, at the same time, don't!

When introducing a new fraction concept, textbooks often revert back to "portions of pie" when it feels convenient and simply drop the "fractions are answers to division problems" definition.

For example, you might remember learning about *equivalent fractions*.

To explain why $\frac{2}{3}$, for instance, is the same as $\frac{4}{6}$ and as $\frac{6}{9}$ textbooks draw pies. Pies might be vertical rectangles now.



Is there a connection to division problems here? It is not obvious if there is one.

So, despite its claims and edicts, the typical school curriculum oscillates between old and new fraction concepts, pulling up whatever seems magically convenient at the time.

But then things get even worse!

## Mysterious "OF" and Magical "KEEP CHANGE FLIP"

Back in your early school days did you hear the mantra: *"of" means multiply*?

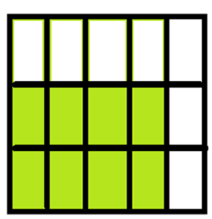
It is natural to use the word "of" when doing work with fractions, at least when we go back to thinking about pie.

For example, we can imagine figuring out something like "two thirds of four fifths of a pie."

To compute it, start with a picture of $\frac{4}{5}$ of a pie.

Then select $\frac{2}{3}$ of that portion of the pie.

If we draw in extra lines, it seems we have a pie that has been divided into $15$ pieces and we've selected $8$ of them. It is natural to say that *two-thirds **of** four-fifths is eight-fifteenths*.



**Question:** Still thinking in terms of portions of pie, does it seem natural to you to say that $\frac{1}{2}$ **of** $\frac{1}{3}$ of a pie is $\frac{1}{6}$ of the pie?

If we are indeed allowed to go back to thinking of fractions as portions of pie, then this use of the word "of" makes perfectly good sense.

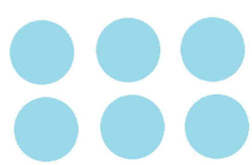
But what does "of" have to do with multiplication?

Well, there seems to be two curriculum answers to this question.

**Answer 1:** *We've been using the word "of" with multiplication all along. Why stop now?*

For example, back with counting numbers we've been reading something as simple as $2 \times 3$ as "two copies **of** three." So, if "3" is represented as three dots

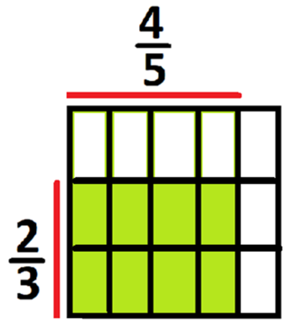
then $2 \times 3$ is six dots.



So, it makes sense that $\frac{1}{2} \times 3$ should be interpreted as "half a copy **of** three dots," and that would match $\frac{3}{2}$, three copies of half a dot.

**Answer 2:** *Look at our pictures of rectangular pie when we calculate fractions of fractions. They look like Rule 8 for chopping up rectangles, which is the area model for multiplication!*

For example, here's our picture of "$\frac{2}{3}$ of $\frac{4}{5}$" again.




It is mighty lucky that we happened to draw a rectangular pie to work this out. (Would we see the same thing if we had been drawing circular pies?)

Authors also point out that, for this example at least, $8 = 2 \times 4$, $15 = 3 \times 5$, and our answer is $\frac{8}{15}$. Curious!

> **Aside:** Allow me to share a personal concern I have about this picture. If we are thinking "pie," then what exactly is our pie here?
>
> We have one thin spaghetti pie for $\frac{4}{5}$, and another thin spaghetti pie for $\frac{2}{3}$, and our final answer is about a rectangle of green pie coming from a two-dimensional rectangular pie.
>
> I thought the edict was: *In a conversation about fractions, the fractions must come from the same whole.*

Also, what happened to our "fractions are answers to division problems" thinking?
What happened to our number line?

The conversations about "of" and "multiplication" ignore what students were just taught.

Some curriculums don't try to justify a multiplication rule for fractions and just assert a rule:

> To multiply two fractions, multiply together the numerators and the multiply the denominators and use those products to create a new fraction: $\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$.

And then comes the division of fractions!

Were you told a rule like this?

**To divide two fractions, just "keep change flip."**

For example, to compute $\frac{2}{3} \div \frac{4}{5}$,

**keep** the first fraction as $\frac{2}{3}$ as it is,
**change** the division symbol to multiplication sign $\times$, and
**flip** the second fraction upside down and make it $\frac{5}{4}$.

That is, to work out $\frac{2}{3} \div \frac{4}{5}$ just compute $\frac{2}{3} \times \frac{5}{4}$ instead.

Got that? Does this make perfectly good logical sense to you?
It doesn't to me! Where is this coming from?

### Where Does this Final Story Leave Us?

At this point one's education on fractions is essentially complete.

But what are we actually left holding on to? I am not sure!

We have some intuitive models of fractions—portions of pie, answers to sharing problems, points on the number line—and were told that these models are not "complete," that fractions are actually numbers and that one can perform the full range of arithmetic with them.

We have a vague sense that, maybe, the arithmetic of fractions is motivated by real-world models, but it all feels haphazard and hazy.

But, in any case, many students by this stage of the story have stopped questioning the haziness and have memorized and mastered the mechanics of fraction arithmetic.

And then we send them off to high school where educators presume fractions are well understood and proceed to speed on to more abstract ideas that make use of fractional quantities.

This reality of K-12 mathematics education deeply perturbs me both as a professional mathematician and as a somewhat functional human being. One needs to look back and take stock of these stories and find a framework to make sense of the jumble of ideas as one whole. (Ha!)

That we don't is a serious disservice to thinking humans. Not "getting" fractions is a natural and valid response to these curriculum stories if they are just left hanging as they are.

So, let's turn matters around together now, in this chapter and in the next.

We'll be gentle and start with one more real-world model of fractions—but use that model as a transition. It will lead us to the true, robust and logically complete mathematical story of fractions.

There is rhyme and reason to fractions.

We'll explore both the rhyme and the reason.

## MUSINGS

**Musing 37.1** It is curious that we need to write fraction, which is a single number, with two numbers—a numerator and a denominator. For example, the number $\frac{5}{7}$ is described with a five and a seven. We also place the two numbers vertically with a horizontal bar between them.

a) Look up the general history of fraction notation. Who first started writing the two numbers to describe a fraction vertically?

b) What is the official name of the horizontal line used in fraction notation? (Most people call it a "fraction bar," but what is its Latin name? Why that name?)

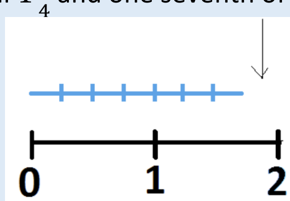
**Musing 37.2** Some curriculums have students play with the division of fractions using a number line. They have students draw a picture and ask: "How many groups of what I am looking for do I see?"

Here are two examples.

**Example:** *Draw a picture to evaluate* $3 \div \frac{2}{3}$.
**Answer:**


We see four-and-a-half copies of $\frac{2}{3}$ in 3. We have $3 \div \frac{2}{3} = 4\frac{1}{2}$.

**Example:** *Draw a picture to evaluate* $2 \div 1\frac{3}{4}$.
**Answer:** We see that if we had one more seventh of the blue segment shown, we'd have a complete "2." Thus, we see one full $1\frac{3}{4}$ and one seventh of $1\frac{3}{4}$ in 2.




This means $2 \div 1\frac{3}{4} = 1\frac{1}{7}$.

Of course, this is another seemingly out-of-the-blue idea to toss into the general jumble of ideas for working with fractions.

a) Do you like what's going on in the two examples?
(The answer can be: "No I don't! I don't get it. I don't like it. And I am turning the page.)

b) Would you like to complete each of these with number-line pictures?
(The answer can be no.)

*i)* $2\frac{1}{2} \div \frac{3}{4}$ *ii)* $\frac{1}{2} \div 3$

# 38. One Somewhat Robust Model of Fractions

Let's really lean into the schoolbook statement:

> **A fraction is a number. It is the answer to a division problem.**

As we saw in section 17, there are three different (but equivalent) ways to think of division. So, let's be specific.

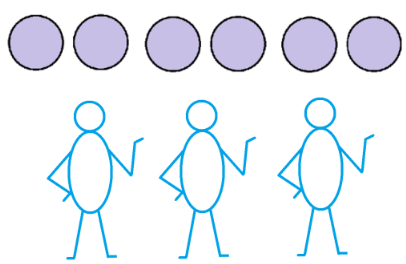
> **A fraction is a number. It is the answer to an equal-sharing problem.**

And what shall we share?

Well, to connect our thinking with standard school mathematics that is obsessed with pies, let's also share pies. We'll share pies equally among students.

Here goes.

> **Question:** Suppose I have 6 pies to share equally among 3 students.
> How many pies per student does that give?

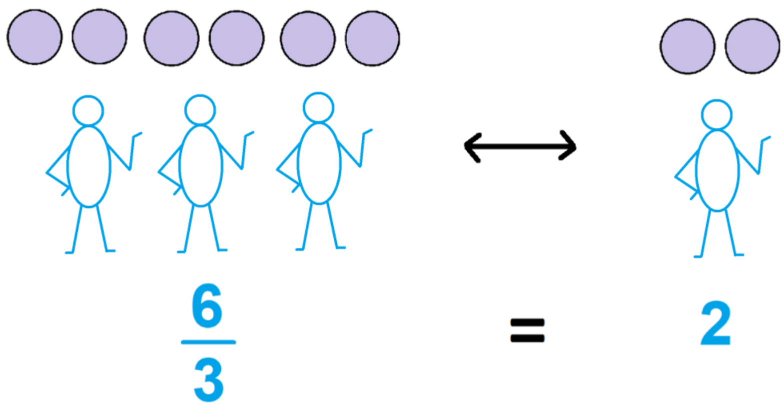


Of course, the answer is 2. Each student gets two pies.

And most people summarize the results of this sharing task by writing $6 \div 3 = 2$.

But let's be direct.

We are saying that a fraction <u>is</u> the answer to a division problem, so let's use fraction notation and write $\frac{6}{3} = 2$ for this sharing task.




In the same way:

Sharing 20 pies equally among 5 students yields $\frac{20}{5} = 4$ pies per student

and

Sharing 100 pies among 2 students yields $\frac{100}{2} = 50$ pies per student

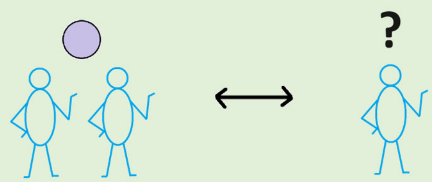
Here's how we now read fraction notation.

## INTUITION CHECK

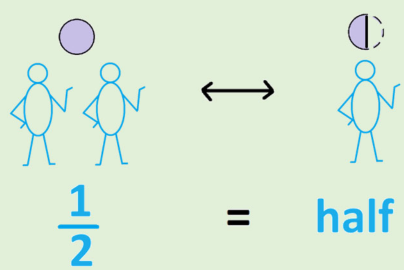
Most people do not think of $\frac{6}{3}$ and $\frac{20}{5}$ and $\frac{100}{5}$ as examples of fractions. The school curriculum starts its story with a numerator smaller than the denominator.

So, what does this pies-per-student model have to say about something we might normally identify as a fraction? For example, what is $\frac{1}{2}$ in this model?

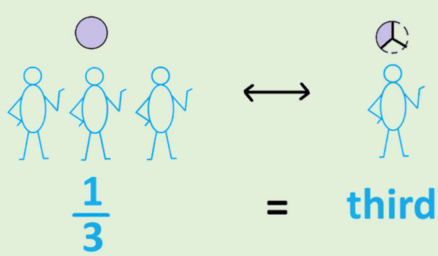
Well, $\frac{1}{2}$ represents the amount of pie each student gets if 1 pie is shared equally among 2 students.

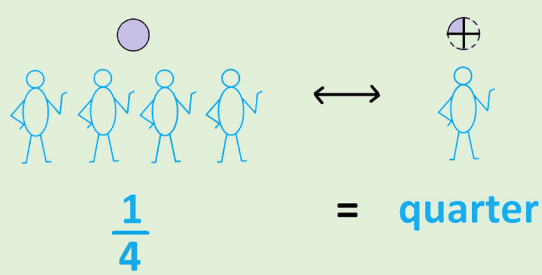



The answer matches what we call a *half* in our early understanding of fractions. This is good!



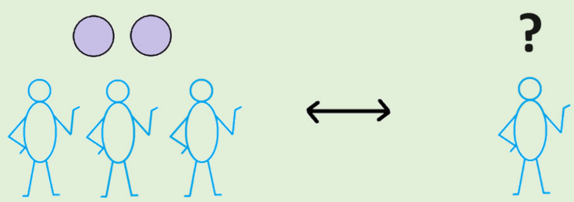

In the same way, $\frac{1}{3}$ matches what we earlier learned to call a *third*, and $\frac{1}{4}$ matches what we earlier learned to call a *quarter*, and so on.



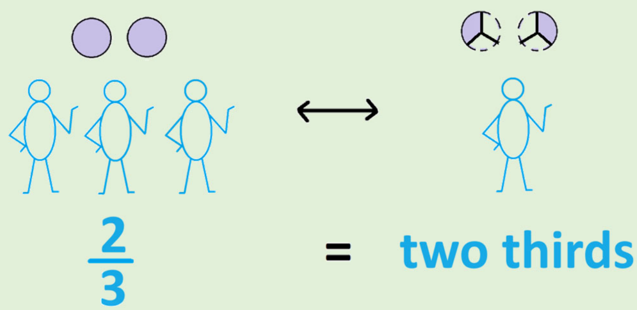
We also interpreted $\frac{2}{3}$ as "two copies of a third."

Does this pies-per-student model interpretation of $\frac{2}{3}$ align with this?

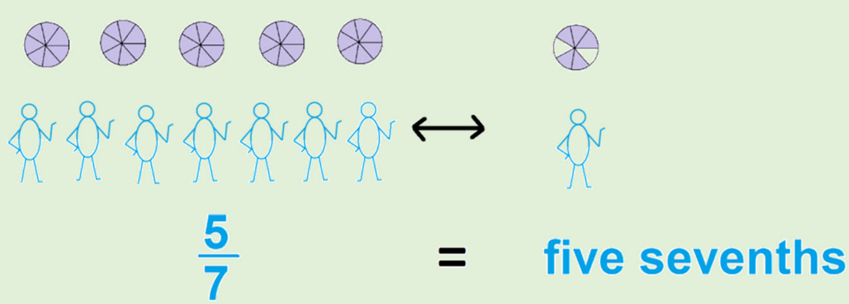



One way to physically share two pies equally among three students is to divide each pie into three equal pieces (thirds) and give each student a slice from each of the pies.

This does indeed give two thirds of a pie to each student.



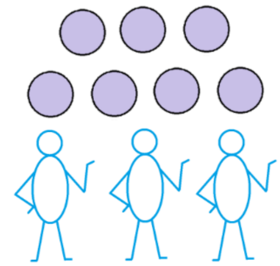

In the same way, $\frac{5}{7}$ would also be "five copies of a seventh:" just divide each of five pies into seven equal-sized pieces (sevenths) and then give each student a slice from each pie. As there are five pies, each student gets five sevenths.



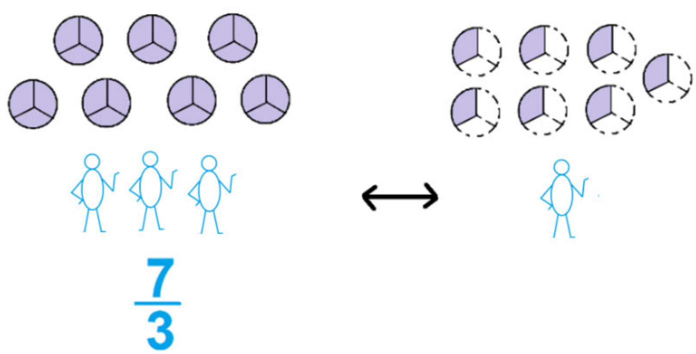

It's comforting that our pies-per-student model is matching our real-world intuition for fractions thus far.

**Question:** How would you personally share 7 pies equally among 3 students?

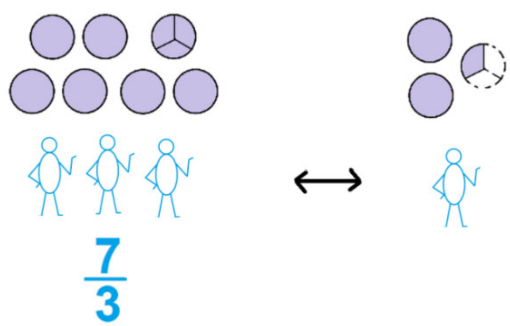
There are at least two ways to complete this task.



**Approach 1**: As per the intuition check, we could slice each pie into three equals parts and give each student a third from each pie. This means that $\frac{7}{3}$ corresponds to seven copes of a third.




**Approach 2:** Give each student two whole pies and then slice the one remaining pie into three equal pieces. This shows that $\frac{7}{3}$ also corresponds to $2 + \frac{1}{3}$.




This shows that answers to sharing problems can come in more than one guise.
This is both a good and an annoying feature of fractions!

## INTUITION CHECK

In the early grades we learned to circle a third of a group of kittens, or half of a set of stars, or identify a quarter of a bag of apples.

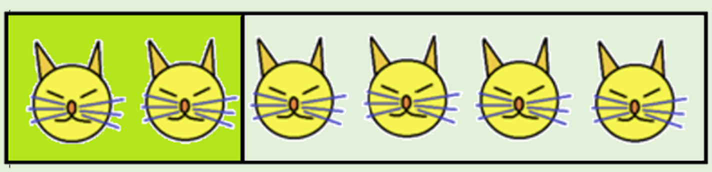
Does our pies-per-student model align with that work too?

Yes … if we regard everything as pie!

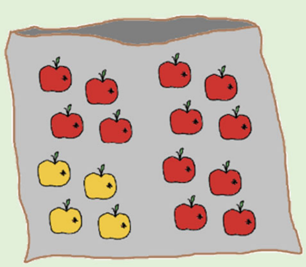
For example, here's a rectangular pie (perhaps cake) that happens to be decorated with six kitten faces.

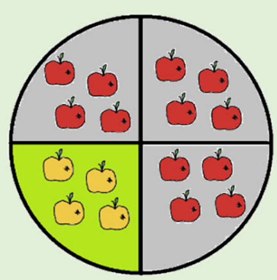
If 1 of these cakes is shared equally among 3 students, how many kitten faces will each student receive?

Well, each students receives a slice of the cake we call a third, $\frac{1}{3}$. And each third has 2 kitten faces.



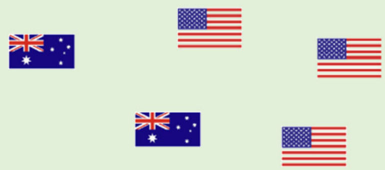
If we regard this bag of apples as a pie, then sharing the 1 pie equally among 4 students does show that the yellow apples match $\frac{1}{4}$ of the pie.





Let’s push this thinking a tad further.

Here’s a collection of flags.

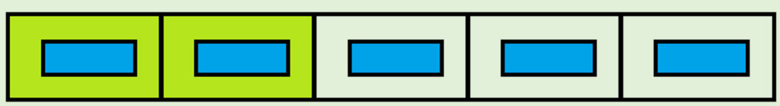


It is natural to say that “two fifths of the flags are Australian” and that “three fifths of them are American.”

Does that language match our pie-sharing thinking?

Yes!

Imagine pies decorated with five generic flags. Now share 2 such pies equally among 5 students. This picture shows what each student will receive. It’s a picture of $\frac{2}{5}$.



Philosophically, this is the same as picture of five flags with two of them identified as special, like the Australian flags in our original picture. (They are special in my mind!)

So yes, it is valid to say that “$\frac{2}{5}$ of the flags are Australian” in our original picture of flags.

And similarly, one can also legitimately say that “$\frac{3}{5}$ of the flags are American.”

Our pie-sharing model—admittedly with some mental contortion—can be seen as aligned with the early-school approach to fractions.

## ASIDE: Some Quirky Fun

Let's be a little bit naughty and get ahead of ourselves and put some non-whole numbers in unexpected places. It gives a hint of how powerful this sharing model is going to be.

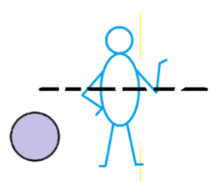
Here's the question.

Can you make sense of $\frac{1}{\frac{1}{2}}$ ?

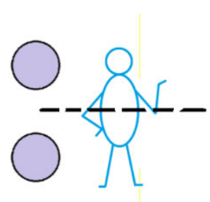
That is, if one pie is "shared equally among" half a student, how much pie does a whole student get?

It's a weird question, but maybe the thinking to go with it is something like the following.

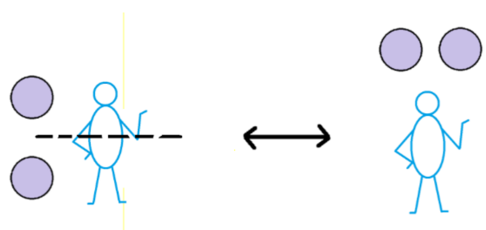
Half a student gets a pie. This picture shows the bottom half of a student getting a pie as dictated.



But there is a top half of the student too. And half a student gets a pie.



So, how much pie does a whole student get? Two pies!

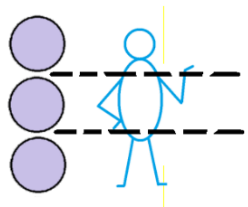


One pie for half a student results in two pies per whole student. Wild!

$$\frac{1}{\frac{1}{2}} = 2$$

What’s the value of $\frac{1}{\frac{1}{3}}$, the result of giving one whole pie to each third of a student?

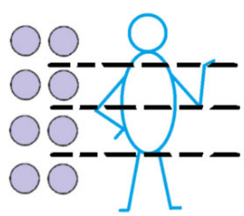
This picture shows it has value 3



Let’s keep going.

What’s the value of $\frac{2}{\frac{1}{4}}$ ?

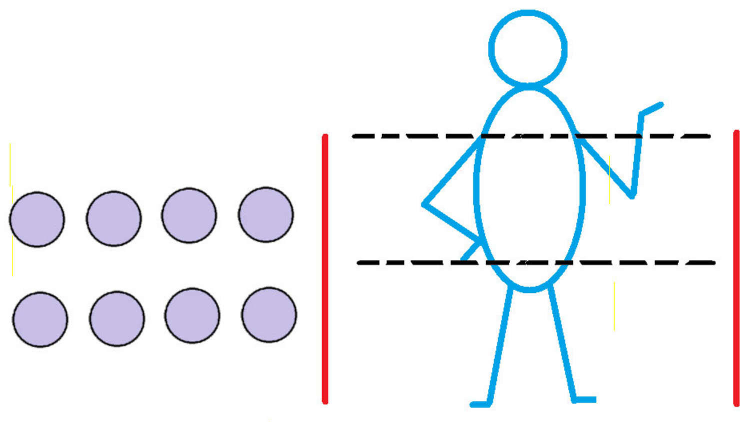
It’s 8!



Here’s a challenge question. Before turning the page can you figure out the answer to this question?

What’s the value of $\frac{8}{\frac{2}{3}}$ ?

Well, if 8 pies are assigned to $\frac{2}{3}$ of a student, then there are 4 pies for each third of the student. This makes for 12 pies in total for one whole student!



This quirkiness is not serious stuff. But it shows we can have some fun with this pies-per-student model if we want to push it.

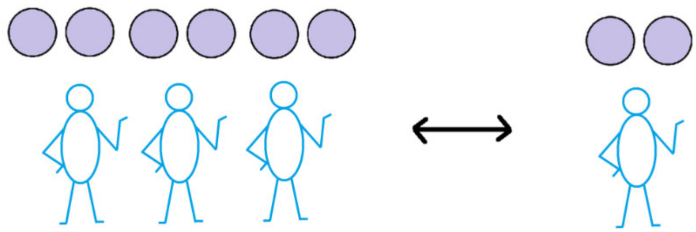
**Question:** Can you reason that $\frac{5}{1/7}$ must be 35?

**An Absurd Challenge not worth Considering:**
Two-and-a-half pies are to be shared equally among four-and-a-half students! How much pie does an individual (whole) student receive?

## Going Backwards

We've been sharing pies equally among students.

And we've been using the notation of fractions to describe that. After all, fractions are answers to sharing problems in this model.



$$\frac{6}{3} = 2$$

Number of pies

$$\frac{a}{b} = \text{amount of pie per student}$$

Number of students

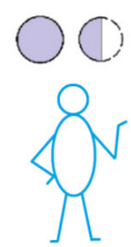
Here's a puzzle.

Some pies were shared equally among some students.

Each student received one full pie and one-half pie.

How many students were there initially? How many pies?

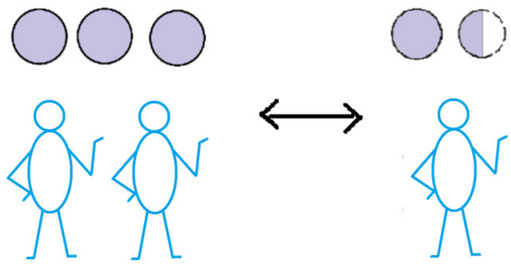
Give one possible answer.



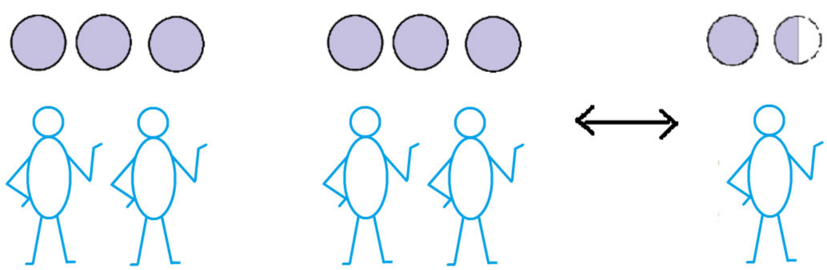
**Amount of pie per student**

We can reason this way.

*Splitting one pie into two equal parts makes us think that there were 2 students. And if each student is getting a whole pie and a half a pie, it must be because there were 3 pies.*



**Practice 38.1:** There are other possible scenarios that could lead to the same amount of pie per student. For example, sharing 6 pies equally among 4 students can give the same final result. (Do you see why?)



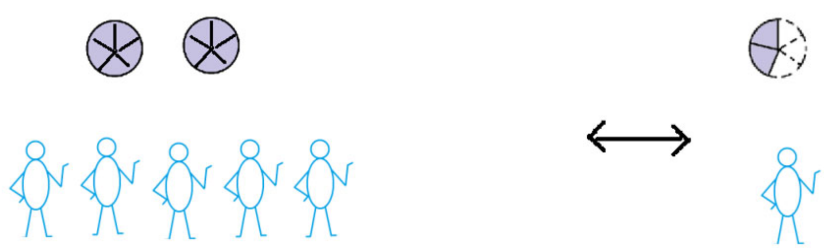
Give a third scenario that leads to the same result again.

Seeing a small portion of pie handed out to a student gives a hint as to how many students there could have been to begin with.

For example, here's a student receiving **two fifths** of a pie. This suggests some pie, or perhaps pies, were divided into five equal parts. Consequently, there could have been 5 students.

And if there were 5 students, how many pies must there have been to get this result?

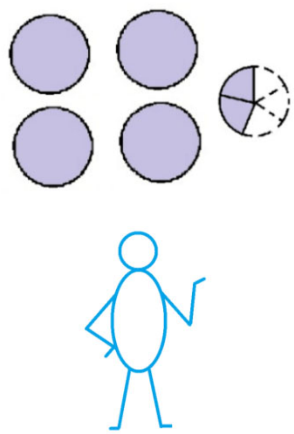
Each slice likely came from a pie. As the student received two slices, we deduce there must have been 2 pies.



Our mathematical notation presents "two fifths" as $\frac{2}{5}$, with a 2 and a 5. The notation itself is describing a scenario that yields the desired result: 2 pies being equally shared among 5 students!

A written fraction describes a working scenario!

**Practice 38.2** Give a number of pies and a number of students that yields 4 whole pies and $\frac{2}{5}$ of a pie for each student when shared equally.

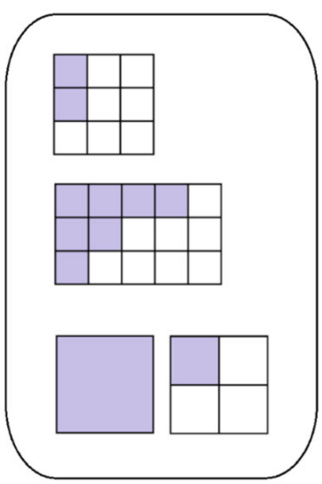


**Practice 38.3** Here's a matching puzzle.

On the left we have some pictures of the amount of square/rectangular pie each student received in a sharing game, and on the right, we have some counts of pies and students.

Match each result on the left with the set-up on the right that produces it.

(Really try to imagine a picture of pies being shared among students each time.)



**5 pies and 4 students**

**2 pies and 9 students**

**7 pies and 15 students**

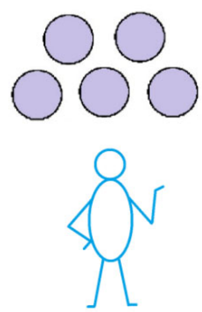
**Practice 38.4:** Is $\frac{10}{13}$ double the value of $\frac{5}{13}$?
If so, how would you justify this thinking about pies and students?

## Some Properties of Fractions

Let's now play with this pies-per-student model to see what it tells us mathematically about fractions.

To start, suppose we have $5$ pies to share among one (lucky) student. How many pies per student is that?

Clearly $5$.



It is tautological, but we have just learned that $\frac{5}{1}$ (five pies for one student) is $5$ (five pies per student).

In the same way, $20$ pies for one (even luckier) student makes for $\frac{20}{1} = 20$ pies per student.
And $4{,}096$ pies for one (burdened?) student makes for $\frac{4096}{1} = 4096$ pies per student.

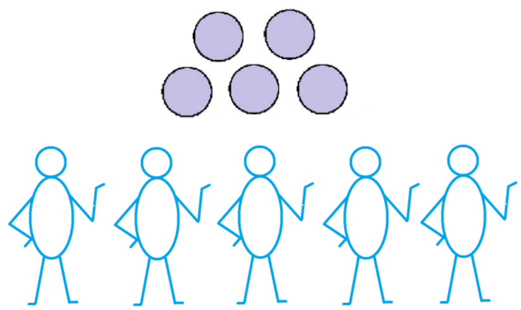
In general, we're seeing:

**FRACTION PROPERTY 1**

$\frac{a}{1} = a$ for each counting number $a$.

**Practice 38.5**: Does this property make sense if $a$ is zero? Does saying that $\frac{0}{1} = 0$ feel right?

Suppose now I share 5 pies equally among 5 students. How many pies per student is that?
It's clearly 1 pie per student.



In the same way, 20 pies shared equally among 20 students makes for 1 pie per student, as does sharing 503 pies among 503 students.

$$\frac{20}{20} = 1$$

$$\frac{503}{503} = 1$$

In general, we're seeing that

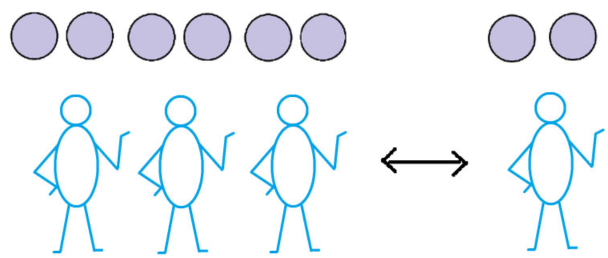
> **FRACTION PROPERTY 2**
> $\frac{a}{a} = 1$ for each counting number $a$ different from zero.

We saw in Section 17 that dividing (sharing) by zero is problematic. Property 2 is avoiding this issue by attending only to nonzero numbers. (After all, if you were tasked to share zero pies equally among no students, what would you do and how much pie would each student get? (What student?).)

Let’s now go back to an early example.

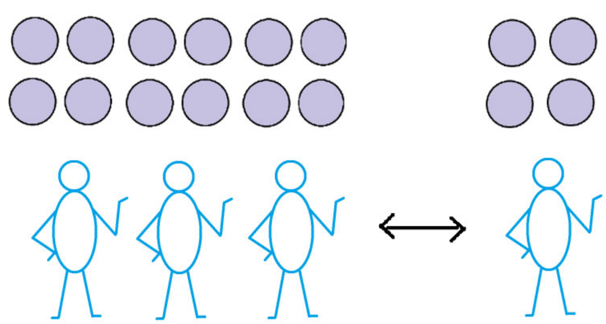
Sharing 6 pies equally among 3 students yields 2 pies per student.

$$\frac{6}{3} = 2$$



Suppose I am feeling generous and want to double the amount of pie each of the three students receives. How might I do that?

Well, I would have to double the count of pies I share. Make it 12 pies instead of 6.



Here’s how to write that mathematically.

$$2 \times \frac{6}{3} = \frac{12}{3}$$

**To double the amount of pie per student, double the number of pies.**

Similarly, to triple the amount of pie each student receives, just triple the number of pies you give out. Or to centuple the amount of pie each student receives, just centuple the number of pies you share.

$$3 \times \frac{6}{3} = \frac{18}{3}$$

$$100 \times \frac{6}{3} = \frac{600}{3}$$

In general, $\frac{a}{b}$ is the amount of pie an individual student receives when $a$ pies are shared equally among $b$ students. To double, triple, quadruple, quintuple, or even centuple the amount of pie each student receives, change the number of pies we give out to $2a$ or $3a$ or $4a$ or $5a$ or $100a$, respectively.

**To change the amount of pie each student receives by a factor $k$, change the number of pies by that factor.**

We have:

**FRACTION PROPERTY 3**

$k \times \frac{a}{b} = \frac{k \times a}{b}$ for each number $k$ and fraction $\frac{a}{b}$.

## INTUITION CHECK

We saw earlier that $\frac{2}{3}$ is usually interpreted as "two copies of $\frac{1}{3}$," that is, as $2 \times \frac{1}{3}$.

Does the mathematics say this too?

It does!

Fraction Property 3 tells us

$$2 \times \frac{1}{3} = \frac{2 \times 1}{3}$$

And the numerator $2 \times 1$ is just 2:

$$2 \times \frac{1}{3} = \frac{2}{3}$$

So, yes, $\frac{2}{3}$ and $2 \times \frac{1}{3}$ are the same.

In the same way,

$$\frac{4}{5} = 4 \times \frac{1}{5}$$

("four fifths" is four copies of a fifth) and

$$\frac{13}{9} = 13 \times \frac{1}{9}$$

("thirteen ninths" is thirteen copies of a ninth).

The mathematics is aligned with our early intuition.

Let's put mathematics to another test.

Rule 3 from our general Rules of Arithmetic says that multiplying a number by zero is sure to give zero. So, for example,

$$0 \times \frac{2}{5} = 0$$

But what does Property 3 say about $0 \times \frac{2}{5}$?

Well,

$$0 \times \frac{2}{5} = \frac{0 \times 2}{5}$$

The numerator here is $0 \times 2$ which is $0$. So, $0 \times \frac{2}{5}$ also equals $\frac{0}{5}$.

We now have two conclusions:

1. $0 \times \frac{2}{5}$ equals $0$ (from Rule 6)

2. $0 \times \frac{2}{5}$ equals $\frac{0}{5}$ (from Property 3)

We conclude that $0$ and $\frac{0}{5}$ must be the same value.

$$\frac{0}{5} = 0$$

Sharing no pies equally among 5 students yields zero pie per student. And that too matches our intuition.

Everything is hanging together!

We are learning that it is possible for a fraction to have a numerator of zero. Such a fraction is sure to have value 0.

$\frac{0}{b} = 0$ for each non-zero counting number $b$ .

But what about a fraction with denominator zero?

Is $\frac{5}{0}$, for instance, a meaningful quantity?
(It doesn't make sense in our pie-sharing scenario.)

Of course, we addressed this issue in Section 17 where we showed that division by zero is undefined in mathematics (and sharing, as we are doing in this section, is an interpretation of division).

But let's have another Intuition Check and discuss denominators of zero.

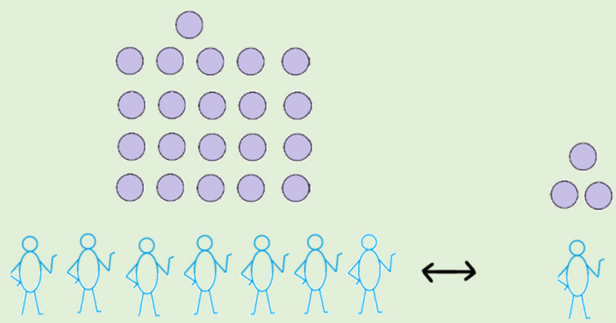
First, this warm-up problem.

**Practice 38.6** In a sharing scenario, some pies were shared equally among 7 students. If each student received 3 pies, how many pies were there in total to begin with?
(That is, if $\frac{a}{7} = 3$, what is $a$?)

## INTUITION CHECK

We know that $\frac{21}{7} = 3$.

If 21pies are shared equally among 7 students, each student will receive 3 pies.



We would then have 7 happy students each with 3 pies, and that accounts for $7 \times 3 = 21$ pies, all of them.

That is: **We can see that $\frac{21}{7} = 3$ is correct because $7 \times 3 = 21$.**

We can also deduce that $\frac{100}{5} = 20$ is correct because $5 \times 20 = 100$. (Yep, if five students each have 20 pies, that accounts for all 100 pies.)

We also deduce that $\frac{42}{6} = 8$ is <u>not</u> correct because $6 \times 8$ is not 42. (If six students each have 8 pies, that makes for 48 pies, not 42 of them.)

We have:

**The statement $\frac{a}{b} = N$ is correct if $b \times N = a$.**

**Practice 38.7:** Use multiplication to check the validity of each statement shown. Which of the statements are incorrect?

$\frac{103}{103} = 1$ $\frac{1000}{125} = 10$ $\frac{999}{1} = 998$ $\frac{5}{0} = 3$

The fourth example, $\frac{5}{0}$, is particularly interesting.

$\frac{5}{0}$ can't be $3$ because $0 \times 3$ is not $5$.

Also

$\frac{5}{0}$ can't be $7$ because $0 \times 7$ is not $5$.

$\frac{5}{0}$ can't be $12{,}002$ because $0 \times 12002$ is not $5$.

There is no value to assign to $\frac{5}{0}$, because zero times any value you care to choose will be $0$, not $5$.

If $a$ is a counting number different from zero, then there is no possible value to assign to $\frac{a}{0}$.

As we saw in Section 17, the expression $\frac{0}{0}$ suffers from a different mathematical challenge.

$\frac{0}{0} = 3$ is seemingly correct because $0 \times 3$ does equal $0$.

$\frac{0}{0} = 7$ is seemingly correct because $0 \times 7$ does equal $0$.

$\frac{0}{0} = 12002$ is seemingly correct because $0 \times 12002$ does equal $0$.

Every value passes the multiplication check $\frac{0}{0}$.

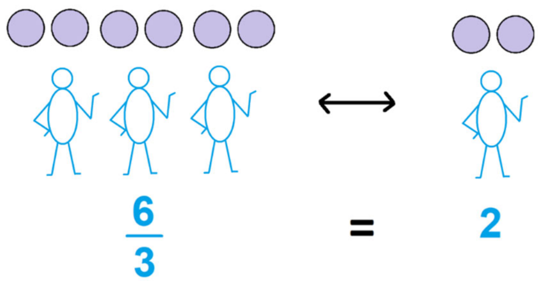
The expression $\frac{0}{0}$ is "indeterminant:" there is no consistent value to assign to it.

In all cases, fractional expressions with a denominator of zero are mathematically inconsistent. (And they are intuitively fraught too: how do you share pies among zero students?)

We are learning that our mathematical theory of fractions must be about quantities of the form $\frac{a}{b}$ with $a$ and $b$ numbers and with $b$ not zero.

## 39. One More Fraction Property

Fractions are answers to division problems. In particular, they are the answers to problems about sharing pies equally among students in this chapter so far.





This model naturally leads to three observed properties of fractions.

**FRACTION PROPERTY 1**

$\frac{a}{1} = a$ for each counting number $a$.

**FRACTION PROPERTY 2**

$\frac{a}{a} = 1$ for each counting number $a$ different from zero.

**FRACTION PROPERTY 3**

$k \times \frac{a}{b} = \frac{k \times a}{b}$ for each number $k$ and fraction $\frac{a}{b}$.

For example,

- Sharing 7 pies among 1 student results in $\frac{7}{1} = 7$ pies for that student
- Sharing 23 pies among 23 students results in $\frac{23}{23} = 1$ pie per student
- To double the amount of pie each individual student receives in a sharing problem, just double the number of pies you share out.

And we've learned to avoid denominators of zero.

Fraction Property 3 allowed us to see “two thirds” as “two copies of a third”

$$\frac{2}{3} = 2 \times \frac{1}{3}$$

Similarly,

$$\frac{4}{5} = 4 \times \frac{1}{5}$$

and

$$\frac{13}{9} = 13 \times \frac{1}{9}$$

But let’s rephrase this observation in a slightly different way. Let’s say that we have “pulled apart” each fraction into an integer multiplied by a more basic fraction.

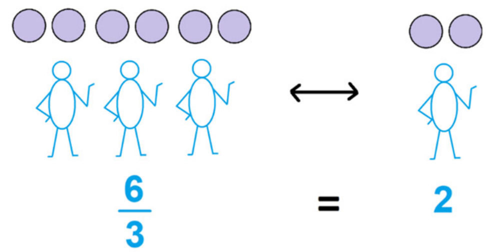
**OBSERVATION 5**

We can always “pull fractions apart.”

$$\frac{a}{b} = a \times \frac{1}{b}$$

There is one more property of fractions to note.

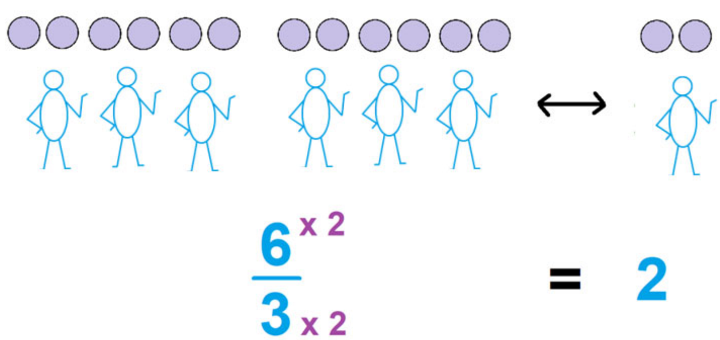
Consider again 6 pies being shared equally among 3 students.




What happens if we double the number of pies **and** double the number of students?

Nothing!

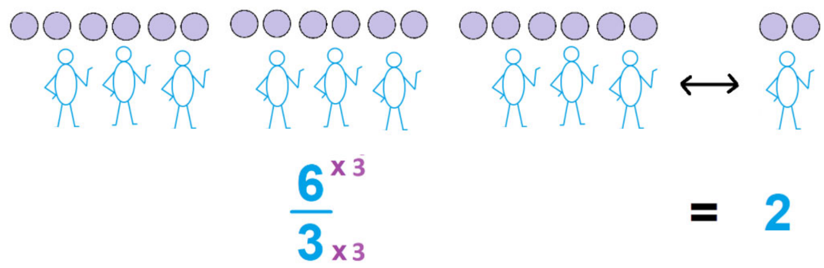
The amount of pie per student doesn't change.




Sharing 12 pies among 6 students, $\frac{12}{6}$, gives the same result as sharing 6 pies among 3 students, $\frac{6}{3}$. We're just doing the same work of giving 2 pies per student, twice over.

What happens if instead we triple the number of pies and triple the number of students?
Do we still get 2 pies per student?

You bet!




We are seeing that $\frac{18}{9}$ and $\frac{12}{6}$ and $\frac{6}{3}$ all give the same result.

Even if we centuple the number of pies and centuple the number of students, $\frac{600}{300}$, we're still handing out 2 pies per student. Nothing changes (except the amount of work we do to get to the same final result!).

Changing the number of pies we have by some factor and changing the number of students we are working with by the same factor changes nothing about the amount of pie each student receives.

$$\frac{a}{b} = \frac{2a}{2b} = \frac{3a}{3b} = \frac{4a}{4b} = \cdots = \frac{100a}{100b} = \cdots$$

This gives us our fourth property of fractions, which I'll add to the list with the other three.

**FRACTION PROPERTY 1**

$\frac{a}{1} = a$ for each counting number $a$.

**FRACTION PROPERTY 2**

$\frac{a}{a} = 1$ for each counting number $a$ different from zero.

**FRACTION PROPERTY 3**

$k \times \frac{a}{b} = \frac{k \times a}{b}$ for each number $k$ and fraction $\frac{a}{b}$.

**FRACTION PROPERTY 4**

$\frac{k \times a}{k \times b} = \frac{a}{b}$ for each number $k$ and fraction $\frac{a}{b}$.

These rules look scary—visually. But the idea is to always "step back" to see through what each is saying.

For example, taking the number $k$ to be $4$ in the fourth rule, we read:

> *Quadrupling the number of pies and the number of students in a sharing problem does not change the final result: each student gets the same amount as pie if the numbers weren't quadrupled.*

**Question:** Which of the following fractions are equivalent to $\frac{3}{7}$?

a) $\frac{6}{14}$ b) $\frac{30}{70}$ c) $\frac{12}{28}$ d) $\frac{33}{77}$

The answer is that they all are. For example, $\frac{12}{28}$ is $\frac{4\times3}{4\times7}$, and so, by Property 4, has the same value as $\frac{3}{7}$.

We have in this question a set of **equivalent fractions**.

(To be clear, two fractional expressions are said to be **equivalent** if they represent the same value.)

Property 4 shows us how to recognize equivalent fractions.

For example,

$\frac{3}{5}$ is equivalent to $\frac{8\times3}{8\times5} = \frac{24}{40}$

$\frac{20}{32}$ is equivalent to $\frac{5}{8}$ (from noticing that $\frac{20}{32} = \frac{4\times5}{4\times8}$)

This second example shows that sharing $20$ pies equally among $32$ students is gives the same amount of pie per student as sharing just $5$ pies among $8$ students. (That second scenario seems more manageable!)

People say that we have **canceled a common factor** from within $\frac{20}{32}$ and, as such, we have just **simplified** the fraction.

$$\frac{20}{32} = \frac{\cancel{4} \text{ x } 5}{\cancel{4} \text{ x } 8} = \frac{5}{8}$$

Students are taught to, and are often required to, always simplify fractions this way. It becomes automatic.

Some people might say instead that we **reduced the fraction** $\frac{20}{32}$.

This term is a little misleading as the fraction $\frac{5}{8}$ is no smaller or bigger than $\frac{20}{32}$. What we have "reduced" is the number of pies and the number of students we are working with (that is, we have reduced the size of the numerator and the denominator). We haven't at all reduced the value of the outcome.

**Practice 39.1:** Select all the fractions that are equivalent to $\frac{140}{490}$.

a) $\frac{14}{49}$ *b)* $\frac{2}{7}$ *c)* $\frac{6}{21}$ *d)* $\frac{1400}{4900}$ *e)* $\frac{15}{35}$

Often students are expected to "simplify as far as possible."

**Example:** Reduce $\frac{280}{350}$ as far as possible.

**Answer:** We can certainly make this fraction look more manageable by noticing that there is a common factor of 10 in the numerator and denominator.

$$\frac{280}{350} = \frac{10 \times 28}{10 \times 35} = \frac{28}{35}$$

We can go further by noticing that 28 and 35 are both multiples of 7.

$$\frac{28}{35} = \frac{7 \times 4}{7 \times 5} = \frac{4}{5}$$

We're seeing that sharing 280 pies among 350 students gives the same result as sharing just 4 pies among 5 students. This is much easier to conceptualize.

$$\frac{280}{350} = \frac{4}{5}$$

As 4 and 5 share no common factors, this is as far as we can go with this example (while staying with whole numbers).

**Question:** *Jennie has read this entire book and has leaned about "mixed numbers" and about being quirky with math. She says that* $\frac{4}{5}$ *does "simplify further" if you are willing to move away from whole numbers. She writes:*

$$\frac{4}{5} = \frac{2 \times 2}{2 \times 2\frac{1}{2}} = \frac{2}{2\frac{1}{2}}$$

*If you feel you understand what she is writing, what do you think? Is she right?*
*Does sharing* $4$ *pies among* $5$ *students yield the same result as sharing* $2$ *pies among* $2\frac{1}{2}$ *students? (And is her answer "simpler"?)*

## Some Logical Consequence of our Fraction Properties

Here's a question:

What is the value of $7 \times \frac{3}{7}$?

I bring this up as I personally have a knee-jerk response to this expression:

*Just cancel the 7s to be left with 3!*

$$\cancel{7} \times \frac{3}{\cancel{7}} = 3$$

This is my school training kicking in. But is this maneuver valid?
Does it follow from one or some of our properties of fractions?

| **FRACTION PROPERTY 1** |
|---|
| $\frac{a}{1} = a$ for each counting number $a$. |

| **FRACTION PROPERTY 2** |
|---|
| $\frac{a}{a} = 1$ for each counting number $a$ different from zero. |

| **FRACTION PROPERTY 3** |
|---|
| $k \times \frac{a}{b} = \frac{k \times a}{b}$ for each number $k$ and fraction $\frac{a}{b}$. |

| **FRACTION PROPERTY 4** |
|---|
| $\frac{k \times a}{k \times b} = \frac{a}{b}$ for each number $k$ and fraction $\frac{a}{b}$. |

and

| **OBSERVATION 5** |
|---|
| $\frac{a}{b} = a \times \frac{1}{b}$ for each fraction $\frac{a}{b}$. |

**Property 3** looks relevant here. It tells us

$$7 \times \frac{3}{7} = \frac{7 \times 3}{7}$$

But what can we do next? What makes us want to "cancel the 7s"?

That would be **Property 4** if we rewrite the denominator as $7 \times 1$.

$$7 \times \frac{3}{7} = \frac{7 \times 3}{7} = \frac{7 \times 3}{7 \times 1} = \frac{3}{1}$$

And **Property 1** tells us that $\frac{3}{1}$ is just $3$.

$$7 \times \frac{3}{7} = \frac{7 \times 3}{7} = \frac{7 \times 3}{7 \times 1} = \frac{3}{1} = 3$$

So, yes! We can just cancel the 7s from the get-go as we were trained in school to do.

$$\cancel{7} \times \frac{3}{\cancel{7}} = 3$$

Since this is such a common practice in working with fractions, let's take note of it and add it to our list of properties. We can then use this property whenever we want and not have to go through this mathematical reasoning every single time.

**LOGICAL CONSEQUENCE 6**

$b \times \frac{a}{b} = a$ for all counting numbers $a$ and $b$ with $b$ not zero.

$$\cancel{b} \times \frac{a}{\cancel{b}} = a$$

## INTUITION CHECK

Our real-world experience tells us that two halves make a whole, three thirds make a whole, four quarters make a whole, and so on.

Logical Consequence 6 is telling us this as well!

$2 \times \frac{1}{2} = 1 \qquad 3 \times \frac{1}{3} = 1 \qquad 4 \times \frac{1}{4} = 1$ etc.

We're now developing a body of fraction properties we can use any time.
They are all aligned with what we were taught in school and with the intuition of fractions we developed from school and everyday life.

**FRACTION PROPERTY 1**

$\frac{a}{1} = a$ for each counting number $a$.

**FRACTION PROPERTY 2**

$\frac{a}{a} = 1$ for each counting number $a$ different from zero.

**FRACTION PROPERTY 3**

$k \times \frac{a}{b} = \frac{k \times a}{b}$ for each number $k$ and fraction $\frac{a}{b}$.

**FRACTION PROPERTY 4**

$\frac{k \times a}{k \times b} = \frac{a}{b}$ for each number $k$ and fraction $\frac{a}{b}$.

**OBSERVATION 5**

$\frac{a}{b} = a \times \frac{1}{b}$ for each fraction $\frac{a}{b}$.

**LOGICAL CONSEQUENCE 6**

$b \times \frac{a}{b} = a$ for all counting numbers $a$ and $b$ with $b$ not zero.

## MUSINGS

**Musing 39.2** Consider the expression $1 \times \frac{20}{1}$.

a) Andre says it has value 20 because you can apply Logical Consequence 6 directly to it.
Do you agree?
b) Andrea has it has value 20 because of Property 1 and the general rule of arithmetic that 1 times a number equals the number itself. Is her reasoning also valid?

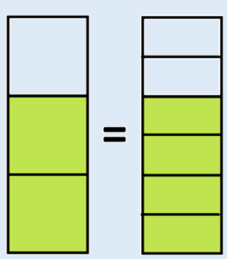
**Musing 39.3** Carefully explain why $\frac{40}{4}$ has the value $10$ using Properties 4 and 1.

**Musing 39.4** For the following questions use the standard Rules of Arithmetic 1-9 and the Properties of Fractions we developed.

a) Explain why $\frac{39}{17} \times 17$ has value $39$.
b) Explain why $\frac{39}{17} \times 34$ has value $78$.
c) Explain why $3 \times \frac{7}{2} \times 12 \times \frac{3}{14} \times \frac{1}{3}$ has value $9$.

**Musing 39.5** Our Fraction Properties can be made logically tighter. We don't actually need to explicitly list Fraction Property 2. It follows as a logical consequence of Properties 4 and 1.
Do you see how?

**Musing 39.6 INTUITION CHECK**
Students are typically asked to construct pictures like this to justify why $\frac{2}{3}$ and $\frac{4}{6}$, for instance, are equivalent fractions. We see that dividing each piece of the left picture in half produces the right picture.

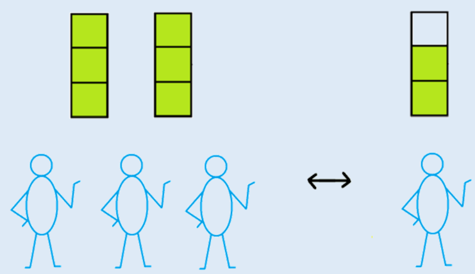



$\frac{2}{3}$ $\frac{4}{6}$

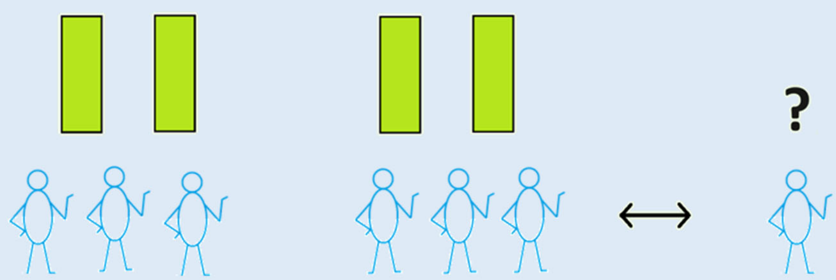
How does this fit with our pies-per-student thinking?

The trick is to draw rectangular pies.

The student picture of $\frac{2}{3}$ is the result of sharing 2 pies equally among 3 students. (Divide each pie into thirds and give a third from each pie to each student.)

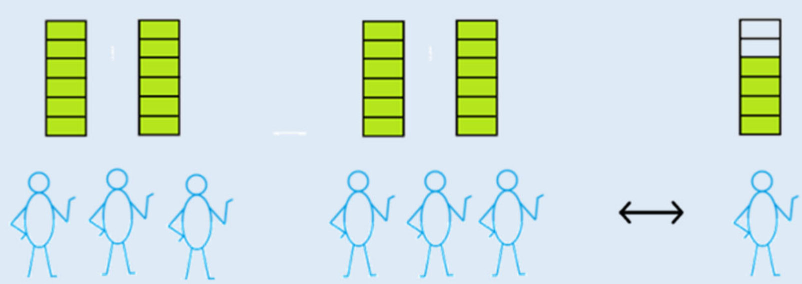


Now we draw a picture of $\frac{4}{6}$. This is the result of sharing 4 pies equally among 6 students.

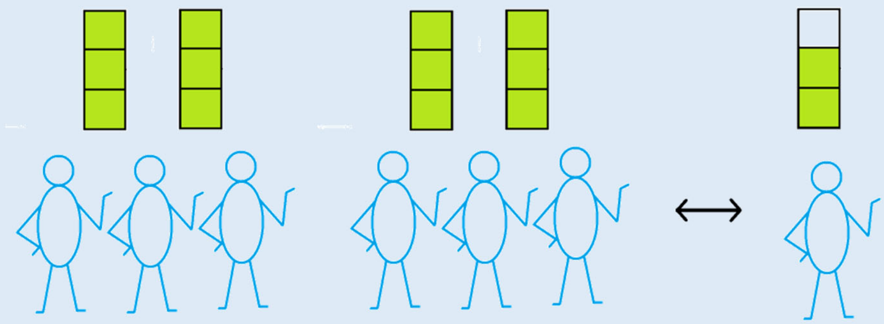



There are at least two ways to complete this task:

**Approach 1:** Divide each pie into sixths and give each student a sixth from each of these four pies. Each student gets four sixths of a pie.



**Approach 2:** Recognize that our task is just a matter of sharing 2 pies equally among 3 students—twice! In which case, each student receives a third from each of two pies, that is, two thirds of a pie.

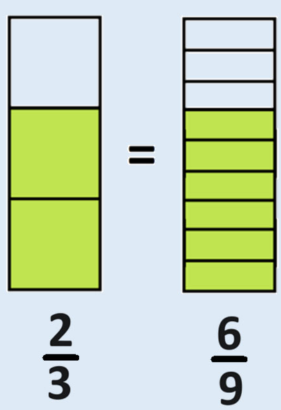


We are seeing $\frac{2}{3}$ and $\frac{4}{6}$ each as the answer to the same sharing problem. They represent the same amount of pie per student and so are equivalent fractions.

Our pies-per-student model for fractions really does yield the same pictures for the "equivalent fractions" pictures students are asked to draw in school.

Your turn:

Draw pictures to show two ways of sharing 6 pies equally among 9 students.
Deduce that $\frac{2}{3}$ and $\frac{6}{9}$ are equivalent fractions.



## 40. Opening Up Our Fraction Rules a Wee Bit

We have six observed properties of fractions from our pies-per-student model.

**FRACTION PROPERTY 1**
$\frac{a}{1} = a$ for each counting number $a$.

**FRACTION PROPERTY 2**
$\frac{a}{a} = 1$ for each counting number $a$ different from zero.

**FRACTION PROPERTY 3**
$k \times \frac{a}{b} = \frac{k \times a}{b}$ for each number $k$ and fraction $\frac{a}{b}$.

**FRACTION PROPERTY 4**
$\frac{k \times a}{k \times b} = \frac{a}{b}$ for each number $k$ and fraction $\frac{a}{b}$.

**OBSERVATION 5**
$\frac{a}{b} = a \times \frac{1}{b}$ for each fraction $\frac{a}{b}$.

**LOGICAL CONSEQUENCE 6**
$b \times \frac{a}{b} = a$ for all counting numbers $a$ and $b$ with $b$ not zero.

We started by stating that **a fraction is a number**, the answer to a division problem, and that we were thinking of division as sharing.

But Consequence 6 is saying something lovely. It brings us right back to division, but now thinking of division as multiplication in reverse.

Let me explain what I mean.

Consider $20 \div 5$.

If we are thinking of division as sharing (our pies-per-student), then $20 \div 5$ is the result of sharing 20 pies equally among 5 students. We called that $\frac{20}{5}$.

If we are thinking of division as multiplication in reverse, then $20 \div 5$ is the number that fills in the blank of this multiplication statement.

$$5 \times \blacksquare = 20$$

And look what Consequence 6 says. It tells us that $5 \times \frac{20}{5} = 20$. The fraction $\frac{20}{5}$ truly is the answer to $20 \div 5$.

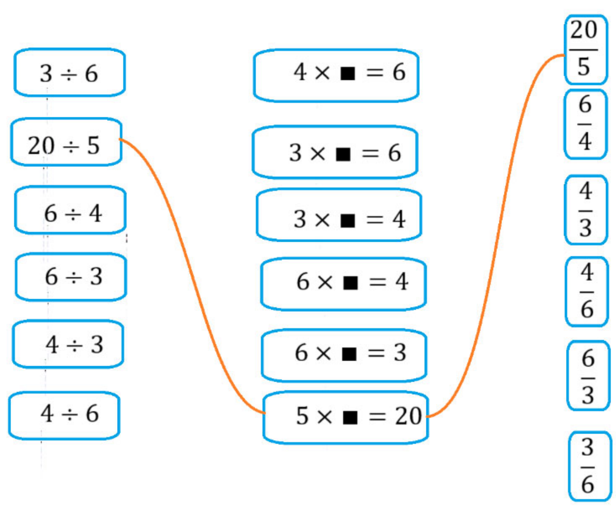
In the same way, $\frac{2}{3}$ truly is the answer to $2 \div 3$, because, by Consequence 6, It is the number that fills in this blank to $3 \times \blacksquare = 2$.

**REALIZATION 7**

The fraction $\frac{a}{b}$ truly is the answer to $a \div b$.

(It is the number that fills in the blank to $b \times \blacksquare = a$.)

**Practice 40.1:** Complete this three-way match table. One triple match is already shown.



Back in Section 38 we were naughty and started putting fractions within fractions. We asked, for example, for the value of $\frac{1}{\frac{1}{2}}$, which we interpreted as the result of sharing one pie for each half of a student.

But now we can make valid mathematical sense of $\frac{1}{\frac{1}{2}}$. Copying what is written in parentheses in the box at this top of the page:

> $\frac{1}{\frac{1}{2}}$ is the answer to the division problem $1 \div \frac{1}{2}$.
>
> It the number that fills in the blank to $\frac{1}{2} \times \blacksquare = 1$.

Can we think of a number that fills the blank?

Well, we do know that

$$2 \times \frac{1}{2} = 1$$

This is just Consequence 6.

But the general rules of arithmetic tell us that we can switch the order of the product of two numbers without any harm. So, we can rewrite this statement as

$$\frac{1}{2} \times 2 = 1$$

This is now saying that the number 2 completes the statement $\frac{1}{2} \times \blacksquare = 1$.

So, $\frac{1}{\frac{1}{2}}$, the number that fills in the blank, is the number 2.

$$\frac{1}{\frac{1}{2}} = 2$$

Wow!

And this matches the answer we deduced when we were being quirky: sharing one pie for each half of a student results in two pies for a whole student.

**Practice 40.3** Explain why the number 3 fills in the blank to $\frac{1}{3} \times \blacksquare = 1$.

What then is the value of $\frac{1}{\frac{1}{3}}$, the answer to $1 \div \frac{1}{3}$?

[**Aside**: If you switch your brain to thinking of division a "division by groups," how many thirds can you find in a whole? Are intuition and math again aligned?]

It looks like we really are allowed to be quirky with fractions and allow the numerators and denominators of fractions to be fractions too! The mathematics we have seems to be robust enough to handle this.

So, let's be bold and open up our five fraction properties to free our numerators and denominators from having to be counting numbers.

Here are our observations again, with each mention of "counting number" replaced with just "number."

**FRACTION PROPERTY 1**
$\frac{a}{1} = a$ for each number $a$.

**FRACTION PROPERTY 2**
$\frac{a}{a} = 1$ for each number $a$ different from zero.

**FRACTION PROPERTY 3**
$k \times \frac{a}{b} = \frac{k \times a}{b}$ for each number $k$ and fraction $\frac{a}{b}$.

**FRACTION PROPERTY 4**
$\frac{k \times a}{k \times b} = \frac{a}{b}$ for each number $k$ and fraction $\frac{a}{b}$.

and

**OBSERVATION 5**
$\frac{a}{b} = a \times \frac{1}{b}$ for each fraction $\frac{a}{b}$.

**LOGICAL CONSEQUENCE 6**
$b \times \frac{a}{b} = a$ for all numbers $a$ and $b$ with $b$ not zero.

We also have our nine general Rules of Arithmetic to play with too.

And we need to keep in mind what Consequence 6 is really saying for us:

**REALIZATION 7**
The fraction $\frac{a}{b}$ truly is the answer to $a \div b$.
(It is the number that fills in the blank to $b \times \blacksquare = a$.)

The power of Realization 7 is this:

**If you happen to know the answer to a division problem, then you know the value of a fraction.**

For example,

$$\frac{12}{3} = 12 \div 3 = 4$$

$$\frac{200}{10} = 200 \div 10 = 20$$

$$\frac{55}{5} = 55 \div 5 = 11$$

I don't have the value of $33 \div 7$ off the top of my head. (Well, it's about four-and-a-half.) Consequently, I can't say too much about the value of $\frac{33}{7}$ (except that it is about four-and-a-half!)

As a final comment, we can point out that Fraction Rule 2 is now, for sure, obsolete:

Dividing a number by itself gives the answer 1.
Knowing this gives:

$$\frac{a}{a} = a \div a = 1$$

## 41. All Fraction Arithmetic in One Hit

Let's just get into it. Let's see how all the fraction arithmetic you were taught to do in school is actually mathematically correct and logically follows from the Fraction Properties we've identified.

Actually, we only need to focus on the following properties and consequences.

**Fraction Property 1**:

**Every number can be written as a fraction with denominator 1.**

$$a = \frac{a}{1}$$

**Fraction Property 4**:

**Multiplying the top and bottom of a fraction by the same number does not change the value of the fraction.**

$$\frac{a}{b} = \frac{k \times a}{k \times b}$$

**Observation 5:**

**We can always "pull apart" a fraction.**

$$\frac{a}{b} = a \times \frac{1}{b}$$

**Consequence 6**:

**Multiplying a fraction by its denominator "cancels that denominator."**

$$\cancel{b} \times \frac{a}{\cancel{b}} = a$$

and **Realization 7:**

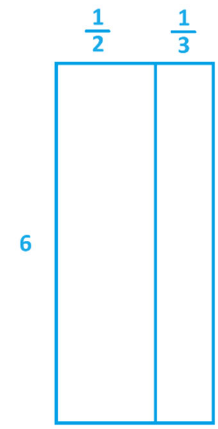
**A fraction is the answer to a division problem. (So, if you happen to know the answer to a division problem, then you know the value of the matching fraction.)**

$$\frac{a}{b} = a \div b$$

(We saw that Fraction Property 2 is obsolete. Fraction Property 3 led us to Observations 5, 6, and 7, but we won't ever need Property 4 itself directly.)

To develop a feel for how all this is going to work, consider these warm-up examples.

**Example:** Show that $6 \times \left(\frac{1}{2} + \frac{1}{3}\right)$ equals 5.




**Answer:** From the general arithmetic principle of "chopping up rectangles," $6 \times \left(\frac{1}{2} + \frac{1}{3}\right)$ is

$$6 \times \frac{1}{2} \; + \; 6 \times \frac{1}{3}$$

We see here two fractions that are pulled apart. We really have:

$$\frac{6}{2} + \frac{6}{3}$$

Each of these fractions matches a division problem whose answer we know. This means that we have

$$3 + 2$$

which equals 5, just as the question suggested.

**Practice 41.1** Show that $5 \times (\frac{1}{5} + \frac{1}{5})$ equals 2.

**Practice 41.2** Determine the value of $100 \times (\frac{1}{20} + \frac{1}{25} + \frac{2}{5})$.

**Example:** Show that $60 \times \frac{1}{3} \times \frac{1}{4}$ equals 5.

**Answer:** It helps to notice that $60 = 3 \times 4 \times 5$, so the product we are asked to look at is really

$$3 \times 4 \times 5 \times \frac{1}{3} \times \frac{1}{4}$$

The general Rules of Arithmetic tell us that it does not matter in which order one computes a string of products. Since we can see $3 \times \frac{1}{3} = 1$ and $4 \times \frac{1}{4} = 1$ within this product (Consequence 6), we can see that our product equals

$$1 \times 1 \times 5$$

which is just 5, as the question suggested.

**Practice 41.3** Determine the value of $100 \times \frac{1}{2} \times \frac{1}{25}$.

**Example:** Find the value of $70 \times \frac{4}{5} \times \frac{2}{7} \times \frac{1}{8}$.

**Answer:** Let's tease each term this product apart. We have

$$70 = 2 \times 5 \times 7$$

$$\frac{4}{5} = 4 \times \frac{1}{5}$$

$$\frac{2}{7} = 2 \times \frac{1}{7}$$

So, the question is asking us to evaluate

$$2 \times 5 \times 7 \times 4 \times \frac{1}{5} \times 2 \times \frac{1}{7} \times \frac{1}{8}$$

The general rules of arithmetic tell us we can compute a string of products in any order we like. Noticing that $5 \times \frac{1}{5} = 1$ and $7 \times \frac{1}{7} = 1$ we see that the product is

$$2 \times 4 \times 1 \times 2 \times 1 \times \frac{1}{8}$$

But, we can read this as

$$8 \times 2 \times \frac{1}{8}$$

and see it as $2 \times 1$, which equals 2.

**Practice 41.4** Determine the value of $360 \times \frac{5}{6} \times \frac{7}{12} \times \frac{2}{25}$.

**Practice 41.5** What's $8 \times \frac{3}{2}$?

Now to the fraction arithmetic we all taught in school.

The general ideas we'll follow are these:

- **It never hurts to "put a number over 1."**
- **Multiplying the top and bottom of a fraction by the same number won't change the value of the fraction.**
- **We can be clever about what to multiply the top and bottom by.**

$$a = \frac{a}{1} \qquad \frac{a}{b} = \frac{k \times a}{k \times b} \qquad \cancel{b} \times \frac{a}{\cancel{b}} = a$$

And let's also remember that if you know the answer to a particular division problem, then you know the value of a fraction. (For example, knowing that $6 \div 3 = 2$ tells me that $\frac{6}{3} = 2$.)

$$\frac{a}{b} = a \div b$$

Also remember we can "pull a fraction apart" if it helps make it easier to see what is going on. (For example, $8 \times \frac{3}{2} = 2 \times 4 \times 3 \times \frac{1}{2} = 4 \times 3 \times 1 = 12$.)

$$\frac{a}{b} = a \times \frac{1}{b}$$

## THIS IS A STAR PAGE!

**It summarizes all you need to actually know to make sense of all the mathematics of fractions.**

Can you put a marker on this page?

## Adding Fractions

Intuition, and school mathematics, tells us that one-fifth plus one-fifth equals two-fifths.

Mathematics agrees with this. (Thank heavens!)

**Example:** Evaluate $\frac{1}{5} + \frac{1}{5}$.

**Answer:** Let's "put this quantity over 1."

$$\frac{1}{5} + \frac{1}{5} = \frac{\frac{1}{5} + \frac{1}{5}}{1}$$

Having fifths in the numerator seems annoying. Multiplying top and bottom each by $5$ will probably help.

$$\frac{1}{5} + \frac{1}{5} = \frac{5 \times (\frac{1}{5} + \frac{1}{5})}{5 \times 1}$$

The numerator is now $5 \times \left(\frac{1}{5} + \frac{1}{5}\right) = 5 \times \frac{1}{5} \; + \; 5 \times \frac{1}{5}$ which is $1 + 1 = 2$.
The denominator is $5 \times 1 = 5$.

So, we have

$$\frac{1}{5} + \frac{1}{5} = \frac{5 \times (\frac{1}{5} + \frac{1}{5})}{5 \times 1} = \frac{1+1}{5} = \frac{2}{5}$$

**Practice 41.6** Follow this technique to show that $\frac{2}{7} + \frac{3}{7}$ equals $\frac{5}{7}$.

**Example:** Evaluate $\frac{2}{9} + \frac{4}{7}$.

**Answer:** Let's "put this quantity over 1."

$$\frac{2}{9} + \frac{4}{7} = \frac{\frac{2}{9} + \frac{4}{7}}{1}$$

Having ninths in the numerator is annoying. Let's multiply top and bottom each by $9$.

$$\frac{2}{9} + \frac{4}{7} = \frac{9 \times (\frac{2}{9} + \frac{4}{7})}{9 \times 1}$$

The numerator is $9 \times \frac{2}{9} \ + \ 9 \times \frac{4}{7}$, which is $2 + 9 \times \frac{4}{7}$, and will still contain sevenths, which are annoying.

Let's go back a step and try this: Let's multiply top and bottom by 9 and by 7 in one hit.

$$\frac{2}{9} + \frac{4}{7} = \frac{9 \times 7 \times (\frac{2}{9} + \frac{4}{7})}{9 \times 7 \times 1}$$

The numerator is $9 \times 7 \times \frac{2}{9} \ + \ 9 \times 7 \times \frac{4}{7}$, which we can see is $7 \times 2 \ + \ 9 \times 4 = 14 + 36 = 50$. The denominator is 63.

We've got it!

$$\frac{2}{9} + \frac{4}{7} = \frac{9 \times 7 \times (\frac{2}{9} + \frac{4}{7})}{9 \times 7 \times 1} = \frac{50}{63}$$

**Practice 41.7** Show that $\frac{3}{4} + \frac{2}{11}$ equals $\frac{41}{44}$.

**Example:** Write $1 + \frac{1}{2}$ as a single fraction.

**Answer:** Well, putting the quantity over 1 does the trick!

$$1 + \frac{1}{2} = \frac{1 + \frac{1}{2}}{1}$$

To handle the annoying half in the numerator, let's double the top and the bottom.

$$1 + \frac{1}{2} = \frac{2 \times (1 + \frac{1}{2})}{2 \times 1}$$

We can now evaluate the numerator and the denominator.

$$1 + \frac{1}{2} = \frac{2 \times (1 + \frac{1}{2})}{2 \times 1} = \frac{2 \times 1 + 2 \times \frac{1}{2}}{2} = \frac{2 + 1}{2} = \frac{3}{2}$$

**Read a long string of equal signs as "… equals … which equals … which equals … which equals …**

(Now, also make sure you see how we got from one "which equals" to the next line the previous example.)

**Practice 41.8** Show that $2 + \frac{4}{9}$ equals $\frac{22}{9}$.

**Practice 41.9:** What is the value of $\frac{3}{19} + \frac{7}{19} + \frac{9}{19}$ ?

**Example:** Show that $\frac{a}{N} + \frac{b}{N}$ equal $\frac{a+b}{N}$.
(We're assuming here that $N$ is a number different than zero.)

**Answer:** Even though this question is dealing with unspecified numbers $a$, $b$, and $N$, we can follow exactly the same procedure as before.

Let's start by putting our given quantity over 1.

$$\frac{a}{N} + \frac{b}{N} = \frac{\frac{a}{N} + \frac{b}{N}}{1}$$

To manage the pesky $N$ths in the numerator, multiply top and bottom by $N$.

$$\frac{a}{N} + \frac{b}{N} = \frac{\frac{a}{N} + \frac{b}{N}}{1} = \frac{N \times (\frac{a}{N} + \frac{b}{N})}{N \times 1}$$

Now, let's just work it all work it all out, reading "which equals" in our minds as we go along.

$$\frac{a}{N} + \frac{b}{N} = \frac{\frac{a}{N} + \frac{b}{N}}{1} = \frac{N \times (\frac{a}{N} + \frac{b}{N})}{N \times 1} = \frac{N \times \frac{a}{N} + N \times \frac{b}{N}}{N} = \frac{a+b}{N}$$

We got the result the question expected!

Looks like we've got the addition of fractions licked!

**Comment:** The last example shows that if we add fractions that all happen to have the same denominator, then we can, in essence, just "add the numerators."

We'll see in the next chapter that this is the typical take followed by schoolbooks when first introducing the addition of fractions. What we are doing now is showing that all the mathematics of fraction arithmetic doesn't actually require memorizing special rules for special cases like these.

Just do the math and all will naturally follow!

## Multiplying Fractions

Let's get straight into it!

**Example:** What does mathematics say is the value of $\frac{1}{2} \times \frac{1}{3}$?

**Answer:** Let's put the quantity over 1 and find out!

$$\frac{1}{2} \times \frac{1}{3} = \frac{\frac{1}{2} \times \frac{1}{3}}{1}$$

We have halves and thirds in the numerator. Let's multiply the top and bottom each by 2 and 3 to handle those.

$$\frac{1}{2} \times \frac{1}{3} = \frac{\frac{1}{2} \times \frac{1}{3}}{1} = \frac{2 \times 3 \times \frac{1}{2} \times \frac{1}{3}}{2 \times 3 \times 1}$$

We see that the numerator is $1 \times 1 = 1$ and that the denominator is $6$.

$$\frac{1}{2} \times \frac{1}{3} = \frac{\frac{1}{2} \times \frac{1}{3}}{1} = \frac{2 \times 3 \times \frac{1}{2} \times \frac{1}{3}}{2 \times 3 \times 1} = \frac{1}{6}$$

**Practice 41.10:** Show that $\frac{1}{3} \times \frac{1}{5} \times \frac{1}{10}$ equals $\frac{1}{150}$.

**Example:** What is the value of $\frac{3}{7} \times \frac{5}{8}$?

**Answer:** Putting the quantity over 1 seems to do the magic.

$$\frac{3}{7} \times \frac{5}{8} = \frac{\frac{3}{7} \times \frac{5}{8}}{1} = \frac{7 \times 8 \times \frac{3}{7} \times \frac{5}{8}}{7 \times 8 \times 1}$$

The numerator is $3 \times 5 = 15$ (do you see that?) and the denominator is $7 \times 8 = 56$. So,

$$\frac{3}{7} \times \frac{5}{8} = \frac{15}{56}$$

**Practice 41.11:** Find the value of $\frac{17}{21} \times \frac{2}{5}$.

**Practice 41.12:** Show that $\frac{10}{64} \times \frac{16}{50}$ equals $\frac{1}{20}$.

**Practice 41.13:** Show, in general, that $\frac{a}{b} \times \frac{c}{d}$ equals $\frac{a \times c}{b \times d}$.

Actually, we can approach these practice problems in a different way if we want.

To set things up, let's first give a name to the $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$ with a numerator of $1$. We'll call them the **basic fractions** as they, well, the basic fractions of everyday life: halves, thirds, quarters, and so on.

Our first example of multiplying fractions was that with two basic fractions. We saw

$$\frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$$

It looks as though we just multiplied together the denominators of two basic fractions to make another basic fraction.

We can prove that multiplying basic fractions this way is valid.

**Multiplying Basic Fractions:** We have that $\frac{1}{n} \times \frac{1}{m}$ equals $\frac{1}{n \times m}$.

**Reason:** Let's put the quantity we're looking at over $1$.

$$\frac{1}{n} \times \frac{1}{m} = \frac{\frac{1}{n} \times \frac{1}{m}}{1}$$

Multiply top and bottom each by $n$ and by $m$.

$$\frac{1}{n} \times \frac{1}{m} = \frac{\frac{1}{n} \times \frac{1}{m}}{1} = \frac{n \times m \times \frac{1}{n} \times \frac{1}{m}}{n \times m \times 1}$$

The numerator is $1 \times 1 = 1$ and the denominator is $n \times m$.

We've got it:

$$\frac{1}{n} \times \frac{1}{m} = \frac{\frac{1}{n} \times \frac{1}{m}}{1} = \frac{n \times m \times \frac{1}{n} \times \frac{1}{m}}{n \times m \times 1} = \frac{1}{n \times m}$$

Now we're set to multiply all fractions just by teasing everything apart.

**Example:** Find the value of $\frac{3}{7} \times \frac{5}{8}$ again.

**Answer:** We can write

$$\frac{3}{7} \times \frac{5}{8} = 3 \times \frac{1}{7} \times 5 \times \frac{1}{8}$$

We can work out a string of products in any order we like.

Noticing that $3 \times 5 = 15$ and $\frac{1}{7} \times \frac{1}{8} = \frac{1}{56}$, we have

$$\frac{3}{7} \times \frac{5}{8} = 3 \times \frac{1}{7} \times 5 \times \frac{1}{8} = 15 \times \frac{1}{56}$$

And $15 \times \frac{1}{56}$ is just the fraction $\frac{15}{56}$ pulled apart

If you are tired of putting a product of fractions over 1, you now have the option of just pulling the product completely apart and computing parts of the product in turn.

**Practice 41.14:** Compute $\frac{10}{64} \times \frac{16}{50}$ again by "pulling apart" the terms of the product. See the answer $\frac{1}{20}$ again.

**Example:** Show that $\frac{37}{97} \times \frac{97}{37}$ is just 1.

**Answer:** We have

$$\frac{37}{97} \times \frac{97}{37} = 37 \times \frac{1}{97} \times 97 \times \frac{1}{37}$$

Since $37 \times \frac{1}{37} = 1$ and $97 \times \frac{1}{97} = 1$, this product is

$$\frac{37}{97} \times \frac{97}{37} = 37 \times \frac{1}{97} \times 97 \times \frac{1}{37} = 1 \times 1 = 1$$

**Example:** Show again that $\frac{a}{b} \times \frac{c}{d}$ equals $\frac{a \times c}{b \times d}$.

**Answer**: We have

$$\frac{a}{b} \times \frac{c}{d} = a \times \frac{1}{b} \times c \times \frac{1}{d}$$

We have in this product $a \times c$ and we have $\frac{1}{b} \times \frac{1}{d}$, which equals $\frac{1}{b \times d}$. We can thus think of the product as

$$\frac{a}{b} \times \frac{c}{d} = (a \times c) \times \frac{1}{b \times d}$$

And this is just a fraction pulled apart. It is the fraction

$$\frac{a \times c}{b \times d}$$

**Practice 41.15** What do you need to multiply $\frac{4}{7}$ by to get the answer $\frac{20}{21}$?

***Example:*** *Ibrahim was asked to compute* $\frac{39}{7} \times \frac{14}{13}$ *and within three seconds he said that the answer was* $6$. *How did he see this so quickly?*

***Answer:*** *Well, we can't say for sure what the fellow saw, but we do have*

$$\frac{39}{7} \times \frac{14}{13} = 39 \times \frac{1}{7} \times 14 \times \frac{1}{13}$$

Since we can multiply a string of products in any order we like, perhaps Ibrahim saw this as

$$39 \times \frac{1}{13} \times 14 \times \frac{1}{7} = \frac{39}{13} \times \frac{14}{7} = 3 \times 2 = 6$$

**Practice 41.16** *What is the value of* $\frac{51}{35} \times \frac{14}{17}$?

**Practice 41.17** *Show that* $\frac{1}{2} \times \frac{2}{3} \times \frac{3}{4} \times \frac{4}{5}$ *equals* $\frac{1}{5}$.

## Dividing Fractions

Remember, a fraction is the answer to a division problem.

$$\frac{a}{b} = a \div b$$

Or, saying this another way, the answer to a division problem is a fraction.

$$a \div b = \frac{a}{b}$$

**Example:** Find the value of $1 \div \frac{1}{7}$.

**Answer:** The answer is the fraction

$$\frac{1}{\frac{1}{7}}$$

To make this look friendlier, let's multiply the top and bottom by 7.

$$\frac{1}{\frac{1}{7}} = \frac{7 \times 1}{7 \times \frac{1}{7}} = \frac{7}{1}$$

And this equals 7.

**Example:** Compute $\frac{3}{5} \div \frac{4}{7}$.

**Answer:** The answer is the fraction.

$$\frac{\frac{3}{5}}{\frac{4}{7}}$$

To make this more tractable, let's handle the awkward fifths and sevenths by multiplying the top and bottom each by 5 and by 7.

We get

$$\frac{5 \times 7 \times \frac{3}{5}}{5 \times 7 \times \frac{4}{7}}$$

Since $5 \times \frac{3}{5} = 3$ and $7 \times \frac{4}{7} = 4$ we really have

$$\frac{7 \times 3}{5 \times 4}$$

giving our final answer

$$\frac{21}{20}$$

So, $\frac{3}{5} \div \frac{4}{7} = \frac{21}{20}$.

**Practice 41.18:** Show that $\frac{2}{3} \div \frac{5}{7}$ equals $\frac{14}{15}$.

**Practice 41.19:** Compute $\frac{3}{4} \div \frac{2}{3}$.

**Practice 41.20:** Show that $1 \div \frac{1}{N}$ equals $N$.

In Section 39, we tried to share "8 pies equally among $\frac{2}{3}$ of a student" (which makes little sense) and concluded that somehow doing so results in giving a whole student $12$ pies.

**Practice 41.21:** Show that $8 \div \frac{2}{3}$ does indeed equal $12$.

## A Comment on Notation

Fractions are written with a horizontal bar (once called a **virga**) to separate numerator from denominator. But writing fractions within fractions is hard and looks confusing.

For example, $\frac{1}{\frac{2}{3}}$, intended to be read as "one over two-thirds," could easily be misread as "one-half over three."

To avoid confusion, a slanted line / (called a **solidus**) is often used instead in fraction notation. For example, writing

$$\frac{1}{2/3}$$

makes clear that we have a fraction as the denominator of a fraction.

Rewriting $\frac{\frac{3}{4}}{\frac{2}{3}}$ as

$$\frac{3/4}{2/3}$$

makes matters clearer, as does rewriting $\frac{\frac{4}{3}}{5}$ as

$$\frac{4/3}{5}$$

**Example:** Figure out $\frac{3}{4} \div \frac{2}{3/5}$.

Here's my approach.

**Answer:**

First of all, I don't like the number $\frac{2}{\frac{3}{5}}$.

I am going to make it look friendlier by multiplying the top and bottom each by $5$. Doing so turns it into $\frac{10}{3}$.

This means that we're really being asked to compute $\frac{3}{4} \div \frac{10}{3}$. The answer to a division problem is a fraction. So, the answer is

$$\frac{\frac{3}{4}}{\frac{10}{3}}$$

To make this tractable, multiply the top and bottom each by $4$ and $3$.

$$\frac{\frac{3}{4}}{\frac{10}{3}} = \frac{3 \times 4 \times \frac{3}{4}}{3 \times 4 \times \frac{10}{3}} = \frac{3 \times 3}{4 \times 10} = \frac{9}{40}$$

And we're done!

$$\frac{3}{4} \div \frac{2}{3/5} = \frac{9}{40}$$

**Practice 41.22:** Show that $1 \div \frac{1}{a/b}$ is $\frac{a}{b}$.

## Fractions and Negative Numbers

You have no doubt noticed that I have skipped over the arithmetic of subtracting fractions. This is because, as you know, I do not believe subtraction exists.

**Subtraction is the addition of the opposite.**

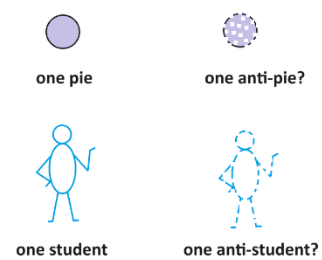
So, we need to be sure we understand "the opposite of a fraction" and how to make sense of fraction that might incorporate negative signs.

**Question:** Are $\frac{-2}{3}$ and $\frac{2}{-3}$ and $-\frac{2}{3}$ the same fraction in different guises, or are they different?

(If you want some fun … Is there any way to make sense of these quantities in the pies-per-student model?

$\frac{-2}{3}$ is result of sharing two anti-pies equally among three actual students.

$\frac{2}{-3}$ is result of sharing two actual pies equally among three anti-students.




$-\frac{2}{3}$ is the opposite of the result of sharing two pies equally among three students.

Do any of these interpretations make sense? If so, do they represent the same final amount of pie (or anti-pie) per student (or is it anti-student?)

Don't take this question seriously!

To remind ourselves, here are the properties of negative numbers we deduced back in Sections 24 and 25.

**Negative Numbers**

- Adding together a number and its opposite produces zero: $a + -a = 0$
- The opposite of the opposite of a number is the original number: $-(-a) = a$
- Multiplying by negative one produces the opposite number: $(-1) \times a = -a$
- The opposite of zero is zero: $-0 = 0$
- We can "pull out" negative signs from products: $(-a) \times b = a \times (-b) = -ab$
- Negative times negative is positive: $(-a) \times (-b) = ab$

Let's see what math has to say about the three quantities $\frac{-2}{3}$ and $\frac{2}{-3}$ and $-\frac{2}{3}$.

For starters, we know

$$-2 = (-1) \times 2$$

and

$$-3 = (-1) \times 3$$

and

$$-\frac{2}{3} = (-1) \times \frac{2}{3}$$

**Let's pull apart $\frac{-2}{3}$:**

$$\frac{-2}{3} = (-2) \times \frac{1}{3} = (-1) \times 2 \times \frac{1}{3}$$

**Let's pull apart $-\frac{2}{3}$:**

$$-\frac{2}{3} = (-1) \times \frac{2}{3} = (-1) \times 2 \times \frac{1}{3}$$

They are the same: $\frac{-2}{3}$ and $-\frac{2}{3}$ are the same quantity in different guises.

**What about $\frac{2}{-3}$?**

$$\frac{2}{-3} = \frac{2}{(-1) \times 3}$$

Multiply the top and bottom each by $-1$.

$$\frac{2}{-3} = \frac{2}{(-1) \times 3} = \frac{(-1) \times 2}{(-1) \times (-1) \times 3}$$

Since $(-1) \times (-1) = 1$ (negative times negative is positive) and $(-1) \times 2 = -2$, this is the same as $\frac{-2}{3}$.

$$\frac{2}{-3} = \frac{2}{(-1) \times 3} = \frac{(-1) \times 2}{(-1) \times (-1) \times 3} = \frac{-2}{3}$$

So, $-\frac{2}{3}$ is the same as $\frac{-2}{3}$, and $\frac{2}{-3}$ is the same as $\frac{-2}{3}$. All three quantities are thus the same!

I don't know what any one of these quantities means in the real world, but that is not the point. We have just proven that they are the same mathematically.

Now we know that we are free to "pull out" a negative sign from a fraction any time.

$$-\frac{a}{b} = \frac{-a}{b} = \frac{a}{-b}$$

**Practice 41.23** Show that $\frac{-a}{-b}$ is the same as $\frac{a}{b}$.

(I can think of two approaches you could take here.

1. Try multiplying the top of bottom of $\frac{-a}{-b}$ each by $-1$.
2. Pull out a negative sign, twice!)

**Practice 41.24** What is the value of $\frac{5}{-2} \times \frac{-3}{10}$?

**Practice 41.25** What is the value of $\frac{2}{9} \div \frac{-2}{3}$?

## Subtracting Fractions

Let's now practice adding the opposite.

**Example:** What is $\frac{5}{7} - \frac{3}{7}$?

Intuition says that "five-sevenths take away three-sevenths is two-sevenths."

**Answer:** Subtraction is the addition of the opposite. We must compute

$$\frac{5}{7} + -\frac{3}{7}$$

This is.

$$\frac{5}{7} + \frac{-3}{7}$$

Now, "putting over 1" does the trick for us.

$$\frac{5}{7} - \frac{3}{7} = \frac{\frac{5}{7} + \frac{-3}{7}}{1} = \frac{7 \times \left(\frac{5}{7} + \frac{-3}{7}\right)}{7 \times 1} = \frac{5 + \ -3}{7} = \frac{2}{7}$$

**Practice 41.26:** Show that $\frac{3}{7} - \frac{5}{7}$ is $-\frac{2}{7}$.

**Example**: Compute $\frac{2}{3} - \frac{1}{4}$.

**Answer**: First, it is really an addition problem.

$$\frac{2}{3} - \frac{1}{4} \quad = \quad \frac{2}{3} + -\frac{1}{4} \quad = \quad \frac{2}{3} + \frac{-1}{4}$$

Now we're set to go

$$\frac{2}{3} + \frac{-1}{4} = \frac{3 \times 4 \times \left(\frac{2}{3} + \frac{-1}{4}\right)}{3 \times 4 \times 1} = \frac{4 \times 2 + 3 \times (-1)}{12} = \frac{8 + \ -3}{12} = \frac{5}{12}$$

**Example**: Compute $1 - \frac{1}{20}$.

**Answer:** Here it is.

$$1 + \frac{-1}{20} = \frac{1 + \frac{-1}{20}}{1} = \frac{20 \times \left(1 + \frac{-1}{20}\right)}{20 \times 1} = \frac{20 + \ -1}{20} = \frac{19}{20}$$

**Practice 41.27**

a) Compute $\frac{12}{11} - \frac{8}{11}$.

b) Show, in general, that $\frac{a}{N} - \frac{b}{N} = \frac{a-b}{N}$.

## MUSINGS

**Musing 41.28** We just went through the basic arithmetic operations of fractions without reference to the usual schoolbook approaches to them.

**Just turn any quantity you have into a fraction (perhaps with a denominator of 1)**
**and multiply top and bottom by what you need to make that fraction more manageable.**

What is your reaction to this?
Are matters clearer and less cluttered? More confusing? Strange and weird? Delightful and freeing?

## MECHANICS PRACTICE

**Practice 41.29** Did you try all 27 practice problems throughout this section?

# Chapter 6

## Fractions:
## Understanding their Schoolbook Arithmetic and The Mathematical Truth

## 42. Fractions: Where are we?

We started the last chapter with an overview of the muddled story of fractions typically presented in most school curriculums. Various real-world explanations are brought at choice moments to "explain" the meaning of fractions and the operations on fractions, and it is never quite clear which real-world model is appropriate to bring in when (nor is it clear why one is allowed to keep switching one's imagery of what a fraction is).

We did identify one real-world model of fractions that seems quite robust: the pies-per-student sharing model. It's a concrete scenario that seems to unify the story of fractions to some degree.

But we are now wise enough to know that no one model of a piece of mathematics can be used to explain all aspects of that mathematics: mathematics sits at a higher plane than any one concrete scenario. Although a topic of mathematics can be applied, when appropriate, to a wide variety of real-world instances, no one real-world instance can "see" and explain all the mathematics of that topic.

So, we worked to let go of the pie-per-student model. We identified a small set of basic properties that arose from the model that seemed to "carry the goods" of all fraction arithmetic. And we saw in Section 41 that they did.

We moved from a real-world story of fractions to a purely mathematical story.

That mathematical story can still be tightened up—and we will indeed wrap it up into an exceptionally tight logical bundle in this chapter—but we still have the weight of schoolbook version of fraction arithmetic on our shoulders.

Can we explain the schoolbook versions of fraction arithmetic as well?

The answer is that we can, and our job this chapter is to do that too.

This chapter is technically irrelevant: you can now do all the mathematics of fractions!

But this chapter is important as it will clear all the hazy thinking of the past and help you see all you were taught in a clear, logical light. There is mathematical validity to it all.

## Were We Are at Right Now

We've been playing with a system of numbers that contains all the integers $-3, -2, -1, 0, 1, 2, 3, \ldots$ (the counting numbers and all their opposites) and includes all the answers to division problems ("fractions"):

> For two numbers $a$ and $b$ in our system of numbers (with $b$ not zero), we write $\frac{a}{b}$ as the answer to the division problem $a \div b$.

$$\frac{a}{b} = a \div b$$

Everything we developed in the last chapter boiled down to accepting just four properties of fractions.

- Every number can be written as a fraction with a denominator of 1. $$a = \frac{a}{1}$$
- Multiplying the top and bottom of a fraction by the same number does not change the value of the fraction. $$\frac{a}{b} = \frac{k \times a}{k \times b}$$
- Multiplying a fraction by its denominator "cancels" the denominator. $$\cancel{b} \times \frac{a}{\cancel{b}} = a$$
- We can "pull apart" a fraction. $$\frac{a}{b} = a \times \frac{1}{b}$$

(We'll see later in this chapter that these four properties all follow from <u>one</u>, even simpler, belief!)

Like the "star page" of Chapter 5, this page shows all the fractions goods needed to make everything work.

## MECHANICS PRACTICE

**Practice 42.1** Which of the following expressions has a value equal to a counting number? Which counting number?

a) $\frac{36}{12}$ b) $3 \times \frac{20}{3}$ c) $\frac{1002}{1}$ d) $\frac{107}{107}$ e) $4 \times \frac{5}{4}$ f) $\frac{0}{8}$ g) $10 \times 6 \times \frac{7}{5}$

# 43. Fractions the School Way: "Parts of a Whole"

As we saw, the story of fractions for students typically begins with developing familiarity with the concept of a **half**, **third**, **quarter** (fourth), **fifth**, **sixth**, and so on, of a given fixed object (a given "whole").

A "half" of a given object is loosely defined as a portion of that object such that two copies of that portion together reconstitute the whole.

A "third" of a given object is loosely defined as a portion of that object such that three copies of that portion together reconstitute the whole.

And so on.

half of a pie

a third of a set of kittens



a quarter of a line segment

Students are introduced to the notation $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \ldots$ for these **basic fractions**.

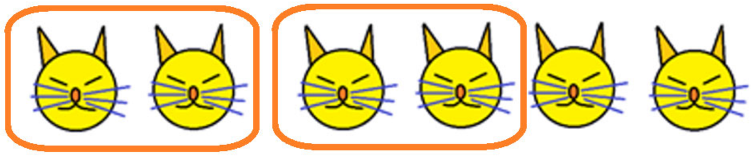
They are also taught to denote the whole in any given context as "1" and the statement "two copies of a half make the whole" is thus written $2 \times \frac{1}{2} = 1$, "three copies of a third make a whole" is written $3 \times \frac{1}{3} = 1$, and so on.

This feels strange and confusing, but the curriculum is trying to help students accept that for each counting number $b$ it is useful to posit the existence of another number, denoted $\frac{1}{b}$, with the property:

$$b \times \frac{1}{b} = 1$$

This statement is aligned with one of our recognized properties of fractions.

Students are also taught that "two thirds," for example, is precisely what it reads: two copies of a third.



**two thirds of a set of kittens**

Similarly, "four fifths" is four copies of a fifth, and so on.

The notation $\frac{a}{b}$, with $a$ a counting number is thus being introduced to mean "$a$ copies of $\frac{1}{b}$."
That is, $\frac{a}{b}$ is shorthand for $a \times \frac{1}{b}$.

$$\frac{a}{b} = a \times \frac{1}{b}$$

This too is one of our recognized properties of fractions.

Just to be clear: In the early grades multiplication is repeated addition and so $\frac{2}{3} = 2 \times \frac{1}{3}$ is interpreted as $\frac{1}{3} + \frac{1}{3}$, just as the picture of kittens above suggests: we have one third of the kittens <u>and</u> one third of the kittens, side by side.

In the same way, $\frac{4}{5} = 4 \times \frac{1}{5} = \frac{1}{5} + \frac{1}{5} + \frac{1}{5} + \frac{1}{5}$, for copies of a fifth of a certain "whole" put side by side.

## Mathematicians’ Start to Fractions

Our mathematical journey back in Chapter 1 started with the **counting numbers** $0, 1, 2, 3, \ldots$. And in that chapter, we showed that all the arithmetic of the counting numbers logically follows from just eight general rules.

> **Rule 1:** For any two numbers $a$ and $b$ we have $a + b = b + a$.
>
> **Rule 2:** For any number $a$ we have $a + 0 = a$ and $0 + a = a$.
>
> **Rule 3:** In a string of additions, it does not matter in which order one conducts individual additions.
>
> **Rule 4:** For any two numbers $a$ and $b$ we have $a \times b = b \times a$
>
> **Rule 5:** For any number $a$ we have $a \times 1 = a$ and $1 \times a = a$.
>
> **Rule 6:** In a string of multiplications, it does not matter in which order one conducts individual multiplications.
>
> **Rule 7:** For any number $a$ we have $a \times 0 = 0$ and $0 \times a = 0$.
>
> **Rule 8:** “We can chop up rectangles from multiplication and add up the pieces.”

These eight rules pinpoint the behavior of addition and multiplication as it applies to the counting numbers. Identifying this behavior then allowed us to extend these operations to a larger class of numbers, the **integers**, and we did that in Chapter 3 with the addition of just one more rule.

> **Rule 9:** For each number $a$, there is one other number “$-a$” such that $a + -a = 0$.

We wanted “opposite numbers,” so we made them happen.

It seems with fractions we again want a new type of opposite numbers—not opposite in the sense of addition, but opposite in a multiplicative sense. We want the basic fractions, numbers that match the statements “two halves make a whole,” “three thirds make a whole,” and so on, to exist in our world of numbers too.

So, let’s make them happen!

> **Rule 10:** For each number $a$ different from zero, there is one other number “$\frac{1}{a}$” such that $a \times \frac{1}{a} = 1$.

(This rule agrees that saying “zero zeroths makes a whole” is meaningless!)

This is how mathematicians start their story of fractions, by expanding the worldview of “number” by declaring the existence of the basic fractions.

And this is in line with the human start to fraction thinking—positing the existence of numbers deserved to be called a half, a third, a fourth, and so on.

We’ll explore the full magic of this Rule 10 at the end of the chapter.

## MUSINGS

**Musing 43.1** Schoolbooks typically define one third, $\frac{1}{3}$, for example as follows:

> *Choose your whole. (Usually a pie.)*
> *Divide that pie into three "equal" pieces.*
> *Then each of those pieces is called a third.*

We've just defined a third a different way.

> *Choose your whole.*
> *A "third" is a portion of that whole with the property that three copies of that portion match the whole.*

Are these two ways of thinking equivalent?

**Musing 43.2** Here's our picture of two-thirds of a set of kittens.

From our study of fractions, we know that $3 \times \frac{2}{3} = 2$.

Is it possible to "justify" this statement using the picture of kittens?

## 44. Adding and Subtracting Fractions the School Way

In the early grades addition is interpreted as the physical act of "putting items together" and recounting.

One apple and two apples make for three apples.
One house and two houses make for three houses.

Consequently, one tenth and two tenths should likewise make for three tenths (especially if one is imagining portions of a given pie).

$$\frac{1}{10}+\frac{2}{10}=\frac{3}{10}$$

Schoolbooks use this physical thinking to "explain" how to add fractions sharing the same denominator (a **common denominator**): just add up the numerators and write their sum over that common denominator.

And this does align with the mathematics of fractions we've seen.

**Example** a) Evaluate $\frac{1}{10}+\frac{2}{10}$ mathematically.

b) Prove that $\frac{a}{N}+\frac{b}{N}+\frac{c}{N}+\frac{d}{N}$ is indeed $\frac{a+b+c+d}{N}$.

**Answer:** a) We have

$$\frac{1}{10}+\frac{2}{10}=\frac{\frac{1}{10}+\frac{2}{10}}{1}=\frac{10\times(\frac{1}{10}+\frac{2}{10})}{10\times 1}=\frac{1+2}{10}=\frac{3}{10}$$

b) And we have

$$\frac{a}{N}+\frac{b}{N}+\frac{c}{N}+\frac{d}{N}=\frac{\frac{a}{N}+\frac{b}{N}+\frac{c}{N}+\frac{d}{N}}{1}=\frac{N\times(\frac{a}{N}+\frac{b}{N}+\frac{c}{N}+\frac{d}{N})}{N\times 1}=\frac{a+b+c+d}{N}$$

Of course, it is much easier to think "one tenth plus two tenths equals three tenths," say, rather than go through the mathematical hoopla of putting over 1, multiplying top and bottom by ten, and so on.

But now that we know that school mathematics is right in what is says about adding fractions with a common denominator, we can make use of this result anytime too and avoid the spelling out the mathematical details each and every time.

**ADDING FRACTIONS: Same Denominator**

$$\frac{a}{N} + \frac{b}{N} = \frac{a+b}{N}$$

But this result now begs the question:

How do we add two fractions whose denominators do not match?
For example, what is the value of $\frac{2}{5} + \frac{4}{9}$?

We computed such sums without a problem in Section 41. (We placed the quantity over 1 and proceeded from there.)

**Practice 44.1:** Compute $\frac{2}{5} + \frac{4}{9}$ vis the technique of Section 41.

But this is not the approach schoolbooks take.

Curriculums have the opportunity, right here, to teach students a meta-lesson: to step back and learn a problem-solving strategy often used by mathematicians

When faced with a challenge, try engaging in **wishful thinking**.

Not knowing how to proceed (having not read the previous chapter), we might muse as follows:

*We know how add two fractions with the same denominator.*

*We want to compute* $\frac{2}{5} + \frac{4}{9}$.

*They don't have a common denominator. Life would be easier if they did.*

*Hmm. How do we make that happen?*

We know how to change the numerators and denominators of fractions without changing the value of the fraction. So, one approach might be to start listing all the equivalent fractions of the two fractions we are considering. Let's double their numerators and denominators, then triple them, then quadruple them, and so on.

$$\boxed{\frac{\mathbf{2}}{\mathbf{5}}} \quad \frac{4}{10} \quad \frac{6}{15} \quad \frac{8}{20} \quad \frac{10}{25} \quad \frac{12}{30} \quad \frac{14}{35} \quad \frac{16}{40} \quad \boxed{\frac{18}{45}} \quad \frac{20}{50} \quad \frac{22}{55} \cdots$$

$$\boxed{\frac{\mathbf{4}}{\mathbf{9}}} \quad \frac{8}{18} \quad \frac{12}{27} \quad \frac{16}{36} \quad \boxed{\frac{20}{45}} \quad \frac{24}{54} \quad \frac{28}{63} \quad \frac{32}{72} \quad \frac{36}{81} \quad \frac{40}{90} \quad \frac{44}{99} \cdots$$

Lo and behold. We see that $\frac{2}{5}$ is the same as $\frac{18}{45}$, and that $\frac{4}{9}$ is the same as $\frac{20}{45}$.

To compute $\frac{2}{5}+\frac{4}{9}$ we can just compute $\frac{18}{45}+\frac{20}{45}$ instead. And that has value $\frac{38}{45}$.

$$\frac{2}{5}+\frac{4}{9}=\frac{38}{45}$$

**Question:** Is this the answer you obtained?

Thank you wishful thinking!

School students are taught …

| **ADDING FRACTIONS: Different Denominators** |
|---|
| Rewrite the fractions to have a common denominator and then add as before. |

**Comment:** Students aren't typically encouraged to write out long lists of equivalent fractions to find two fractions with the same denominator. For example, realizing that multiplying the numerator and denominator of $\frac{2}{5}$ each by $9$, and multiplying the numerator and denominator of $\frac{4}{9}$ each by $5$, will yield two fractions with the same denominator of $45$. Of course, feel free to employ such swiftness anytime you want to too.

**Example:** Evaluate $\frac{3}{4} + \frac{11}{10}$ the schoolbook way.

**Answer:** Let's create fractions with the same denominators by multiplying the tops and bottoms of the fractions with each other's denominators.

$$\frac{3}{4} + \frac{11}{10} = \frac{10 \times 3}{10 \times 4} + \frac{4 \times 11}{4 \times 10} = \frac{30}{40} + \frac{44}{40}$$

We now see that the sum is $\frac{74}{40}$.

Some school courses won't accept this as a final answer as we can "reduce" this fraction. Those curricula often insist that we must reduce if we can.

$$\frac{74}{40} = \frac{2 \times 37}{2 \times 20} = \frac{37}{20}$$

Some hyper-fussy school curricula will insist that students be as efficient as possible from the get-go.

They would want them to realize that we could rewrite $\frac{3}{4}$ and $\frac{11}{10}$ with a common denominator smaller than $40$, and say that they should! (I don't know why.)

$$\frac{3}{4} + \frac{11}{10} = \frac{5 \times 3}{5 \times 4} + \frac{2 \times 11}{2 \times 10} = \frac{15}{20} + \frac{22}{20}$$

The answer $\frac{37}{20}$ pops out again.

**Question:** Does the phrase **least common multiple** ring a bell for you?

Here's the thing. There is no need to be hyper-efficient in mathematics work. All good and correct approaches and answers are good and correct!

If you want to work out the sum of two fractions the schoolbook way, great!
If you want to work it out by putting the sum over 1 and going from there, also great!

If you get an answer that you might later need to rewrite in a different form, rewrite it later.

If you don't ever need to rewrite an answer, then the answer you first obtain is fine as it is.
(Though test writers often object to this final comment.)

**Example:** What is $\frac{12}{17} - \frac{8}{17}$?

Thinking concretely of portions of pie, schoolbooks encourage students to think "12 slices take away 8 slices obviously leaves 4 slices." We have $\frac{12}{17} - \frac{8}{17} = \frac{4}{17}$.

And we know this aligns with solid mathematics.

$$\frac{12}{17} - \frac{8}{17} = \frac{12}{17} + \frac{-8}{17} = \frac{\frac{12}{17} + \frac{-8}{17}}{1}$$

$$= \frac{17 \times (\frac{12}{17} + \frac{-8}{17})}{17 \times 1}$$

$$= \frac{12 + \ -8}{17}$$

$$= \frac{4}{17}$$

**Practice 44.2** Show, in general, that $\frac{a}{N} - \frac{b}{N}$ equals $\frac{a-b}{N}$.

School intuition and solid mathematics do align. We have, and can use anytime, …

| **SUBTRACTING FRACTIONS: Same Denominator** |
|---|
| $$\frac{a}{N} - \frac{b}{N} = \frac{a-b}{N}$$ |

You can guess how schoolbooks advice students to subtract fractions whose denominators do not match.

For instance, most people would compute $\frac{2}{3} - \frac{1}{4}$ by saying:

> $\frac{2}{3}$ *is really* $\frac{8}{12}$.
>
> $\frac{1}{4}$ *is really* $\frac{3}{12}$.
>
> *Eight twelfths take away three twelfths is five twelfths.*
> *Done!*

$$\frac{2}{3} - \frac{1}{4} \quad = \quad \frac{8}{12} - \frac{3}{12} \quad = \quad \frac{5}{12}$$

**Question** What is the value of $\frac{5}{6} - \frac{1}{2}$?

a) $\frac{4}{12}$ b) $\frac{2}{6}$ c) $\frac{1}{3}$ d) All of these

We can see this problem as $\frac{10}{12} - \frac{6}{12} = \frac{4}{12}$ or as $\frac{5}{6} - \frac{3}{6} = \frac{2}{6}$, and both these values equal $\frac{1}{3}$.

Students are taught …

| **SUBTRACTING FRACTIONS: Different Denominators** |
|---|
| Rewrite the fractions to have a common denominator and then subtract as before. |

Of course, you also have the option to put the quantity over 1 and just work on from there.

**Example:** Compute $\frac{7}{12} - \frac{2}{5} + \frac{13}{6} - 1$ the schoolbook way.

**Answer:** With denominators of 12, 5, and 6, I am thinking 60ths might be good for this challenge.

Also, think of "1" as the fraction $\frac{60}{60}$. (1 is the answer to $60 \div 60$.)

$$\frac{5 \times 7}{5 \times 12} - \frac{12 \times 2}{12 \times 5} + \frac{10 \times 13}{10 \times 6} - \frac{60}{60} \quad = \quad \frac{35}{60} - \frac{24}{60} + \frac{130}{60} - \frac{60}{60} \quad = \quad \frac{81}{60}$$

## MUSINGS

**Musing 44.3** We showed that $\frac{2}{5}+\frac{4}{9}$ equals $\frac{38}{45}$. We did this purely by what mathematics led us to do.

But can we make sense of this with our pies-and-student model?

Is sharing 2 pies equally among 5 students together with sharing 4 pies equally among 9 students somehow connected with sharing 38 pies equally among 45 students?

I personally do not see how, but maybe you do?

> **Comment:** This question illustrates the challenge (absurdity, actually) of expecting one real-world experience to model absolutely everything mathematics has to offer. Mathematics sits at a different level to real-world applications and has relevance to many, many different scenarios. It is not the other way round.

**Musing 44.4** In school, with our concrete thinking, we were taught that adding fractions was just a matter of bringing portions of pie together on plate. We have that $\frac{2}{5}$ of a pie and $\frac{4}{9}$ of a pie together make $\frac{38}{45}$ of a pie.

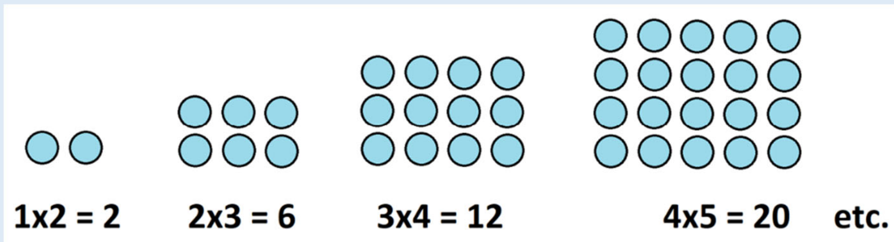
What picture would you have drawn back in your school days to verify this particular example?

**Musing 44.5** Match each arithmetic statement on the left with its value on the right.

| | |
|---|---|
| $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}$ | $\frac{3}{16}$ |
| $\frac{1}{4}-\frac{1}{5}$ | $\frac{15}{16}$ |
| $1-\frac{1}{20}$ | $\frac{1}{20}$ |
| $\frac{9}{32}-\frac{1}{8}+\frac{1}{32}$ | $\frac{6}{20}$ |
| $\frac{3/4}{5}+\frac{3}{20}$ | $\frac{19}{20}$ |

**Musing 44.6** The **oblong numbers** are the numbers that arise from arranging dots into rectangular arrangements with one side one unit longer than the other.

The first oblong number is $1 \times 2 = 2$. The second oblong number is $2 \times 3 = 6$. And so on.




a) What's the thirtieth oblong number?

b) What is the value of $\frac{1}{30} - \frac{1}{31}$?

c) What's the tenth oblong number and what is the value of $\frac{1}{10} - \frac{1}{11}$?

In the previous question you matched $\frac{1}{4} - \frac{1}{5}$ with $\frac{1}{20}$, and 20 is the fourth oblong number.

d) If you are feeling game and particularly mathy, perform some mathematics to show how $\frac{1}{n} - \frac{1}{n+1}$ is connected to the $n$th oblong number. (But do this only if this is fun for you!)

**Musing 44.7**

a) Compute $\frac{a}{b} + \frac{c}{d}$ by putting this quantity over 1 and working from there. Show that you get the (unfriendly) answer $\frac{ad+bc}{bd}$.

b) Compute $\frac{a}{b} + \frac{c}{d}$ again, but now the schoolbook way by working with a common denominator. Show that you again get the answer $\frac{ad+bc}{bd}$.

Once more, all correct approaches to mathematics match and are good and correct!

## MECHANICS PRACTICE

**Practice 44.8** Did you do the matching exercise in Musing 43.5? (I guess that is technically a Yes/No question!)

| | |
|---|---|
| $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}$ | $\frac{3}{16}$ |
| $\frac{1}{4}-\frac{1}{5}$ | $\frac{15}{16}$ |
| $1-\frac{1}{20}$ | $\frac{1}{20}$ |
| $\frac{9}{32}-\frac{1}{8}+\frac{1}{32}$ | $\frac{6}{20}$ |
| $\frac{3/4}{5}+\frac{3}{20}$ | $\frac{19}{20}$ |

**Practice 44.9** Compute each of the following.

a) $\frac{6}{11}+\frac{7}{10}$ b) $\frac{5}{14}-\frac{1}{70}$ c) $\frac{4}{3}-\frac{4}{9}$ d) $2+\frac{7}{5}-\frac{3}{7}$

**Practice 44.10** Compute each of these.

a) $\frac{1}{19}-\frac{1}{21}$

b) $1+\frac{1}{2}+\frac{1}{4}-\frac{7}{2}$

c) $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}$

## 45. Multiplying Fractions the School Way: The Word "Of"

We saw how to multiply two fractions in Section 41.

**MULTIPLYING FRACTIONS**

$$\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d}$$

**Practice 45.1** Do you remember how to obtain this result?

a) Use the technique of Section 41 to find the value of $\frac{1}{3} \times \frac{1}{7}$.

b) Find the value of $\frac{2}{3} \times \frac{4}{7}$.

c) Show that $\frac{a}{b} \times \frac{c}{d}$ does indeed equal $\frac{a \times c}{b \times d}$.

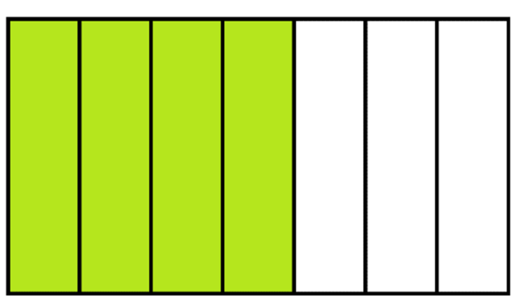
But schoolbooks do not typically show the approach of Section 41 to justify this multiplication rule. Instead, they appeal to a curious aphorism: "of means multiply" (or, perhaps it should be "multiply means of"?)

What exactly is the link between the everyday use of the word "of" and the multiplication of fractions?

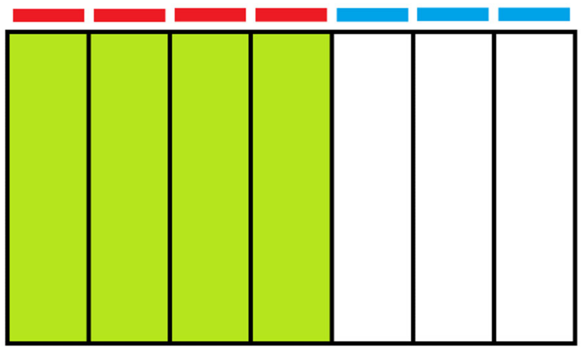
Let's start slowly by first asking …

What do we mean by "four sevenths <u>of</u> a pie"?

Well, everyday practice suggests we divide the pie into seven equal pieces and select four of them. For ease of drawing, let's draw rectangular pie.



But we let me point out that in dividing the rectangle into seven equal pieces and selecting four of them, we've also divided the top edge of the pie into seven equal pieces as well and selected four of those segments.



Or we could look at this picture and think of it as coming from having divided the top edge of the pie into seven equal pieces first, selecting four of them, and then using them to guide to how to divide the whole rectangular pie.

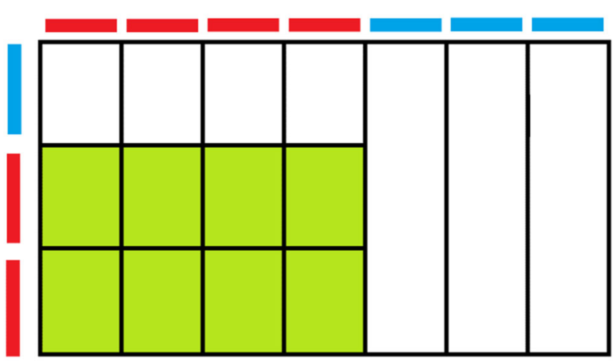
> There is a natural correspondence between portions of the side of a rectangular pie and portions of the whole pie itself.

Now let's ask:

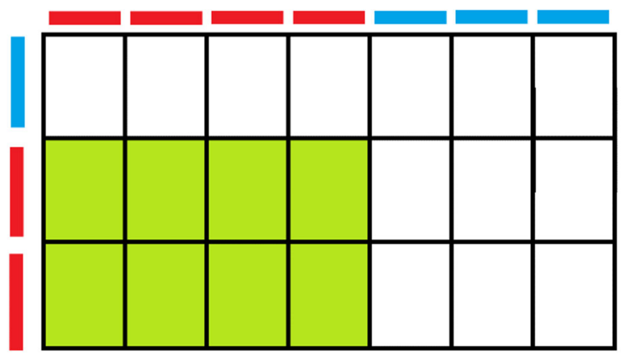
> What do we mean by "two thirds of four sevenths of a pie"?

The green part of the picture above, which represents four fifths of our pie, looks like a rectangular pie in-and-of itself. The question seems to be asking us to identify two-thirds of this green rectangular portion.

Here it is.



Drawing in extra lines helps makes sense of this picture.



Our "everyday thinking" here has led us to a picture of rectangular pie subdivided into 21 pieces with 8 of them highlighted. We are led to say:

*Two thirds of four sevenths of a rectangular pie is eight twenty-firsts of the pie.*

**Practice 45.2 OPTIONAL** Can you draw a picture of two thirds of four sevenths of a circular pie? Can you see eight twenty-firsts of the pie as a result of what you draw?
[Rectangular pies are so much friendlier to draw!]

This work suggests the following:

$$\frac{2}{3} \text{ of } \frac{4}{7} \text{ equals } \frac{8}{21}$$

Schoolbooks tend to suppress the words "of pie," which is naughty. The statement really should read: "$\frac{2}{3}$ of $\frac{4}{7}$ of a pie equals $\frac{8}{21}$ of the pie."

And one cannot help but notice that $8 = 2 \times 4$ and $21 = 3 \times 7$.

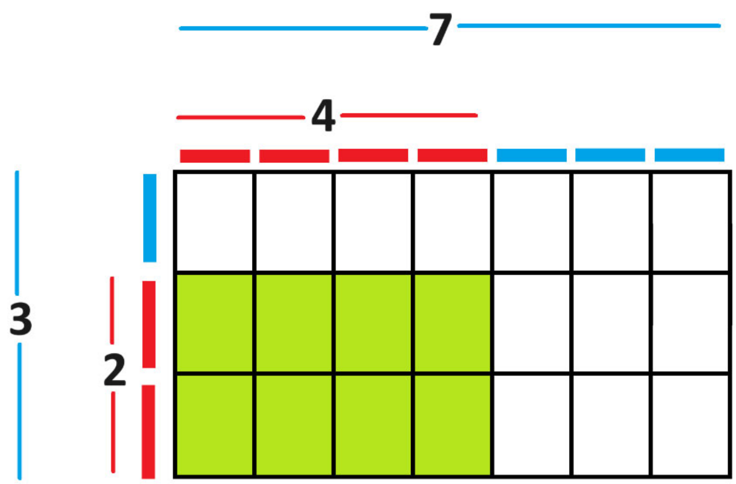
$$\frac{2}{3} \text{ of } \frac{4}{7} \text{ equals } \frac{2\times4}{3\times7}$$

Let’s be clear.

The operation “of” in this context of slicing pie is a physical action.
It is not a mathematical operation.

Nonetheless, we can use mathematical thinking to understand why a numerator of $8$ and a denominator of $21$ appeared when answering this particular practical problem.

In computing $\frac{2}{3}$ of $\frac{4}{7}$ of a rectangular pie, we naturally see the numbers $2$, $3$, $4$, and $7$ arise in the picture we draw.




The shaded pieces form a 2-by-4 rectangle, and so there are $2 \times 4 = 8$ of them.

These shaded pieces sit inside a 3-by-7 rectangle. There are $3 \times 7 = 21$ pieces in total.

Our final answer is thus naturally $\frac{2\times 4}{3\times 7}$.

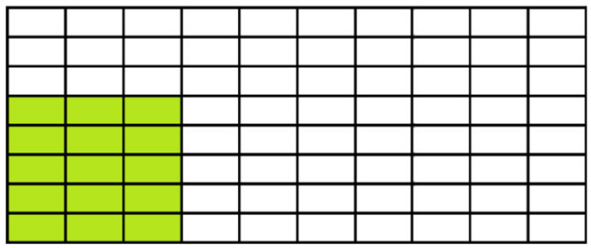
What is remarkable is that this answer happens to match the mathematical answer to $\frac{2}{3} \times \frac{4}{7}$.

What a lovely coincidence!

**Practice 45.3** The picture shows the result of another "portion of a portion" problem.

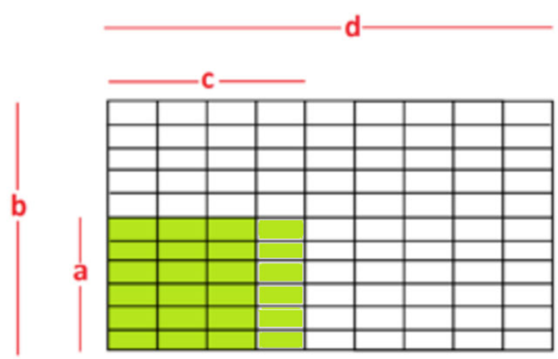
Fill in the blanks to show the coincidence continues.

$$\frac{\square}{8} \text{ of } \frac{\square}{\square} \text{ equals } \frac{\square \text{ x } 3}{8 \text{ x } \square}$$



Of course, this is not a coincidence. This general picture of a portion of a portion of a rectangular pie shows that the result will always be a fraction that happens to be given by the same formula we derived in Section 41 for the product of two fractions.

$$\frac{a}{b} \text{ of } \frac{c}{d} \text{ of a pie is } \frac{a \times c}{b \times d} \text{ of the pie.}$$




And $\frac{a \times c}{b \times d}$ is the mathematical answer to $\frac{a}{b} \times \frac{c}{d}$.

Schoolbooks lean into this alignment.

Since we often read $2 \times 5$, say, as **two groups <u>of</u> five** it feels reasonable to say that the word "of" is synonymous with the word "multiply."

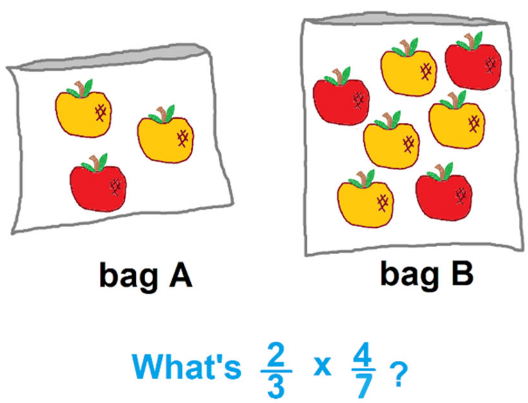
And drawing pictures of portions or portions of rectangular pie, which very much look like pictures of the area model for multiplication, gives the feeling of justifying why "of means multiply."

But the truth is that fractions have their own mathematics, and that mathematics tells us how to define fraction multiplication.

And, yes, it is absolutely lovely that this definition matches how people think of portions of portions of pie. But it is the math that comes first and the connection to real-world thinking that comes second.

Most schoolbooks muddle this up.

Not all real-world scenarios have a place or need for fraction multiplication. For example, in thinking about bags of yellow and red apples, trying to make sense of $\frac{2}{3} \times \frac{4}{5}$ is meaningless and silly.




Just because you and I know there is a mathematical answer to $\frac{2}{3} \times \frac{4}{5}$, it doesn't mean that the answer is meaningful for a particular scenario.

Math has the answers for all the places where it is meaningful and relevant to ask particular math questions. That's the beauty and power of mathematics.

It's always at-the-ready for us to use.

But it is up to us to recognize when a piece of mathematics does or doesn't have relevance to a particular scenario.

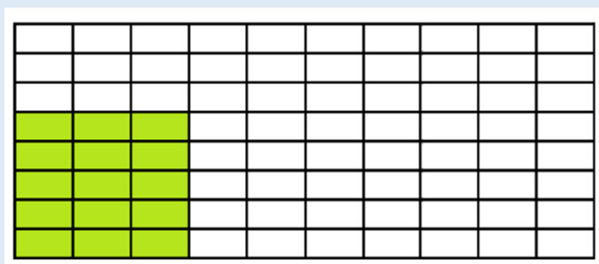
## MUSINGS

**Musing 45.4** In this section we showed that "$\frac{5}{8}$ of $\frac{3}{10}$" of a pie matches $\frac{5}{8} \times \frac{3}{7}$ of the pie.

But it is possible to switch multiplications around: $\frac{5}{8} \times \frac{3}{10}$ is mathematically the same as $\frac{3}{10} \times \frac{5}{8}$. This means we should be able to switch the order of the fractions around in a portion of a portion problem and obtain the same result.

> Do "$\frac{3}{10}$ of $\frac{5}{8}$ of a pie" and "$\frac{5}{8}$ of $\frac{3}{10}$ of a pie" represent the same amount of pie?
> It is not *a priori* obvious that they should!

Look at this picture.



a) Explain how to view this picture as representing "$\frac{5}{8}$ of $\frac{3}{10}$ of a pie."

b) Explain how to also view the same picture as representing "$\frac{3}{10}$ of $\frac{5}{8}$ of a pie."

Conclude that yes, we can switch the order of the fractions in a "portion of a portion" of statement.

**Musing 45.5 Is "two-thirds of three-fifths of seven-ninths of a pie" the same amount as "seven-ninths of two-thirds of three-fifths" of the same pie?**

## MECHANICS PRACTICE

Of course, one doesn't want to always want to think of the word "of" when multiplying fractions. Try these problems purely as arithmetic problems with no regard to pie!

**Practice 45.6**

a) *Show that* $\frac{407}{933} \times \frac{933}{407}$ *equals* $1$.

b) *More generally, show that multiplying a fraction* $\frac{\mathrm{a}}{\mathrm{b}}$ *by its* ***reciprocal*** $\frac{\mathrm{b}}{\mathrm{a}}$ *is sure to give* $1$.

$$\frac{a}{b} \times \frac{b}{a} = 1$$

**Practice 45.7** If $n = p \times q$, what is the value of $n \times \frac{p}{q}$?

**Practice 45.8** What is $\frac{-8}{9} \times \frac{2}{-5}$?

**Practice 45.9** What do you need to multiply $\frac{4}{7}$ by to get the answer $\frac{1}{2}$?

**Practice 45.10** What is the value of this unfriendly quantity?

$$\frac{7}{12} \times \frac{2}{5} \times 90 \times \frac{10}{21} \times \frac{3}{19} \times \frac{1}{3} \times \frac{57}{9}$$

## 46. Dividing Fractions the School Way: "Keep, Change, Flip" and the word "Of" again!

Dividing fractions in Section 41 was a not really an issue.

**Example:** Compute $\frac{5}{3} \div \frac{4}{7}$.

**Answer:** A fraction is an answer to a division problem. So, the answer to this division problem is the fraction

$$\frac{\frac{5}{3}}{\frac{4}{7}}$$

Our job is to make this fraction look friendlier.

Multiplying top and bottom each by 3 and by 7 yields

$$\frac{3 \times 7 \times \frac{5}{3}}{3 \times 7 \times \frac{4}{7}} = \frac{7 \times 5}{3 \times 4} = \frac{35}{12}$$

That's it,

$$\frac{5}{3} \div \frac{4}{7} = \frac{35}{12}$$

But let's pause for a moment. At one point along the way we wrote:

$$\frac{5}{3} \div \frac{4}{7} = \frac{7 \times 5}{3 \times 4}$$

We can switch the order of multiplication in the numerator and write instead

$$\frac{5}{3} \div \frac{4}{7} = \frac{5 \times 7}{3 \times 4}$$

And this answer can be thought of as the result of multiplying our first fraction $\frac{5}{3}$ by our second fraction $\frac{7}{4}$ but written upside down!

$$\frac{5}{3} \div \frac{4}{7} = \frac{5}{3} \times \frac{7}{4}$$

To divide these two fractions we, in effect, **KEPT** the first fraction as it was, **CHANGED** the division symbol to a multiplication symbol, and **FLIPPED** the second fraction upside down, and computed that product instead (a.k.a. the KCF rule).

**Practice 46.1** Prove that the KCF trick is valid in the general setting.

a) Compute $\frac{a}{b} \div \frac{c}{d}$

b) Compute $\frac{a}{b} \times \frac{d}{c}$

and show these have the same answer.

This **KCF** trick from schoolbooks can indeed be mathematically justified. But it really is unnecessary to memorize such a trick.  (Moreover, it is deeply sad and disturbing if this trick is the only thing offered to students in a discussion on the division of fractions.)

We simply looked at the expression $\frac{\frac{5}{3}}{\frac{4}{7}}$, balked at how confusing it looks, and followed our mathematical noses to make it appear more tractable. The answer $\frac{35}{12}$ naturally popped out. Moreover, if anyone would challenge our answer, we can explain why it is so!

**Practice 46.2:** Match each quantity on the left with its value on the right.
(You see that there are more options than correct answers!)

| | |
|---|---|
| | $1$ |
| $\frac{1}{2} \div \frac{1}{3}$ | $2$ |
| | $5$ |
| $\frac{4}{5} \div \frac{3}{7}$ | $\frac{2}{3}$ |
| | $\frac{3}{2}$ |
| $\frac{2}{3} \div 5$ | $\frac{5}{13}$ |
| $\frac{45}{45} \div \frac{902}{902}$ | $\frac{2}{15}$ |
| $\frac{10}{13} \div \frac{2}{13}$ | $\frac{28}{15}$ |
| | $\frac{15}{28}$ |

**Practice 46.3:** Did you catch in the previous question that $\frac{45}{45} \div \frac{902}{902}$ is just $1 \div 1$ in disguise and so has answer 1? (I love it when you can just use common sense and cut through clunky work!)

Computing $\frac{10}{13} \div \frac{2}{13}$ in Practice problem 46.2 is also curious. Its answer happens to be the same as the answer to $10 \div 2$ from just ignoring the common denominator of 13.

**Example:** Show that $\frac{a}{N} \div \frac{b}{N}$ and $a \div b$ are sure to have the same answer.

Some textbooks call this observation the **common denominator division method.**

**Answer:** The answer to the division problem $\frac{a}{N} \div \frac{b}{N}$ is the fraction

$$\frac{\frac{a}{N}}{\frac{b}{N}}$$

The answer to the division problem $a \div b$ is the fraction $\frac{a}{b}$.
Are these the same fraction?

It seems natural to multiply top and bottom of the complicated fraction each by $N$. This gives

$$\frac{\frac{a}{N}}{\frac{b}{N}} = \frac{N \times \frac{a}{N}}{N \times \frac{b}{N}} = \frac{a}{b}$$

Yes! They are the same fraction!

Wow!

Here's an example of how schoolbooks suggest using this "common denominator method."

**Example:** Compute $\frac{3}{4} \div \frac{2}{3}$ via the common denominator method.

**Answer:** We need to express each fraction with a shared denominator.

Since we have fourth and thirds, it seems that working with twelfths could be good.

$$\frac{3}{4} \div \frac{2}{3} = \frac{9}{12} \div \frac{8}{12}$$

Via the common denominator method, this equals $9 \div 8$, which is $\frac{9}{8}$.

**Practice 46.4:** Follow your instincts and show that $\frac{\frac{3}{4}}{\frac{2}{3}}$ does indeed equal $\frac{9}{8}$.

Truly, there is no need to memorize any rule or method, including this "common denominator method."
(But it is fun to try to explain why all these various curious methods actually work!)

**Practice 46.5**: Show that $\frac{12}{15} \div \frac{3}{5}$ equals $\frac{4}{3}$.

There is something extra strange about this example.

Notice that

$$12 \div 3 = 4$$

and

$$15 \div 5 = 3$$

and

$$\frac{12}{15} \div \frac{3}{5} = \frac{4}{3} = \frac{12 \div 3}{15 \div 5}$$

It is as though to compute $\frac{12}{15} \div \frac{3}{5}$ we just divided the numerators and divided the denominators separately.

Could it be true that $\frac{a}{b} \div \frac{c}{d}$ equals $\frac{a \div c}{b \div d}$? That is, to divide two fractions, just divide the numerators and divide the denominators individually?

That seems weird!

**Practice 46.6**: Show that $\frac{32}{35} \div \frac{8}{7}$ does equal $\frac{4}{5}$ (and again this is just $\frac{32 \div 8}{35 \div 7}$).

**Practice 46.7**:

a) The answer to $\frac{a}{b} \div \frac{c}{d}$ is $\frac{\frac{a}{b}}{\frac{c}{d}}$. Rewrite this expression to show that it equals $\frac{a \times d}{b \times c}$.

b) The quantity $\frac{a \div c}{b \div d}$ is $\frac{\frac{a}{c}}{\frac{b}{d}}$. Rewrite this expression to show that it also equals $\frac{a \times d}{b \times c}$.

This allows us to conclude that $\frac{a}{b} \div \frac{c}{d}$ does indeed equal $\frac{a \div c}{b \div d}$.

## “Of” and Division

People also often use the word “of” when they mean, or perhaps are just thinking, division.

For example, someone might say

> “half of six is three,” while thinking $6 \div 2 = 3$
>
> “a third of twelve is four,” while thinking $12 \div 3 = 4$.

But if we’re think “of means multiply” from our schoolbook training, then

> “half of six is three” is saying $\frac{1}{2} \times 6 = 3$.
>
> “a third of twelve is four” is saying $\frac{1}{3} \times 12 = 4$.

Is everything the same here? That is,

> Is multiplication by a basic fraction the same as division?

To think through this, let's play with

$$\frac{1}{2} \times 6$$

a wee bit.

By the general rules of arithmetic, we can switch the order of multiplication.

$$6 \times \frac{1}{2}$$

Now we just see this as a fraction "pulled apart." It is

$$\frac{6}{2}$$

But a fraction is the answer to a division problem: $\frac{6}{2}$ is the answer to $6 \div 2$.

Putting this all together we get

$$\frac{1}{2} \times 6 = 6 \times \frac{1}{2} = \frac{6}{2} = 6 \div 2$$

Yes! "A half of six" interpreted as multiplication by a fraction is the same as interpreting the expression as dividing by 2.

> **Practice 46.8** Show that interpreting "a third of twelve" as a statement of multiplication by a fraction ($\frac{1}{3} \times 12$) is equivalent to interpreting it as a statement of division ($12 \div 3$).

In general,

> $a \div N$ is the fraction $\frac{a}{N}$
>
> $\frac{1}{N} \times a$ equals $a \times \frac{1}{N}$, which is also $\frac{a}{N}$

So

> $a \div N$ and $\frac{1}{N} \times a$ are the same!

**Multiplying a quantity by $\frac{1}{N}$ is the same as dividing that quantity by $N$.**

## MUSINGS

**Musing 46.9**

a) Find the value of $\frac{1}{\frac{8}{7}}$.

b) In general, what is the value of $\frac{1}{\frac{a}{b}}$?

**Musing 46.10** Mathematicians say that "Division does not exist: it is multiplication by the reciprocal." This comes from noting that

**To "divide" a quantity by a number $N$ is to <u>multiply</u> that quantity by $\frac{1}{N}$.**

(We saw why this is so on the previous page.)

Consequently, mathematicians don't use the division symbol ÷ (the **obelus**) and will always write a quantity multiplied by a fraction instead.

Musing 17.2 had us consider the following ambiguous expression that regularly makes the rounds on the internet

$$8 \div 2(2+2)$$

a) Rewrite the expression in terms "multiplication by a fraction" that unambiguously evaluates to 1.
b) Rewrite the expression in terms "multiplication by a fraction" that unambiguously evaluates to 16.

## MECHANICS PRACTICE

**Practice 46.11** Compute each of the following.

a) $\frac{1}{2} \div \frac{1}{3}$ b) $\frac{4}{5} \div \frac{3}{7}$ c) $\frac{2}{3} \div \frac{1}{5}$

**Practice 46.12** Make the following look much friendlier.

$$\frac{8/5}{2/\left(\frac{5}{4}\right)}$$

## 47. Mixed Numbers

We have in our mathematical world positive whole numbers, negative whole numbers, zero, and fractions.

And we know that every integer can be thought of as a fraction too: rewrite it by introducing a denominator of 1. For example,

**3 is the fraction $\frac{3}{1}$.**

**$-20$ is the fraction $\frac{-20}{1} = -\frac{20}{1} = \frac{20}{-1}$.**

**Practice 47.1** What fraction is zero?

This means we can add together integers and fractions.

**Example:** What is the value of $3 + \frac{1}{2}$?

**Answer:** We have

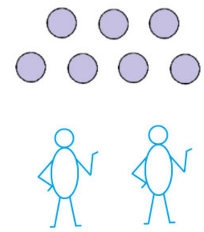
$$3 + \frac{1}{2} = \frac{3}{1} + \frac{1}{2}$$

The schoolbook approach has us find a common denominator.

$$3 + \frac{1}{2} \quad = \quad \frac{3}{1} + \frac{1}{2} \quad = \quad \frac{6}{2} + \frac{1}{2} \quad = \quad \frac{7}{2}$$

**Practice 47.2** Show again that $3 + \frac{1}{2} = \frac{7}{2}$ by writing $\frac{3+\frac{1}{2}}{1}$ and doubling top and bottom.

Going back to the practical world for a moment …



**Question:** If 7 pies are shared equally among 2 students, does each student get three whole pies and half a pie?

The answer is yes.

I brought up this “real-world” example as most people would prefer to think of and work with the expression $3+\frac{1}{2}$ (“three and a half”) rather than $\frac{7}{2}$ (“seven halves”) in the real world. It’s a common practice in society to mix counting numbers and fractions.

Let’s pin this idea down.

The counting number 3 and the fraction $\frac{1}{2}$ added together give the “mixed number” $3+\frac{1}{2}$. People usually omit the plus sign in such an expression and just write $3\frac{1}{2}$, and still read it out loud as even “three **and** a half.”

$5\frac{4}{17}$ means $5+\frac{4}{17}$ (five **and** four seventeenths).

$200\frac{1}{200}$ means $200+\frac{1}{200}$ (two hundred **and** one two-hundredth).

But society is fussy about what it will permit as a “valid” mixed number. (Are you surprised?)

**Example:** Show that mixed number $6\frac{4}{3}$ is the same as mixed number $7\frac{1}{3}$.

**Answer:** We have that $6\frac{4}{3}$ is

$$6+\frac{4}{3}$$

and $7\frac{1}{3}$ is

$$7+\frac{1}{3}$$

Are these the same?

Well, we can rewrite $7+\frac{1}{3}$ as

$$6+1+\frac{1}{3}$$

which is a little closer to making the two look alike.

The next thing is to realize that $\frac{4}{3}$ and $1+\frac{1}{3}$ are the same number.

We can think of $\frac{4}{3}$ as $\frac{3+1}{3}=\frac{3}{3}+\frac{1}{3}$ by doing addition in reverse. And this is indeed $1+\frac{1}{3}$.

Swiftly …

$$6\frac{4}{3} \;=\; 6+\frac{4}{3} \;=\; 6+\frac{3+1}{3} \;=\; 6+\frac{3}{3}+\frac{1}{3} \;=\; 6+1+\frac{1}{3} \;=\; 7+\frac{1}{3} \;=\; 7\frac{1}{3}$$

**Practice 47.3:** Show that mixed number $9\frac{17}{5}$ is the same as mixed number $12\frac{2}{5}$.

**Practice 47.4:** Explain why the mixed number $2\frac{0}{9}$ is just the number 2.

**Example:** Show that the mixed number $4\frac{-2}{3}$ is the same as the mixed number $3\frac{1}{3}$.

**Answer:** Let's be swift from the get-go this time. Make sure you follow what is happening from one line to the next.

We have

$$\begin{aligned} 4\frac{-2}{3} &= 4+\frac{-2}{3} \\ &= 3+1+\frac{-2}{3} \\ &= 3+\frac{3}{3}+\frac{-2}{3} \\ &= 3+\frac{3+\;-2}{3} \;=\; 3+\frac{1}{3} \;=3\frac{1}{3} \end{aligned}$$

**Practice 47.5** Match each quantity on the left with a quantity on the right.

| | |
|---|---|
| $\frac{20}{7}$ | $7$ |
| $0\frac{7}{20}$ | $7\frac{1}{20}$ |
| $7\frac{0}{10}$ | $7\frac{1}{2}$ |
| $8\frac{-1}{2}$ | $\frac{7}{20}$ |
| $7\frac{1/2}{10}$ | $2\frac{6}{7}$ |

Each expression on the right is either a positive whole number, a fraction with numerator and denominator each a counting number with numerator smaller than denominator, or a counting number and such a fraction added together.

Society won't accept any of the expressions on the left as a "valid" mixed number, but it will accept each answer on the right as either a whole number, a "proper" fraction, or as a valid mixed number.

Here's the societal definition of a mixed number.

> A ***mixed number*** is an expression of the form $a\frac{b}{c}$ with $a$, $b$, and $c$ is each a non-zero counting number and with $b$ smaller than $c$.
>
> Such an expression is really the number $a + \frac{b}{c}$.

No zeros allowed within a mixed number expression.
No negative integers are allowed either.
And no fractional expressions with numerators larger than their denominators are permitted as well.

Demanding!

Let's practice doing some arithmetic with mixed numbers.

**Example:** What is $6\frac{2}{7} + 3\frac{3}{4}$? Write your answer as a societally acceptable mixed number.

**Answer:** We must work with

$$6 + \frac{2}{7} + 3 + \frac{3}{4}$$

Since we can conduct a string of summations in any order we like, this is

$$9 + \frac{2}{7} + \frac{3}{4} \;=\; 9 + \frac{8}{28} + \frac{21}{28} \;=\; 9 + \frac{29}{28}$$

But we have a fraction with numerator larger than denominator. We must keep going!

$$9 + \frac{29}{28} = 9 + \frac{28+1}{28} = 9 + \frac{28}{28} + \frac{1}{28} = 10 + \frac{1}{28}$$

So,

$$6\frac{2}{7} + 3\frac{3}{4} = 10\frac{1}{28}$$

(This is an absurd amount of work just to appease societal expectations!)

**Practice 47.6:** Compute $15\frac{1}{4} + 5\frac{4}{5}$ and write your answer as a societally acceptable mixed number.

**Example:** Write $\frac{16}{3}$ as a societally acceptable mixed number.

**Answer:** We have

$$\frac{16}{3} = \frac{15+1}{3} = \frac{15}{3} + \frac{1}{3} = 5 + \frac{1}{3} = 5\frac{1}{3}$$

(Make sure you followed each step here.)

**Practice 47.7:** Write $\frac{400}{99}$ as a societally acceptable mixed number.

**Example:** Write $7\frac{4}{9}$ as a single fraction.

**Answer:** We have

$$7 + \frac{4}{9} = \frac{7}{1} + \frac{4}{9} = \frac{63}{9} + \frac{4}{9} = \frac{67}{9}$$

(Again, make sure you followed each step here.)

**Alternatively …**

$$7 + \frac{4}{9} = \frac{7 + \frac{4}{9}}{1} = \frac{9 \times (7 + \frac{4}{9})}{9 \times 1} = \frac{63 + 4}{9} = \frac{67}{9}$$

(Does this make sense too?)

**Practice 47.8:** Write $200\frac{1}{200}$ as a single fraction.

**Practice 47.9** What is $5\frac{3}{4}$ quadrupled?

## Negative Mixed Numbers

Even though we can make sense of mixed numbers with negative numbers as entries, the textbook definition won't allow them.

Nonetheless, you will see in textbooks quantities such as the following.

$$-5\frac{3}{4}$$

**Question:** How do you think school students are expected to interpret this expression?

As …

a) $-5+\frac{3}{4}$ b) $5-\frac{3}{4}$ c) $-5-\frac{3}{4}$ d) $-5+\frac{-3}{4}$ e) Some other way

Matters are confusing.

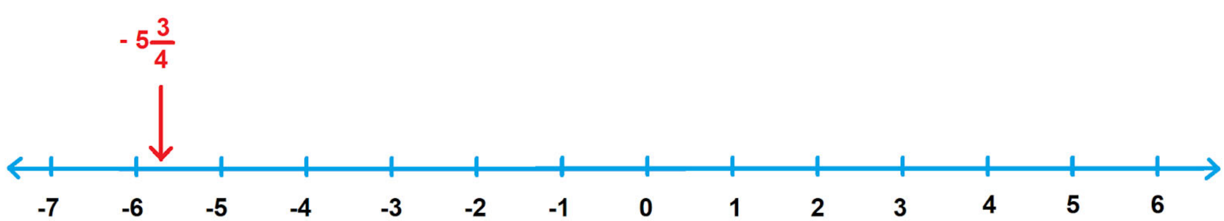
The convention is that, since mixed numbers are written using only positive whole numbers, a negative sign in front of a mixed number applies to the mixed number altogether, not just to part of the mixed number.

$$\boxed{-a\frac{b}{c} \text{ is taken to mean } -\left(a+\frac{b}{c}\right).}$$

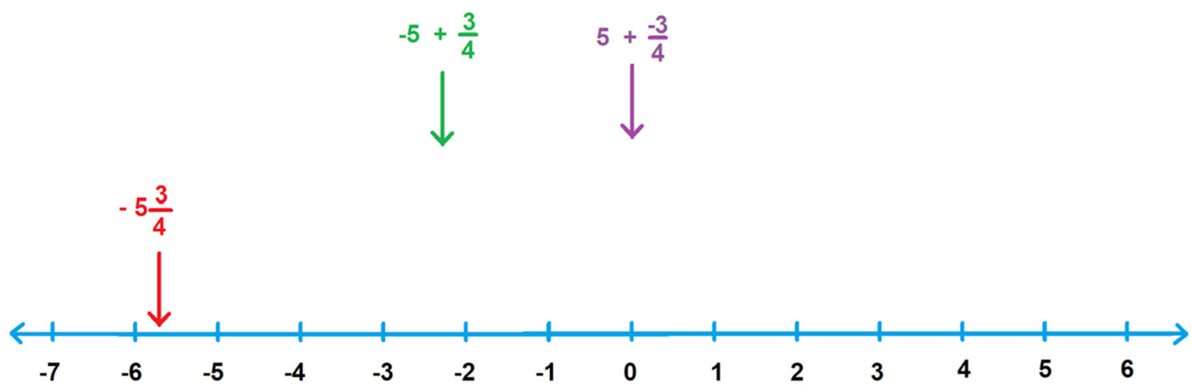
And by distributing the negative sign (Section 21), this means:

$$-a\frac{b}{c} = -a - \frac{b}{c}$$

So, $-5\frac{3}{4}$ means $-5-\frac{3}{4}$ (which happens to be the same as $-5+\frac{-3}{4}$, making options c) and d) both correct in the question above). Consequently, $-5\frac{3}{4}$ is value between $-5$ and $-6$ on the number line.



**Practice 47.10:** Where on the number line is $-5 + \frac{3}{4}$? Where is $5 + \frac{-3}{4}$? Can you imagine the correct placements of the green and purple arrows?



## Subtracting Mixed Numbers

Here's a subtraction problem.

> Compute $13\frac{3}{7} - 10\frac{5}{7}$.

Think for a moment about how you might work through this question.

This question represents a good moment **not** to believe in subtraction!
Regard subtraction as the addition of the opposite.

$$13\frac{3}{7} - 10\frac{5}{7} = 13\frac{3}{7} \ + \ -10\frac{5}{7}$$

And what is $-10\frac{5}{7}$? It's $-10 - \frac{5}{7}$.

So

$$13\frac{3}{7} - 10\frac{5}{7} \ = \ 13 + \frac{3}{7} \ - 10 - \frac{5}{7}$$

This gives us

$$3 + \frac{-2}{7}$$

(Do you see this?)

Now let's fix up this answer for society. We have

$$3 + \frac{-2}{7} = 2 + \frac{7}{7} + \frac{-2}{7} = 2 + \frac{5}{7}$$

Thus

$$13\frac{3}{7} - 10\frac{5}{7} = 2\frac{5}{7}$$

**Practice 47.11** Match time!

| | |
|---|---|
| $2\frac{2}{5} + 3\frac{4}{5}$ | $6\frac{1}{5}$ |
| $10\frac{7}{10} - 4\frac{3}{10}$ | $6\frac{2}{5}$ |
| $8\frac{5}{12} - 1\frac{2}{3}$ | $6\frac{1}{2}$ |
| $102\frac{2}{5} - 95\frac{9}{10}$ | $6\frac{3}{4}$ |

Of course, we can always sidestep all this mixed number work and convert each number into a single fraction if we like.

**Example:** Compute $5\frac{1}{3} - 1\frac{1}{2}$ by turning each mixed number into a single fraction first.

(Aside: It is sometimes fun—and good!—to get an estimate of the answer to a calculation first. This question is essentially "$5 - 1$,", which is $4$, plus the difference of two fractions. One should obtain an answer close to $4$.)

**Answer:** Writing each mixed number as a fraction gives

$$3\frac{1}{3} = \frac{15}{3} + \frac{1}{3} = \frac{16}{3}$$

and

$$1\frac{1}{2} = \frac{2}{2} + \frac{1}{2} = \frac{3}{2}$$

So, we need to compute $\frac{16}{3} - \frac{3}{2}$. Going for a common denominator, we get

$$\frac{16}{3} - \frac{3}{2} = \frac{32}{6} - \frac{9}{6} = \frac{23}{6}$$

Thus, $5\frac{1}{3} - 1\frac{1}{2} = \frac{23}{6}$.

There is no need to do anything further with this answer (the question had no further instructions), but if you like you can rewrite $\frac{23}{6}$ as the mixed number

$$3\frac{5}{6}$$

(And yes, this is an answer close to $4$.)

**Practice 47.12** What is $90\frac{1}{6} - 100\frac{2}{3}$. Write your answer as a negative mixed number.

## Dividing Mixed Numbers

There is nothing new for us here. We just have to remember that the answer to a division problem is a fraction.

Back in Section 39 when we were playing with pies and students and, while being quirky, asked for the value of

$$2\frac{1}{2} \div 4\frac{1}{2}$$

That is, we were asking for the result of "sharing two-and-a-half pies equally among four-and-a-half students."

The answer is this fraction:

$$\frac{2\frac{1}{2}}{4\frac{1}{2}}$$

It feels compelling to double the numerator and denominator here. That should clear away those annoying halves.

$$\frac{2 \times (2 + \frac{1}{2})}{2 \times (4 + \frac{1}{2})} = \frac{2 \times 2 + 2 \times \frac{1}{2}}{2 \times 4 + 2 \times \frac{1}{2}} = \frac{4 + 1}{8 + 1} = \frac{5}{9}$$

And there we have it

$$2\frac{1}{2} \div 4\frac{1}{2} \quad = \quad \frac{5}{9}$$

Sharing two-and-a-half pies equally among four-and-a-half students is equivalent to sharing five pies among nine students. Each student gets five-ninths of a pie. That's much more manageable to envision.

It was hard to nut our way through this problem when we were working with the "real-world" scenario of sharing pies with students. But mathematics set free from real-world limitations is often much easier to do and is more universal: we can likely apply the result to more than one concrete scenario.

**Practice 47.13** Rewrite $\frac{3\frac{1}{3}}{\frac{2}{3}}$ as a much simpler number. That is, compute $3\frac{1}{3} \div \frac{2}{3}$.

**Example:** Show that $\frac{1\frac{4}{5}}{2\frac{1}{6}}$ is equivalent to $\frac{54}{65}$.

**Answer:** We have

$$\frac{1\frac{4}{5}}{2\frac{1}{6}} = \frac{1+\frac{4}{5}}{2+\frac{1}{6}}$$

Let's contend with the fifths first by quintupling the numerator and the denominator.

$$\frac{5\times(1+\frac{4}{5})}{5\times(2+\frac{1}{6})} = \frac{5+5\times\frac{4}{5}}{10+5\times\frac{1}{6}} = \frac{5+4}{10+\frac{5}{6}}$$

Let's now multiply top and bottom each by 6.

$$\frac{6\times(5+4)}{6\times(10+\frac{5}{6})} = \frac{30+24}{60+6\times\frac{5}{6}} = \frac{54}{65}$$

And there it is.

Okay. Time for an extra juicy example.

**Example:** Show that $7\frac{2}{3} \div 5\frac{3}{4}$ is $1\frac{1}{3}$.

**Answer:** The answer to a division problem is a fraction. So, our goal is to show that

$$\frac{7\frac{2}{3}}{5\frac{3}{4}} = \frac{7+\frac{2}{3}}{5+\frac{3}{4}}$$

is $1\frac{1}{3}$ in disguise.

We have an expression involving thirds and fourths. Multiplying top and bottom of this expression each by 3 and then by 4 would be helpful.

Let's start by tripling top and bottom.

$$\frac{3 \times (7 + \frac{2}{3})}{3 \times (5 + \frac{3}{4})} = \frac{21 + 2}{15 + \frac{9}{4}}$$

Now quadruple top and bottom.

$$\frac{84 + 8}{60 + 9} = \frac{92}{69}$$

So, $7\frac{2}{3} \div 5\frac{3}{4}$ is $\frac{92}{69}$.

This doesn't look like $1\frac{1}{3}$. Hmm.

Well, we can write $\frac{92}{69}$ as a mixed number, at least.

$$\frac{92}{69} = \frac{69 + 23}{69} = 1 + \frac{23}{69}$$

Is $\frac{23}{69}$ just one-third in disguise?

Yes.

$$\frac{23}{69} = \frac{1 \times 23}{3 \times 23} = \frac{1}{3}$$

Great!

Putting it all together …

$$\frac{7\frac{2}{3}}{5\frac{3}{4}} = 1 + \frac{23}{69} = 1 + \frac{1}{3} = 1\frac{1}{3}$$

Phew!

The previous example involved a lot of arithmetic.

But the technique was simply to "follow your mathematical nose" all the way through it.

**Practice 47.14** When faced with $\frac{7+\frac{2}{3}}{5+\frac{3}{4}}$, did it cross your mind to multiply top and bottom each by $3 \times 4$ right off the bat? If you do that, do you see that you get $\frac{84+8}{60+9} = \frac{92}{69}$ as we had earlier?

**Practice 47.15**: True or False? All three of these quantities have the same value.

$$\frac{3\frac{1}{3}}{12\frac{1}{2}} \qquad \frac{\frac{4}{3}}{5} \qquad \frac{2}{3} \div \frac{5}{2}$$

## MUSINGS

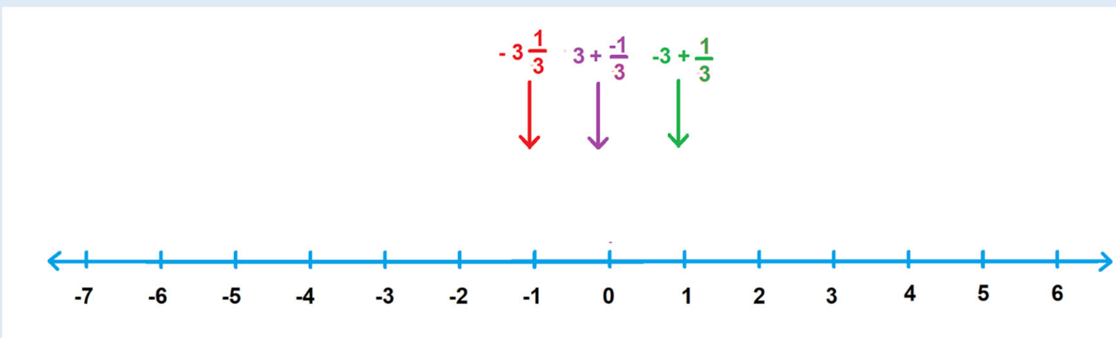
**Musing 47.16**

a) What number must we add to $2\frac{5}{13}$ to get the value $10$?

b) What number must we add to $-2\frac{5}{13}$ to get the value $10$?

c) What number must we subtract from $-10$ to get the value $2\frac{5}{13}$?

**Musing 47.17** Place each of these arrows in their correct location on the number line.




**Musing 47.18** MULTIPLYING MIXED NUMBERS

Consider $3\frac{1}{3} \times 4\frac{1}{2}$.

Without thinking too deeply, many students say that this product has value $12\frac{1}{6}$.

a) Do you see why this is a tempting answer to give?

We can check this answer by converting each mixed number into single fractions and then multiplying the fractions.

We have

$$3\frac{1}{3} = \frac{9}{3} + \frac{1}{3} = \frac{10}{3}$$

$$4\frac{1}{2} = \frac{8}{2} + \frac{1}{2} = \frac{9}{2}$$

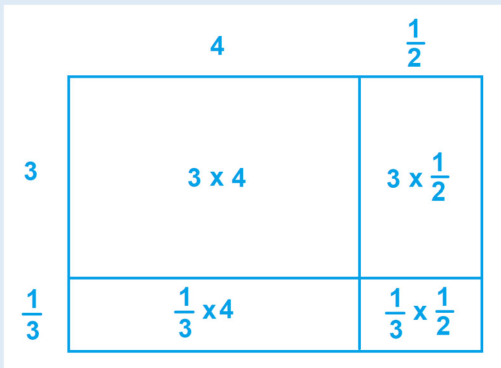
and so

$$3\frac{1}{3} \times 4\frac{1}{2} = \frac{10}{3} \times \frac{9}{2} = \frac{90}{6} = 15$$

This is not $12\frac{1}{6}$. What went wrong?

b) Let's compute $3\frac{1}{3} \times 4\frac{1}{2}$ via the area model, chopping a rectangle into four pieces.

Show that $12 + \frac{4}{3} + \frac{3}{2} + \frac{1}{6}$ does indeed sum to 15. (This is the correct value of $3\frac{1}{3} \times 4\frac{1}{2}$.)




c) When a student says that $3\frac{1}{3} \times 4\frac{1}{2}$ equals $12\frac{1}{6}$, which pieces of the chopped-up rectangle did the student fail to consider in their calculation?

d) Compute $1\frac{2}{3} \times 2\frac{1}{2}$ by

i) converting each mixed number into a fraction and then multiplying
ii) the area model

Do your answers agree? (They should!)

## MECHANICS PRACTICE

**Practice 47.19** Write each of these expression as a societally acceptable mixed number

a) $\frac{8}{5}$ b) $\frac{100}{13}$ c) $\frac{200}{199}$ d) $\frac{199\frac{1}{2}}{199}$

**Practice 47.20** Write each expression as single fraction.

a) $7\frac{2}{9}$ b) $2\frac{3}{4}+5\frac{2}{7}$ c) $300\frac{299}{300}$ d) $2\frac{3}{4}-5\frac{2}{7}$ e) $873$

**Practice 47.21**

a) What is $9\frac{1}{2}$ doubled?

b) What is $-8\frac{1}{3}$ tripled?

**Practice 47.22** Compute as many of these as you have the patience for.

a) $2\frac{2}{5}+\frac{2}{3}$ b) $10\frac{1}{7}-9\frac{3}{7}$ c) $6\frac{3}{4}-5\frac{8}{9}$ d) $199\frac{1}{3}-198\frac{1}{2}$

**Practice 47.23** Make each of the following look much friendlier.

a) $\frac{3\frac{1}{2}}{1\frac{1}{2}}$ b) $\frac{4\frac{2}{3}}{5\frac{1}{3}}$ c) $\frac{2\frac{1}{5}}{2\frac{1}{4}}$ d) $\frac{1\frac{4}{7}}{2\frac{3}{10}}$ e) $\frac{3/5}{4/7}$

# 48. Percentages

Let's start with a little piece of history.

Two thousand years ago, emperor Augustus (25 BCE - 14 CE) was cash-strapped. He needed funds to support the expansion of the Roman Empire.

As expansion was supposedly for the good of the Roman citizenry, he came up with the idea of levying a tax on Roman citizens to pay for this enterprise. He decreed that "one part per hundred" of all monetary transactions that occurred in markets were to be transferred to the Empire.

Consequently, if 100 **aureus** (gold coins) were passed between the hands of two citizens making a sale, one coin was given to the Empire. If they exchanged a thousand coins, ten were given to the Empire.

The Romans also used silver, bronze, and copper coins each worth fractional amounts of gold coins. For example, 25 **denarius** (silver coins) were equivalent to 1 aureus.

> **Question 48.1** If two Roman citizens exchanged 4 aureus, how many denarius were handed to the Empire?

From the Latin term *per centum* meaning "part per hundred" we obtained the word **percentage**.

We are still being charged taxes to this day and tax rates are still presented in terms of parts per one hundred (as are interest rates and store discounts and other transactional quantities).

### The Mathematical Meaning of Percentage

Quite simply, a **percentage** is a fraction expressed as a part per hundred. That is, it is a fraction with denominator $100$.

But annoyingly, we don't write percentages explicitly as fractions, and use the curious symbol % to mean "and the denominator is $100$."

For example,

> 17% is the fraction $\frac{17}{100}$.
>
> 50% is the fraction $\frac{50}{100}$ (and this is the same as $\frac{50\times1}{50\times2} = \frac{1}{2}$).
>
> 300% is the fraction $\frac{300}{100}$ (and this is the same as the number $3$).

In general, N% is just the fraction with numerator $N$ and denominator $100$.

$$N\% = \frac{N}{100}$$

**Practice 48.2** Explain why 25% is the fraction $\frac{1}{4}$

**Example:** Explain why $\frac{1}{2}\%$ is the fraction $\frac{1}{200}$.

**Answer:** We have that $\frac{1}{2}\%$ is the fraction

$$\frac{\frac{1}{2}}{100}$$

Multiplying top and bottom each by $2$ shows that this is the fraction

$$\frac{1}{200}$$

**Practice 48.3** What fraction is $\frac{1}{10}\%$?

During the Renaissance in Europe, Italy developed robust financial institutions. Accountants and financiers, writing in Italian rather than Latin, wrote *per cento* in their legers for "one part per hundred."

But they started to abbreviate this to phrase to just the letter "p" with a curly tail. (Perhaps the tail represented the "c" or the "o" from the word *cento*.) Eventually this shorthand for *per cento* transmuted into the symbol % we use today.

**Practice 48.4** The term *per mille* is denoted ‰ and is sometimes used today. Make a guess as to what this term means. (Hint: How many years are in a millennium?)

To convert a fraction into a percentage we need to rewrite the fraction to have a denominator of $100$.

For example, $\frac{1}{5}$ can be rewritten as $\frac{1\times20}{5\times20}=\frac{20}{100}$ and so

$$\frac{1}{5}=20\%$$

And, $\frac{3}{2}$ can be rewritten as $\frac{3\times50}{2\times50}=\frac{150}{100}$ and so

$$\frac{3}{2}=150\%$$

**Practice 48.5** What is $\frac{7}{10}$ as a percentage?

Of course, most fractions are not this friendly to work with.

Let's rewrite $\frac{15}{32}$ as a percentage. (I am not sure why we would want to in real life! We're just doing it now for practice.)

We can get a denominator close to 100 by tripling the top and bottom.

$$\frac{15\times3}{32\times3}=\frac{45}{96}$$

and so, as a percentage, the fraction is close to 45%. (Will the exact value be bigger or smaller than this?)

Let's find the exact percentage.

We want to convert the fraction $\frac{15}{32}$ to one with a denominator of 100. But that denominator of 32 is annoying! I wish we had a nicer denominator.

Well, we can always create a denominator of 1.

$$\frac{15}{32} = \frac{\frac{15}{32}}{1}$$

And to make a denominator of 100, let's multiply top and bottom by 100.

$$\frac{15}{32} = \frac{\frac{15}{32} \times 100}{1 \times 100} = \frac{\frac{1500}{32}}{100}$$

So,

$$\frac{15}{32} = \frac{1500}{32}\%$$

My calculator says that $\frac{1500}{32}$ is a value between 46 and 47. So ,yes, this is close-ish to 45%.

**Practice 48.5** Write $\frac{1500}{32}$ as a mixed number. Show that $\frac{15}{32} = 46\frac{7}{8}\%$.

**Practice 48.6** Many schoolbooks tell students a rule.

*To convert a fraction $\frac{a}{b}$ into a percentage, multiply the fraction by 100 and slap on a percentage sign.*

Did we, in effect, do just that when we converted $\frac{15}{32}$ into a percentage?

My advice is to **not** memorize a rule like this. Memorizing items that feel random is just too hard!

Instead, simply understand that a percentage is just a fraction with denominator 100. And your task is thus to do whatever you can to create a denominator of 100! (You always have the option to start by creating a denominator of 1.)

**Practice 48.7.6** What is $\frac{5}{6}$ as a percentage? Write your final answer as a mixed number.

**Example:** What fraction is $2\frac{1}{2}\%$?

**Answer:** This percentage is the fraction $\frac{2+\frac{1}{2}}{100}$. Doubling top and bottom we see that this is

$$\frac{4+1}{200} = \frac{5}{200} = \frac{5 \times 1}{5 \times 40} = \frac{1}{40}$$

This next question feels strange.

What is the number $1$ as a percentage?

To answer this we need to write $1$ as a fraction with denominator $100$. Writing

$$1 = \frac{100}{100}$$

does the trick.

We see that

$$1 = 100\%$$

People say, "If you've got the whole of something, then you've got 100% of it," and the number $1$ represents "the whole" they have in mind in this statement.

**Practice 48.8** What is the number $30$ as a percentage?

a) $30\%$ a) $300\%$ a) $3{,}000\%$ a) $30{,}000\%$

**Example:** What is $71\frac{3}{7}\%$ as a fraction?

**Answer:** It's

$$\frac{71\frac{3}{7}}{100}$$

To make this fraction look friendlier, let's multiply top and bottom each by 7. This gives

$$\frac{490+7+3}{700}=\frac{500}{700}=\frac{5}{7}$$

It's the fraction five sevenths!

**Practice 48.9** Match each quantity on the left with its matching quantity on the right.

| | |
|---|---|
| $\frac{13}{2}$ | $\frac{1}{100}$ |
| $1000\%$ | $\frac{3}{20}$ |
| $\frac{4}{5}$ | $116\frac{1}{3}\%$ |
| $1\%$ | $10$ |
| $15\%$ | $80\%$ |
| $\frac{7}{6}$ | $650\%$ |

## MUSINGS

**Musing 48.10** Speaking of the Romans …

The Romans of ancient times used words to describe fractions.

A twelfth of a unit of weight, $\frac{1}{12}$, for instance, was called the **uncia** (from which we obtained the word "ounce") and $\frac{11}{12}$ was called **deunx**, short for *de uncia* meaning "one twelfth taken away."

Two twelfths, $\frac{1}{6}$, was called **sextans**, and three twelfths, $\frac{1}{4}$, **quadrans**.

a) Makes some guesses: Which fraction is **dextans**? Which fraction is **dodrans**?

b) Make some more guesses: Which fraction is **quinque unciae**? Which is **septem unciae**?

Six twelfths, $\frac{1}{2}$, was called **semis.**

c) Make a guess: Which fraction was **semuncia**?

## MECHANICS PRACTICE

**Musing 48.10** Recall that the multiplication of fractions matches society's use of the word "of."

For example, "two-thirds of 90" can legitimately be calculated as

$$\frac{2}{3} \times 90$$

which equals $2 \times \frac{1}{3} \times 90 = 2 \times \frac{90}{3} = 2 \times 30 = 60$.

In the same way, computing "105% of 400" requires computing the product

$$\frac{105}{100} \times 400$$

It gives the answer $420$.

a) What is $15\%$ of $80$?
b) What is $80\%$ of $15$?

c) What is $75\%$ of $50$?
d) What is $50\%$ of $75$?

e) What is $25\%$ of $300$?
f) What is $300\%$ of $25$?

Can you explain why "$a\%$ of $b$" and "$b\%$ of $a$" are sure to have the same value?

# 49. Tips, Percentage Increase, and Percentage Decrease

The Mechanics Practice problem of the previous page reminded us that the multiplication of fractions matches society's use of the word "of."

**Example:** What is $17\frac{1}{2}\%$ of $160$?

To answer this, we need to compute

$$\frac{17\frac{1}{2}}{100} \times 160$$

which, to be frank, doesn't seem fun.

**Practice 49.1** Try computing $17\frac{1}{2}\%$ of $160$ before reading on.
(Or feel free to say "NO. I WON'T!")

Problems of this nature arise often in everyday life, most notably, when computing tips on services provided. And to handle such computation, people typically **don't** pull out pencil-and-paper and multiply awkward fractions.

Instead, they use the "anchor point' of 10% of the bill to help them do a mental calculation.

**Example:** You just had a very fancy dinner and a bill of \$160 has come your way. You want to add a 15% tip. How much is such a tip?

The tip is "$15\%$ of $160$," that is, it's

$$\frac{15}{100} \times 160$$

This is tricky to work out in your head.

But working out "10% of 160" instead is fairly easy. It's $\frac{10}{100} \times 160$ and that's $\frac{1}{10} \times 160 = 16$.

**$10\%$ of a quantity is that quantity divided by ten.**

We have that $10\%$ of $\$160$ is $\$16$.

If we halve this, we then deduce that $5\%$ of $\$160$ is $\$8$.

$$\boxed{\mathbf{10\%} \longleftrightarrow \mathbf{\$16}}$$

$$\mathbf{5\%} \longleftrightarrow \mathbf{\$8}$$

Our $15\%$ must thus be $16 + 8 = 24$ dollars.

> **Example:** You are feeling generous and decide to instead leave a $20\%$ tip instead on your $\$160$ bill. What's your new tip?

Let's just double our anchor point of $10\%$.

$$\mathbf{20\%} \longleftrightarrow \mathbf{\$32}$$

$$\boxed{\mathbf{10\%} \longleftrightarrow \mathbf{\$16}}$$

$$\mathbf{5\%} \longleftrightarrow \mathbf{\$8}$$

Your tip is now $32.

> **Example:** You've changed your mind: a tip of 20% feels a bit too much and a tip of 15% too small. You decide to leave a tip of $17\frac{1}{2}\%$.
>
> What is $17\frac{1}{2}\%$ of $\$160$?

Let's keep halving from our 10% anchor point.

$$\mathbf{20\%} \longleftrightarrow \mathbf{\$32}$$

$$\boxed{\mathbf{10\%} \longleftrightarrow \mathbf{\$16}}$$

$$\mathbf{5\%} \longleftrightarrow \mathbf{\$8}$$

$$\mathbf{2\tfrac{1}{2}\%} \longleftrightarrow \mathbf{\$4}$$

We can build the number $17\frac{1}{2}$ as $10 + 5 + 2\frac{1}{2}$.

This means that our $17\frac{1}{2}$% tip on $160 must be $16 + 8 + 4 = 28$ dollars!

**Practice 49.2** What is $47\frac{1}{2}$% of $160?

**Keep in mind an "anchor point" of 10% of a bill**
**and use it to compute the tip amount for all standard percentages.**

**Practice 49.3** What is 18% of $310? (Can you reason out an answer?)

Let's now consider another confusing matter, namely, statements like these you could read in a news article.

*The average price of a home in the metropolitan area has increased 300% over the past four years.*

What does this mean?

Let's say that the average home in the area cost $200,000 four years ago.

"Three hundred percent" represents the fraction $\frac{300}{100}$. Consequently "300% of $200,000" means

$$\frac{300}{100} \times 200{,}000 = 600{,}000$$

It's the figure, tripled.

But this is just the price increase!

So, the new price of the house is $\$200{,}000 + \$600{,}000 = \$800{,}000$.

The actual home price has quadrupled!

The language of “percentage increase” and “percentage decrease” is always horribly confusing.

But it is helpful to realize the percentage referred to is always (or should always be) **based on the initial figure before it changes.**

For instance, consider this problem.

> **Example:** A sofa normally costs $600, but it is now on sale for $420.
> What percentage decrease in cost is that?

Let’s work through this slowly.

We start by making sense of the term “percentage decrease.”

This term sounds like it wants the decrease in terms of a percentage. Great start!

*What is the decrease?*

Let’s just answer that in the obvious way. The price went from $600 down to $420. That's a decrease of $180. But we want to express $180 as a percentage.

*A percentage of what?*

We have two choices: of $600 or of $420.

A percentage increase or decrease should always refer to the initial figure.
So, we want $180 as a percentage of $600.

Now, a percentage is just a fraction.

*What fraction is $180 of $600?*

It’s $\frac{180}{600}$.

*What’s* $\frac{180}{600}$ *written as a percentage?*

It’s $\frac{180}{600} = \frac{\frac{1}{6}\times 180}{\frac{1}{6}\times 600} = \frac{30}{100} = 30\%$

A change of price from $600 to $420 represents a 30% decrease in price.

**Example Continued:** Even though you have a 30% discount on your \$600 sofa, you still have to pay a sales tax of 15%.

What will be your total payment for the sofa?

**Answer:** The sales tax is 15% of \$420.

Now,

10% of \$420 is \$42

and

5% of \$420 is half of this, \$21.

So, you need to pay $42 + 21 = 63$ dollars in sales tax.

The total cost of your sofa is \$483.

**Example Still Continued:** The shop clerk decides to apply 15% sales tax to the original price of \$600 first, and then reduce the total amount you owe by 30%.

Should you complain?

**Answer:** We have that 15% of \$600 is $60 + 30 = 90$ dollars.

Now, a 30% discount to this total amount, $600 + 90 = 690$ dollars, corresponds to a discount of

$$\frac{30}{100} \times 690 = \frac{30 \times 690}{100} = 3 \times 69 = 207$$

dollars.

Your final bill from the clerk is $690 - 207 = 483$ dollars again!

Her approach made no difference!

This is surprising! Adding 15% of value to a quantity and then reducing the total amount by 30% has the same effect as reducing the quantity by 30% first and then increasing that result by 15%.

Hmm!

Language really matters in percentage increase or percentage decrease statements.

**Practice 49.4:** Consider these two statements.

a) I got a raise! My salary has increased by 120%.
b) I got a raise! My salary has increased to 120% of what it was.

Which of these situations would you prefer to be in?

Let's end this Section with one more typical (and a bit annoying) textbook example.

**Practice 49.5** A town's population has changed from 12,500 to 14,000 over the past three months.

a) What percentage increase is that?
b) What is the population of the town now as a percentage of its previous population figure?

The mathematics of percentage problems, in and of itself, isn't really the challenge. It's figuring out what mathematics one is expected to do that is the tricky part.

In the end, you really just have to follow your common sense of how people use language and go from there.

## MUSINGS

**Musing 49.6**

a) Explain why applying a 25% discount to a price is the same as multiplying that price by $\frac{3}{4}$.

b) Explain why applying a 20% increase to a price is the same as multiplying that price by $\frac{6}{5}$.

**Musing 49.7**

a) Show that increasing 240 by 20% and then decreasing the result by 25% yields the same value as decreasing 240 by 25% first and then increasing the result by 20%.

b) **OPTIONAL CHALLENGE**: Can you explain why, in general, increasing a number $N$ by $a\%$ and then decreasing the result by $b\%$ yields the same value as decreasing $N$ by $b\%$ and then increasing the result by $a\%$?

## MECHANICS PRACTICE

**Practice 49.8** What is $7\frac{1}{2}\%$ of 480?

**Practice 49.9** What's a $22\frac{1}{2}\%$ tip on $\$20$?

**Practice 49.10**

a) What is a $32\frac{3}{5}\%$ tip on $100?

b) What is a $32\frac{3}{5}\%$ tip on $200?

**Practice 49.11** What is $70\%$ of $12{,}000$?

**Practice 49.12** Consider this sad statement.

> *My salary has been reduced by* 100%.

a) What is your salary now?

b) What if the statement was, instead "My salary has been reduced to 100% of what it was before"? Would you still panic?

# 50. Comparing Fractions

To get started, let's make sure we understand the mathematics of comparing numbers in general.

**Practice 50.1** Which of these numbers is the smallest in value?

a) 917 b) 719 c) 197 d) 791 e) 179

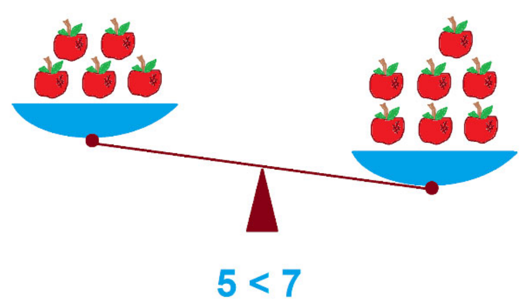
We say that a counting number $a$ is **smaller than** or **less than** a counting number $b$, written $a < b$, if we can add a counting number to $a$ to get the value $b$. (The symbol $<$ is called an **inequality sign**.)

$a < b$ if $a + n = b$ for some (non-zero) counting number $n$.

This is motivated by real-world thinking. For example, "5 is less than 7" because a set of five apples is less in count than a set of seven apples since we need to add two more apples to pile of 5 apples to make a pile of 7 apples.

Or, if we assume all the apples are identical, we notice that 5 apples together weigh less than 7 apples.

People often draw balance scales – well, unbalanced scales! – to visualize an inequality.

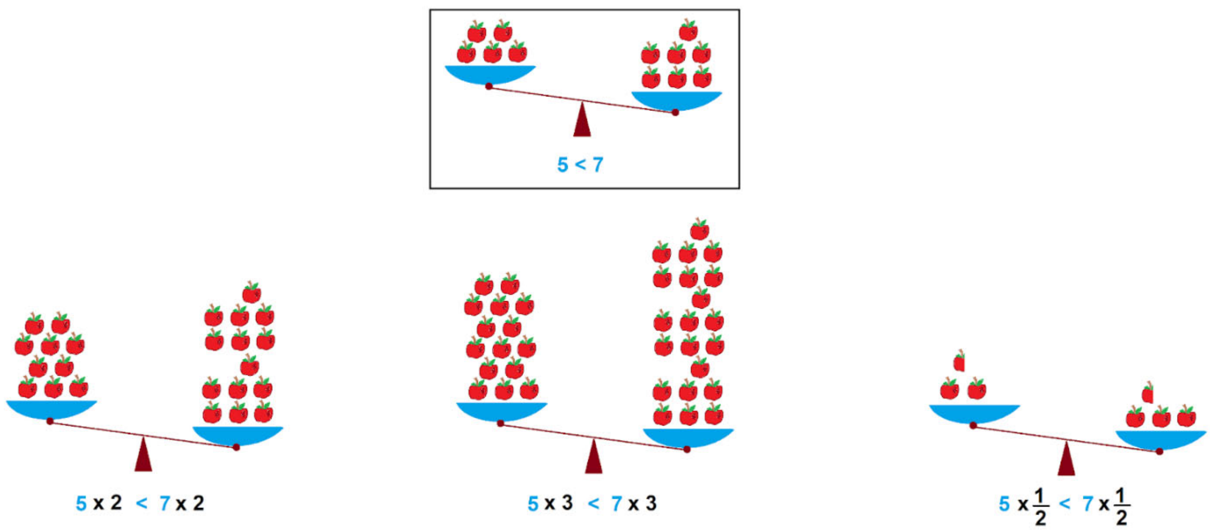



**Something Cute:** Imagine an equality sign with its lower horizontal bar titled down in the direction of the balance scale. That creates the inequality sign.

An **inequality** is any statement about one number being larger in value than another.

Each inequality can be written two ways. For instance, the statement $5 < 7$ can also be written $7 > 5$.

If we have a statement about inequality, then we like to believe we can double, triple, or halve all the numbers mentioned and not affect the inequality. This is motivated by real-world experience: double a light weight will still be less in weight than double a heavier weight, and so on.




This suggests a fundamental rule of inequalities: scaling two quantities by the same (positive) factor does not change the inequality.

If

$$a < b$$

then

$$k \times a \quad < \quad k \times b$$

for any positive number $k$.

And it is this fundamental belief about inequality that allows us to compare the sizes of fractions.

For example, consider the true statement $5 < 7$. Scale each number by an eleventh. We get

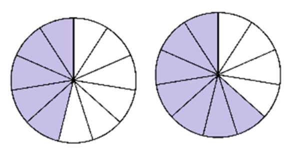
$$\frac{1}{11} \times 5 < \frac{1}{11} \times 7$$

which tells us that

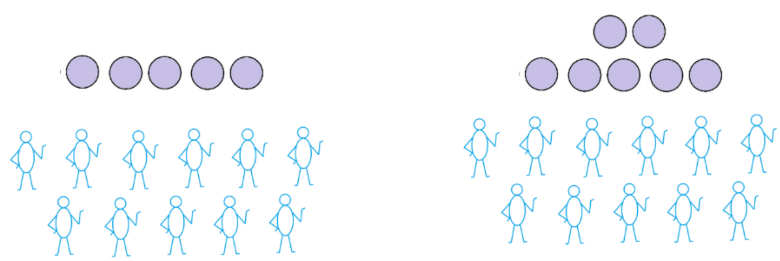
$$\frac{5}{11} < \frac{7}{11}$$

**Practice 50.2**

a) Does this fit with schoolbook intuition? Is five-elevenths of a pie less than seven elevenths of a pie?



b) Does this also fit with pies-per-student thinking? Which gives more pie per student: sharing 5 pies equally among 11 students or sharing 7 pies equally among 11 students?



In general, if $a < b$, then multiplying each term by $\frac{1}{n}$ gives

$$\frac{a}{n} < \frac{b}{n}$$

We have the schoolbook rule:

*If two fractions have the same denominator, then the fraction with the smaller numerator is the smaller fraction.*

Did you learn this rule?

This rule makes intuitive sense with our various "real-world" models of fractions. But it is good to see that it has a logical mathematical basis too.

Let's go back to the statement $5 < 7$ and this time let's scale each number by $\frac{1}{5}$ and then by $\frac{1}{7}$. This gives:

$$\frac{1}{7} \times \frac{1}{5} \times 5 \quad < \quad \frac{1}{7} \times \frac{1}{5} \times 7$$

Using that $\frac{1}{5} \times 5 = 1$ and $\frac{1}{7} \times 7 = 1$ , this reads

$$\frac{1}{7} < \frac{1}{5}$$

So, one seventh is a smaller fraction than one fifth.

**Practice 50.3** Show that one-thirteenth is smaller than one tenth. (Start with $10 < 13$ and scale the numbers in this inequality.)

Going back to $\frac{1}{7} < \frac{1}{5}$, we can now multiply through by $61$, say, to obtain

$$\frac{61}{7} < \frac{61}{5}$$

(sixty-one sevenths is smaller than sixty-one fifths).

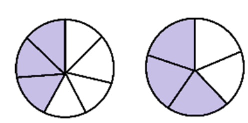
Or we can multiply instead through by $3$ to obtain

$$\frac{3}{7} < \frac{3}{5}$$

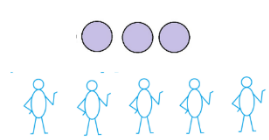
(three sevenths is less than three fifths).

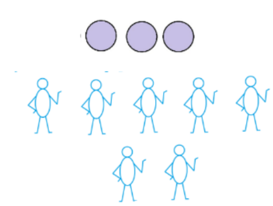
**Practice 50.4**

a) Does this too fit with schoolbook intuition? Is three-sevenths of a pie less pie than three-fifths of one?



b) Does this also fit with pies-per-student thinking? Which gives more pie per student: sharing 3 pies equally among 7 students or sharing 3 pies equally among 5 students?





In general, we have that $\frac{n}{b} < \frac{n}{a}$ if $b > a$.

This is the schoolbook rule:

*If two fractions have the same numerator, then the fraction with the bigger denominator is the smaller fraction.*

Is this something also familiar to you as a rule about fractions?

**Practice 50.5** Arrange these three fractions in order from smallest to biggest: $\frac{7}{11}, \frac{9}{7}, \frac{7}{12}, \frac{9}{11}$.

Here's a more challenging question.

Which of these two fractions is the larger?

$$\frac{5}{9} \qquad \frac{6}{11}$$

The trouble—or delight—of this question is that the two fractions don't have the same numerator, nor do they have the same denominator. They are hard to compare!

One approach is to rewrite the fractions to have the same … err …. which?

Let's go for the same numerator. Why not?

We currently have numerators of 5 and 6. This suggests going for a common numerator of $5 \times 6 = 30$.

We have

$$\frac{5}{9} = \frac{6 \times 5}{6 \times 9} = \frac{30}{54}$$

and

$$\frac{6}{11} = \frac{5 \times 6}{5 \times 11} = \frac{30}{55}$$

We see that $\frac{5}{9} = \frac{30}{54}$ is the larger of the two fractions.

**Practice 50.6** Rewrite $\frac{5}{9}$ and $\frac{6}{11}$ with a common denominator of 99. Deduce again that $\frac{5}{9}$ is the larger of the two fractions.

**Practice 50.7** Arrange the fractions $\frac{5}{9}$, $\frac{6}{11}$, and $\frac{15}{28}$ from smaller to largest.

Young students "dream" of being able to add fractions just by adding together their numerators and their denominators.

For example, they would love $\frac{4}{5} + \frac{7}{10}$ to be $\frac{11}{15}$. But unfortunately, the addition of fractions does not work this way.

**Practice 50.8** What is correct value of $\frac{4}{5} + \frac{7}{10}$?

**Practice 50.9** How does the dream answer of $\frac{11}{15}$ compare to the two original fractions $\frac{4}{5}$ and $\frac{7}{10}$? Please arrange the three fraction is order from smallest to largest.

**Practice 50.10** Adding $\frac{1}{2}$ and $\frac{1}{3}$ incorrectly gives $\frac{2}{5}$. Arrange these three fractions from smallest to largest.

If you are game, try to show why the incorrect addition of two fractions is sure to always produce a result that sits between the two original fractions.

$$\boxed{\frac{a}{b}} < \boxed{\frac{a+c}{b+d}} < \boxed{\frac{c}{d}}$$

## MUSINGS

**Musing 50.11** Recall that saying $a < b$ for two numbers $a$ and $b$ means that we can find a positive number $n$ that makes $a + n$ equal to $b$. ("We need to add more to $a$ to get $b$.")

$$a + n = b$$

Multiplying everything in this equation by two tells us that $2a + 2n = 2b$.
This is saying that we need to "add more to $2a$ to get $2b$." We conclude that $2a < 2b$.

a) From $a < b$, explain why we also have $3a < 3b$.
b) From $a < b$, explain why we also have $50a < 50b$.
c) From $a < b$, explain why we also have $k \times a < k \times b$, for any positive number $k$.

This means we don't have to rely on "real-world experience" to justify our rule for inequality. It too can be justified purely by mathematics.

**Musing 50.12**
a) Explain why a fraction $\frac{a}{b}$, with $a$ and $b$ both positive integers, is bigger than 1 if $a > b$.
b) Explain why a fraction $\frac{a}{b}$, with $a$ and $b$ both positive integers, is smaller than 1 if $a < b$.

**Musing 50.13** *Throughout this question $N$ represents a positive number.*

People say that multiplying a (positive) quantity by a number bigger than 1 gives an answer larger than the original quantity..

a) Show that $\frac{5}{4} \times 100$ is larger than $100$.

b) Show that $\frac{5}{4} \times N$ is larger than $N$.

We can prove the general claim as follows:

*Suppose $k$ is a number bigger than $1$.*
*Then we have $k > 1$.*

*Scale both sides of the inequality by $N$. This gives*

$$k \times N > 1 \times N$$

*which reads $k \times N > N$.*

People also say that multiplying a (positive) quantity by a number smaller than 1 (but still positive) gives an answer smaller than the original quantity.

c) Show that $\frac{4}{5} \times 100$ is smaller than $100$.

d) Show that $\frac{4}{5} \times N$ is smaller than $N$.

e) Suppose $k$ is a positive number and $k < 1$. Explain what $k \times N$ is sure to be smaller than $N$.

What about dividing by quantities that are bigger or smaller than $1$?

e) Show that $\frac{100}{5/4}$ is smaller than $100$.

f) Show that $\frac{N}{5/4}$ is smaller than $N$.

g) Show that $\frac{100}{4/5}$ is bigger than $100$.

h) Show that $\frac{N}{4/5}$ is bigger than $N$.

i) If you are game, try to prove that $\frac{N}{k}$ is sure to be smaller than $N$ if $k > 1$, and bigger than $N$ if $k < 1$ (and is still positive).

## MECHANICS PRACTICE

**Practice 50.14**

a) Which is bigger: $\frac{2}{3}$ or $\frac{3}{4}$?

b) Which is bigger: $\frac{9}{11}$ or $\frac{11}{12}$?

c) Which is bigger: $\frac{15}{22}$ or $\frac{16}{23}$?

d) Which is bigger: $\frac{40}{9}$ or $\frac{41}{10}$?

## 51. Fractions and Rational Numbers: Some Schoolbook Fussiness

We've developed a system of arithmetic that is based on the **counting numbers** $0, 1, 2, 3, 4, \ldots$ and extended it to include numbers denoted "$-a$" and numbers denoted "$\frac{a}{b}$" with $a$ and $b$ other numbers within our system ($b$ not zero).

People call each (non-zero) counting number, $1, 2, 3, 4, \ldots$, a **positive integer** and the opposite of each of these, $-1, -2, -3, -4, \ldots$, a **negative integer**. (Recall that zero is neither positive nor negative.)

We've also described fractions as positive and negative: $\frac{2}{3}$ is a positive fraction and $-\frac{2}{3}$ is a negative fraction, for instance. (And zero, thought of as a fraction, is neither positive or negative.) Speaking this way feels very natural.

But some schoolbooks are fussy. They like to use the term "fraction" only for what we've been thinking of as positive fractions and give a different name for the class of fractions as a whole.

**(Some) Schoolbooks:**
A **fraction** is a number that can be written in the form $\frac{a}{b}$ with $a$ and $b$ each a <u>positive</u> integer.
A **rational number** is any number that can be written in the form $\frac{a}{b}$ with $a$ and $b$ each an integer, but necessarily positive. (We just need $b$ to be non-zero).

Every fraction is also a rational number.

For example, in schoolbook world:

$\frac{2}{3}$ is a fraction. It's a positive integer over a positive integer.

$-\frac{2}{3} = \frac{-2}{3}$ is not a fraction, but it is a rational number.

$\frac{-2}{-3}$ is equivalent to $--\frac{2}{3} = \frac{2}{3}$, and so can be written as a positive integer over a positive integer. It is a fraction.

$\frac{7}{1/2}$ can be rewritten as $\frac{2\times 7}{2\times\frac{1}{2}} = \frac{14}{1}$, a positive integer over a positive integer. It is a fraction.

$\frac{1-\frac{4}{3}}{1+\frac{3}{5}}$ is equivalent to $\frac{15-20}{15+9} = \frac{-5}{24}$ is a rational number, but not a fraction.

**Practice 51.1** According to schoolbook fussiness …

a) Explain why 3 is allowed to be called a fraction.

b) Explain why 0 is a rational number but is not allowed to be called a fraction.

c) Explain why $-3$ is a rational number but is not allowed to be called a fraction

Mathematicians might think this schoolbook fussiness is too fussy to bother with, but they will agree to call any quantity that is equivalent to a number of the form $\frac{a}{b}$ with $a$ and $b$ each an integer (positive, negative, or zero—except $b$ cannot be zero) a **rational number**.

Every integer is a rational number, because we learned that an integer $a$ can be written as $\frac{a}{1}$.

The set of all rational numbers is denoted $\mathbb{Q}$ for "quotients."
We have:

$$\mathbb{N} = \{0, 1, 2, 3, 4, \ldots.\} \text{ or } \{1, 2, 3, 4, \ldots.\}$$

$$\mathbb{Z} = \{\ldots. -3, -2, -1, 0, 1, 2, 3, \ldots\}$$

$$\mathbb{Q} = \text{all the fractional quantities (which includes all the integers)}$$

Rational numbers can come in crazy guises. For example, can you see that this number

$$\frac{2 + \dfrac{1}{1 - \dfrac{1}{5}}}{1 + \dfrac{1}{3} - \dfrac{1 + \dfrac{1}{3}}{1 - \dfrac{6}{5}}}$$

is really $\frac{13}{32}$ in disguise and so is a rational number (a "positive fraction").

**Aside:** Mathematicians will call a fraction **positive** if it can be written in the form $\frac{a}{b}$ with $a$ and $b$ both positive integers, **negative** if it can be written instead in the form $-\frac{a}{b}$. (The number 0 is still neither positive nor negative!)

Some schoolbooks won't say "negative fraction."

Getting ahead of ourselves with decimals and square roots …

The quantity $\frac{1.2}{3}$ is a (positive) fraction. It is equivalent to $\frac{10\times1.2}{10\times3} = \frac{12}{30}$.

The quantity $\frac{2\sqrt{5}}{3\sqrt{5}}$ is also a fraction. It is equivalent to $\frac{2}{3}$.

So, even if the numerator and denominator of a fractional quantity are not themselves integers, it is possible that the quantity is still a fraction in disguise.

**Example:** Explain why the product of two rational numbers is sure to again be a rational number.

**Answer:** If the two numbers are rational, then they can be written as $\frac{a}{b}$ and $\frac{c}{d}$ for some integers $a$, $b$, $c$, and $d$ (with $b$ and $d$ not zero).
|
Their product is

$$\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d}$$

which is “an integer over an integer,” and so is a also rational number.

Right now, every number in our mathematical universe is a fraction (rational number).

It was a real shocker when some 2500 years ago scholars realized that not all numbers that arise in the “real world” are fractions!

We’ll realize that too in the next chapter.

## MUSINGS

**Musing 51.2**
a) Explain why the sum of two fractions is sure to be a fraction.
b) Explain why if we divide two fractions the answer is sure to be another fraction.

## 52. The Mathematical Truth about Fractions: Rule 10

We've been following the human progress of developing mathematics: using concrete, real-world scenarios to guide us as to what numbers exist (the counting numbers and zero, integers, and fractions) and how they should behave, and then pulling away from the concrete to let the logic of mathematics propel us to beyond that real-world thinking.

For example:

> *It is cumbersome (perhaps impossible) to give a convincing "real world" explanation of why negative times negative is positive, yet the logic of mathematics tells us this must be so.*
>
> *It makes no sense to multiply together portions of pie ("What's half a pie times a third of a pie?"), yet the logic of mathematics tells us how to multiply two fractions.*

Nonetheless, mathematics so often harks back to real-world contexts:

> *The product of two fractions matches the use of the word "of" in society's understanding of a fraction of a fraction.*

But in the end, the driving force behind all of our mathematics is mathematics itself, not the real-world examples that suggested it. (School mathematics tries to force everything to be "real world" justified and thus often gets muddled.)

We've made significant strides following the sheer logical power of math. And we can go a step further with this within this story of fractions.

As mentioned in Section 43, all our fraction work logically follows just by adding <u>one</u> additional rule to our list of general rules of arithmetic. This is the "Rule 10" we mentioned there.

So, let's now properly introduce Rule 10.

We've created universe of numbers that contains (and is in fact based on) the **counting numbers**:

$$0, 1, 2, 3, \ldots.$$

And within this universe there are two number operations—**addition** and **multiplication**. These two operations behave exactly as we expect them to when applied just to the counting numbers. But these operations go further and extend to all the numbers in our universe.

The two operations follow these nine rules of arithmetic.

**Rule 1:** For any two numbers $a$ and $b$ we have $a + b = b + a$.

**Rule 2:** For any number $a$ we have $a + 0 = a$ and $0 + a = a$.

**Rule 3:** In a string of additions, it does not matter in which order one conducts individual additions.

**Rule 4:** For any two numbers $a$ and $b$ we have $a \times b = b \times a$

**Rule 5:** For any number $a$ we have $a \times 1 = a$ and $1 \times a = a$.

**Rule 6:** In a string of multiplications, it does not matter in which order one conducts individual multiplications.

**Rule 7:** For any number $a$ we have $a \times 0 = 0$ and $0 \times a = 0$.

**Rule 8:** "We can chop up rectangles from multiplication and add up the pieces."

**Rule 9:** For each number $a$, there is one other number "$-a$" such that $a + -a = 0$.

Rule 9 created the "opposite numbers" for us and shows how they behave with respect to addition.

In section 43 we saw it behooves us to create another kind of "opposite number," but opposite in a multiplicative sense this time.

So, let's add to our list Rule 10 that creates for us the basic fractions: $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ and shows how they behave with respect to multiplication.

> **Rule 10:** For each number $a$ different from zero, there is one other number "$\frac{1}{a}$" such that $a \times \frac{1}{a} = 1$.

**Question:** Did we also just create $\frac{1}{1}$?

This rule tells us that there is a number called $\frac{1}{2}$ with the property that $2 \times \frac{1}{2} = 1$, and a number called $\frac{1}{3}$ with the property that $3 \times \frac{1}{3} = 1$, and so on.

Another way of stating Rule 10 is to say that each non-zero number $a$, there is a number that fills in the blank for this multiplication statement. We call that number $\frac{1}{a}$.

$$a \times \blacksquare = 1$$

For those who like the fancy language the quantity $\frac{1}{a}$ is called the **multiplicative inverse** of $a$.

(We called $-a$ the **additive inverse** of $a$).

### Why $a$ can't be zero in Rule 10?

Rule 10 steers clear of trying to create $\frac{1}{0}$, a quantity with denominator zero. The reason for this is that we would create a contradiction with Rule 7 which states that any quantity times zero is zero.

To explain …

If $\frac{1}{0}$ were to exist, then Rule 7 says that

$$0 \times \frac{1}{0} = 0$$

But Rule 10 says that

$$0 \times \frac{1}{0} = 1$$

The existence of $\frac{1}{0}$ would force us to conclude that zero equals one.

$$0 = 1$$

And then adding one to these numbers forces us to also conclude that one equals two.

$$1 = 2$$

And then adding one again has us conclude that two equals three.

$$2 = 3$$

And so on. All numbers would be equal, and mathematics would collapse to a universe of everything being the same!

To get the full picture of fractions, beyond just the basic fractions, we need an addendum to Rule 10. It takes the intuitive idea of being able to "pull fractions apart" and makes it fundamental.

> **Addendum:** We take an expression of the form $\frac{a}{b}$ to mean $a \times \frac{1}{b}$ (the basic fraction $\frac{1}{b}$ multiplied by $a$).

$$\frac{a}{b} = a \times \frac{1}{b}$$

So, for example, $\frac{2}{3}$ is understood to be shorthand for $2 \times \frac{1}{3}$, and $\frac{-7}{5}$ as shorthand for $(-7) \times \frac{1}{5}$.

This establishes for us the basic mathematical function we want fractions to have, namely, being answers to division problems.

$$\frac{a}{b} = a \div b$$

Here's how:

The mathematical definition of division is multiplication in reverse.

Specifically, for two numbers $a$ and $b$, with $b$ not zero, the number $a \div b$ is defined as the value that fills in the blank to the statement

$$b \times \blacksquare = a$$

And we can see that $\frac{a}{b} = a \times \frac{1}{b}$ fills in the blank. Using that $b \times \frac{1}{b} = 1$, as per Rule 10, we get

$$b \times \frac{a}{b} = b \times a \times \frac{1}{b} = a \times 1 = a$$

So, yes, $\frac{a}{b}$ is the answer to $a \div b$.

Since mathematicians tend to avoid using the division symbol $\div$, let's rephrase what we have just established solely in terms of reverse multiplication.

We have established:

**For two numbers $a$ and $b$, with $b$ not zero, $\frac{a}{b}$ is the number that fills in the blank to**

$$b \times \blacksquare = a$$

That is, we have:

$$\cancel{b} \times \frac{a}{\cancel{b}} = a$$

In particular, $\frac{1}{1}$ is the number that fills the blank to

$$1 \times \blacksquare = 1$$

But 1 also fits that blank!

So $\frac{1}{1}$ must be 1.

And $\frac{1}{2 \times 3}$ is the number that fills in the blank to

$$2 \times 3 \times \blacksquare = 1$$

But $\frac{1}{2} \times \frac{1}{3}$ also fills in this blank. So

$$\frac{1}{2} \times \frac{1}{3} = \frac{1}{2 \times 3}$$

All the properties and all the play of fractions we brought in over the past two chapters are starting to emerge from this Rule 10 and its addendum.

Let's keep going and show that <u>every</u> fraction property we highlighted these past two chapters follows from Rule 10 and its addendum.

**Practice 52.1** To get a feel for how this will proceed, perhaps try each of these problems on your own before reading on.

a) According to the addendum, what is $\frac{20}{1}$? Explain why $\frac{20}{1}$ must be $20$.

b) According to the addendum, what is $\frac{20}{20}$? Explain why $\frac{20}{20}$ must be $1$.

c) What is the value of $20 \times \frac{1}{4} \times \frac{1}{5}$? Explain why $\frac{1}{4} \times \frac{1}{5}$ must be $\frac{1}{20}$.

d) Explain why $2 \times \frac{10}{3}$ and $\frac{20}{3}$ must have the same value. (What does the addendum have to say about each of these quantities?)

e) Explain why $\frac{12}{20}$ and $\frac{3}{5}$ must have the same value.

## The Magic of Rule 10 and its Addendum

**Example:** For every number $a$ we have $a = \frac{a}{1}$.

$$a = \frac{a}{1}$$

**Reason:**

The addendum says that $\frac{a}{1}$ is just shorthand for $a \times \frac{1}{1}$.

We just saw that $\frac{1}{1}$ is $1$.

So, $\frac{a}{1}$ is really just $a \times 1$, which is $a$.

**Example**: For every non-zero number $a$ the quantity $\frac{a}{a}$ is just $1$.

$$\frac{a}{a} = 1$$

**Reason:**

Again, $\frac{a}{a}$ is shorthand for $a \times \frac{1}{a}$. By Rule 10, this is $1$.

Alternatively:
$\frac{a}{a}$ is the number that fills in the blank to $a \times \blacksquare = a$.
Clearly $1$ does the trick too!

**Example**: We have that $k \times \frac{a}{b}$ and $\frac{k \times a}{b}$ are the same for numbers $a$, $b$, and $k$ with $b$ non-zero.

$$k \times \frac{a}{b} = \frac{k \times a}{b}$$

**Reason**:

By the addendum...

$k \times \frac{a}{b}$ is really

$$k \times a \times \frac{1}{b}$$

and $\frac{k \times a}{b}$ is really

$$k \times a \times \frac{1}{b}$$

These are indeed the same!

**Example:** For non-zero numbers $k$ and $b$, we have:

$$\frac{1}{k} \times \frac{1}{b} = \frac{1}{k \times b}$$

$$\frac{1}{k} \times \frac{1}{b} = \frac{1}{k \times b}$$

**Reason**:

By Rule 10, $\frac{1}{k \times b}$ is the number that fills in this blank.

$$k \times b \times \blacksquare = 1$$

But $\frac{1}{k} \times \frac{1}{b}$ fills in the blank too!

$$k \times b \times \frac{1}{k} \times \frac{1}{b} = 1 \times 1 = 1$$

This means that $\frac{1}{k} \times \frac{1}{b}$ is $\frac{1}{k \times b}$.

**Example**: We have that $\frac{k \times a}{k \times b}$ and $\frac{a}{b}$ are the same for numbers $a$, $b$, and $k$, with $b$ and $k$ nonzero.

$$\frac{a}{b} = \frac{k \times a}{k \times b}$$

**Reason:**

From the addendum, $\frac{k \times a}{k \times b}$ is shorthand for

$$k \times a \times \frac{1}{k \times b}$$

And we've just seen that $\frac{1}{k \times b}$ is $\frac{1}{k} \times \frac{1}{b}$. So, we this reads

$$k \times a \times \frac{1}{k} \times \frac{1}{b}$$

But $k \times \frac{1}{k} = 1$. So, we actually have

$$1 \times a \times \frac{1}{b}$$

which is

$$a \times \frac{1}{b}$$

And according to the addendum this as $\frac{a}{b}$.

So $\frac{k \times a}{k \times b}$ and $\frac{a}{b}$ are the same quantity in disguise!

**Example:** For numbers $a$ and $b$, with $b$ nonzero, we have

$$-\frac{a}{b} = \frac{-a}{b} = \frac{a}{-b}$$

**Reason:** We have from our properties of negative numbers that $-n = (-1) \times n$.

So, using this and the addendum,

$$-\frac{a}{b} = (-1) \times a \times \frac{1}{b}$$

Also,

$$\frac{-a}{b} = (-a) \times \frac{1}{b} = (-1) \times a \times \frac{1}{b}$$

So $-\frac{a}{b}$ and $\frac{-a}{b}$ are the same.

Also, by the previous example, we can multiply the top and bottom of a fraction by $-1$ and not affect its value.

$$\frac{a}{-b} = \frac{a}{(-1) \times b} = \frac{(-1) \times a}{(-1) \times (-1) \times b} = \frac{-a}{b}$$

(Recall we did prove in Chapter 3 that $(-1) \times (-1) = 1$.)
Thus $\frac{-a}{b}$ and $\frac{a}{-b}$ are also the same.

All three quantities are equivalent.

That's it!

Everything we did in the previous chapter (in particular in Section 41) and in this chapter follows logically and mathematically from all we've just now presented. And all that we just presented follows logically and mathematically from just that one additional rule, Rule 10, with the understanding that $\frac{a}{b}$ is shorthand for $a \times \frac{1}{b}$.

Fractions are logical and tight and mathematically meaningful. And the mathematics of fractions is beautifully aligned with each and every real-world context schoolbooks present to students.

But again, the mathematics of fractions sits at a higher plane to any one real-world context. No one real-world context "sees" the full mathematics of fractions, yet the mathematics of fractions has something to say about each real-world context.

One final example that looks out of place.

**Example:** Show if two numbers $a$ and $b$ have a product of zero,

$$a \cdot b = 0$$

then it is because at least one of the numbers is zero.

Schoolbooks usually cite this a special rule of arithmetic.

**Reason:** Suppose we do have two numbers $a$ and $b$ that do multiply together to make zero.

If the first number $a$ is already zero, then we're set: one of the numbers is indeed zero!

But what if $a$ is not zero?

Then look at the fraction $\frac{1}{a}$ and let's work out $\frac{1}{a} \times a \times b$.

We can look at this as

$$\frac{1}{a} \times a \times b = 1 \times b = b$$

and as

$$\frac{1}{a} \times a \times b = \frac{1}{a} \times 0 = 0$$

But we just looked at the same quantity two different ways. We have to conclude than that if $a$ is not zero, then $b$ is the same as $0$!

## MUSINGS

**Musing 52.2** Mathematicians will likely quibble with how I phrased Rule 10.

> For each number $a$ different from zero, there is one other number "$\frac{1}{a}$" such that $a \times \frac{1}{a} = 1$.

They will say that "there is **one other** number" should be replaced with "there is **a** number" because logic dictates that there cannot be two or more numbers that fill in the blank to $a \times \blacksquare = 1$.
(So, if you've got "a number" that works, then it is the only "one number" that works.)

Here's the logic.

*Consider a non-zero number $a$.*

*Suppose $b$ and $c$ are two different numbers that both fill in the blank to $a \times \blacksquare = 1$.*
*So,*

$$a \times b = 1$$
$$a \times c = 1$$

*Let's now work out $a \times b \times c$.*

*We can work out a string of products in any order we like, so we can think of this product as*

$$(a \times b) \times c = 1 \times c = c$$

*We can also think of it as*

$$(a \times c) \times b = 1 \times b = b$$

*So, this one product equals both $b$ and $c$.*

*So $b$ and $c$ can't be two different numbers after all!*

My questions:

i) Did that line of logical reasoning make sense to you?
ii) What do you think of this degree of logical fussiness?

## 53. All the Rules of Arithmetic in One Place

Here's absolutely everything we have learned about arithmetic all in one spot.

We have a mathematical universe of numbers, which includes the counting numbers

$$0, 1, 2, 3, \ldots$$

There are two operations—**addition** and **multiplication**—on these numbers which behave just as we expect them to when applied to the counting numbers. But they also apply to all numbers in our number universe and behave as follows:

**Rule 1:** For any two numbers $a$ and $b$ we have $a + b = b + a$.

**Rule 2:** For any number $a$ we have $a + 0 = a$ and $0 + a = a$.

**Rule 3:** In a string of additions, it does not matter in which order one conducts individual additions.

**Rule 4:** For any two numbers $a$ and $b$ we have $a \times b = b \times a$

**Rule 5:** For any number $a$ we have $a \times 1 = a$ and $1 \times a = a$.

**Rule 6:** In a string of multiplications, it does not matter in which order one conducts individual multiplications.

**Rule 7:** For any number $a$ we have $a \times 0 = 0$ and $0 \times a = 0$.

**Rule 8:** "We can chop up rectangles from multiplication and add up the pieces."

**Rule 9:** For each number $a$, there is one other number "$-a$" such that $a + -a = 0$.

**Some Logical Consequences of Rule 9:** For any two numbers $a$ and $b$

i) $-0 = 0$
("The opposite of zero is zero")

ii) $- - a = a$
("The opposite of the opposite is back to the original")

iii) $-(a + b) = -a + -b$
(We can "distribute a negative sign")

iv) $(-a) \times b$ and $a \times (-b)$ and $-(a \times b)$ all have the same value
(We can "pull out a negative sign")

v) $(-1) \times a = -a$
("Multiplying by $-1$ gives you the opposite number")

**Rule 10:** For each non-zero number $a$ there is one number, $\frac{1}{a}$ such that $a \times \frac{1}{a} = 1$. (That is, we can fill in the blank to $a \times \blacksquare = 1$.)

**Convention:** For a number $a$ and a non-zero number $b$, the notation $\frac{a}{b}$ is shorthand for $a \times \frac{1}{b}$.

**Some Logical Consequences of Rule 10:**

i) $\frac{\boldsymbol{a}}{\mathbf{1}} = \boldsymbol{a}$ ("We can put numbers over 1.")

ii) $\frac{\boldsymbol{a}}{\boldsymbol{a}} = \mathbf{1}$ for $a$ not zero ("We can write 1 in many forms.")

iii) $k \times \frac{a}{b} = \frac{ka}{b}$ for $b$ not zero

iv) $\boldsymbol{b} \times \frac{\boldsymbol{a}}{\boldsymbol{b}} = \boldsymbol{a}$ for $b$ not zero ("We can cancel a denominator.")

v) $\frac{\boldsymbol{ka}}{\boldsymbol{kb}} = \frac{\boldsymbol{a}}{\boldsymbol{b}}$ for $k$ and $b$ each not zero ("We can simplify fractions.")

vi) **ADDING/SUBTRACTING FRACTIONS**: $\frac{a}{N} \pm \frac{b}{N} = \frac{a \pm b}{N}$ for $N$ not zero

vii) **MULTIPLYING FRACTIONS**: $\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d}$ for $b$ and $d$ each not zero

viii) **PULLING OUT NEGATIVE SIGNS:** $-\frac{a}{b} = \frac{-a}{b} = \frac{a}{-b}$ for $b$ non-zero

**ADDING/SUBTRACTING FRACTION WITH DIFFERENT DENOMINATORS**
Either rewrite each fraction using v) to create a common denominator, or just put the quantity over 1 and use v).

**DIVIDING FRACTIONS** Rewrite $\frac{a}{b} \div \frac{c}{d}$ as $\frac{\frac{a}{b}}{\frac{d}{c}}$ and use v).

Finally …

ix) **$a \div b$ and $\frac{a}{b}$ are the same number**, assuming $b$ is not zero, or course.

Point ix) is just a reinterpretation of iv).

Allow me to reiterate one more time …

**Division does not really exist. Division is just multiplication by a fraction.**

For any two numbers $a$ and $b$, with $b$ not zero, we have by Rule 10, consequence iv) that $\frac{a}{b}$ is the number that fills in the blank to

$$b \times \blacksquare = a$$

Schoolbooks call this number $a \div b$ (using the "reverse multiplication" understanding of division.)

But $\frac{a}{b}$ is $a \times \frac{1}{b}$, or, switching the order of the multiplication, $\frac{a}{b}$ is $\frac{1}{b} \times a$.

We have

**To divide a quantity by a value $b$, just multiply that quantity by $\frac{1}{b}$.**

$$a \div b = \frac{a}{b} = \frac{1}{b} \times a$$

So, in our mathematical universe, there are only two operations: addition and multiplication.
Subtraction is just the addition of the (additive) opposite.
Division is just multiplication by the (multiplicative) opposite.

And the full mathematical story of arithmetic follows from just ten basic rules.

# Chapter 7

## Decimals
## and Scientific Notation,
## Rounding, and Significant Figures

# 53. Discovering Decimals

Let's go back to the machines in Chapter 4 and their *Exploding Dots*. Something about them has been bothering me all this time.

Recall, no matter the machine, we had boxes going to the left as far as we pleased.


But that seems awfully lopsided! Why can't we have boxes going infinitely far to the right as well?

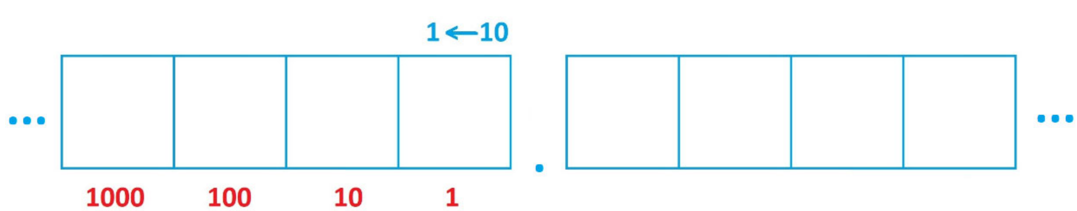
Mathematicians like symmetry and so let's follow suit and now make all our machines symmetrical. Let's have boxes going to the left and to the right.

But the challenge now is to figure out what those boxes to the right mean.

## Focusing on the 1 ← 10 machine.

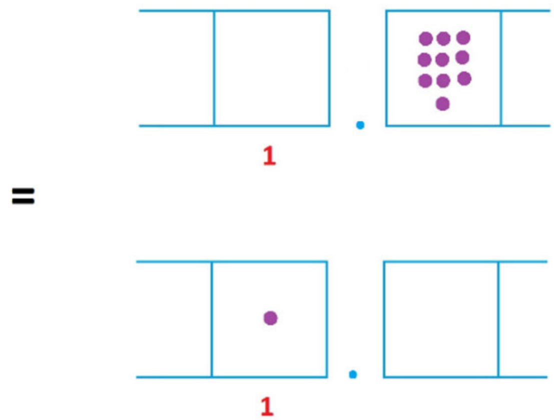
Let's focus on a 1 ← 10 machine and see what boxes to the right could mean for that machine.

To keep the left and right boxes visibly distinct, we'll separate them with a point. (Society calls this point—for base ten, at least—a **decimal point**.)

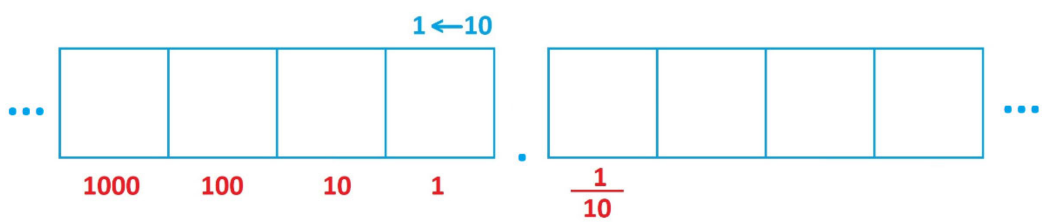



So, what does it mean to have dots in the right boxes? What are the values of dots in those boxes?

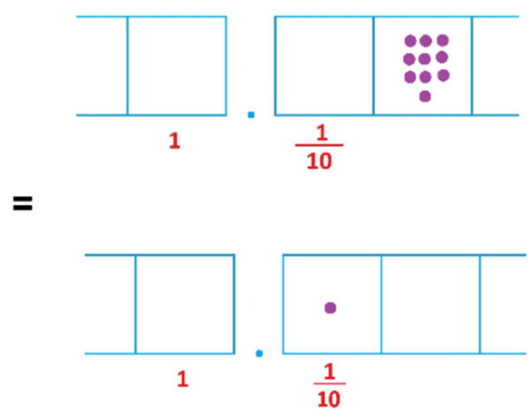
Since this is a $1 \leftarrow 10$ machine, we do know that ten dots in any one box explode to make one dot one place to the left. So, ten dots in the box just to the right of the decimal point are equivalent to one dot in the 1s box. Each dot in that box must be worth one tenth. (And yes, ten tenths is one: $10 \times \frac{1}{10} = 1$.)




We have our first place-value to the right of the decimal point.

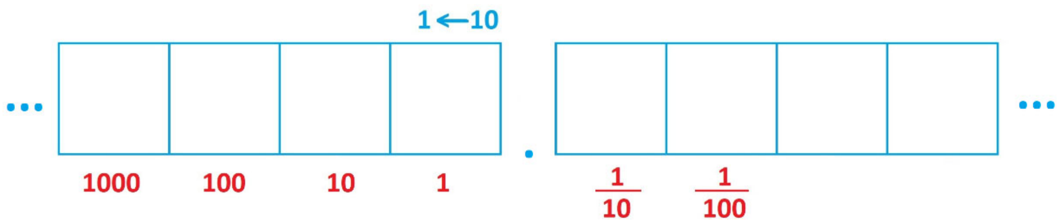



In the same way, ten dots in the next box over are worth one dot in the one tenth place.

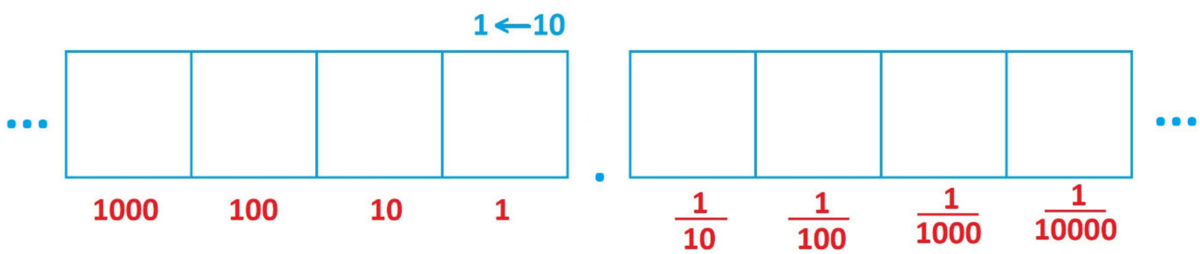



And so, each dot in that next box over must be worth one tenth of one tenth. That's one hundredth. (Yes, ten one hundredths do make one tenth: $10 \times \frac{1}{100} = \frac{10}{100} = \frac{1}{10}$.)

We now have two place values to the right of the decimal point.




And we can keep going: ten one-thousandths make a hundredth, and ten ten-thousandths make a thousandth, and so on.




**Practice 53.1** Show that $10 \times \frac{1}{1{,}000} = \frac{1}{100}$ and $10 \times \frac{1}{10{,}000} = \frac{1}{1{,}000}$.

We see that the boxes to the left of the decimal point represent place values given by tens multiplied together and boxes to the right of the decimal point represent place values given by tenths multiplied together.

**Practice 53.2** Are you clear on why $\frac{1}{10} \times \frac{1}{10} \times \frac{1}{10} = \frac{1}{1{,}000}$ and $\frac{1}{10} \times \frac{1}{10} \times \frac{1}{10} \times \frac{1}{10} = \frac{1}{10{,}000}$?

We have just discovered the decimal places!

**Practice 53.3** What does the prefix deci- or deca- mean in English?
(How many years are in a decade? In geometry, how many sides does a decagon have?)

**Practice 53.4** What might we call the point that separates left and right boxes if we were doing this work in a $1 \leftarrow 2$ machine instead? (There isn't an official name for one, but can you see why it shouldn't be called a <u>deci</u>mal point?)

**Practice 53.5** Do all cultures use a point to separate boxes?

When people write $0.3$, for example, in base ten, they mean the value of three dots placed in the first box after the decimal point.

**.3 =** 

We see that $0.3$ equals three tenths: $0.3 = \frac{3}{10}$.

Seven dots in the third box after the decimal point is seven thousandths: $0.007 = \frac{7}{1000}$.

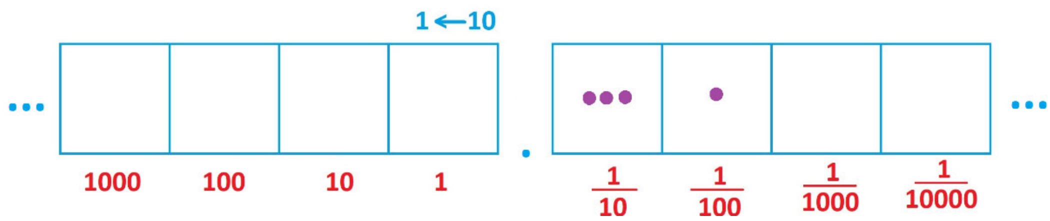
**.007 =** 

**Practice 53.6** What fraction is $0.00008$?

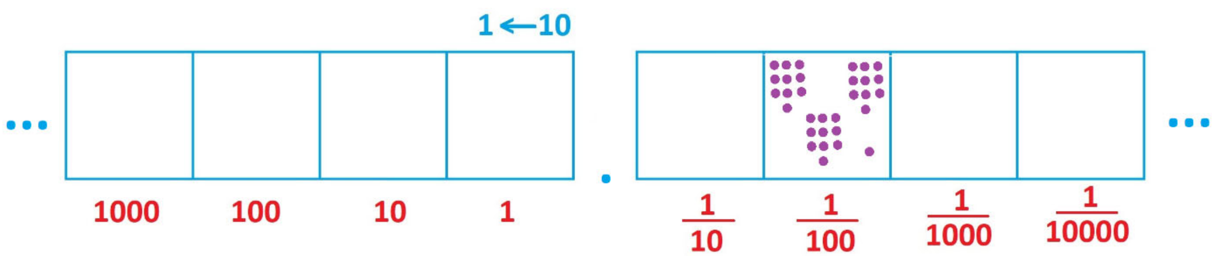
**Comment**: Some people might leave off the beginning zero and just write $.007$ rather than $0.007$. This is just a matter of personal taste. I've already used both styles of presentation on this page.

**Question:** Some people read $0.6$ out loud as "point six" and others read it out loud as "six tenths." Which is more helpful for understanding what the number really is?

There is a possible source of confusion with a decimal such as $0.31$. This is technically three tenths and one hundredth: $0.31 = \frac{3}{10} + \frac{1}{100}$.




But some people read $0.31$ out loud as “thirty-one hundredths,” which looks like this.




Are these the same thing?

Well, yes! With three explosions we see that thirty-one hundredths becomes three tenths and one hundredth.

**Some Language**

People are a little loose in how they describe a number written with a decimal point and some digits to the right of the decimal point.

They might say that the number has been written in **decimal notation** or that it has been expressed simply as a **decimal**. One might call for the **decimal representation** of a number, meaning that one is meant to express a given number in decimal notation.

The term **decimal number** means any number that is expressed via decimal notation.

One final thought.

**Question:** What does $-0.31$ mean?

**Answer:** Well, $0.31$ is the fraction $\frac{31}{100}$ and so $-0.31$ is this fraction made negative. It's $-\frac{31}{100}$.

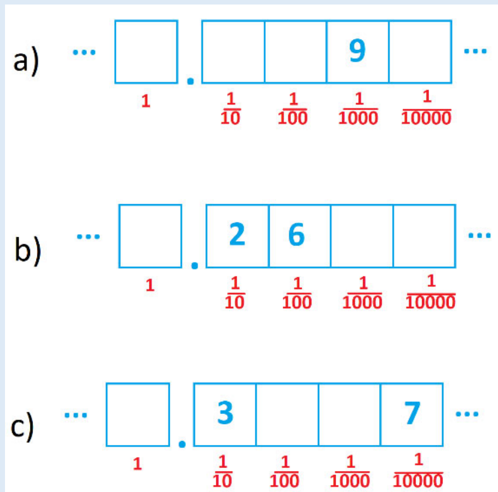
People like to think of negative sign in front of decimal as applying to all of the decimal number. For example, if you are thinking of $0.31$ as $\frac{3}{10}+\frac{1}{100}$, then $-0.31$ is

$$-\left(\frac{3}{10}+\frac{1}{100}\right)=\frac{-3}{10}+\frac{-1}{100}$$

**Question:** Can you see that $\frac{-3}{10}+\frac{-1}{100}$ is the same as $\frac{-31}{100}$, which is $-\frac{31}{100}$?

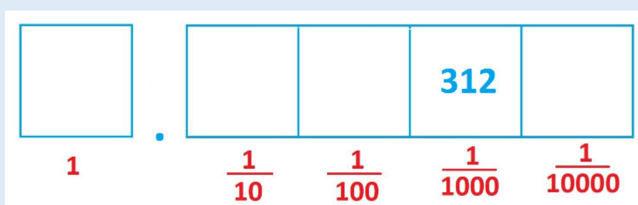
## MUSINGS

**Musing 53.7** For each picture, write the decimal number the picture represents and the fraction that that decimal equals. (For example, anwer to part a is $0.009$, which is $\frac{9}{1000}$.)

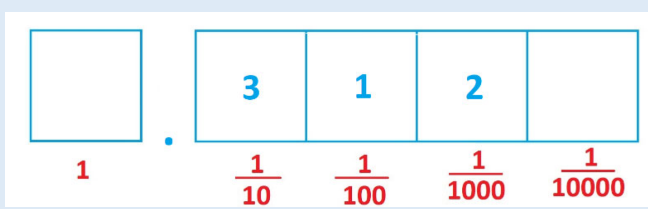



**Musing 53.8** A schoolteacher asked his students to each draw a $1 \leftarrow 10$ machine picture of the fraction $\frac{312}{1000}$.

JinJin drew:




Subra drew:



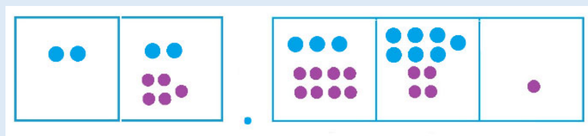

The teacher marked both students as correct.

Are both of these responses indeed valid? Explain your thinking.

**Musing 53.9** Aparna was asked to compute $22.37 + 5.841$. She wrote this answer for her professor.

$$\begin{array}{r} 22.3\ 7\phantom{\ 1} \\ +\ \ 5.8\ 4\ 1 \\ \hline 2|7.11|11|1 \end{array}$$

Her professor was confused, so she added this picture to her page.



Her professor was still a bit puzzled, but she had an idea now as to what Aparna might be thinking. The professor said “I was expecting to see the answer $28.211$. Does your work lead to that answer?”

What could Aparna do next to show her professor that $2|7.11|11|1$ is indeed the number $28.211$ in disguise?

**Musing 53.10** Tijana said that $23.56 \times 11$ equals $22|33.55|66$.
Can you explain what she is thinking and how to fix her answer to one that society understands?

**Musing 53.11** Can you explain using dots and boxes in a $1 \leftarrow 10$ machine why $22.37 \times 10$ equals $223.7$? (It looks like the decimal point shifted one place. Did it really?)

**Musing 53.12** James, feeling naughty, wrote the decimal number $3|-4|0\,.\,5|-7|0|-1$.
What is a societally acceptable version of this decimal number?

## MECHANICS PRACTICE

**Musing 53.13**

a) The decimal $0.23$ equals which of the following?

a) $\frac{23}{10}$ b) $\frac{23}{100}$ c) $\frac{23}{1000}$ d) $\frac{23}{10000}$

b) The decimal $0.0409$ equal which of the following?

a) $\frac{409}{100}$ b) $\frac{409}{1000}$ c) $\frac{409}{10000}$ d) $\frac{409}{100000}$

**Musing 53.14** Draw $1 \leftarrow 10$ machine pictures of each of these quantities.

a) $\frac{66}{100}$ b) $\frac{6}{10} + \frac{6}{100}$ c) $1\,\frac{66}{100}$ d) $\frac{66}{50}$

**Musing 53.15**

a) This picture for a $1 \leftarrow 10$ machine is equivalent to which of the following numbers?


a) $31.2$ b) $3.12$ c) $.312$ d) $.0312$

b) Can you see that this picture is the decimal $34.32$in disguise?


(This is technically a YES/NO question. It would be nice if the answer were YES.)

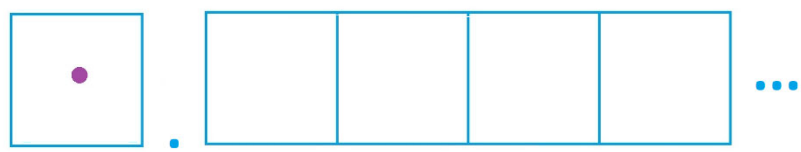
## 54. Fractions as Decimals

We saw in chapters 5 and 6 that fractions are numbers that match answers to division problems.

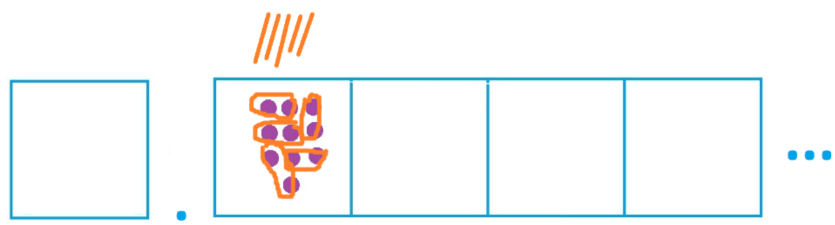
For example, $\frac{2}{3}$ is the answer to $2 \div 3$, and $\frac{1}{2}$ is the answer to the division problem $1 \div 2$.

Moreover, as we saw in Chapter 4, we can compute answers to division problems in a $1 \leftarrow 10$ machine, even a division problems like $1 \div 2$. All we have to do is make use of the boxes to the right of the decimal point. The division process is exactly the same.

For instance, to compute $1 \div 2$ we need to identify groups of two in this picture with just one dot.



No groups of two can be seen at present, so let's unexplode. Doing so reveals five groups of two at the tenths level.



We have that $\frac{1}{2}$ is $0.5$ as a decimal. (And as a check, $\frac{5}{10}$ does indeed equal $\frac{1}{2}$.)

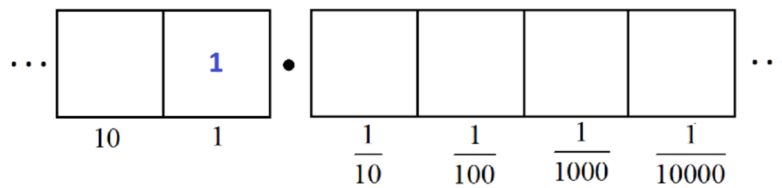
**Practice 54.1:** Write $\frac{1}{4}$ as a decimal by computing $1 \div 4$. Do you get $0.25$?

**Practice 54.2:** Write $\frac{1}{5} = 1 \div 5$ as a decimal.

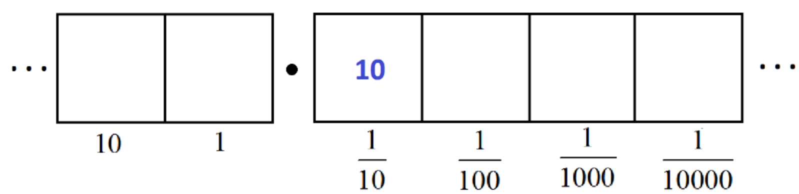
**Practice 54.3:** Write $\frac{1}{10} = 1 \div 10$ as a decimal. (Why should you get the answer $0.1$?)

Another example: Let's write $\frac{1}{8}$ as a decimal. We need to compute $1 \div 8$ in a $1 \leftarrow 10$ machine.

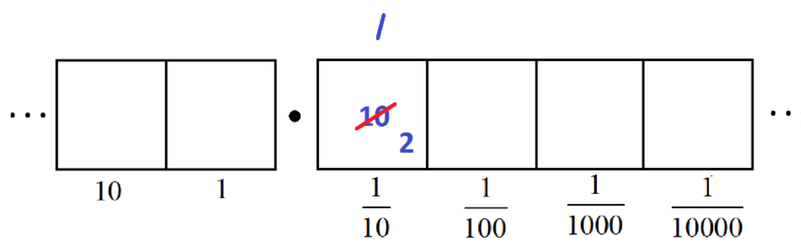
We seek groups of eight in the following picture. (I won't draw dots this time and just write numbers.)



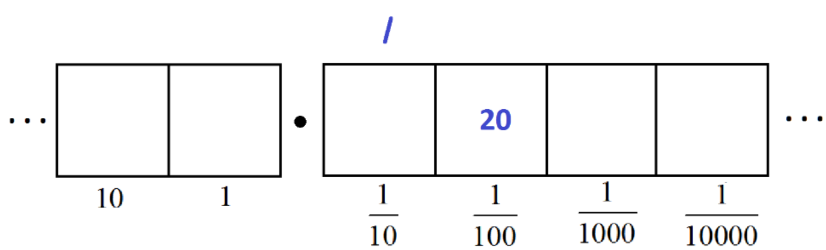

None are to be found right away, so let's unexplode.



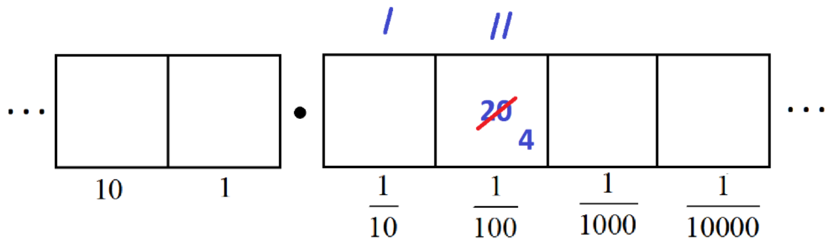

We have one group of 8, leaving two behind.



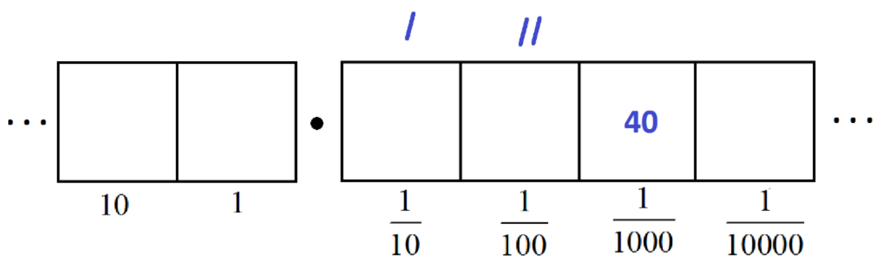

Two more unexplosions.



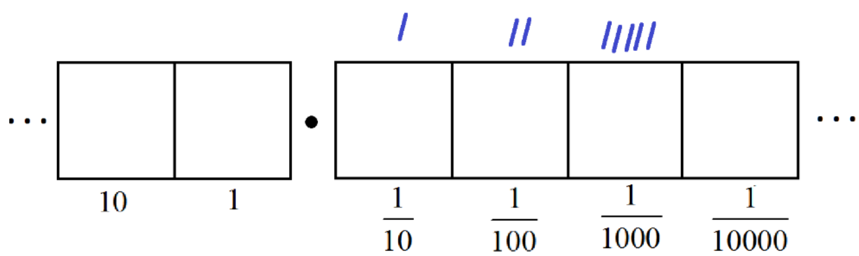
This gives two more groups of 8 leaving four behind.




Unexploding again




reveals five more groups of 8 leaving no remainders.




We see that, as a decimal, $\frac{1}{8}$ turns out to be $0.125$. And as a check we have

$$0.125 = \frac{125}{1000} = \frac{25}{200} = \frac{5}{40} = \frac{1}{8}$$

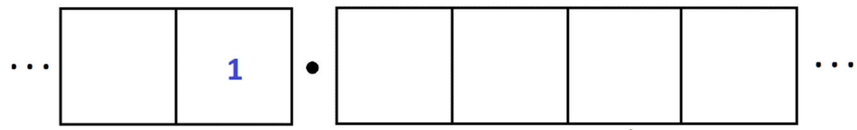
Super!

(Did you follow all the steps on this and the previous pages?)

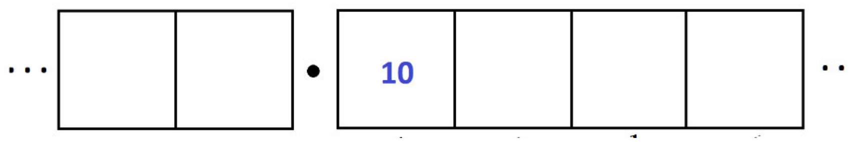
**Practice 54.4:** Write $\frac{1}{40} = 1 \div 40$ as a decimal.
(Can you compute this in a $1 \leftarrow 10$, writing numbers instead of drawing dots?)

Not all fractions lead to simple decimal representations. For example, consider the fraction $\frac{1}{3}$.

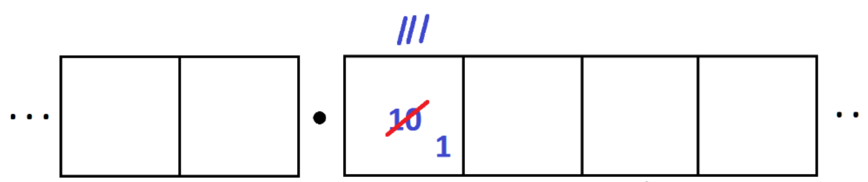
To compute it, we seek groups of three in the following picture.



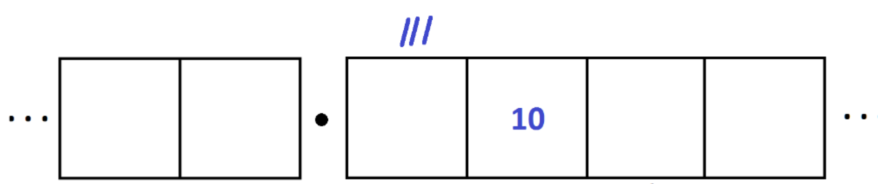

Let's unexplode.



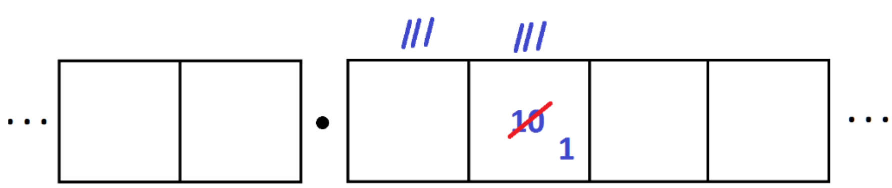

We see three groups of 3 leaving one behind.



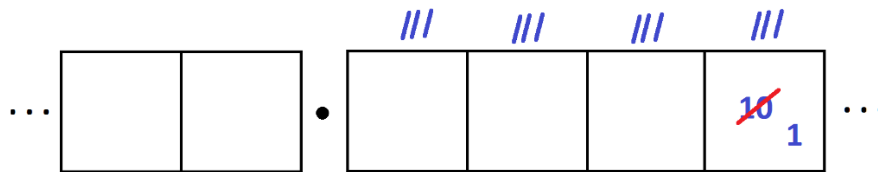

Unexploding gives another ten dots to examine.




We find another three groups of 3 leaving one behind.




And so on. We are caught in an infinite repeating cycle.




This puts us in a philosophically interesting position. As human beings we cannot conduct this, or any, activity for an infinite amount of time. But it seems very tempting to write

$$\frac{1}{3} = 0.33333333\ldots$$

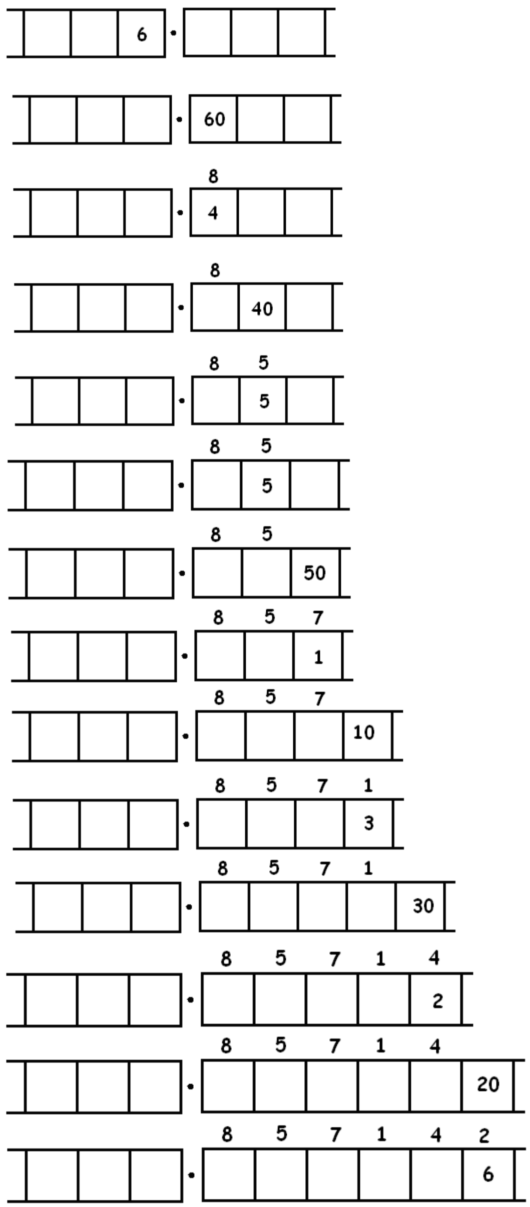
with the ellipsis representing the instruction "keep going with this pattern forever."

In our minds we can almost imagine what this means. But as a practical human being it is beyond our abilities: one cannot actually write down those infinitely many 3s represented by the ellipsis.

Nonetheless, many people choose not to contemplate what an infinite statement like this means and just carry on and say that some decimals are infinitely long and not worry about it. The fraction $\frac{1}{3}$ is one of those fractions whose decimal expansion goes on forever.

**Practice 54.5:** Write $\frac{1}{9}$ as an infinitely long decimal.

Here's a complicated example. Work through it if you are game! Here we convert the fraction $\frac{6}{7}$ to an infinitely long decimal.




Do you see with this 6 in the final rightmost box that we have returned to the very beginning of the problem? This means that we shall simply repeat the work we have done and obtain the same sequence "857142" of answers again, and then again, and then again.

We have

$$\frac{6}{7} = 0.857142\ \ 857142\ \ 857142\ \ 857142\ \ldots$$

**Practice 54.6:** Write $\frac{2}{15} = 2 \div 15$ as a decimal. Does it too fall into an infinitely repeating pattern?

**Practice 54.7:** Write $\frac{1}{6} = 1 \div 6$ as a decimal. Does it too fall into an infinitely repeating pattern? What about $\frac{2}{6}$?

**Challenge 54.8:** Which of the following fractions give infinitely long decimal expansions? (We've done some of these already.)

$$\frac{1}{2} \quad \frac{1}{3} \quad \frac{1}{4} \quad \frac{1}{5} \quad \frac{1}{6} \quad \frac{1}{7} \quad \frac{1}{8} \quad \frac{1}{9} \quad \frac{1}{10} \quad \frac{1}{11} \quad \frac{1}{12}$$

## MUSINGS

**Musing 54.9** We saw that $\frac{1}{3} = 0.333333\ldots$. is an infinitely long decimal.

a) What must $\frac{2}{3}$ be as a decimal?

b) Compute $\frac{4}{3}$ as a decimal. Is what you get the same as $0.12\,|\,12\,|\,12\,|\,12\,|\,12\,|\ldots$ ?

c) Sona says that every fraction of the from $\frac{N}{3}$ is sure to be an infinitely long decimal. (Here $N$ is some number for the numerator.) Is Sona right?

**Musing 54.10** We have that $\frac{1}{3} + \frac{1}{6} = \frac{1}{2}$. (Check this fraction arithmetic.)

If you haven't already (but surely you have by now!) work out $\frac{1}{3} = 1 \div 3$ and $\frac{1}{6} = 1 \div 6$ as decimals.

Add your two answers together. Do you get $0.5$?

(You don't actually! This example shows that there is something philosophically deep we need to attend to in this non-human play of infinitely long decimal representations.)

**Musing 54.11**

a) Compare the decimal representations of $\frac{1}{2}$ and $\frac{1}{20}$. What do you notice?

b) Compare the decimal representations of $\frac{1}{5}$ and $\frac{1}{50}$. What do you notice?

c) Compare the decimal representations of $\frac{1}{3}$ and $\frac{1}{30}$. What do you notice?

## MECHANICS PRACTICE

**Musing 54.12** Performing the division in a $1 \leftarrow 10$ machine show that $\frac{3}{5}$ is $0.6$ as a decimal.

**Musing 54.13** Compute $\frac{4}{7}$ as an infinitely long repeating decimal.

**Musing 54.14** If you haven't already, compute $\frac{1}{11}$ as an infinitely long repeating decimal.

## BONUS (completely optional) MUSING

(as if anything in this book is compulsory!)

**Musing 54.15** Here's a very strange way to divide a number by 9.
We'll illustrate it with a specific example.

*To divide $312$ by $9$, write out the partial sums of its digits, computed from left to right*

$$\begin{aligned} 3 &= 3 \\ 3+1 &= 4 \\ 3+1+2 &= 6 \end{aligned}$$

*and then read off the answer:*

$$312 \div 9 = 34\ R\ 6$$

*(And indeed, $312 = 9 \times 34 + 6$.)*

In the same way:

For the number $1221$ we get the sums $1 = \mathbf{1}$, $1+2 = \mathbf{3}$, $1+2+2 = \mathbf{5}$ and $1+2+2+1 = \mathbf{6}$. This bizarre method suggests

$$1221 \div 9 = 135\ R\ 6$$

which turns out to be correct.

and

For the number 20,000 we get the sums 2= $\mathbf{2}$, 2+0 $=\mathbf{2}$, 2+0 $+0=\mathbf{2}$, $2+0+0+0=\mathbf{2}$ and 2+0 $+0+0+0=\mathbf{2}$ . This method suggests

$$20{,}000 \div 9 = 2222\ R\ 2$$

which also turns out to be correct.

One might have to perform some explosions along the way and deal with extra-large remainders.

For instance, the method suggests that $5623 \div 9 = 5|11|13$ with a remainder of 16. (Do you see this?) With explosions, this gives

$$5623 \div 9 = 623\ \ R\ \ 16$$

But a remainder of "16" corresponds to one extra group of 9 and a remainder of 7. So, we really have

$$5623 \div 9 = 624\ \ R\ \ 7$$

a) Before reading on, can you explain why this strange method works?

One way to explain the puzzle is to write $\frac{1}{9}$ as an infinitely long decimal.

b) If you haven't done so already, show that $\frac{1}{9} = 0.111111\ldots..$

| $\frac{1}{9}$ = | | | . | 1 | 1 | 1 | 1 | 1 | 1 | … |
|---|---|---|---|---|---|---|---|---|---|---|

If we double this expression, and triple it, and so forth, we get

$$\frac{1}{9} = 0.1111\ldots \quad \frac{2}{9} = 0.2222\ldots \quad \frac{3}{9} = 0.3333\ldots \quad \frac{4}{9} = 0.44444\ldots$$

and so on. In general, $\frac{N}{9} = 0.N|N|N|N|N$ ... for any given number $N$ for a numerator.

To compute $312 \div 9$ we need to compute $\frac{312}{9}$. This is a decimal with 312 dots in each box to the right of the decimal point.

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $312 \times \frac{1}{9} =$ | | | . | 312 | 312 | 312 | 312 | 312 | 312 | ⋯ |

We have a lot of explosions to do!

c) Perform the explosions, working left to right.

Do you see the numbers $3$ and $3+1$ and $3+1+2$ playing a key role along the way?

Do you see the answer $34.666666\cdots$appear?

d) What is $0.66666\cdots$ as a fraction?

e) Can you see that you have just established that $312 \div 9$ is $34$ with a remainder of $6$?

Repeat this argument to show that

f) $214 \div 9$ equals $23 + \frac{7}{9}$, with the numbers $2$, $2+1$, and $2+1+4$ playing a key role along the way.

g) $432 \div 9$ equals $47 + \frac{9}{9}$ (also known as $48$), with the numbers $4$, $4+3$, and $4+3+2$ playing a key role.

## ANOTHER BONUS!

Let's attend to an age-old question.

**Is $0.9999999\ldots$ equal to 1or is it not?**

Many people argue that this quantity must equal 1 because of what we observed in the previous Bonus Musing. There we saw

$$\frac{1}{9} = 0.1111\ldots \quad \frac{2}{9} = 0.2222\ldots \quad \frac{3}{9} = 0.3333\ldots \quad \frac{4}{9} = 0.44444\ldots$$

It follows that

$$\frac{9}{9} = 0.99999999\ldots$$

and we know the value of $\frac{9}{9}$. It's one!

Others argue that it cannot be 1.

$0.9 = \frac{9}{10}$ is smaller than 1
$0.99 = \frac{99}{100}$ is also smaller than 1
$0.999 = \frac{999}{1000}$ still is smaller than 1

even

$0.9999999999 = \frac{9999999999}{10000000000}$ is smaller than 1

We humans can only ever write down a finite number of 9s and every time we do so we see a value smaller than 1. Surely, with an infinite number of 9s (if we could write them down) we still have a value smaller than 1?

The issue is that we are playing a mind game. With that ellipsis, we are never actually writing down the number we are talking about!

We humans will only ever be able to experience a finite number of 9s (to give a value smaller than 1), but no experience is about the number 0.9999 .... itself.

The jury has to remain "out" on any conclusions about 0.9999 ... via this argument.

Most people feel like they can imagine the quantity $0.9999\ldots$ and feel like it should have a meaningful value.

If you are one of those people, then mathematics is suggesting it has value $1$ (look at $\frac{9}{9}$ again).

Some people are more cautious about assuming a mind-game quantity like $0.9999\ldots$ really "exists" in the first place and will argue that asking for its value is a moot: $0.9999\ldots$ doesn't exist!

Mathematicians take a different approach. They interpret an ellipses as indicating "journey," not a destination.

The numbers we humans can write down—$0.9$ and $0.99$ and $0.999$ and $0.9999$ and $0.99999$ and so on—are certainly getting closer and closer to the value $1$. So, let's interpret an ellipsis as

*the value that the decimals we humans can experience, as indicated by the decimal number, seem to approach.*

For example, $0.3 = \frac{3}{10}$ and $0.33 = \frac{33}{100}$ and $0.333 = \frac{333}{1000}$ and so on, are getting closer and closer to the value one third. So, mathematicians are happy to write

$$0.3333333\ldots = \frac{1}{3}$$

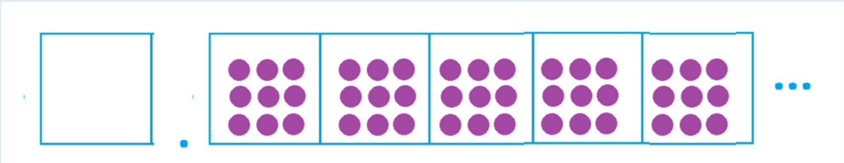
Mathematicians have checked that this line of thinking is consistent with all our rules of arithmetic, and so no surprises and contradictions will result if you play with infinite decimals via this mindset.

**Question:** What is $3 \times 0.333333\ldots$? Do we get an answer consistent with $0.99999\ldots$ having value $1$?

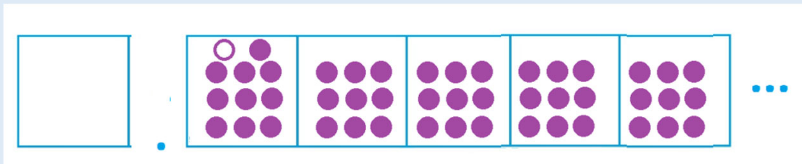
So, yes, $0.9999\ldots$ exists and "has" value $1$ if you interpret the infinite decimal as a journey: Where do the numbers $0.9$, $0.99$, $0.999$ , $0.9999$, and so on, seem to be taking you?

**Musing 54.16** Even Exploding Dots shows that $0.9999\ldots$ suggests a journey to the number $1$.

Here's a picture of $0.99999\ldots$ in a $1 \leftarrow 10$ machine.



Place a dot and an antidot in the first decimal place. This is technically adding nothing to the picture.



Perform an explosion. What picture do you get?

Now place a dot and antidot in the second decimal place and perform another explosion.

Do the same for the third decimal place, the fourth one, the fifth one.

What "final" picture do we seem to be approaching if we keep going?

## 55. Finite Decimals

The fractions $\frac{3}{10}$, $\frac{47}{100}$, and $\frac{813}{10000}$ have what are called **finite decimal expressions**: one only needs a finite number of digits to express each of them as a decimal number.

$$\frac{3}{10} = 0.3$$

$$\frac{47}{100} = 0.47$$

$$\frac{813}{10{,}000} = 0.0813$$

(Do you agree with these representations?)

In general, any fraction with denominator of either $10$, $100$, $1000$, … has a decimal representation that stops after a finite number of places to the right of the decimal point.

Actually, any fraction that is equivalent to a fraction with denominator of either 10 or 100 or 1000 or 10000, and so on, has a finite decimal expression. For instance,

$\frac{7}{20}$ is equivalent to $\frac{7\times5}{20\times5} = \frac{35}{100}$ and so, as a decimal, is $0.35$,

$\frac{131}{500}$ is equivalent to $\frac{131\times2}{500\times2} = \frac{262}{1000} = 0.262$,

and

$\frac{1}{2} = \frac{1\times5}{2\times5} = \frac{5}{10} = 0.5.$

**Practice 55.1:** Write $\frac{3}{5}$, $\frac{21}{250}$, and $\frac{3}{125}$ each as finite decimals.

**Practice 55.2:** Write $\frac{1}{2\times2\times2\times2\times2\times5\times5\times5}$ as a finite decimal.

**MUSINGS**

**Musing 55.3** Do you think the reverse is true? If a number can be written as a decimal with only a finite number of non-zero digits to the right of the decimal point, must that number be a fraction with denominator 10 or 100 or 1000, and so on?

**Musing 55.4** Do you think $0.7$ and $0.70000\ldots$ represent the same number?

# 56. Decimal Arithmetic

Decimals have two advantages when thinking about doing arithmetic with them.

- Decimals are numbers written as codes from a $1 \leftarrow 10$ machine and so one can use all the tricks and tools of place value to conduct computations.

- Decimals can often be rewritten as fractions and so one can use all the tools of fraction arithmetic to help you out too.

For example, let's add, subtract, multiply, and divide the two numbers $0.05$ and $0.006$.

Addition, via place value, is straightforward.

$$\begin{array}{rl} & 0.\,0\,5 \\ + & 0.\,0\,0\,6 \\ \hline = & 0.\,0\,5\,6 \end{array}$$

**Practice 56.1:** As fractions, $0.05$ is $\frac{5}{100}$ and $0.006$ is $\frac{6}{1000}$. Check that $\frac{5}{100} + \frac{6}{1000}$ is indeed $\frac{56}{1000}$.

Subtracting these two numbers requires an unexplosion.

$$\begin{array}{rl} & 0.\,0\,5 \\ - & 0.\,0\,0\,6 \\ \hline = & 0.\,0\,5\,{-6} \end{array} \quad = \quad 0.\,0\,4\,\overset{10}{-6}$$

$$= \quad 0.\,0\,4\,\,4$$

**Practice 56.2:** Check that $\frac{5}{100} - \frac{6}{1000}$ is indeed $\frac{44}{1000}$.

**Question:** Which is easier for you here: place-value thinking or fraction thinking?

Multiplication this way, however, seems awkward. Hmm.

$$\begin{array}{r} 0.05 \\ \times\ 0.006 \\ \hline = \ ??? \end{array}$$

But fraction-thinking makes it fairly straightforward.

$$\begin{aligned} 0.05 \times 0.006 &= \frac{5}{100} \times \frac{6}{1000} \\ &= \frac{30}{100000} \\ &= \frac{3}{10000} \qquad = 0.0003 \end{aligned}$$

**Practice 56.3:**
a) What is $23 \times 37$?

b) What is $0.023$ as a fraction? What is $0.37$ as a fraction?

c) Your work in parts a) and b) show that $0.023 \times 0.37$ equals $\frac{851}{100{,}000}$. Do you see how? What is $0.023 \times 0.37$ as a decimal?

d) What is $2.3 \times 3.7$ and $230 \times 0.037$?

**If you want to multiply two decimal numbers by hand, it does seem like a good move to convert the numbers into fractions first.**

One typically doesn't want to do such work by hand. But every now and then you might find yourself doing more tractable problems this way.

**Practice 56.4:** What is $0.7 \times 0.004$? (Do you see "28" in your head right away?)

Dividing our two decimal numbers directly seems awful.

$$0.006\overline{)\,0.05}$$

**ICK!**

But doing it via fraction thinking is fine!

$$0.05 \div 0.006 = \frac{0.05}{0.006} \quad = \frac{\frac{5}{100}}{\frac{6}{1000}}$$

$$= \frac{\frac{5}{100} \times 1000}{\frac{6}{1000} \times 1000}$$

$$= \frac{50}{6} \quad = \frac{25}{3} \quad = 8\frac{1}{3}$$

And we can write the final answer as $8.33333\ldots$ if we wish.

**It's a good move too to convert decimal numbers into fractions first if you want to divide them.**

(Of course, the best move of all most of the time is to just use a calculator!)

**Practice 56.5**

a) What is $0.21 \div 0.003$?

b) What is $0.021 \div 0.3$?

c) What is $2.1 \div 3$?

Some more practice.

**Practice 56.6:** Compute $13.276 + 5.94$ and $13.276 - 5.94$.

**Practice 56.7:**
a) Agatha says that computing $0.0348 + 0.0057$ is essentially a matter of adding 348 and 57. What does she mean by this? Is she right?

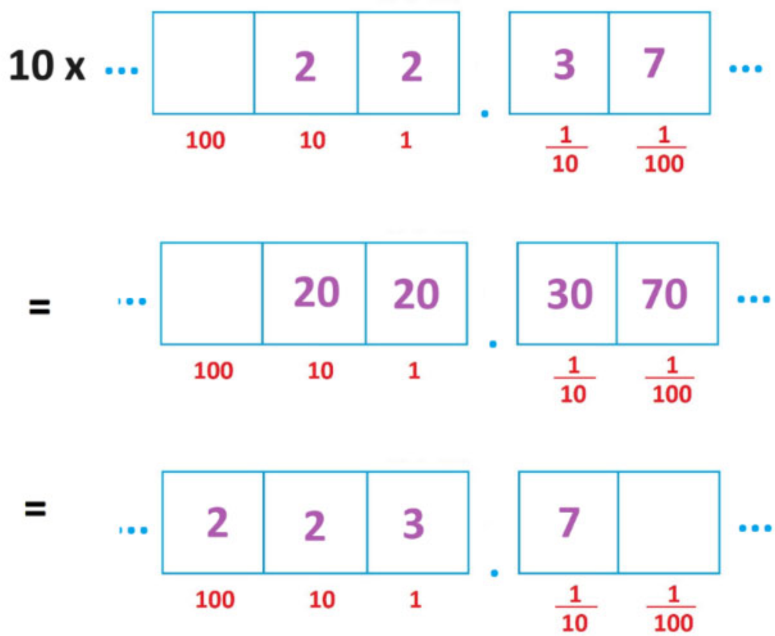
b) Percy says that computing $0.0852 + 0.037$ is essentially a matter of adding 852 and 37. He is not right. What is wrong with Percy's thinking?

**Practice 56.8:**
a) What is $\frac{1}{5} \times 0.02$?
b) What is $\frac{1}{5} \div 0.02$?

## Multiplying and Dividing Decimals by $10$.

Musing 53.11 had us multiply 22.37 by 10. We got the answer 223.7, making it appear that the decimal point magically shifted place.

Of course, it is really the explosions of dots in a $1 \leftarrow 10$ machine that give this illusion.




We can also see this if we write out the number 22.37 in full "expanded form" (as schoolbooks call it).

$$22.37 \times 10 = \left(20 + 2 + \frac{3}{10} + \frac{7}{100}\right) \times 10 = 200 + 20 + 3 + \frac{7}{10} = 223.7.$$

We can also readily divide 22.37 by ten (that is, multiply it by $\frac{1}{10}$) this way too.

$$22.37 \div 10 = \left(20 + 2 + \frac{3}{10} + \frac{7}{100}\right) \times \frac{1}{10} = 2 + \frac{2}{10} + \frac{3}{100} + \frac{7}{1000} = 2.237.$$

Both answers involve the same digits 2-2-3-7 in order.

**To multiply or divide decimal numbers by 10 and 100 and 1000 and so on, it's always good write out the decimals as fractions.**

**Example:** Compute $0.9 \div 100$.

**Answer:** This is $\frac{0.9}{100} = \frac{\frac{9}{10}}{100} = \frac{\frac{9}{10}\times 10}{100\times 10} = \frac{9}{1000} = 0.009$.

**Practice 56.9:** Consider the "abstract" decimal $0.abc$.

a) Be clear on why $0.abc \div 10$ equals $0.0abc$.
b) Be clear on why $0.abc \div 100$ equals $0.00abc$.
c) Be clear on why $0.abc \div 0.1$ equals $a.bc$.
d) Compute $0.abc \div 0.01$.
e) Compute $0.abc \times 100$.
f) Compute $0.abc \times 0.01$.

Multiplying a number by $10$ should make it ten times as big. (That's one of those "duh" comments!)

Dividing a number by $10$ (that is, multiplying it by $\frac{1}{10}$) will make it ten times as small.

So …

A number in the $200$s multiplied by ten will be in the $2000$s.

A number in the $400$s divided by ten will be in the $40$s.

A number close to $15$ multiplied by ten will be close to $150$.

A number close to $15$ divided by ten will be close to $1.5$.

Also, we've seen in a $1 \leftarrow 10$ machine that multiplying a decimal number by ten produces an answer with the same digits—the digits "shifted" because of explosions. (Actually, we can do unexplosions too in order to divide by ten and see a shift of digits again.)

We can see this to by working directly with fractions. For example,

$$5.67 \times 10 = \left(5 + \frac{6}{10} + \frac{7}{100}\right) \times 10$$

$$= 50 + 6 + \frac{7}{10} = 56.7$$

$$5.67 \div 10 = \left(5 + \frac{6}{10} + \frac{7}{100}\right) \times \frac{1}{10}$$

$$= \frac{5}{10} + \frac{6}{100} + \frac{7}{1000} = 0.567$$

These observations allow us to deduce the values of decimal numbers multiplied or divided by ten quite swiftly.

**Example:** What is $307.231 \times 10 \times 10$? What is $307.231 \times \frac{1}{10}$?

**Answer:** Both problems will produce answers that involve the digits 3-0-7-2-3-1 in order.

We are starting with a number in the 300s.

Multiplying by ten twice should give us a number in the 30,000s. We deduce

$$307.231 \times 10 \times 10 = 30{,}723.1$$

Dividing by ten once should give an answer the 30s. We deduce

$$307.231 \times \frac{1}{10} = 30.7231$$

**Example:** What is $307.231 \times 10 \times 10 \times 10 \times 10 \times 10$? (That is, what is $307.231 \times 100{,}000$?)

**Answer**: We have a number in the 300s being multiply by ten five times.

This means we'll have an answer in the ~~3,000~~ ~~30,000~~ ~~300,000~~ ~~3,000,000~~ 30,000,000s.

The answer must be 30,723,100.

**Practice 56.10:** Write down the values of each of these computations.

a) $483.014 \times 10$
b) $483.014 \times 10 \times 10$
c) $483.014 \times 10{,}000$
d) $483.014 \times \frac{1}{10}$
e) $483.014 \times \frac{1}{10} \times \frac{1}{10}$
f) $483.014 \times \frac{1}{10{,}000}$

Let's end this Section with something quirky.

**Example:** Find 99 fractions that lie between $\frac{1}{11}$ and $\frac{1}{12}$.

**Answer:** Here are some that work!

$$\frac{1}{11.01} \quad \frac{1}{11.02} \quad \frac{1}{11.03} \quad \cdots \quad \frac{1}{11.99}$$

(Are these in increasing or decreasing in size when reading left to right?)

If you don't like how these look, you can always rewrite them as more traditional fractions. For instance,

$$\frac{1}{11.01} = \frac{1}{11 + \frac{1}{100}} = \frac{1 \times 100}{(11 + \frac{1}{100}) \times 100} = \frac{100}{1100 + 1} = \frac{100}{1101}$$

**Practice 56.11:** Write down 999 fractions that lie between $\frac{1}{11}$ and $\frac{1}{12}$. List them in increasing order.

**Practice 56.12:** Compute each the following. (Or not! These each look very ugly!)

a) $0.3 \times (5.37 - 2.07) + \frac{0.75}{2.5}$

b) $\frac{0.1+(1.01-0.1)}{0.11+0.09}$

c) $\frac{(0.002+0.2\times 2.02)(0.22-0.02)}{2.22-0.22}$

Actually, this final practice problem brings up a good point.

The fraction bar looks like a vinculum, but, historically, it isn't.
Nonetheless, we follow the convention of treating it like one, as its own symbol of grouping.

**When presented with a complicated expression in the form of a fraction, treat the numerator as a quantity to be computed in its own right and treat the denominator as a quantity to be computed in its own right.**

You probably naturally did this if you tried the practice problem.

## MECHANICS PRACTICE

**Practice 56.13** Ask Siri or Alexa or some virtual friend to compute each of the following.

$$11.9 + 1.7 \qquad 11.9 - 1.7 \qquad 11.9 \times 1.7 \qquad 11.9 \div 1.7$$

Feel free to ty them by hand too to see if you get the same answers. (Actually, can you do the first two in your head?)

**Practice 56.14** Compute each of the following.
(Before you commence, make an estimate of each answer you should get.)

$$\begin{array}{r} 33.156 \\ +\ 8.814 \\ \hline = \qquad\quad \end{array} \qquad\qquad \begin{array}{r} 2.03 \\ -\ 0.777 \\ \hline = \qquad\quad \end{array}$$

**Practice 56.15** What is a natural way to see that $1.45 - 0.96$ is $0.49$?

**Practice 56.16**
a) What is $0.41$ as a percentage?
b) What is $58\%$ as a decimal?
c) What is $0.001$ as a percentage?
d) What fraction is $33\frac{1}{3}\%$ ?
e) What fraction is $0.12\%$?

**Practice 56.17** Compute each of the following

a) $1.5 \times 1.5$
b) $1.4 \times 1.4$
c) $1.45 \times 1.45$
d) $1.42 \times 1.42$

What I am trying to do here is come up with a decimal number with the property that if I square it, it will give the answer $2$.

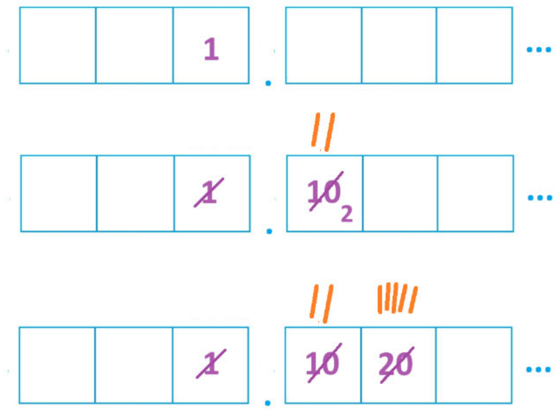
e) Do you see that $1.4$ is too small a number for this and $1.42$ is too big? What about $1.41$? Is that too big or too small?

f) Care to pin down what the digit in the third decimal place for such a number must be?

## 57. Every Fraction is a Repeating Decimal

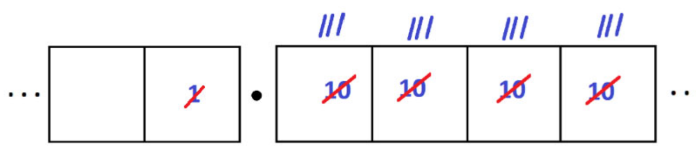
We've used division in a $1 \leftarrow 10$ machine to rewrite fractions as decimals.

For example, we saw that $\frac{1}{4}$, computed as $1 \div 4$, has decimal representation $0.25$.



Other fractions have infinitely long decimal expansions. For example, we computed $\frac{1}{3}$ as $1 \div 3$ and saw

$$\frac{1}{3} = 0.33333\ldots$$



And we saw too that

$$\frac{6}{7} = 0.857142\ \ 857142\ \ 857142\ \ \ldots$$

with a have a repeating pattern of "857142."

Back in Section 8 we got ahead of ourselves and talked about the use of the **vinculum** (a horizontal bar) throughout mathematics. We mentioned its use in decimals with repeating patterns.

Rather than write out blocks of digits that repeat a few times and slapping on an ellipsis to mean "keep this pattern going," people sometimes put a vinculum over the repeating group with the understanding that group of digits is repeating indefinitely.

For example, folk write

$$\frac{6}{7} = 0.\overline{857142}$$

and

$$\frac{1}{3} = 0.\overline{3}$$

for the two fractions we just considered.

An expression such as $0.38\overline{142}$ means "repeat the group 142 indefinitely after the beginning hiccup of 38."

$$0.38\overline{142} = 0.38\ \ 142\ \ 142\ \ 142\ \ 142\ \ldots$$

All the examples of fractions with infinitely long decimal expansions we've seen so far fall into a repeating pattern. This is curious.

We can even say this is the case too for our finite decimal examples: they fall into a repeating pattern of zeros after an initial start.

$$\frac{1}{4} = 0.2500000\ldots = 0.25\overline{0}$$

$$\frac{1}{2} = 0.50000\ldots = 0.5\overline{0}$$

(After all, one quarter is 2 tenths and 5 hundredths and 0 of every other decimal place-value thereafter, and one half is indeed 5 tenths and 0 of every other decimal place-value.)

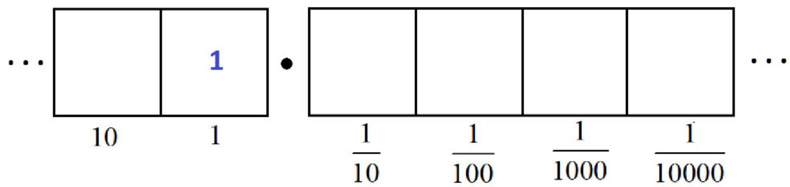
This begs the question:

> Does every fraction have a decimal representation that eventually repeats (allowing repeating zeros)?

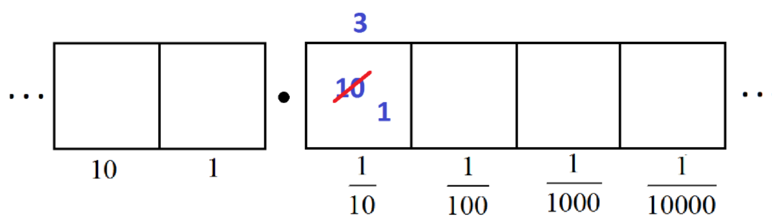
The answer to this question, surprisingly, is yes, and our method of division explains why.

Let's go through the division process again, slowly, first with a familiar example. Let's compute the decimal expansion of $\frac{1}{3}$ again in a $1 \leftarrow 10$ machine.

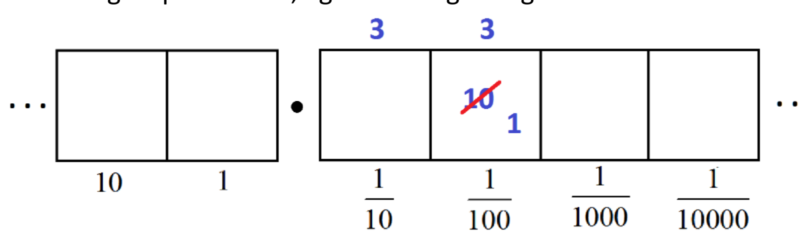
We think of $\frac{1}{3}$ as the answer to the division problem $1 \div 3$, and so we need to find groups of three within a diagram of one dot.




We unexplode the single dot to make ten dots in the tenths position. There we find three groups of three leaving a remainder of 1 in that box.



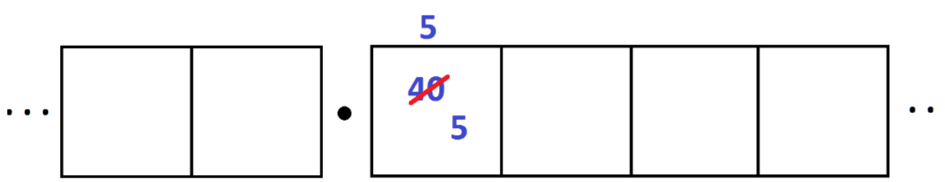

Now we can unexploded that single dot in the tenths box and write ten dots in the hundredths box. There we find three more groups of three, again leaving a single dot behind.



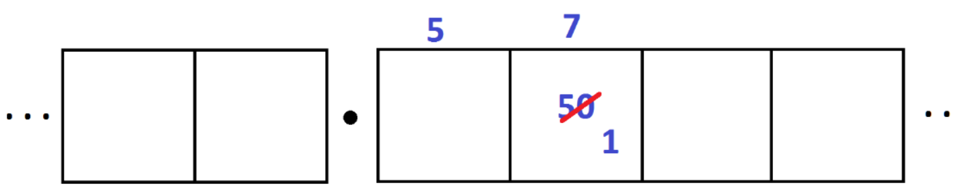

And so on. We are caught in a cycle of having the same remainder of one dot from cell to cell, meaning that the same pattern repeats. Thus, we conclude $\frac{1}{3} = 0.333.\ldots$ The key point is that the same remainder of a single dot kept appearing.

Let's compute the decimal expansion of $\frac{4}{7}$ in the $1 \leftarrow 10$ machine. That is, let's compute $4 \div 7$ and be sure to take note of the remainders that occur.

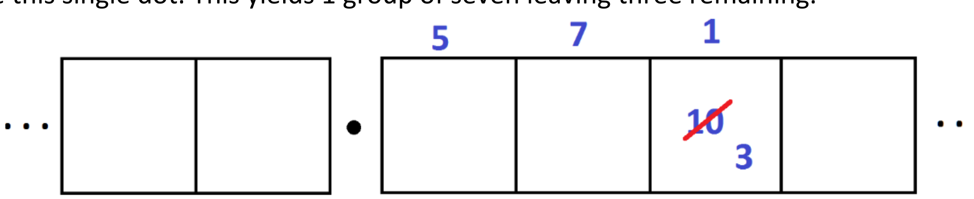

We start by unexploding the four dots to give 40 dots in the tenths cell. There we find 5 groups of seven, leaving five dots over.



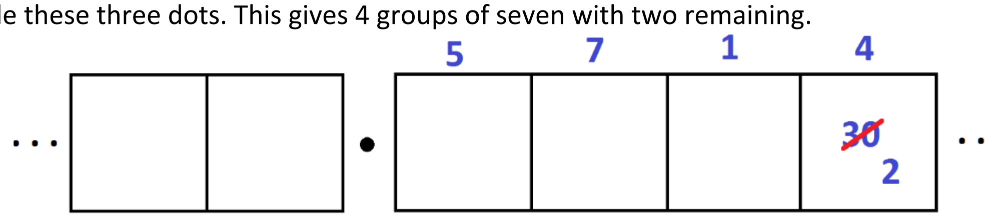

Now unexplode those five dots to make 50 dots in the hundredths position. There we find 7 groups of seven, leaving one dot over.



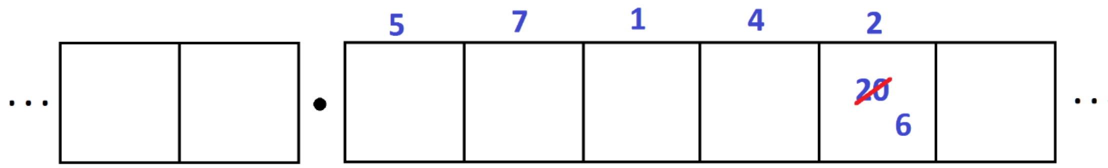

Unexplode this single dot. This yields 1 group of seven leaving three remaining.



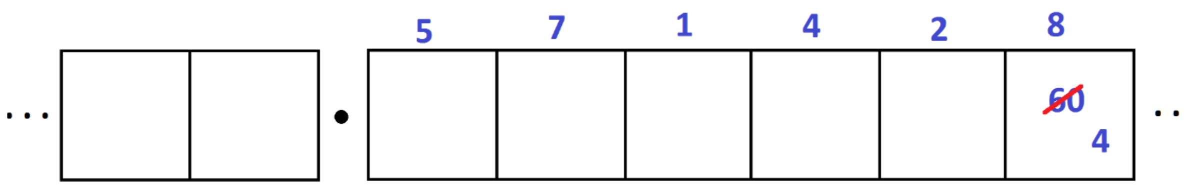

Unexplode these three dots. This gives 4 groups of seven with two remaining.



Unexplode the two dots. This gives 2 groups of seven with six remaining.




Unexplode the six dots. This gives 8 groups of seven with four remaining.




But this is the predicament we started with: four dots in a box!

So now we are going to repeat the pattern and produce a cycle in the decimal representation. We have

$$\frac{4}{7} = 0.571428\ \ 571428\ \ 571428\ \ \ldots$$

Stepping back from the specifics of this problem, it is clear now that one must be forced into a repeating pattern. In dividing a quantity by seven, there are only seven possible values for a remainder number of dots in a cell—0, 1, 2, 3, 4, 5, or 6—and there is no option but to eventually repeat a remainder and so enter a cycle.

In the same way, the decimal expansion of $\frac{18}{37}$ must also cycle. In doing the division, there are only thirty-seven possible remainders for dots in a cell (0, 1, 2, …, 36). As we conduct the division computation, we must eventually repeat a remainder and again fall into a cycle.

We have just established a very interesting fact.

| |
|---|
| Every fraction has a decimal representation that falls into a repeating pattern.<br>(A pattern of repeating zeros is allowed.) |

**Practice 57.1** As a check, conduct the division procedure for the fraction $\frac{1}{4}$. Make sure to understand where the cycle of repeated remainders commences.

## MUSINGS

**Musing 57.2** Find the decimal representation of $\frac{23}{45}$. (After a "hiccup," its decimal representation repeats just one digit over and over again. Which digit?)

**Musing 57.3** The fraction $\frac{1}{7}$ has a repeating decimal representation with a repeating block of six digits.

$$\frac{1}{7} = 0.\overline{142857}$$

Do you think it is possible for a fraction of the form $\frac{b}{7}$ (with $b$ a counting number) to have a decimal representation with a repeating block of digits ten digits long? Eight digits long? Seven digits long?

**Musing 57.4** **BACKWARDS: Is Every Repeating Decimal a Fraction?**

Consider the repeating decimal $0.6363636....$ . Is this number a fraction? If so, which one?

A popular technique for attending to this issue starts by giving the quantity a name and to repeatedly multiply the quantity by ten. Let's call the decimal Cecile. (Why not?).

We have

$$\begin{aligned} C &= 0.63636363\ldots \\ 10 \times C &= 6.36363636\ldots \\ 100 \times C &= 63.63636363\ldots \end{aligned}$$

Let's stop here since the infinite parts of $C$ and $100 \times C$ align perfectly.
Let's subtract them.

$$\begin{aligned} 100 \times C &= 63.63636363\ldots \\ C &= 0.63636363\ldots \\ \hline 99 \times C &= 63.00000000\ldots \end{aligned}$$

We see that one hundred $C$s take away one $C$, that's ninety nine $C$s, must equal 63.

$$99 \times C = 63$$

Ahh! $C$ must be the fraction $\frac{63}{99} = \frac{9\times7}{9\times11} = \frac{7}{11}$.

a) Use this technique to show that $0.111111$ .... is the fraction $\frac{1}{9}$.

b) Show that $0.213213213213213$ ... is the fraction $\frac{213}{999}$

c) What fraction is $0.2111111$ ... ?
(Keep multiplying this number by $10$ until you have two decimal parts that align.)

d) What fraction is $2.8213213213213213$ ...?

This technique shows that if a decimal number (eventually) has a repeating pattern, then we can keep multiplying that number by ten and until we find two multiples whose decimal parts align perfectly. Subtraction then allows us to identify that decimal number as a fraction.

## 58. A Decimal that Does Not Repeat is not a Fraction

We established in the previous section

> **Every fraction has a decimal representation that falls into a repeating pattern (perhaps a repeating pattern of zeros).**

People don't usually bother writing out a repeating pattern of zeros: writing $\frac{1}{2} = 0.5$ , for instance, rather than $\frac{1}{2} = 0.50000 \ldots$ or $\frac{1}{2} = 0.5\overline{0}$.

People call decimals that have repeating zeros **finite decimals** because they can be expressed with only a finite number of digits.

For example, $276\frac{1}{2}$ and $0.0000000000000000000000000000000001$ are finite decimals.

> **Practice 58.1** Do you think the number 3 could be called a finite decimal?

This now opens up a curious idea.

| A quantity given by a decimal expansion that does not repeat cannot be a fraction. |
|---|

Pause! Do you get the logic here?

> **Question:** Consider the statement: "Every crow is a black bird."
> Does it logically follow that if a bird is not black, it is not a crow? (Are there albino crows?)
>
> If the statement "Every Australian is cheery" is true, then what can you say about a non-cheery person you meet?
>
> If "Every fraction has a repeating decimal expansion" is true (and it is!), what can you say about a number that has a decimal expansion that never repeats?

Consider this decimal number

$$0.101100111000111100000111110000000\ldots$$

Even though we see a pattern to its decimal expansion (which allows us to figure out any particular decimal digit we want just by writing out the pattern far enough), it is not a repeating pattern.

This means that this number cannot be a fraction!

**Question:** Whoa! Pause again! Take this in.

The quantity $0.10110011100011110000\ldots$ is a bit bigger than $0.1$, which is $\frac{1}{10}$, and so is just to the right of one tenth on the number line. It's a number!

Yet it's a number that cannot be a fraction: it doesn't have a repeating decimal representation.

Do you find this freaky?

We can invent all sorts of numbers that can't be fractions.

For example,

$$0.102030405060708090100110120130140150\ldots$$

and

$$0.303003000300003000003000000 3\ldots$$

are numbers that are not fractions.

(Do you see a pattern in each of these examples? Do you see that neither is a repeating pattern?)

Recall that people call numbers that are either positive fractions or negative fractions **rational numbers**. Any number that cannot be a fraction, like the ones we are creating now, are called **irrational numbers**.

Irrational numbers can be positive and be negative. For example, $-0.10110011100011110000\ldots$ is a negative irrational number.

## MUSINGS

**Musing 58.2** Write down two infinitely long decimal expansions that you personally know cannot be rational numbers.

**Musing 58.3** Write down a number slightly larger than $\frac{1}{3}$ that is not a fraction.

**Musing 58.4** Could a number slightly larger than $\frac{1}{3}$ that is not a fraction and number slightly smaller than $\frac{1}{3}$ that is also not a fraction add to a number that is a fraction?

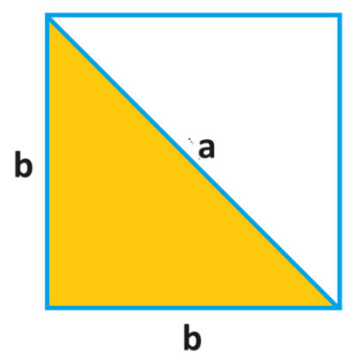
## 59. VERY OPTIONAL ASIDE: A Historically Famous Example a Number that is not a Fraction

The Pythagoreans of some 2500 years ago believed that all that is good and harmonious in the world can be expressed mathematically via a counting number or via a comparison of two counting numbers.

For example, simultaneously plucking two identical strings under the same tension, but with one string twice the length of the other produces two notes that sound harmonious to the ear. (They make the musical interval of one *octave*.) Two identical strings, one $3$ units long and the other $2$ units long plucked simultaneously produce a pleasing *perfect fifth*, and one $5$ units long the other $4$ units long a pleasing *major third*, and so on.

Mathematics was central to the worldview of the Pythagoreans. They sought to describe the universe in terms of number (counting numbers) and geometry to such a degree that many people today regard their academic pursuits as tied to a religious cult.

A fundamental shape in geometry is a square. Surely, the two fundamental lengths in a square—the length of any one of its sides and the length of its diagonal—are “in harmony”? That is, if you choose the right basic unit of length, surely you can say the diagonal of the square is $a$ units long while its sides are $b$ units long with $a$ and $b$ both counting numbers?




The Pythagoreans were utterly shocked to eventually learn that this is not so!

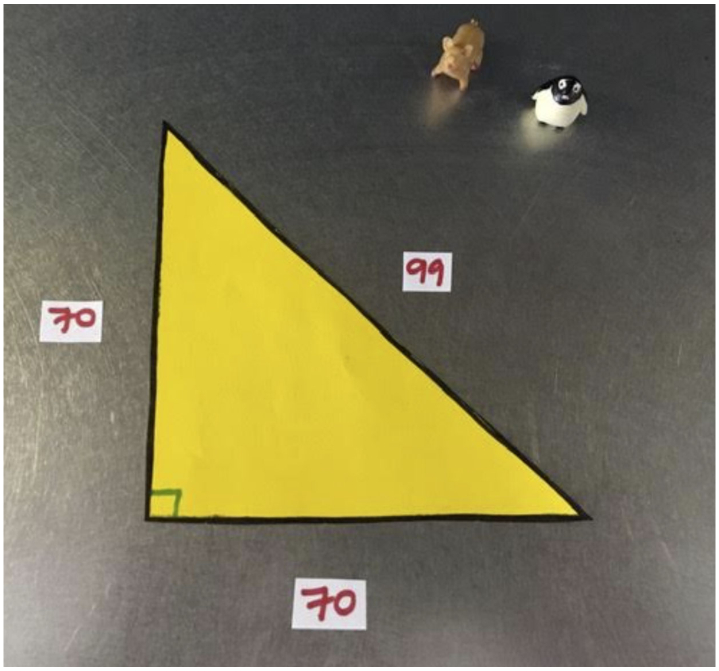
Their worldview was truly shattered.

There are a number of ways to show that there cannot be two counting numbers that describe the two fundamental lengths in a square.

My personal favorite approach is a physical one—cutting out a square in paper, folding it, and thinking about what the folding means.

I did this as an activity and took photos of my work. I started with paper square 70 centimeters in side length, which I then cut diagonally in half. (Two figurines, a pig and a penguin, assisted me.)

What's lovely about this choice of side length is that the diagonal is close, very close, to being 99 centimeters long, so close that my human eye cannot tell that it is not.

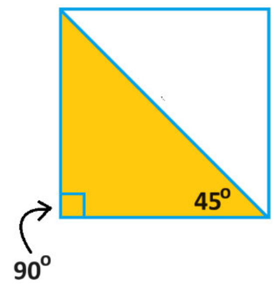



Our goal is to show that believing the diagonal has length a counting number too, such as 99, just cannot be so.

We need to show that something is terribly wrong with this picture.

**Comment:** As we proceed, we will need to one fact from geometry class, namely:

Any triangle that comes from cutting a square in half diagonally has a 90-degree angle and a 45-degree angle. And, in reverse, any triangle that has a 90-degree angle and a 45-degree angle is half a square.




I hope this at least feels intuitively right to you.

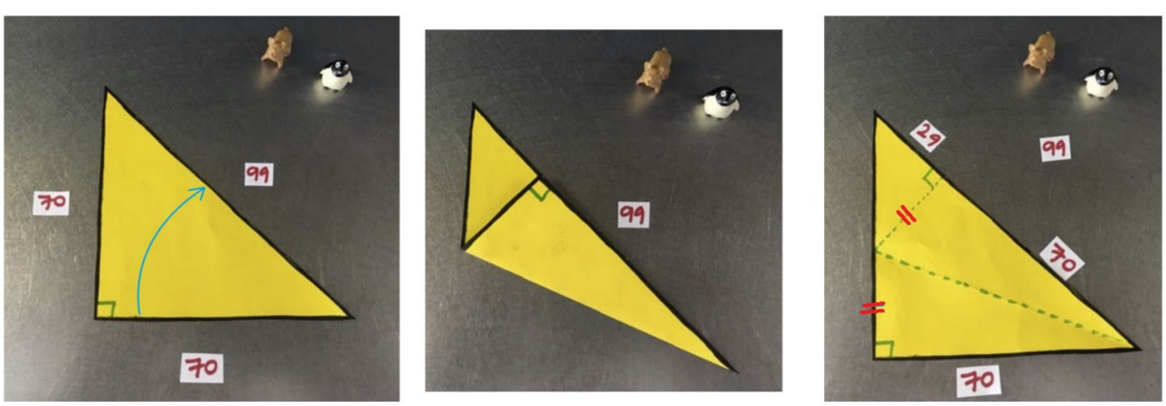
Okay, back to our triangle, which is half a square with sides allegedly of lengths 70, and 70, and 99 centimeters.

Fold the bottom edge of the triangle up to the hypotenuse of the triangle and draw in the lines to show the edges we created when doing that.

Notice two things:

- The two lengths marked with the double dashes in the third photo below are the same length. (We see that from the folding in the middle photograph.)

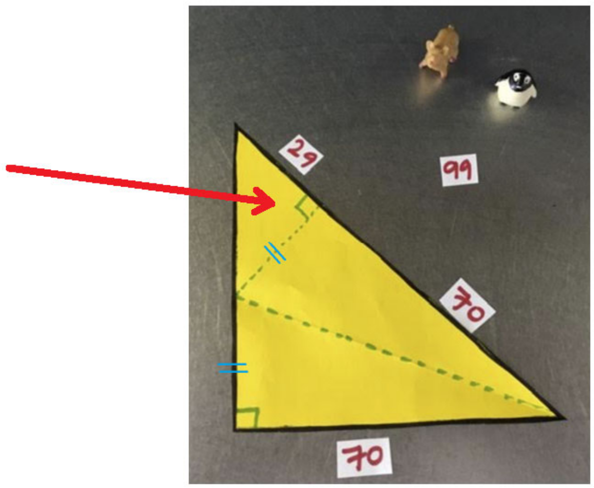
- We can identify an edge of length 29 centimeters because we folded an edge of length 70 centimeters up against an edge of length of 99 centimeters. (Again, look at the middle photograph.)



Actually, there is a third important thing to notice.

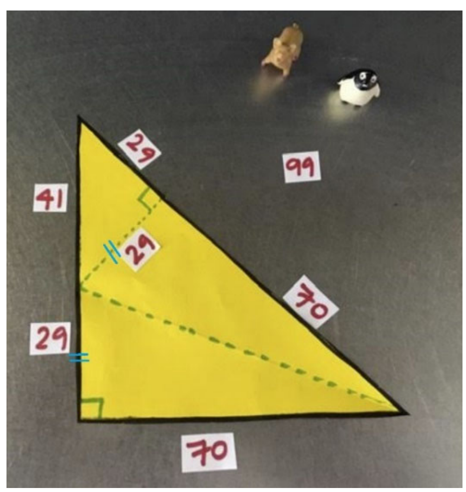
We see in our third photograph a little triangle with top angle $45$ degrees (from the original big triangle that is half a square) and with a $90$-degree angle (from folding a $90$-degree angle up to the hypotenuse of the big triangle.)

This little triangle at the top must be half of its own square.




One side of the smaller square is 29 centimeters long. So, the other side of the square is 29 centimeters long as well.

And since the two lengths marked with a double dash are the same, we have three lengths of 29 centimeters in our picture!



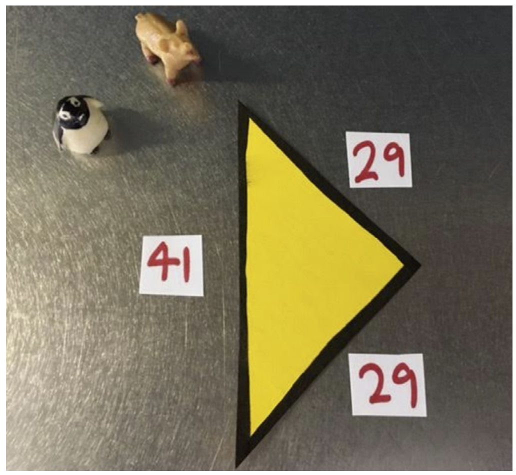

And we can mark a length of 41 centimeters as well: the left edge of the big triangle is 70 centimeters long, and $70 - 29 = 41$.

Everything we just did came from combining the two counting numbers 99 and 70 we started with via subtraction to produce new counting numbers.

$$29 = 99 - 70$$

$$41 = 70 - 29$$

And these new, smaller counting numbers are the side-lengths of another triangle that is half a square. And we see from the paper, this new triangle is much smaller than the original triangle.




**Your Turn:**

Cut out a 7 inch by 7 inch square from a piece of paper.
Cut your square in half diagonally to make a triangle.

With a ruler, verfiy that the length of the long edge of your triangle is very close to 10 inches.

Using the numbers 7, 7, and 10, follow the folding and thinking outlined above to create a smaller traingle that must also be half a square. If we believe those counting numbers are accurate what are the side lengths of your smaller half square? (You should be reasoning that they are 3, 3, and 4 inches long.)

What we are doing is worrisome.

We are seeing that if you have a triangle that is half a square with side lengths each a counting number, then you can fold that triangle to create a smaller half square also counting numbers as side lengths.

And each new half-square triangle we create is demonstrably smaller than the triangle we started with.

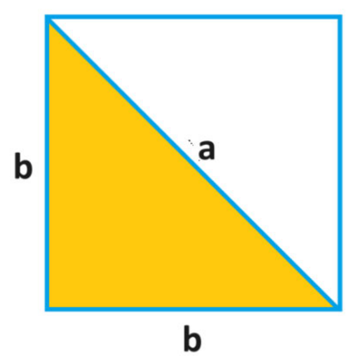
> **Practice 59.1** Right now, in my photos, we allegedly have a half square with side lengths 29, 29, and 41 centimeters—all counting numbers.
>
> If we apply the folding procedure on this triangle, what are the (alleged) dimensions of the even smaller half square we create?

If we do this folding process over and over and over and over and over again will eventually obtain a half-square smaller than an atom, all the while giving us counting number side lengths. The side-lengths can't ever be zero—we do have a triangle of some size—but no triangle smaller than an atom can have counting number side lengths!

Believing that we had counting number side lengths to begin with puts us into a logical pickle. Something is indeed terribly wrong!

The only way out of this pickle is to conclude that believing we had nothing but counting numbers to begin with is wrong. (Even though we assumed we had the counting numbers 70, 70, and 99, any set of beginning counting numbers will lead to this pickle. We'd again create a triangle smaller than an atom but still with counting number side lengths, allegedly!)

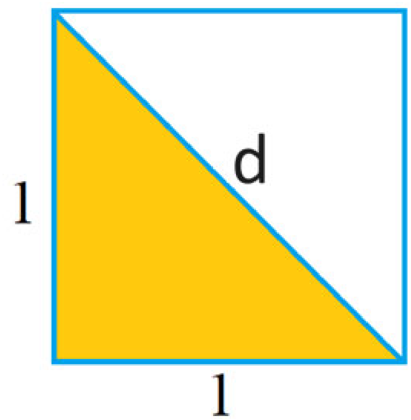
There are no counting numbers $a$ and $b$ that describe the lengths in this picture.




The side length of a square and the diagonal length of a square are in discord!

Let's rephrase what we just concluded in our modern setting.

Consider a square with side length $1$ unit. Then the diagonal has some length. Call it $d$ units.




Now, $d$ is a number. It has some value.

**Question:** Getting ahead of ourselves again …
Do you remember the Pythagorean Theorem from geometry class?

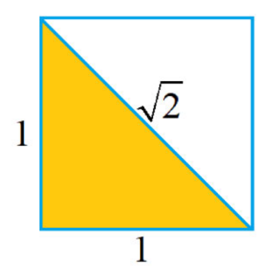
We have a right triangle in our picture, and we see by the Pythagorean Theorem that

$$1^2 + 1^2 = d^2$$

This tells us that $d^2 = 2$.

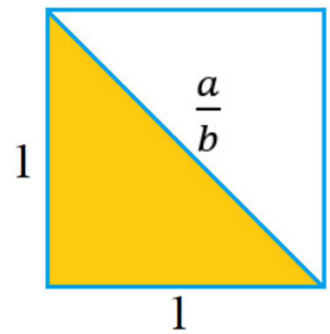
Thus, $d$ is a number that multiplies by itself to give the value $2$.
People call that the "square root" of $2$.

$$d = \sqrt{2}$$

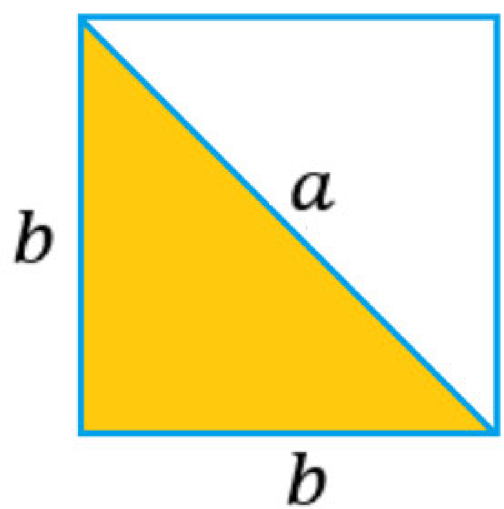


We have essentially just demonstrated that the number $d$, whatever it is, cannot be a fraction.

For if $d = \frac{a}{b}$ for two counting numbers $a$ and $b$,




then we can scale our picture up by a factor $b$ (have all the lengths grow to $b$ times as big: $b \times 1 = b$ and $b \times \frac{a}{b} = a$) and obtain a half square with counting numbers lengths.



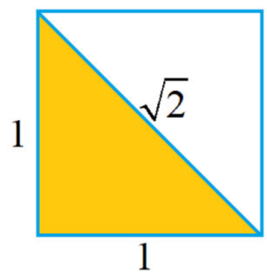

And we just proved that that cannot be!

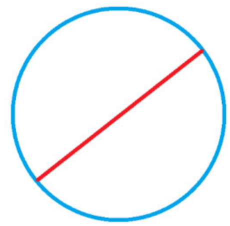
The number $d$, the length of the diagonal of a square with side length 1 unit, whatever that value is, must be an irrational number.

**Irrational numbers exist in the real world!**

**Side Comment:** Schoolbooks want students to "know" that two famous numbers in mathematics are irrational. These numbers are:

- $\sqrt{2}$, the length of the diagonal of a square with side length one unit
- $\pi$ (pi) the number that arises if you take the circumference of a circle and divide it by its width.



$$\pi = \frac{\text{circumference}}{\text{diameter}}$$

We just went to an awful lot of effort to demonstrate that $\sqrt{2}$ is an irrational number. It is not at all "obvious" that $\sqrt{2}$ is not a fraction.

Matters are more challenging for the number $\pi$.

In fact, scholars wondered for millennia whether or not $\pi$ is a fraction. They calculated its decimal expansion to many hundreds of counts of digits and saw no pattern or structure to those digits. They found fractions that approximated the value of $\pi$ very closely, and developed methods for creating more fractions that would approximate it as closely as one pleases. But whether or not $\pi$ itself is a fraction (with some gigantically large numerator and some gigantically large denominator) remained a frustrating mystery for centuries and centuries.

It wasn't until around the year 1761 that Swiss mathematician Johann Lambert was finally able to establish, once and for all, that $\pi$ is an irrational number. It is not a fraction.

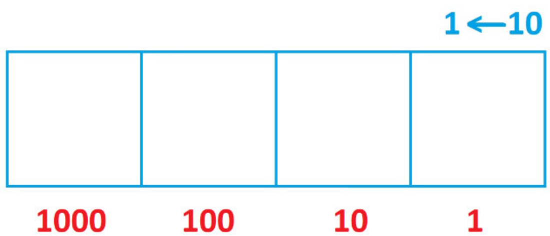
The proof of this is very hard, and well beyond the school curriculum. It is extremely far from "obvious" that $\pi$ is not a fraction.

So, when school curriculums say "Students should know that $\sqrt{2}$ and $\pi$ are examples of irrational number" they really mean, "Students should be told that $\sqrt{2}$ and $\pi$ are examples of irrational number."

**Question:** Did your schoolbooks ever explain why $\sqrt{2}$ is an irrational number?

## 60. The Powers of Ten

We saw that dots in a $1 \leftarrow 10$ machine are worth $1$ in the rightmost box and then have values $10$ , $100$, $1000$, and so on, as we move through the places to the left. These values grow by a factor of ten from one box to the next.




We have

$$\begin{aligned}
&1 = 1\\
&10 = 1 \times 10\\
&100 = 1 \times 10 \times 10\\
&1000 = 1 \times 10 \times 10 \times 10\\
&10000 = 1 \times 10 \times 10 \times 10 \times 10
\end{aligned}$$

and so on.

Rather than repeatedly write out products of ten, the mathematics community has settled on using superscripts to denote the result of repeatedly multiplying the number $1$ by a fixed value.

> If $n$ is a counting number, then for any number $a$ the notation $a^n$ means
>
> $$1 \times \overbrace{a \times a \times \cdots \times a \times a}^{n \text{ of these}}$$
>
> We read $a^n$ as $a$ **raised to the $n$th power**.

For example, two raised to the third power is the number $1$ doubled three times,

$$2^3 = 1 \times 2 \times 2 \times 2 = 8$$

and ten raised to the sixth power is the number $1$ increased by a factor of ten, six times

$$10^6 = 1 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10 = 1{,}000{,}000$$

This gives a million.

We saw this notation way back in Section 10 when we were chopping up squares and cubes.

We had

$$5^2 = 5 \times 5$$

saying "five squared," and

$$5^3 = 5 \times 5 \times 5$$

saying "five cubed." (And we didn't talk about $5^4$ and $5^5$, and so on because we don't have everyday language for ideas that go beyond the third dimension!)

**Question:** Do we have a discrepancy here?

Shouldn't we have $5^2 = 1 \times 5 \times 5$ and $5^3 = 1 \times 5 \times 5 \times 5$?

Does it matter if we ignore the 1 that is meant to be up front?

You might see in some schoolbook authors defining $a^n$ as

$$\overbrace{a \times a \times \cdots \times a \times a}^{n \text{ of these}}$$

without a 1 up front. This makes no difference if $n$ is a counting number different from zero.

**Practice 60.1**

a) Does defining $a^n$ as $1 \times \overbrace{a \times a \times \cdots \times a \times a}^{n \text{ of these}}$ make sense if $n$ happens to be zero? (If so, what is the value of $a^0$?)

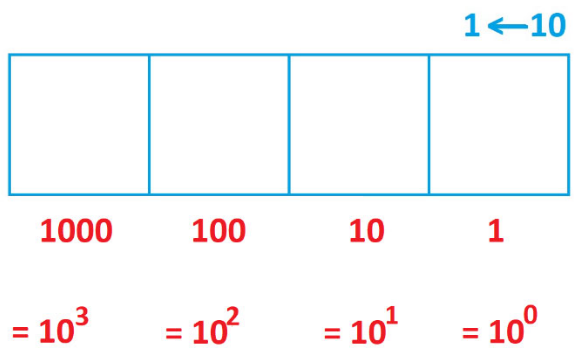
a) Does defining $a^n$ as $\overbrace{a \times a \times \cdots \times a \times a}^{n \text{ of these}}$ make sense if $n$ happens to be zero? (If so, what is the value of $a^0$?)

Thinking of $10^n$ as

$$1 \times \overbrace{10 \times 10 \times \cdots \times 10 \times 10}^{n \text{ of these}}$$

it is clear, for us, that $10^0$ is $1$.

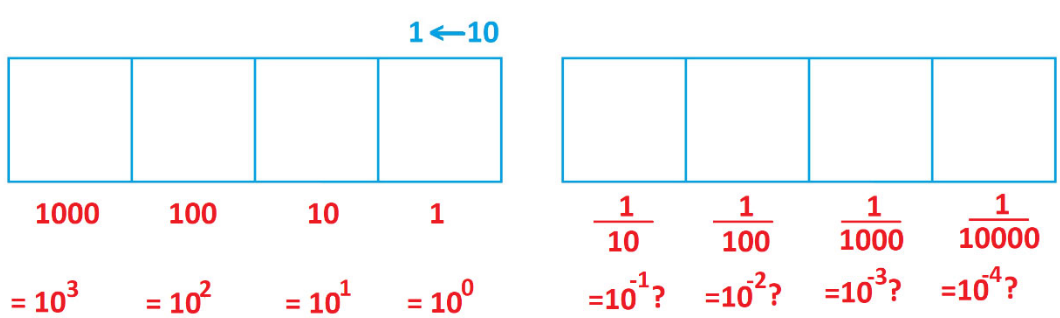
And this is nice as we can now say that all the place values in our $1 \leftarrow 10$ machine are powers of ten.




But what about the boxes to the right, the decimal places?

It seems irresistible then to keep the powers-of-ten pattern going: from $10^3$, $10^2$, $10^1$, and $10^0$ down into $10^{-1}$, $10^{-2}$, $10^{-3}$, and so on.

Can we say this?



Mathematicians have settled on a second piece of convenient notation.

If $n$ is a counting number, then for any number $a$ (not zero) the notation $a^{-n}$ means

$$1 \times \overbrace{\frac{1}{a} \times \frac{1}{a} \times \cdots \times \frac{1}{a} \times \frac{1}{a}}^{n \text{ of these}}$$

We read $a^{-n}$ as $a$ **raised to the negative $n$th power**.

This notation is motivated by the following idea:

Since $10^1$ means (for us) "multiply the number $1$ by ten," it feels like $10^{-1}$ should be the opposite of this, which would be: "divide the number $1$ by ten." And dividing by ten, as we know, is the same as multiply by $\frac{1}{10}$.

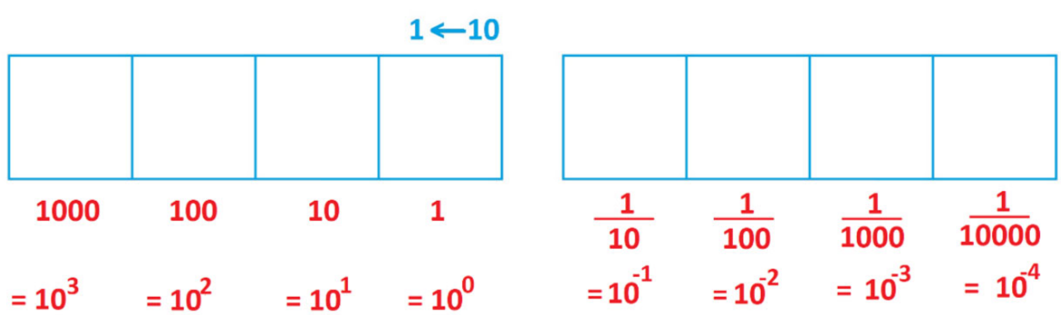
$$10^{-1} = 1 \times \frac{1}{10} = \frac{1}{10}$$

And since $10^2$ means "multiply the number $1$ by ten, twice," it feels like $10^{-2}$ should be the opposite of this, "divide the number $1$ by ten, twice." That would be multiplying the number $1$ by $\frac{1}{10}$, twice.

$$10^{-2} = 1 \times \frac{1}{10} \times \frac{1}{10} = \frac{1}{100}$$

And so on.

This notation is convenient as we can now represent each place value in our $1 \leftarrow 10$ machine using this notation.



**Comment:** We are not doing any mathematics with these powers $10^n$ and $10^{-n}$. Both are just shorthand for writing out repeated multiplications

$$10^n = 1 \times \overbrace{10 \times 10 \times \cdots \times 10 \times 10}^{n \text{ of these}}$$

$$10^{-n} = 1 \times \overbrace{\frac{1}{10} \times \frac{1}{10} \times \cdots \times \frac{1}{10} \times \frac{1}{10}}^{n \text{ of these}}$$

But there is a slew of mathematics one can explore with such repeated products, and that mathematics agrees with the notation we happen to be using here. (We'll talk about the mathematics of "powers," for sure, in our next volume of chapters.)

For us now, this notation will just be notation, nothing more.

And this notation will help us write really really big numbers and really really small numbers in a $1 \leftarrow 10$ machine with ease.

**MUSINGS**

**Musing 60.2** Recall that nine-year-old Milton Sirotta in 1938 coined the term *googol* for the number 1 with one-hundred zeros after it ($10000 \cdots 00$) and the term *googolplex* for the number 1 followed by a googol zeros.

a) Write down a googol as a power of ten.
b) Write down a googolplex as a power of ten.

**Musing 60.3**
Draw a $1 \leftarrow 10$ machine picture of each of these quantities. What number do they each represent?

a) $3 \times 10^5 + 2 \times 10^4 + 7 \times 10 + 5 \times 10^{-2}$

b) $17 \times 10^3 + 82 \times 10^2 + 90 \times 10 + 76 \times 1 + 23 \times 10^{-1} + 48 \times 10^{-2}$

**Musing 60.4** We have:

$10^3 = 1{,}000$ is called a **thousand**.
$10^6 = 1{,}000{,}000$ is called a **million** (it's a thousand thousands).
$10^9 = 1{,}000{,}000{,}000$ is called a **billion** (it's a thousand millions).

a) What number is a **trillion**? A **quadrillion**?

b) In the past, a million million was called *billion.* What number is that as a power of ten?

In the past, a million billion was called *trillion.* What number is that as a power of ten?

In the past, a million trillion was called *quadillion.* What number is that as a power of ten?

c) The prefixes "bi," "tri," and "quad" mean *two*, *three*, and *four*, respectively.
Do these prefixes make sense for names of the numbers *billion*, *trillion*, and *quadrillion*?

d) What is a *milliard* ?

Do you care to look up the history of the names for these big numbers?
When did their meaning change? What instigated the change?

**Musing 60.5** How many bytes is a gigabyte? Express your answer as a power of ten.

**Musing 60.6** Are you a billion seconds old?

## MECHANICS PRACTICE

**Practice 60.7** What number do each of these quantities represent?

a) $2^2$ b) $2^{-2}$ c) $3^4$ d) $1^{506}$ e) $1^{-2}$ f) $0^{20}$

**Practice 60.8**

a) What is $0.00001$ as a power of ten?

b) What is $64$ has a power of four?

c) What is $\frac{1}{8}$ as a power of two?

d) If $a^3 = \frac{8}{125}$, what is $a$?

**Practice 60.9**

a) Can you see that $10^7 \times 10^8$ has to be $10^{15}$?

b) What is $10^3 \times 10^5 \times 10^6$ as a power of ten?

c) What is $10^5 \times 10^{-2}$ as a power of ten?

d) What is $10^{-3} \times 10^{-4}$ as a power of ten?

## 61. Scientific Notation

As we noted in Section 56,

- Multiplying a number by 10 gives an answer ten times as big.
- Dividing a number by 10 (that is, multiplying it by $\frac{1}{10}$) gives an answer ten times as small.

And multiplying a number written in base ten either by 10 or $\frac{1}{10}$ gives an answer with the same digits as the original number, in the same order. (Well, you might introduce some zeros.)

For example,

$$5.67 \times 10 \times 10 \times 10 = \left(5 + \frac{6}{10} + \frac{7}{100}\right) \times 10 \times 10 \times 10 = 5000 + 600 + 70 = 5670$$

$$5.67 \times \frac{1}{10} \times \frac{1}{10} \times \frac{1}{10} = \left(5 + \frac{6}{10} + \frac{7}{100}\right) \times \frac{1}{10} \times \frac{1}{10} \times \frac{1}{10} = \frac{5}{1000} + \frac{6}{10000} + \frac{7}{100000} = 0.00567$$

These observations allow us to deduce the values of decimal numbers multiplied or divided by ten quite swiftly.

**Example:** What is $7.03 \times 10^6$? What is $7.03 \times 10^{-2}$?

**Answer:** Both problems will produce answers that involve the digits 7-0-3.
And we are starting with a number that is close to 7.

Now, $10^6$ is a million, so $7.03 \times 10^6$ must be a value close to seven million.
We must have

$$7.03 \times 10^6 = 7{,}030{,}000$$

Also, $10^{-2}$ is one hundredth, so $7.03 \times 10^{-2}$ must be a value close to seven hundredths.
We must have

$$7.03 \times 10^{-2} = 0.00703$$

**Example:** What is $307.231 \times 10^5$?

**Answer**: We have a number in the $300$s being multiply by ten five times. (Do you see that?)

This means we'll have an answer in the $\cancel{3{,}000}$ $\cancel{30{,}000}$ $\cancel{300{,}000}$ $\cancel{3{,}000{,}000}$ $30{,}000{,}000$s.

The answer must be $30{,}723{,}100$.

(Did this feel like *déjà vu*?)

**Practice 61.1:** Write down the values of each of these computations.

a) $52.004 \times 10$
b) $52.004 \times 10 \times 10$
c) $52.004 \times 10^6$
d) $52.004 \times \frac{1}{10}$
e) $52.004 \times \frac{1}{10} \times \frac{1}{10}$
f) $52.004 \times 10^{-4}$

We humans are not at all good at comprehending very very big numbers nor very very small numbers.

For example,

> Do you have a sense for how much time has passes in 78840000 seconds?
>
> Do you have a feel for the length 0.0000000892306 kilometers?

Even just saying these numbers is hard!

The first number has a lot of digits. Folks in the western world often use commas to separate long numbers into sets of three digits to help us think of the numbers in terms of thousands, millions, billions, and so on. Our count of seconds reads

$$78{,}840{,}000$$

We see immediately that we are talking about millions of seconds. We can even say that we're talking about roughly 79 million second, or maybe saying 80 million seconds is good enough.

Saying out loud just the first one or two digits of a large number—as millions or thousands and such—makes the number feel more manageable. (Though I still don't have a sense of how long a time about eighty million seconds actually is!)

**Practice 61.2:** In India the noun *lakh* is used for one-hundred thousand.
About how many lakh is the number 78840000?

**Practice 61.3:** Show that 78840000 seconds is about 2.5 years.

A number with a large number of decimal places is equally hard for us to wrap our brains around.

Rounding to just one or two digits usually makes matters feel better:

> $0.0000000892306$ kilometers is approximately $0.00000009$ kilometers.

And we can get a feel for this number if we change the units.

With a thousand meters in a kilometer, multiplying by a thousand gives the count of meters in this measurement.

$$0.00000009 \times 1000 = 0.00000009 \times 10 \times 10 \times 10 = 0.00009 \text{ meters}$$

Since there are 100 centimeters in a meter, multiplying this value by one hundred gives the count of centimeters in this length.

$$0.00009 \times 100 = 0.00009 \times 10 \times 10 = 0.009 \text{ centimeters}$$

There are 10 millimeters in a centimeter, so this is

$$0.009 \times 10 = 0.09 \text{ millimeters.}$$

And this looks and feels more manageable. (And just so you have it, 0.09 mm is a typical width of a human hair.)

Adjusting very big numbers and very small numbers by powers of ten is a common practice to get a manageable sense of the number. For instance, we see that

$$\begin{aligned} 78840000 &= 7884 \times 10{,}000 \\ &= 788.4 \times 10 \times 10{,}000 \\ &= 78.84 \times 10 \times 10 \times 10{,}000 \\ &= 7.884 \times 10 \times 10 \times 10 \times 10{,}000 \\ &= 7.884 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10 \\ &= 7.884 \times 10^7 \end{aligned}$$

and $7.884 \times 10^7$ is about $7.9 \times 10^7$ or $8 \times 10^7$, depending on how much you want to round.

Also,

$$0.0000000892306 = 8.92306 \times \frac{1}{10} \times \frac{1}{10} \times \frac{1}{10} \times \frac{1}{10} \times \frac{1}{10} \times \frac{1}{10} \times \frac{1}{10} \times \frac{1}{10} = 8.92306 \times 10^{-8}$$

which we might round to $8.9 \times 10^{-8}$or $9 \times 10^{-8}$.

Writing numbers like $7.9 \times 10^7$ and $9 \times 10^{-8}$ makes measurements look much more tractable.

A number rewritten to be a single non-zero digit followed by some decimal places and then multiplied by a power of ten,

$$a.bcd\ldots \times 10^m$$

is said to be written in **scientific notation**.

Often humans find it convenient to round a number written in scientific notation with either no, or just one or two digits after the decimal point.

Also, people tend to have certain "powers of ten landmarks" in their heads:

$10^3$ corresponds to thousands
$10^6$ corresponds to millions
$10^9$ corresponds to billions

and

$10^{-2}$ corresponds to hundredths (a hundredth of a meter is a centimeter)
$10^{-3}$ corresponds to thousandths (a thousandth of a kilometer is a meter)

and so on.

This helps with developing an intuitive feel for the number.

## MECHANICS PRACTICE

**Practice 61.4**

a) What is the approximate value of the number $8.02 \times 10^6$ in words?
b) What is the approximate value of the number $7.983 \times 10^3$ in words?
c) Approximate $2.01 \times 10^{-2}$ as a fraction.

**Practice 61.5**

The average distance to the Moon is $384400$ km. Write this number in scientific notation.

**Practice 61.6**

a) Write each of these numbers in scientific notation.

$$6539 \quad 750000000000 \quad 0.0004 \quad 212.872$$

b) Write each of the following as ordinary decimal numbers.

$$7.27 \times 10^2 \quad 7.27 \times 10^{-2} \quad 7.27 \times 10^5 \quad 7.27 \times 10^{-5}$$

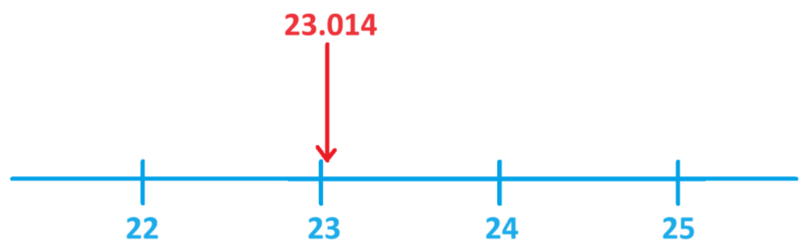
**Practice 61.7** Write the answer to each of these computations in scientific notation.

A. $4.4 \times 10^6 \ + \ 2.2 \times 10^6$
B. $4.4 \times 10^6 \ + \ 2.2 \times 10^7$
C. $3 \times (2.2 \times 10^6)$
D. $5 \times (2.2 \times 10^6)$
E. $(2 \times 10^{18}) \times (5.5 \times 10^3)$

(Doing arithmetic with numbers presented in scientific notation can be annoying.
How did you handle part b)?)

# 62. Rounding

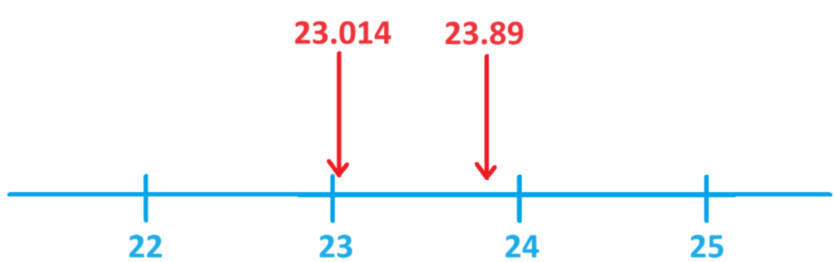
The decimal number 23.014, we all would agree, is close to the whole number 23 on the number line.




So too are the decimal numbers 23.09 and 23.1 and 23.26.

**Practice 62.1** What about 23.4789? Which whole number is it closest to on the number line?

But 23.89 is closer to 24 on the number line than it is to 23.



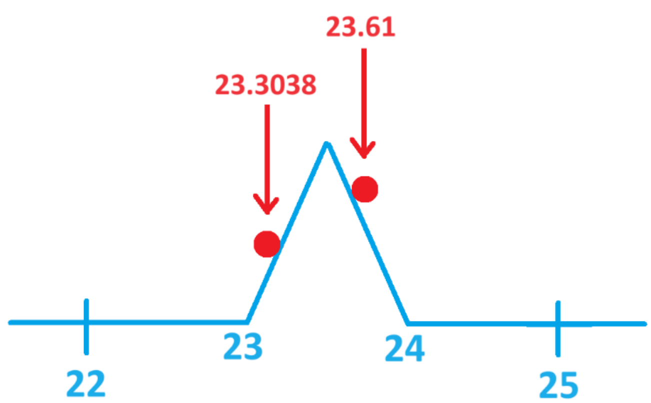

The number 23.5, on the other hand, is equally distant from the whole numbers 23 and 24.

We learned in school to **round** a decimal number composed of a whole number and some decimal digits by looking for the whole number (without any decimal places) that is closest to it on the number line.

For example,

> The decimal number $23.3038$ rounds (down) to the whole number $23$.
> The decimal number $23.61$ rounds (up) to the whole number $24$.

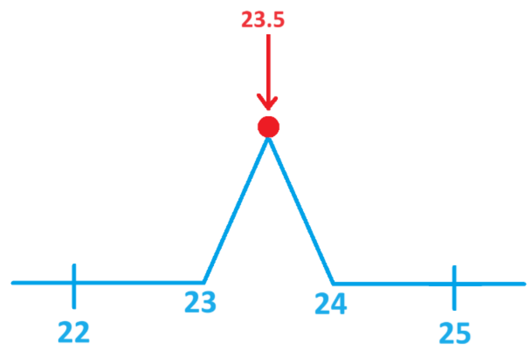
A lovely visual for this is to imagine a number line with a kink in it. Place a ball at the location of the decimal number in question and see to which whole number the ball rolls!




This leads to the schoolbook rule:

> To round decimal number to a whole number …
>
> Look at the first decimal place digit.
>
> If that digit is a $0$, $1$, $2$, $3$, or $4$, then the ball will roll to the left. You will thus “round down” to a whole number.
>
> If, on the other hand, that digit is a $5$, $6$, $7$, $8$, or $9$, then the ball will roll to the right. You will thus “round up” to a whole number.

The only ambiguous number in our specific example is $23.5$. A ball sitting at this position on the number line is directly on the apex of the kink – and it could roll either direction!




Just to make the rounding rule consistent ("round up" if the first decimal digit is $5$, $6$, $7$, $8$, or $9$), the world has decided to have the ball roll to right in this ambiguous case.

Convention: **$23.5$ rounds (up) to $24$**

**Practice 62.2** Which of these decimal numbers round to $8045$?

$8045.3909725555$

$8045.7097$

$8044.49999$

$8045.50001$

$8044.60986654$

$8045.060986654$

$8044.5$

$8045.5$

$8045.0$

## Rounding to hundreds, thousands, tenths, and more

Let's start with an example.

> **Example:** The average distance to the Moon is 384400 kilometers.
> Round that number to the nearest thousand kilometers.

Let's try to make sense of what is being asked of us.

For starters, we apparently should be thinking in terms of "thousands of kilometers."
So, let's try rewriting the figure given in terms of thousands. (Recall: A thousand can be written as $10^3$.)

We have

$$384400 = 384.4 \times 10 \times 10 \times 10 = 384.4 \times 10^3$$

So, the distance to the Moon is $384.4$ thousands of kilometers.

We're asked to round this number. Well, $384.4$ rounds to $384$ and so our answer must be: $384$ thousands of kilometers. And this measurement is

$$384 \times 10^3 = 384{,}000$$

kilometers.

> **Example:** Round the measurement of the distance to the Moon to the nearest hundred thousand kilometers.
>
> **Answer:** No worries!
>
> One hundred thousand is $10^5$ and so let's work with
>
> $$384400 = 3.844 \times 10^5$$
>
> So, we have $3.844$ hundreds of thousands of kilometers.
>
> Rounding $3.844$ gives $4$.
>
> And so, the distance to the Moon rounded to the nearest hundred thousand kilometers is
>
> $$4 \times 10^5 = 400{,}000$$
>
> kilometers.

Another example.

**Example:** Round $0.5670764$ to the nearest hundredth.

**Answer:** A hundredth is $10^{-2}$ so let's rewrite this number to make hundredths explicit. We have

$$0.5670764 = 56.70764 \times \frac{1}{10} \times \frac{1}{10} = 56.70764 \times 10^{-2}$$

Okay. We have $56.70764$ hundredths, which rounds to 57 of them.

Our appropriately rounded number is

$$57 \times 10^{-2} = 0.57$$

We've got a procedure here!

If asked to round a measurement the nearest hundred or million or thousandth …

Rewrite the given figure in terms of a decimal number multiplied by the appropriate power of ten. (This tells you how many hundreds or millions or thousandths you actually have.)

Round that decimal number.

Work out the value of that rounded value times the power of ten you have.

Done!

**Practice 62.3** Round $24{,}506.089$ to

a) the nearest whole number.
b) the nearest hundred
c) the nearest ten thousand
d) the nearest hundredth
e) the nearest tenth

Some schoolbooks have students use the following technique for rounding.

*To round the number in this question, say, to the nearest thousand, underline all the place values for thousands and up and then round according to the digit to the right of them.*

$$\underline{24},506.089$$

*The digit to the right is a five, so we round up and get* $25{,}000$ *as our rounded value.*

Here are the answers to Practice 62.3 in turn:

$$\underline{24,506}.089 \rightarrow 24,506$$

$$\underline{24,5}06.089 \rightarrow 24,500$$

$$\underline{2}4,506.089 \rightarrow 20,000$$

$$\underline{24,506}.\underline{08}9 \rightarrow 24,506.09$$

$$\underline{24,506}.\underline{0}89 \rightarrow 24,506.1$$

Do you see that this is the same technique we've been following without mention of scientific notation?

## MUSINGS

**Musing 62.4** Write down a number smaller than $27{,}000$ that, when rounded to the nearest thousand, rounds to $27{,}000$.

**Musing 62.5** What is the largest whole number which, when rounded to the nearest hundred, gives $50{,}000$?

**Musing 62.6**
a) Which number on the number line is equally distant from $5{,}700$ and from $5{,}800$.

b) A number $N$ between $5{,}700$ and $5{,}800$ on the number line lies to the right of the number you gave in part a). What is $N$ rounded to the nearest hundred? How do you know?

**Musing 62.7** What, do you think, is $-35{,}483$ rounded to the nearest hundred?

## MECHANICS PRACTICE

**Musing 62.8** What is the average distance to Moon rounded to the nearest tens of kilometers?

**Musing 62.9** Round $8{,}383{,}838.3838$ to

a) the nearest million
b) the nearest thousand
c) the nearest hundredth
d) the nearest thousandth

## 63. Significant Figures

One meter is divided into one hundred **centimeters**.
Each centimeter is divided into ten **millimeters** (and so a full meter is divided into one thousand millimeters).

> **Question:** Do the prefixes *centi*- and *milli*- make sense in this context?

If I gave you a length of string to measure with a meter stick just marked with centimeters, you would probably round the length you measure to the nearest centimeter.

> "The string is about 78 centimeters long" you might say.

The true length of the string might be a little less than 78 centimeters, or a little more, but it will be a value within a centimeter range of 78 centimeters.

If I then asked you to repeat the task, but this time using a meter stick with each centimeter length on it divided into millimeters, you would then likely give me an answer rounded to the nearest millimeter (tenth of a centimeter).

> "The string is about 78.2 centimeters long" you might say this time.

In science, the level of precision marked on an instrument tends to determine the level of precision to which we make measurements.

But this idea can lead to a mathematical curiosity.

For example, suppose a botanist measured the length of a reed stalk and wrote in her paper that it was 0.190 meters tall.

What is she telling us?

By giving us the values of three digits after the decimal point, she is saying that she measured the length as 1 tenth of a meter and 9 hundredths of a meter and 0 thousandths of a meter, thereby informing the readers of her paper that her measuring tool went to the thousandths (millimeters, in this case).

Her measurement is thus a number rounded to the nearest millimeter. So, the true height of the stalk is close to 19.0 centimeters with no more than a millimeter of inaccuracy.

Now, of course, mathematically the number 0.190 is no different than the number 0.19.

If the botanist decided to write in her paper that her measurement was $0.19$ meters (mathematically the same number), then readers will presume something else, that she measured the length of the stalk only to the nearest hundredth of a meter (that is to the nearest centimeter).

We still conclude that the stalk is about $19$ centimeters tall, but we'd now think this value could have up to a centimeter of inaccuracy to it.

Even though the expressions $0.190$ and $0.19$ represent the exact same number mathematically, to a scientist, the expressions "$0.190$ meters" and "$0.19$ meters" tell us more than just a numerical value. They indicate the level of accuracy associated with the numerical value in a practical context.

**Practice 63.1:** A scientist records in an experiment a temperature of $713.020$ degrees Celsius. What was the accuracy of his measuring instrument?

The botanist who measured the stalk length as $0.190$ meters published this number in scientific notation.

$1.90 \times 10^{-1}$ meters

Again, she included the decimal digit of zero to indicate the level of precision she conducted in her measurement. (She measured zero thousandths of a meter.) Every digit she wrote down in scientific notation was deliberate and significant.

**Significant Figures:** If scientific measurements are recorded in scientific notation, then all of the digits written to the left of the power of ten are considered "significant." They were measured by an instrument that has a certain degree of accuracy. (So, if some of the digits written down are zero, that is deliberate—they were measured to be zero!)

The **number of significant figures** in a recorded measurement is the number of digits written to the left of the power of ten.

For example, the recorded measurement $1.90 \times 10^{-1}$ meters has three significant figures.

Another botanist measuring the same stalk with a different instrument writes its height as $1.89701 \times 10^{-1}$meters, giving a recorded result with six significant figures. She measured to the nearest millionth of a meter.

A third botanist writing the height as $2 \times 10^{-1}$ meters is recording the measurement with just one significant figure. He measured to the to the nearest tenth of a meter. (His meter stick must have been divided into just ten equally spaced marks.)

Writing measurements in scientific notation avoids confusion.

**Example:** Using estimation techniques, I determined the population of a certain town to be 34,000 residents.

Which of the five digits in the measurement are significant? (That is, which of the five digits did I actually measure?)

**(Non) Answer:**

I recorded the digit 3, so I was certainly measuring to the accuracy of ten-thousand residents.

I recorded the digit 4, so I was actually doing better, measuring at least to the nearest thousand residents.

Did I record that middle zero? Was I able to estimate to the nearest 100 residents?
You can't tell.

Maybe I was able to measure to the nearest 10 residents and two of the zeros I recorded are "genuine."

Or maybe I counted every single resident and got the figure 34,000 on the nose? (All five digits are valid.)

It is impossible for you to say what exactly I mean by the zeros in my recorded result.

To obviate confusion, let me present my measurement in scientific notation instead.

**Example Continued:** I intend to publish this measurement as

$$3.400 \times 10^4$$

residents.

Now you can see that I was measuring to the nearest ten residents. The town population might well be 34,003 residents, or 33,998 residents, for instance. But we can be assured that the town population is 34,000 with an accuracy range of 10 residents.

For such clarity, scientists record their results in scientific notation.

## MECHANICS PRACTICE

**Practice 63.2**

a) A geographer determines the population of a county to be 310,000 people. He says he counted to the nearest thousand people. Rewrite his count in scientific notation, displaying the appropriate number of significant figures.

b) Later, he said he mis-spoke: actually measured to the nearest 100 people. Adjust your answer to part a) appropriately.

**Practice 63.3:** As part of a cleanliness study, a scientist measured the width of a dust particle to be $0.000100$ millimeters. The scientific journal in which he wants to publish his results wants all measurements to be given in scientific notation.

a) How does his measurement appear when written in scientific notation?
b) He later learns that the journal wants all measurements to be given in units of kilometers, not millimeters (and still be in scientific notation). Now how will his measurement appear?

# 64. Order of Magnitude

If you earn a six-digit salary (lucky you!), you are earning an annual salary in the hundreds of thousands of dollars. Perhaps it is $\$240{,}000$ dollars a year, or $\$598{,}764$ dollars a year. (Salaries are usually given in units of dollars, not to the level of cents.)

Someone earning a seven-digit salary is even luckier and has an annual income in the millions.

> **Practice 64.1:** I am currently earning a six-digit salary. But if I earned just one more dollar per year, I'd be earning a seven-digit salary. What is my current salary?
>
> (By the way … the premise of this question is not true!)

We often speak in terms of the "order of magnitude" of a number, giving a sense of the size of a number without specifying the number directly. There are two typical ways this is done in society. They aren't quite the same mathematically.

**Order of Magnitude 1:** If the number is a counting number, then just say the number of digits of the number.

For example, most everyone earns a five-digit salary.

**Order of Magnitude 2:** Write the number in scientific notation: $a.bcd\ldots \times 10^m$. Then $m$, the power of ten mentioned, is the order of the magnitude of the number. This gives a natural means to talk about the order of magnitude of decimal numbers too.

For example, in this second setting

$7.12 \times 10^3$ has order of magnitude **three**. (It is a number in the thousands.)

$4.8 \times 10^{-5}$ has order of magnitude $-5$. (It's a number "talking about" one-hundred thousandths.)

$0.0074$ has order of magnitude $-3$. (We're talking about thousands, essentially.)

$6.9872 \times 10^6$ has order of magnitude $6$. (It's a number in the millions.)

**Practice 64.2:**
a) How many digits does the number $7.12 \times 10^3$ have when written out as a counting number?

What is its order of magnitude according to the first definition and what is its order of magnitude according to the second?

b) How many digits does the number $6.9872 \times 10^6$ have when written out as a counting number?

We see that these two ways of expressing an order of magnitude of a large number are slightly different.

## Order of Magnitude as Scales

Many natural phenomena, such as earthquake vibrations and sound intensities, occur in a vast array of strengths.

For example, the sound intensity of a NASA Saturn V rocket launch is about $100{,}000{,}000{,}000{,}000{,}000{,}000 = 10^{20}$ times more intense than the sound of a pin drop.

It would be very annoying to have a measurement scale for sound intensity that starts close to zero (for a pin drop) and goes to such huge multi-digit numbers (for rocket launches). For this reason, the **Bel scale** for sound intensity is based on an order of magnitude rather than a direct intensity measure. (To be clear, it is the second definition of "order of magnitude" being used here.)

The sound of a pin drop measures 1 Bel and the sound of the rocket launch measures 20 Bel.

> **Aside:** Actually, it has become standard to speak in terms tenths of Bels—**decibels**. A pin drop is 10 dB and a rocket launch 200 dB.

The **Richter scale** for earthquake intensity is also a scale based on the second definition of order of magnitude. An earthquake of measure 6 on the Richter scale (it's $10^6$ fundamental units of measurement) is one unit of magnitude up from one of measure 5 (which is $10^5$ fundamental units of measurement). Consequently, the earthquake of magnitude 6 is ten times as strong as the one of magnitude $5$.

MUSINGS

**Musing 64.3:** Two earthquakes measure $4$ and $7$, respectively, on the Richter scale. By what factor is the second quake stronger than the first?

**Musing 64.4** The number $4.1 \times 10^9$ is a number in the billions. What is its order of magnitude according to the first definition? According to the second?

**Musing 64.5:**
a) Do the numbers $999{,}999$ and $1{,}000{,}000$ seem significantly different to you?
b) What is the order of magnitude of each of these numbers according to the first definition?
c) What is the order of magnitude of each of these numbers according to the second definition?

> **Comment:** We're seeing that a concept of the "order of magnitude" of a number is just a quick and rough attempt to give an intuitive sense of the size of the number. There are always going to be examples that show our estimations are a tad questionable. But that's not a real concern. We really are just going for rough-and-ready intuitive sense of the size of numbers.

**Musing 64.6** VERY OPTIONAL

Scientists (in particular, astronomers) consistently working with very big numbers often represent *all* their numbers solely in terms of powers of ten. This means that they are willing to work with strange values for the powers.

We'll make sense of this when we discuss powers in proper detail in the next volume. But for now, let's just play with our calculators, even if we are not sure we understand what our calculators mean in what they are showing.

We know, for example, that

$$10^6 = 1{,}000{,}000$$

and

$$10^7 = 10{,}000{,}000$$

Now $3{,}300{,}00$, for instance, is a value between one million and ten million. It is possible to imagine that there is a power of ten, between $6$ and $7$, that gives this value.

Experimentation with a calculator shows that $10^{6.519}$ is about this value. (This is weird, I know. But try entering $10^{6.519}$ on a calculator.)

In the same way, we see $3{,}100{,}000$ seems to be about $10^{6.491}$.

Here's a third convention for stating what the "order of magnitude" of a number is.

**Order of Magnitude 3:** Write the given number solely as a power of ten, most likely with a decimal as the power: $10^r$. Then round $r$ up or down to the nearest integer. Call that rounded value the order of magnitude of the number.

For example,

$3{,}300{,}000 \approx 10^{6.519}$ and $6.519$ rounds to 7. So $3{,}300{,}000$ has order of magnitude 7 in this third definition.

$3{,}100{,}000 \approx 10^{6.491}$ and $6.491$rounds to 6. So $3{,}100{,}000$ has order of magnitude 6 in this third definition.

Confusing!

a) What do you think? Do $3.1$million and $3.3$ million seem like they should have different orders of magnitude to you?

b) What is the order of magnitude of the number $254$ according to this third definition?

c) What is the order of magnitude of the number $0.0033$ according to this third definition?

# Chapter 8

## Beyond Base 10
## All Bases, All at Once

## 65. A Famous Mystery About Prime Numbers

Throughout the millennia, scholars and enthusiasts have loved playing with mathematics. What seems "incidental" and frivolous at first can often lead to deep joy, curious mathematical mystery, and even to immense practical use. A prime example of this (a pun is intended) is some play with the **doubling numbers,** play that started some 400 years ago.

We met the doubling numbers via a mind-reading trick at the start of Chapter 4. They are the numbers that result from repeatedly doubling the number 1.

**1 2 4 8 16 32 64 128 256 ...**

These are, of course, also the **powers of two**.

$$1 = 2^0$$
$$2 = 1 \times 2 = 2^1$$
$$4 = 1 \times 2 \times 2 = 2^2$$
$$8 = 1 \times 2 \times 2 \times 2 = 2^3$$
$$16 = 1 \times 2 \times 2 \times 2 \times 2 = 2^4$$

and so on.

These numbers get big quickly. For example, $2^{10}$ equals 1,024 and $2^{300}$ is a number 91 digits long. (Its value is a little over two *novemvigintillion*.)

A French monk by the name Merin Mersenne (1588-1648) noticed something curious if you subtract 1 from each doubling number. Doing so gives this list of numbers.

**0 1 3 7 15 31 63 127 255 ...**

Actually, Mersenne ignored the zero at the start and focused on this list of numbers instead.

$$1 \quad \boxed{3} \quad \boxed{7} \quad 15 \quad \boxed{31} \quad 63 \quad \boxed{127} \quad 255 \quad \ldots$$

Here observed:

3 is a prime number and 3 is the **second** number in this list.
7 is a prime number and 7 is the **third** number in this list.
31 is a prime number and 31 is the **fifth** number in this list.
127 is a prime number and 127 is the **seventh** number in this list.

It looks like prime numbers are appearing in every prime position!

**Practice 65.1** Show, on the other hand, that 15, 63, and 255 are each a composite number. (Also remember, the number 1 is deemed neither prime nor composite.)

We can rewrite this observation in terms of the powers of two.

2 is prime and $2^2 - 1 = 3$ is prime
3 is prime and $2^3 - 1 = 7$ is prime
5 is prime and $2^5 - 1 = 31$ is prime
7 is prime and $2^7 - 1 = 127$ is prime

Of course, Mersenne wondered

Is $2^{prime} - 1$ always a prime number?

**Practice 65.2** The next prime value to consider is 11 and $2^{11} - 1 = 2047$.
Is 2047 a prime number? (Ask Siri or Alexa or some virtual friend?)

It turns out that the pattern we, and Mersenne, are observing is a false one.

> 2047 is composite. We have $2047 = 23 \times 89$.

But Mersenne wondered:

> When is $2^{prime} - 1$ a prime number?
> Is it often a prime number?
> There are infinitely many prime numbers, so maybe $2^{prime} - 1$ also prime infinitely often?

He could only find eleven examples of primes that arise this way (using only pencil and paper back in the early 1600s!),

2 is a prime number and $2^2 - 1 = 3$ is prime
3 is a prime number and $2^3 - 1 = 7$ is prime
5 is a prime and number $2^5 - 1 = 31$ is prime
7 is a prime number and $2^7 - 1 = 127$ is prime
13 is a prime number and $2^{13} - 1 = 8191$ is prime
17 is a prime number and $2^{17} - 1 = 13107$ is prime
19 is a prime number and $2^{19} - 1 = 524287$ is prime
31 is a prime number and $2^{31} - 1 = 2147483647$ is prime
67 is a prime number and $2^{67} - 1 = 2147483647$ is prime
127 is a prime number and $2^{127} - 1 = 2305843009213693951$ is prime
257 is a prime number and $2^{257} - 1 = 618970019642690137449562111$ is prime

and people have been looking for more examples ever since the time of Mersenne.

Prime numbers that happen to be of the form $2^{prime} - 1$ are today called **Mersenne primes**.

**A FAMOUS UNSOLVED QUESTION**

No one on this planet currently knows whether or not there are an infinite number of Mersenne primes to be found. Perhaps there is just a finite count of them?

.

As of the time of writing this Section (April 2024), only 51 examples of Mersenne primes are known, the most recent discovered being $2^{82,589,933} - 1$, a number over 24 million digits long. (And 85,589,933 is a prime number, by the way.) It was discovered in October 2020.

Is there a $52^{\text{nd}}$ prime of this form to be found? No one knows!

**Optional Challenge:** For world fame, find a 52nd prime number that is one less than a power of two. (Or prove that no other Mersenne primes exist.)

**Comment:** You can join the *Giant Internet Mersenne Prime Search* (GIMPS) and have your computer hunt for a 52nd Mersenne prime!

Computers work in binary (base 2) and are thus quite adept at looking for prime numbers connected to the powers of 2. In fact, all the exceptionally large prime numbers we know today come from playing with the powers of 2.

And why do we care about finding larger and larger prime numbers?

Well, it turns out that all our computer encryption codes for financial services, military services, and the like, are based on knowing prime numbers and the larger the prime numbers used, the much harder those code are to crack. (It is thus likely that the financial institutions and the military might be aware of some large prime numbers not known to the public.)

But what is remarkable about this story is that Mersenne's casual play with the doubling numbers about 400 years ago led to a practical application of vital relevance to the 21st century!

Mathematicians, and the institutions that support them, thoroughly value the art of mathematical play and general investigation. You never know what curious questions might open up and what immensely practical applications might result. A new result might be "frivolous," but the new mathematical tools and ideas that led to it often turn out to be of immense value.

But let's move on.

We have seen that a number of the form $2^{prime} - 1$ could be prime or could be composite.

But what about numbers of the form $2^{composite} - 1$? Could they ever be prime?

The three examples we have in our list of numbers are each composite.

$2^4 - 1 = 15$ is composite
$2^6 - 1 = 63$ is composite
$2^8 - 1 = 255$ is composite

**Practice 65.3** Evaluate $2^9 - 1$, $2^{10} - 1$, and $2^{12} - 1$ and show they too are each composite.

Mersenne was able to prove that a number of the form $2^{composite} - 1$ will never prime: such numbers will always have a proper factor.

That's a goal of this chapter: to develop the mathematics that will allow us to prove that too.

As a practice challenge:

Find a proper factor of $2^{300} - 1$ to show that this number is not prime.

(By the way, it is not possible to work with this number on a calculator. It is 91 digits long, after all!)

## MUSINGS

**Musing 65.4** Look at the doubling numbers again:

# 1 2 4 8 16 32 64 128 256 ...

The first is one more than a multiple of 3. The second is one less. The third is one more. The fourth is one less. The fifth is one more.

It seems that this pattern persists.

a) If this pattern is truly valid (and is not a false pattern), would $2^{300}$ be one more or one less than a multiple of 3?

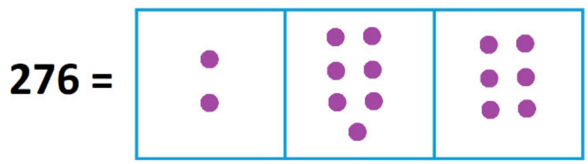
b) If you trust this pattern, deduce that $2^{300} - 1$ is divisible by 3 and so is not prime.

c) Should you trust the pattern?

# 66. Revisiting Division

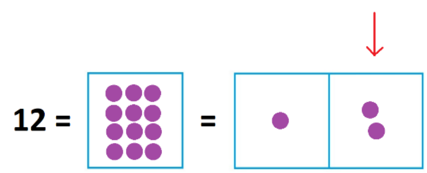
We learned how to conduct long division in a $1 \leftarrow 10$ machine back in Section 35.

For instance, to compute $276 \div 12$ we started by drawing a picture of the number 276.

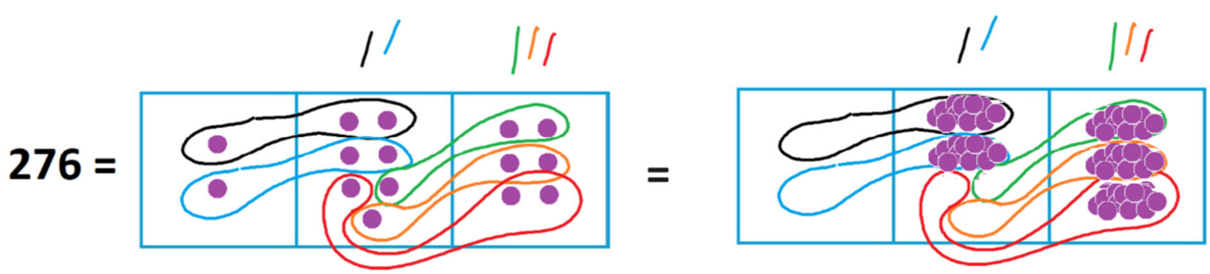



Our job is to then work with this diagram to see “what got multiplied by 12” to produce it.

A group of 12 appears as one dot next to two dots, but it is really twelve dots in one box. (Society insists on conducting the explosions!)




And we can identify groups of 12 in the picture: two at the tens level and three at the ones level.




We conclude that the number 23 was multiplied by 12 to give our picture of 276.

Thus,

$$276 \div 12 = 23$$

**Question:** Do you recall this process?
Would you like to practice it be showing that $27999 \div 132$ equals 212 with remainder of 15?

## Division in Any Base

Let's now live up to the promise of Section 36 where we claimed that this long division is high-school algebra in disguise.

The only thing to realize is that there is nothing special about a $1 \leftarrow 10$ machine.

We could be doing all our arithmetic in a $1 \leftarrow 2$ machine if we desired, or a $1 \leftarrow 5$ machine, or even a $1 \leftarrow 37$ machine. The math doesn't care in which machine we conduct it. It is only us humans with a predilection for the number ten that draws us to the $1 \leftarrow 10$ machine.

So, let's now be bold and do our work in all possible machines, all at once!

That sounds crazy, but it is surprisingly straightforward.

What I am going to do is draw the a picture of a machine, but I am not going to tell you which machine it is. It could be a $1 \leftarrow 10$ machine again, I am just not going to say. Maybe it will be a $1 \leftarrow 2$ machine, or a $1 \leftarrow 4$ machine or a $1 \leftarrow 13$ machine. You just won't know as I am not telling. It's the mood I am in!

Now, in school algebra there seems to be a favorite letter of the alphabet to use for a quantity whose value you do not know. It's the letter $x$. Always the letter $x$. (It's a weird obsession.)

> **Question:** What can you find on the internet about why the letter $x$ is the favored letter to represent an unknown quantity in mathematics?
>
> (Watch out! There are multiple thoughts, theories, and combinations of details. Don't fully believe the first explanation you encounter.)

So, let's work with an $1 \leftarrow x$ machine with the letter $x$ representing some number whose actual value we do not know.

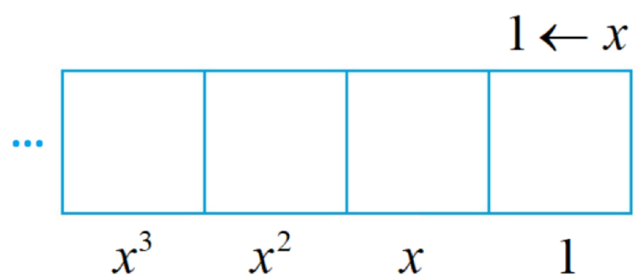
In Section 60, we saw that the place values of the boxes in a $1 \leftarrow 10$ machine are powers of ten.

$$\mathbf{1} = 10^0 \quad \mathbf{10} = 10^1 \quad \mathbf{100} = 10^2 \quad \mathbf{1,000} = 10^3 \quad \ldots$$

The place values of the boxes in a $1 \leftarrow 2$ machine are powers of two.

$$\mathbf{1} = 2^0 \quad \mathbf{2} = 2^1 \quad \mathbf{4} = 2^2 \quad \mathbf{8} = 2^3 \quad \mathbf{16} = 2^4 \quad \ldots$$

In an $1 \leftarrow x$ machine, the place values of the boxes will be the powers of $x$.




This $1 \leftarrow x$ machine represents all machines, all at once!

If I tell you that I am thinking of $x$ as $10$ in this picture, then you can see the picture as a $1 \leftarrow 10$ machine.

If I change my mind and tell you that I am actually thinking of $x$ as $2$ in this picture, then you will see a $1 \leftarrow 2$ machine.

(Or $x$ could be $3$ to give a $1 \leftarrow 3$, machine, or $7$ for a $1 \leftarrow 7$ machine, and so on.)

**Question 66.1:** In a $1 \leftarrow 10$ machine …

Ten ones give ten: $10 \times 1 = 10$
Ten tens give a hundred: $10 \times 10 = 100$
Ten hundreds give a thousand: $10 \times 100 = 1000$

and so on.

In a $1 \leftarrow 2$ machine …

Two ones give two: $2 \times 1 = 2$
Two twos give four: $2 \times 2 = 4$
Two fours give eight: $2 \times 4 = 8$

and so on.

For an unspecified number $x$, can we still say …

$$x \times 1 = x$$

and

$$x \times x = x^2$$

and

$$x \times x^2 = x^3$$

and so forth?

The answer to Question 67.1 is yes (can you explain why?), which affirms that an $1 \leftarrow x$ machine really does match a base machine whenever you think of $x$ as an actual specific number.

Okay. Out of the blue; with everything now set up; here's an advanced algebra problem.

> **Exercise:** Compute
> $$(2x^2 + 7x + 6) \div (x + 2)$$

Go!

Well, actually, let's pause and not "Go!" just yet. I don't know you about you, but I am having a bit of an emotional reaction right now.

Moments ago we were doing schoolbook arithmetic and now we're suddenly being thrust into something that looks very strange and very scary.

Deep breath!

When faced with a challenge in math—and in life—there are two fundamental steps to problem solving one must start with, yet no one seems to talk about.

> **Step 1:** Be human and acknowledge your honest human reaction to the challenge.

If the problem looks weird, say "This is weird!"
If it looks scary, acknowledge that you are nervous.
If the challenge looks curiously quirky, acknowledge you are intrigued.

Whatever your human reaction is to the problem, take note of it.

Then, take a deep breath, and move to …

> **Step 2:** Do something! ANYTHING!

When faced with an emotional reaction, many people shut down. But doing something—anything—no matter how tiny or indirect it might feel helps you get past an emotional impasse.

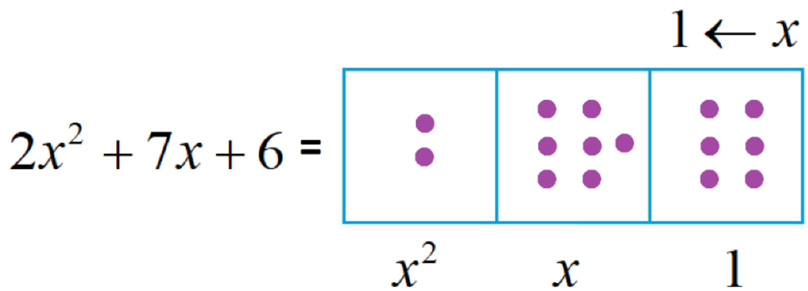
> Could you draw a picture? Could you draw a picture perhaps relevant to the problem?
>
> Could you underline some words in the problem statement – the scary words, or perhaps all the words that begin with a vowel?
>
> Could you reread the problem statement three times fast and then go for a short walk and not think about the problem—but with the promise you'll read it a fourth time, slowly, upon your return?

Okay. Now that I've acknowledged that I am nervous and have taken a deep breath, let me attempt to DO SOMETHING for the problem. We need to make sense of

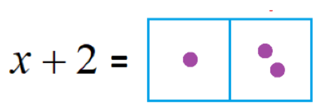
$$(2x^2 + 7x + 6) \div (x + 2)$$

Well. I can at least draw a picture for the challenge in an $1 \leftarrow x$ machine.

We have that $2x^2 + 7x + 6$ is two $x^2$, seven $x$s, and six ones.



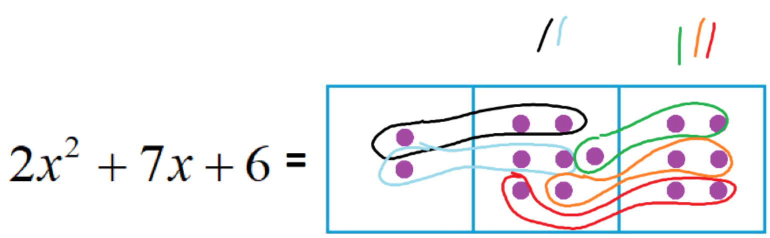

Here's what $x + 2$ looks like.




Great. I've done something!

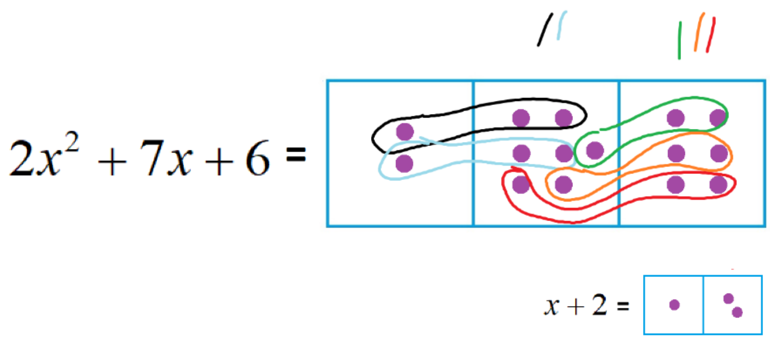
And I now feel like I know what to do next.

The division problem $(2x^2 + 7x + 6) \div (x + 2)$ is asking us to find copies of $x + 2$ in the picture of $2x^2 + 7x + 6$.





I see two copies of $x + 2$ at the $x$ level and three copies at the $1$ level.




The picture is showing me that answer must be $2x + 3$!

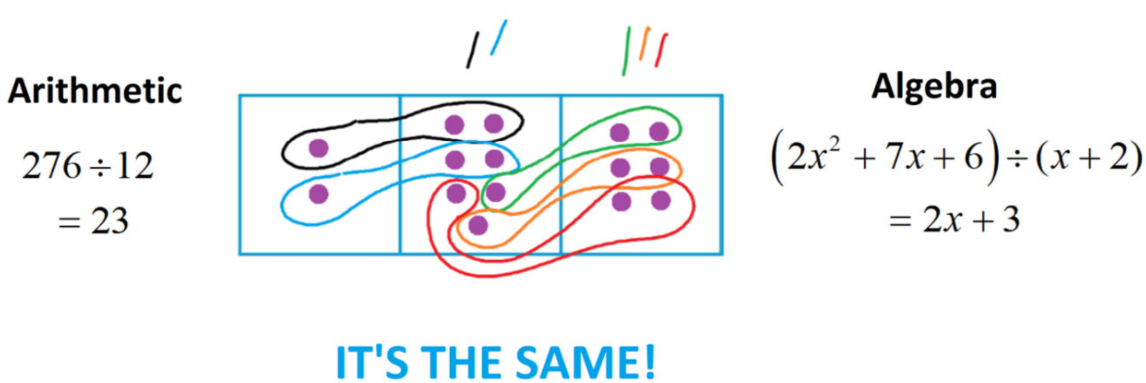
$$(\mathbf{2x^2 + 7x + 6}) \div (x + 2) = 2x + 3$$

We did it!

Now …

> Stare at the picture above showing $(2x^2 + 7x + 6) \div (x + 2) = 2x + 3$.
> Does it look familiar?

Look back at our picture for $276 \div 12$ we created on the first page of this section.
We have identical pictures!




We've just completed an advanced algebra problem as though it is nothing more than an early-grade school arithmetic problem.

Whoa! What's going on?

Suppose I told you that $x$ really was 10 in my head all along though this work. Then

$\mathbf{2x^2 + 7x + 6}$ is the number $2 \times 10^2 + 7 \times 10 + 6$, which is $200 + 70 + 6 = \mathbf{276}$.

$\mathbf{x + 2}$ is the number $10 + 2 = \mathbf{12}$.

$\mathbf{2x + 3}$ is the number $20 + 3 = \mathbf{23}$.

and

$(2x^2 + 7x + 6) \div (x + 2) = 2x + 3$ is the statement $\mathbf{276} \div \mathbf{12} = \mathbf{23}$, which is exactly what we did on the first page of this Section!

Indeed, we really have just repeated a school arithmetic problem if I happen to declare that $x$ was the number 10 in my head all along.

But here's the wonderful thing.

The statement

$$(2x^2 + 7x + 6) \div (x + 2) = 2x + 3$$

is really an infinitude of school arithmetic problems completed all in one hit!

For instance, suppose I tell you that $x$ is actually the number 2 (not ten). Then

$2x^2 + 7x + 6$ is $\quad 2 \times 4 + 7 \times 2 + 6 = \mathbf{28}$

$x + 2$ is $\quad 2 + 2 = \mathbf{4}$

$2x + 3$ is $\quad 2 \times 2 + 3 = \mathbf{7}$

and we've just ascertained that **28** ÷ **4 = 7**, which is correct!

Or suppose I tell you that $x$ represents the number 5. Then

$2x^2 + 7x + 6$ is $\quad 2 \times 25 + 7 \times 5 + 6 = \mathbf{91}$

$x + 2$ is $\quad 5 + 2 = \mathbf{7}$

$2x + 3$ is $\quad 2 \times 5 + 3 = \mathbf{13}$

and we've just ascertained that **91** ÷ **7 = 13**, which is correct!

Algebra really is the art of doing an infinite number of arithmetic problems all in one hit.

**Practice 66.2** What is the statement $(2x^2 + 7x + 6) \div (x + 2) = 2x + 3$ saying if $x$ represents the number 3?

**Practice 66.3**:
a) Compute

$$(2x^3 + 5x^2 + 5x + 6) \div (x + 2)$$

in an $1 \leftarrow x$ machine. (You should get the answer $2x^2 + x + 3$.)

b) If I tell you that $x$ is the number $10$ in my mind, what school arithmetic problem have you just answered?

Quantities expressed as codes in an $1 \leftarrow x$ machine are called **polynomials**.

These codes are just like numbers expressed in base 10, except now they are “numbers” expressed in base $x$. (And if someone tells you $x$ is actually $10$, then they really are base-ten numbers!)

The work one learns to do in high-school “polynomial algebra” is essentially just a repeat of early-grade base-ten school arithmetic.

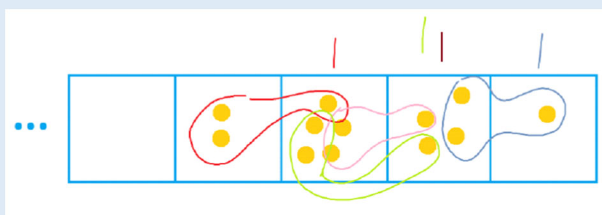
## MUSINGS

**Musing 66.4**
a) Draw a picture of $x^3 + 2x^2 + 3x + 1$ in an $1 \leftarrow x$ machine.
Add $3x^3 + 7x^2 + 4x + 1$ to your picture.
What then is $(x^3 + 2x^2 + 3x + 1) + (3x^3 + 7x^2 + 4x + 1)$ according to your picture?

b) If $x$ is the number $10$ throughout part a), what ordinary arithemetic problem have you conducted?

**Musing 66.5** What do you think $(5x^2 + 4x + 9) - (2x^2 + x + 6)$ equals?
What does this translate to if $x$ is the number $10$?

**Musing 66.6** Kennedy drew this picture when she was working on a division problem.



a) If this is a picture of a division problem in a $1 \leftarrow 10$ machine, what division problem is she conducting and what is its answer?
b) If this is a picture of a division problem in an $1 \leftarrow x$ machine, what division problem is she conducting and what is its answer?

**Musing 66.7**
a) Compute $(2x^4 + 3x^3 + 5x^2 + 4x + 1) \div (2x + 1)$.
b) Compute $(x^4 + 3x^3 + 6x^2 + 5x + 3) \div (x^2 + x + 1)$.

If $x$ is the number $10$ in both these problems, what two division problems in ordinary arithmetic have you just computed?

**Musing 66.8** Here's a polynomial division problem written in fraction notation. Can you compute its value? (Is there something tricky to watch out for?)

$$\frac{x^4 + 2x^3 + 4x^2 + 6x + 3}{x^2 + 3}$$

**Musing 66.9**

a) Show that $(x^4 + 4x^3 + 6x^2 + 4x + 1) \div (x + 1)$ equals $x^3 + 3x^2 + 3x + 1$.

b) What is this saying for $\mathrm{x} = 10$?

c) What is this saying for $\mathrm{x} = 2$?

d) What is this saying for x equal to each of 3, 4, 5, 6, 7, 8, 9, and 11?

e) What is this saying for $\mathrm{x} = 0$?

## 67. A Problem!

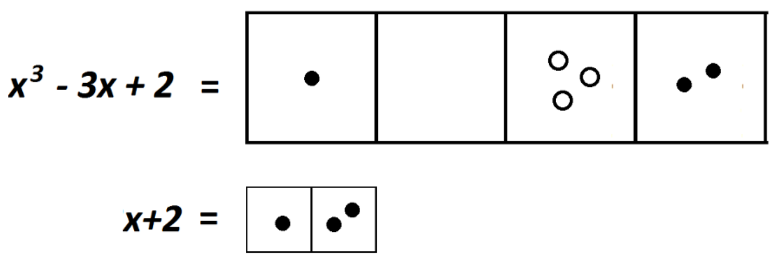
Okay. Now that we are feeling really good about doing advanced algebra, I have a confession to make. I've been hiding a problem. A serious problem!

I've been choosing examples that only use dots. What about antidots?

Consider, for example,

$$\frac{x^3 - 3x + 2}{x + 2}$$

Here's what we have as a picture in an $1 \leftarrow x$ machine.




We seek sets of one-dot-next-to-two-dots in this picture of $x^3 - 3x + 2$.

Do you see any single dots right next to a pair of double dots? I don't!

When we were in a predicament like this back in Section 35, we thought to unexplode dots. Perhaps we can take that leftmost dot in the picture and unexplode it into … umm … how many dots?

That's the snag!
Since we don't know the value of $x$, we don't know how many dots to draw when we unexplode.

Bother!

It seems like we are stuck.

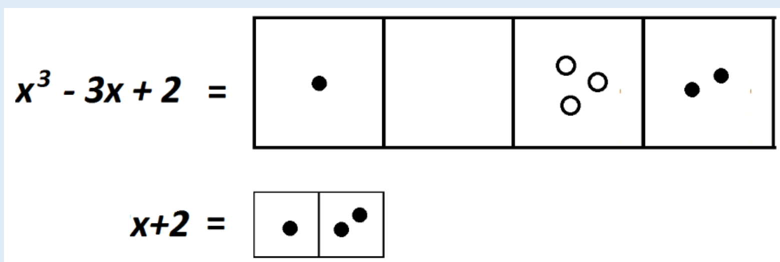
Either we need to conclude that any polynomial division that involves negative numbers just can't be done, or we need some amazing flash of insight that allows us to move forward.

## MUSINGS

**Musing 67.1** What's your take on this?

Are we at an impasse? Or do you think there is a way to move forward with a problem like this?

$$\frac{x^3 - 3x + 2}{x + 2}$$



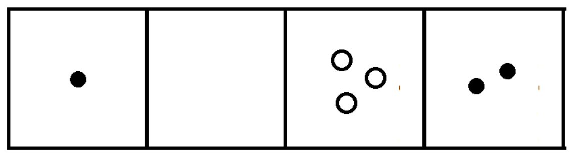
# 68. Resolution

We are stuck computing this division problem.

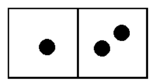
$$\frac{x^3 - 3x + 2}{x + 2}$$

Here's a picture of the problem in an $1 \leftarrow x$ machine.

$x^3$ - $3x + 2$ =



$x+2$ =



We seek copies of $x + 2$—one dot right next to two dots—anywhere in the picture of $x^3 - 3x + 2$. But there are none!

And we can't unexplode dots to help us out as we don't know the value of $x$. (We don't know how many dots to draw when we unexplode.)

The situation seems hopeless at present.

But I have a piece of advice for you, a general life lesson in fact. It's this:

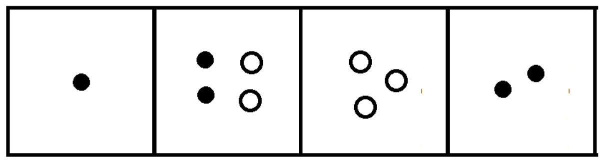
> **IF THERE IS SOMETHING IN LIFE YOU WANT … MAKE IT HAPPEN!**
> **(And deal with the consequences.)**

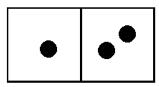
Right now, is there anything in life we want?

Look at that single dot way out to the left. Wouldn't it be nice if we had two dots in the box next to it, to make a copy of $x + 2$?

Well, if there is something you want … Make it happen!
Let's just put two dots into that empty box!

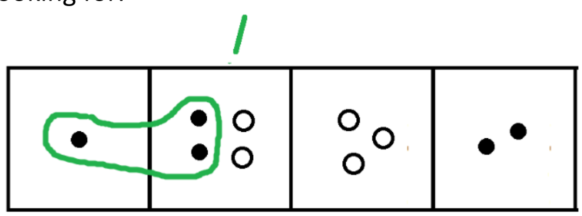
But we have to deal with the consequences. That box is meant to be empty and we can't just willy-nilly change it. So, in order to keep it empty, let's put in two antidots as well.

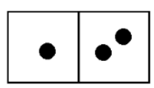
$x^3 - 3x + 2 =$ 

$x+2 =$ 

Brilliant!

We now have one copy of what we're looking for.

$x^3 - 3x + 2 =$ 

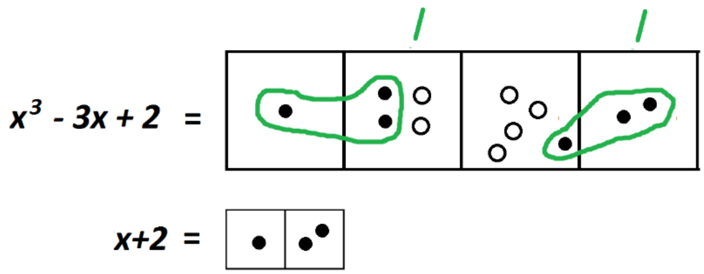
$x+2 =$ 

But there is still the question: Is this brilliant idea actually helpful?

Hmm.

Well. Is there anything else in life you want right now? Can you create another copy of $x + 2$ anywhere?

I'd personally like a dot to the left of the pair dots in the rightmost box. I am going to make it happen! I am going to insert a dot and antidot pair. Doing so finds me another copy of $x + 2$.

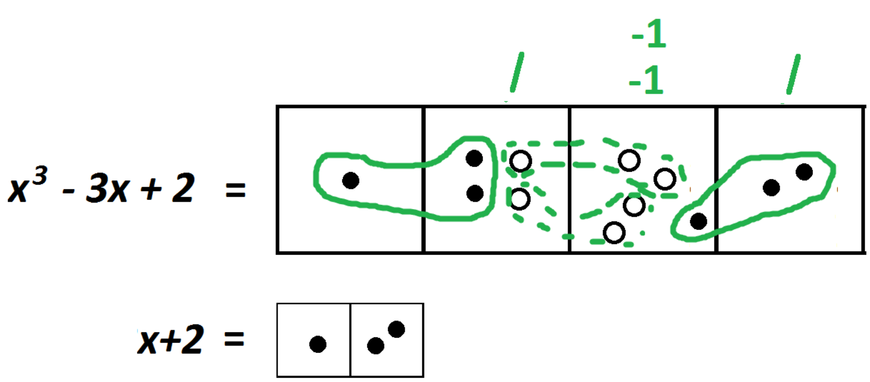



This is feeling good!

But … err … are we now stuck?
Maybe this brilliant idea isn't helpful.

> Stare at this picture.
> Do you notice anything?

Do you see copies of the exact opposite of what we're looking for? Instead of one dot next to two dots, there are copies of one antidot next to two antidots! We have two anti-copies!




Whoa! It looks like we did it.

How do we read the answer?
We see that

$$(x^3 - 3x + 2) \div (x + 2) = x^2 - 2x + 1$$

Fabulous!

So, actually, we <u>can</u> do all polynomial division problems with this dots-and-boxes method, even problems that involve negative numbers.

**Practice 68.1** Try computing

$$\frac{x^4 - 1}{x - 1}$$

Can you get the answer $x^3 + x^2 + x + 1$?

**Practice 68.2** Show that

$$\frac{x^6 + 1}{x^2 + 1}$$

equals $x^4 - x^2 + 1$.

**Practice 68.3:** Play with

$$(4x^4 - 7x^3 + 10x^2 - 4x + 2) \div (x^2 - x + 1)$$

to see that it equals $4x^2 - 3x + 3$ with a remainder of $2x - 1$ yet to be divided by $x^2 - x + 1$.

(People typically write this answer as $\frac{4x^4-7x^3+10x^2-4x+2}{x^2-x+1} = 4x^2 - 3x + 3 + \frac{2x-1}{x^2-x+1}$.)

## MUSINGS

**Musing 68.4** In this Section we showed

$$(x^3 - 3x + 2) \div (x + 2) = x^2 - 2x + 1$$

If $x$ is actually the number $10$, what ordinary arithmetic problem does this represent?

**Musing 68.5** Compute $\frac{x^3-3x^2+3x-1}{x-1}$.

**Musing 68.6** Try computing $\frac{4x^3-14x^2+14x-3}{2x-3}$.

**Musing 68.7** If you can compute this problem, you can probably do any problem!

$$\frac{4x^5 - 2x^4 + 7x^3 - 4x^2 + 6x - 1}{x^2 - x + 1}$$

**Musing 68.8** We compute $(2x^2 + 7x + 6) \div (x + 2)$ to be $2x + 3$.

Can you predict what the answer to $(2x^2 + 7x + 7) \div (x + 2)$ will be?

**Musing 68.9** Compute $\frac{x^4}{x^2-3}$.

**Musing 68.10** Try this crazy one: $\frac{5x^5-2x^4+x^3-x^2+7}{x^3-4x+1}$.

If you do this one with paper and pencil, you will find yourself trying to draw 84 dots at some point. Is it swift and easy just to write the number "84"? In fact, how about just writing numbers and not bother drawing dots at all?

In general, is there a swift way to conduct polynomial division with ease on paper? Rather than draw boxes and dots, maybe work with tables that keep track of numbers?

(The word **synthetic** is often used for algorithms one creates that are a step or two removed from that actual process at hand. Some school curriculums teach students a process called **synthetic division** which is really just our dots-and-boxes method in disguise.)

**Musing 68.11** What do you think are the answers to each of these polynomial algebra questions?

$$(x^3 - 3x^2 + 3x - 1) + (x^2 - 2x + 1)$$

$$(x^3 - 3x^2 + 3x - 1) - (x^2 - 2x + 1)$$

$$(x^3 - 3x^2 + 3x - 1) \times (x^2 - 2x + 1)$$

$$(x^3 - 3x^2 + 3x - 1) \div (x^2 - 2x + 1)$$

**Musing 68.12**

a) Lara was asked to compute $(x^2 + 7x + 6) \div (x + 6)$ and reasoned as follows:

> "If $x = 10$, then this reads $176 \div 16$, which has the answer $11$.
> So $(x^2 + 7x + 6) \div (x + 6)$ must be $x + 1$."

Is it?

b) Lara was also asked to compute $(3x^2 + x + 2) \div (x + 3)$. She reasoned that since $312 \div 13 = 24$ the answer must be $2x + 4$.

It's not!

What is the correct answer and what went wrong?

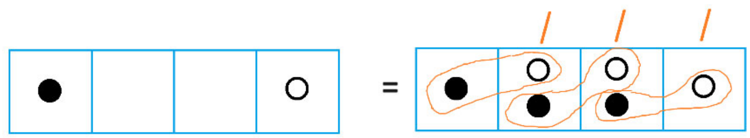
## 69. The Opening Mystery

This picture shows that

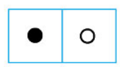
$$(x^3 - 1) \div (x - 1)$$

equals $x^2 + x + 1$.

$x^3 - 1$ =



$x - 1$ =



Remember that a division problem like $(x^3 - 1) \div (x - 1)$ is asking:

*What is multiplied by $x - 1$ to give $x^3 - 1$ ?*

And we see that $x^2 + x + 1$ is what must have been multiplied by $x - 1$ to give $x^3 - 1$

$$x^3 - 1 = (x - 1) \times (x^2 + x + 1)$$

But let's lose some detail and just write

$$x^3 - 1 = (x - 1) \times (something)$$

**Practice 69.1**

a) Show that $(x^4 - 1) = (x - 1) \times (something)$.
b) Show that $(x^7 - 1) = (x - 1) \times (something)$.
c) Show that $(x^{100} - 1) = (x - 1) \times (something)$.

The algebra is showing us that

$$\mathrm{x}^{\text{counting number}} - 1 = (\mathrm{x} - 1) \times (\text{something}).$$

and the "something" is always friendly: It's $x^3 + x^2 + x + 1$ or $x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$ or $x^{99} + x^{98} + \cdots + x^2 + x + 1$, or some such. If $x$ represents a counting number, then the "something" will be a counting number too (a sum of counting numbers, in fact.)

Let's have some fun with this observation.

**Example:** Show that $17^6 - 1$ is sure to be a multiple of 16.

**Answer:** We know that

$$x^7 - 1 = (x - 1) \times (something)$$

If $x$ represents the number 17, then this is saying

$$17^6 - 1 = (17 - 1) \times (something)$$

That is, $17 = 16 \times (something)$ and so is a multiple of 16.

**Practice 69.2** Explain why $637^{52} - 1$ must be a multiple of 636.

**Practice 69.3** Explain why $8^{100} - 1$ must be a multiple of 7.

Can we use this technique to show that $2^{300}-1$ is a composite number?
This was a challenge from earlier on.

Let's try!

Our work gives

$$\begin{aligned} 2^{300}-1 &= (2-1)\times(\mathit{something}) \\ &= 1\times(\mathit{something}) \end{aligned}$$

Hmm. Not very helpful. We knew that already!

Here's something very sneaky.

Rather than think of $2^{300}$ as a product of three hundred 2s, think of it as a product of one hundred-and-fifty 4s by noticing that $2\times 2=4$.

$$\begin{aligned} 2^{300} = 1 &\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2) \\ &\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2) \\ &\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2) \\ &\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2) \\ &\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2) \\ &\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2) \\ &\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2) \\ &\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2) \\ &\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2) \\ &\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2) \\ &\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2) \\ &\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2) \\ &\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2) \\ &\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2) \\ &\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2) \end{aligned}$$

$$\begin{aligned} = 1 &\times(4)\times(4)\times(4)\times(4)\times(4)\times(4)\times(4)\times(4)\times(4)\times(4) \\ &\times(4)\times(4)\times(4)\times(4)\times(4)\times(4)\times(4)\times(4)\times(4)\times(4) \\ &\times(4)\times(4)\times(4)\times(4)\times(4)\times(4)\times(4)\times(4)\times(4)\times(4) \\ &\times(4)\times(4)\times(4)\times(4)\times(4)\times(4)\times(4)\times(4)\times(4)\times(4) \\ &\times(4)\times(4)\times(4)\times(4)\times(4)\times(4)\times(4)\times(4)\times(4)\times(4) \\ &\times(4)\times(4)\times(4)\times(4)\times(4)\times(4)\times(4)\times(4)\times(4)\times(4) \\ &\times(4)\times(4)\times(4)\times(4)\times(4)\times(4)\times(4)\times(4)\times(4)\times(4) \\ &\times(4)\times(4)\times(4)\times(4)\times(4)\times(4)\times(4)\times(4)\times(4)\times(4) \\ &\times(4)\times(4)\times(4)\times(4)\times(4)\times(4)\times(4)\times(4)\times(4)\times(4) \\ &\times(4)\times(4)\times(4)\times(4)\times(4)\times(4)\times(4)\times(4)\times(4)\times(4) \\ &\times(4)\times(4)\times(4)\times(4)\times(4)\times(4)\times(4)\times(4)\times(4)\times(4) \\ &\times(4)\times(4)\times(4)\times(4)\times(4)\times(4)\times(4)\times(4)\times(4)\times(4) \\ &\times(4)\times(4)\times(4)\times(4)\times(4)\times(4)\times(4)\times(4)\times(4)\times(4) \\ &\times(4)\times(4)\times(4)\times(4)\times(4)\times(4)\times(4)\times(4)\times(4)\times(4) \\ &\times(4)\times(4)\times(4)\times(4)\times(4)\times(4)\times(4)\times(4)\times(4)\times(4) \end{aligned}$$

We have

$$4^{150} - 1 = (4 - 1) \times (something)$$
$$= 3 \times (something)$$

which is really saying that $2^{300} - 1$ equals $3 \times (something)$ and so is a multiple of three.

It is a composite number!

**Practice 69.4**
a) Explain why $2^{300}$ and why $8^{100}$ are the same number.
b) Explain why $2^{300} - 1$ is also a multiple of $7$.

**Practice 69.5**
a) Explain why $2^{300}$ and why $16^{75}$ are the same number.
b) Explain why $2^{300} - 1$ is also a multiple of $15$.

**Practice 69.6**
a) Explain why $2^{300} - 1$ is also a multiple of $31$.
b) Explain why $2^{300} - 1$ is also a multiple of $63$.

Okay. I think we have well and truly established that the 91-digit long number $2^{300} - 1$ is not a prime number!

## MUSINGS

**Musing 69.7**

a) Why must $2^{44} - 1$ be multiple of $15$?
b) Why must $2^{55} - 1$ be multiple of $31$?
c) Why must $2^{2222} - 1$ be multiple of $3$?
d) Why must $2^n - 1$ be a multiple of $3$ if $n$ is an even number?
e) Why must $2^n - 1$ be a multiple of $7$ if $n$ is multiple of three?

**Musing 69.8**

a) Compute $\frac{x^3+1}{x+1}$ and $\frac{x^5+1}{x+1}$ and $\frac{x^7+1}{x+1}$.

b) Make a guess as to what $\frac{x^{107}+1}{x+1}$ equals.

c) Why must $78^{107} + 1$ be a multiple of $79$?

**Musing 69.9** CHALLENGE

Let's complete Mersenne's work. Let's see if we can show that if $n$ is a composite number, then $2^n - 1$ is sure to be a composite number as well.

Since $n$ is composite, we have $n = ab$ for two counting numbers $a$ and $b$, each different from $1$.

This means, $2^n - 1 = 2^{ab} - 1$.

a) Can you explain why $2^{ab}$ is the same as $M^b$, with $M$ being the number $2^a$?

We know that $M^b - 1$ is a multiple o $M - 1$.

b) Why then is $M - 1$ a factor of $2^n - 1$?

This shows that $2^n - 1$ has a proper factor and so is composite.

## 70. “Infinite Polynomials”

We’ve seen that the fraction $\frac{1}{9}$ has an infinite (repeating) decimal expansion.

$$\frac{1}{9} = 0.111111\ldots.$$

> **Practice 70.1** Feel free to compute the division problem $1 \div 9$ in a $1 \leftarrow 10$ machine to remind yourself of this.

.

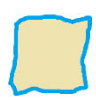
Given that the decimal places represent tenths, hundredths, thousandths, and so forth, this is really a statement about an infinite sum. It is saying

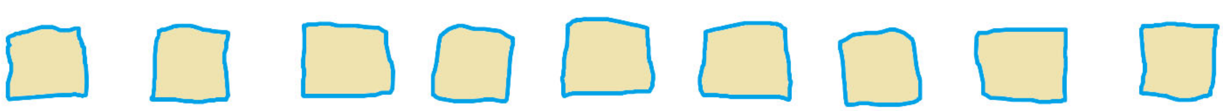
$$\frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + \frac{1}{10000} + \cdots = \frac{1}{9}$$

Here’s a fun, “practical” way to see this infinite sum in action.

Imagine sharing a piece of paper with nine of your friends. You divide the paper into tenths and give each of your friends a tenth of the paper ($\frac{1}{10}$) and keep a tenth for yourself.

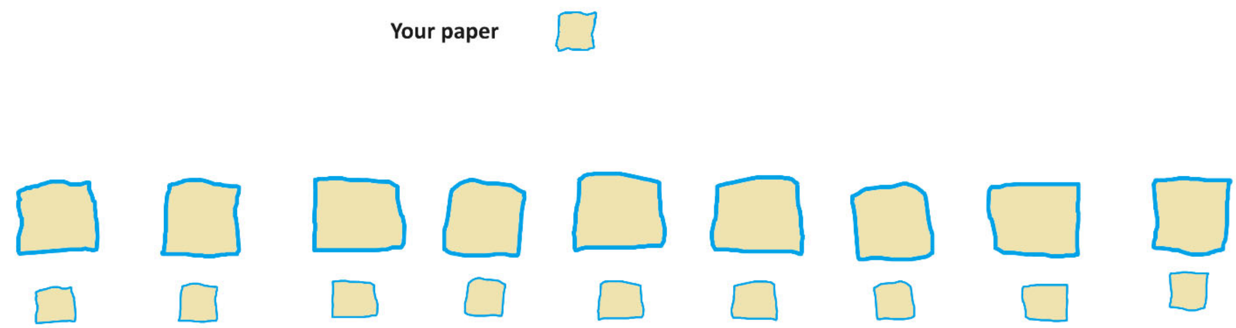
**Your paper**



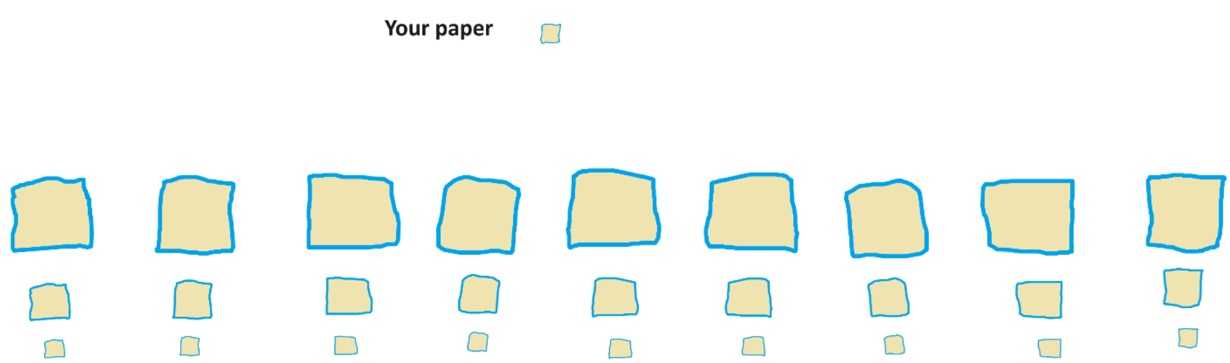


But you’re feeling generous and decide to share your tenth as well.

You divide it into ten parts and give each of your friends a tenth of that tenth (that’s $\frac{1}{100}$ of the original sheet) and keep one part for yourself, a hundredth.




You share again! Dividing your hundredth of the paper into ten parts, you give each of your friends a tenth of a hundredth ($\frac{1}{1000}$ of the original sheet) and keep a thousandth for yourself.




And you keep doing this … forever!

*Once you've reached the end of time … how much paper will you hold?*

None of it. You'll have given it all away.

*And where did the paper go?*

The sheet of paper was equally distributed among your nine friends.

So, each of your friends has $\frac{1}{9}$ of the original sheet.

But consider how they each received that paper. They each received it as

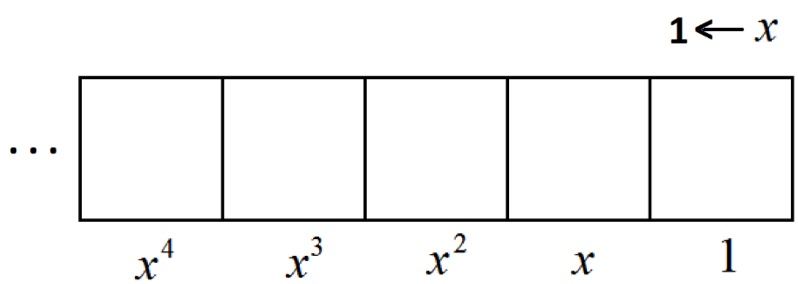
$$\frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + \cdots$$

It must be that this infinite sum corresponds to $\frac{1}{9}$ of the paper!
This is, of course, a mind game (as is the nature of infinite decimals!). But it feels like we can imagine an infinite sharing process like this going on forever and having a sense of what "final" result all is heading to.

> **Practice 70.2 OPTIONAL** Imagine repeatedly sharing in this way a sheet of paper equally among you and three friends. Can you see that $\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\frac{1}{80}+\cdots$ wants to be $\frac{1}{4}$?

Here's another way to play with the infinite.

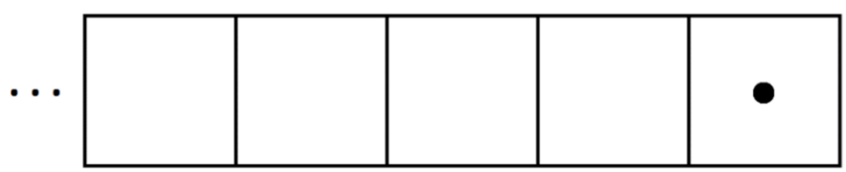
Consider the $1 \leftarrow x$ machine again with boxes heading infinitely far to the left.

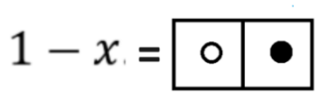



Let's use this machine to compute the strange division problem

$$\frac{1}{1-x}$$

It is the number $1$, just a single dot in the machine, divided by $1-x$.
If we think of $1-x$ as $-x+1$ we can see it as an antidot next to a dot.



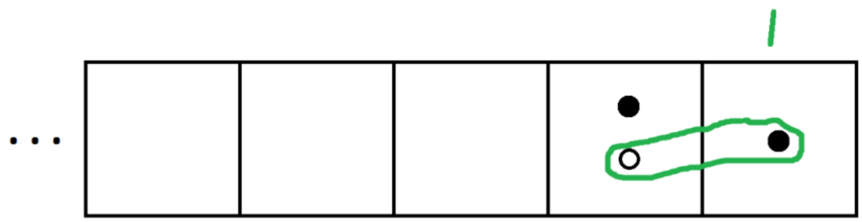


Do you see any antidot-dot pairs in this picture of just one dot?
I don't.

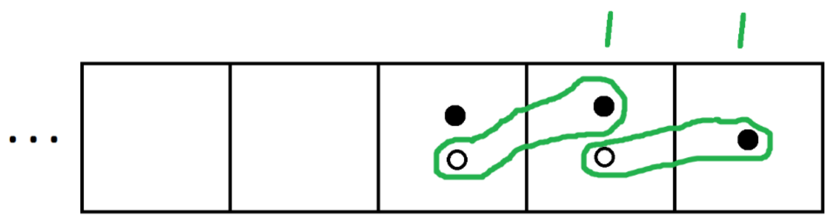
But remember

> If there is something in life you want, make it happen!
> (And deal with the consequences.)

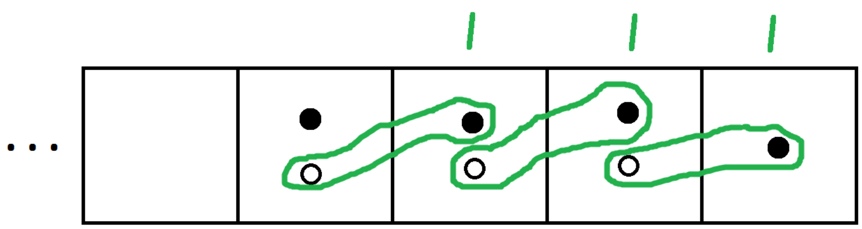
Let's create we want by adding a dot-antidot pair to the picture.



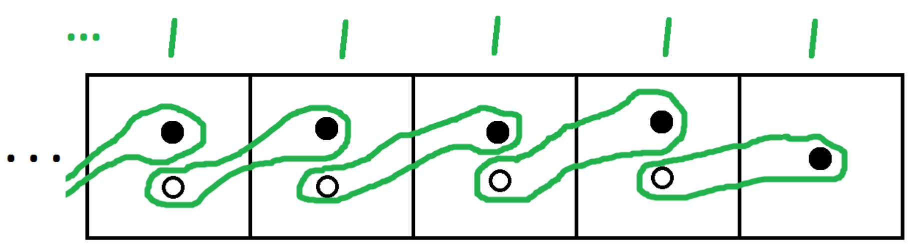
And let's do it again.



And again.



And we can see that we'll be doing this forever.



Whoa!

How do we read this answer?

Well, we have one antidot-dot pair at the $1$ level, one at the $x$ level, one at the $x^2$ level, one at the $x^3$ level, and so on and so on.

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots$$

The answer is an infinite sum.

**Practice 70.3** Let's imagine that $x$ represents the number $\frac{1}{10}$.

a) Do you see that $1 + x + x^2 + x^3 + x^4 + \cdots$ is then $1 + \frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + \frac{1}{10000} + \cdots$?

b) Make $\frac{1}{1-\frac{1}{10}}$ look friendlier.

c) Do a) and b) together show (again) that $\frac{1}{9} = \frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + \frac{1}{10000} + \cdots$?

The equation we obtained is a famous formula in mathematics. It is called the **geometric series formula** and it is often given in many upper-level high school text books for students to use. But textbooks often write the formula the other way round, and with the letter $r$ rather than the letter $x$.

$$1 + r + r^2 + r^3 + \cdots = \frac{1}{1-r}$$

**Practice 71.4** Suppose $x$ represents the number 2.

Do you believe the equation $\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots$ in this case?

The previous two practice problems show that playing with the infinite is dangerous.
The formula $\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots$ seems believable for some values of $x$ (like $\frac{1}{10}$) but not for others (like 2).

All through the 1600s and beyond scholars tried to understand what formulas involving the infinite could mean and when, exactly, they are to believed. This helped spur on the famous subject called **calculus**.

Just so you have it, scholars have come to understand that the geometric series formula

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots$$

is consistent with arithmetic if $x$ represents a number between $-1$ and $1$ on the number line.

## MUSINGS

Continuing the dangerous play with the infinite …

**Musing 70.5** Show, according to an $1 \leftarrow x$ machine, that $\frac{1}{1+x}$ equals

$$1 - x + x^2 - x^3 + x^4 - \cdots$$

**Musing 70.6:** Compute $\frac{x}{1-x^2}$. Do you get a sum of odd powers of $x$?

**Musing 70.7:** Compute $\frac{1}{1-x-x^2}$ and discover a very famous sequence of numbers. (Draw very big boxes for this one. The picture gets messy quickly!)

# 71. COMPLETELY OPTIONAL AND COMPLETELY WILD: Base One-and-a-Half

In all our base machines, a set number of dots "explode," to disappear to be replaced **one** dot, one place to their left.

> In a $1 \leftarrow 10$ machine, groups of **ten** dots are replaced by **one** dot.
> In a $1 \leftarrow 2$ machine, groups of **two** dots are replaced by **one** dot.
> In a $1 \leftarrow 3$ machine, groups of **three** dots are replaced by **one** dot.

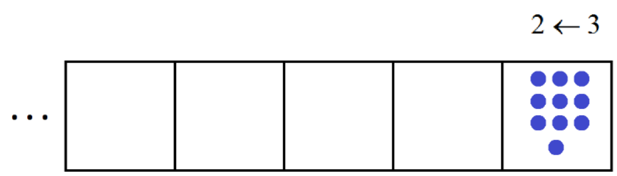
And so forth.

What if we mix things up a bit?

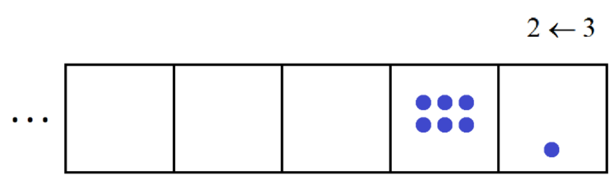
> Consider a $2 \leftarrow 3$ machine.
>
> This machine replaces **three** dots in any one box with **two** dots one place to their left.

Curious!

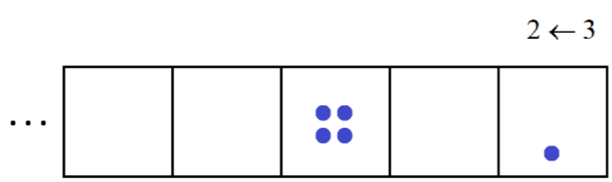
To get a feel for the machine, let's try putting ten dots into the machine.




We immediately get three explosions creating two and two and two dots in the next box over.

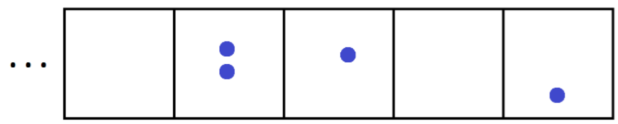



Now there are two more explosions creating two and two dots.




One more explosion.

$2 \leftarrow 3$



We see the code $2101$ appear for the number ten in this $2 \leftarrow 3$ machine.

**Practice 71.1** Show that the number thirteen has code $2121$.

Here are the $2 \leftarrow 3$ codes for the first fifteen numbers. (Check these!)

| | | |
|---|---|---|
| **1: 1** | **6: 210** | **11: 2102** |
| **2: 2** | **7: 211** | **12: 2120** |
| **3: 20** | **8: 212** | **13: 2121** |
| **4: 21** | **9: 2100** | **14: 2122** |
| **5: 22** | **10: 2101** | **15: 21010** |

**Practice 71.2** Does it make sense that only the digits $0$, $1$, and $2$ appear in these codes? Explain why you won't see a digit of $3$ or higher in any $2 \leftarrow 3$ machine code.

**Practice 71.3** Does it make sense to you that the final digits of these codes cycle $1, 2, 0, 1, 2, 0, 1, 2, 0, \ldots$?

Even if we don't know what these codes mean, we can still do arithmetic in this weird system!

For example, ordinary arithmetic says that $6 + 7 = 13$, and the codes in this machine say the same thing too.

Six has code $210$.
Seven as code $211$.
Adding gives $2121$, which is the code for thirteen.

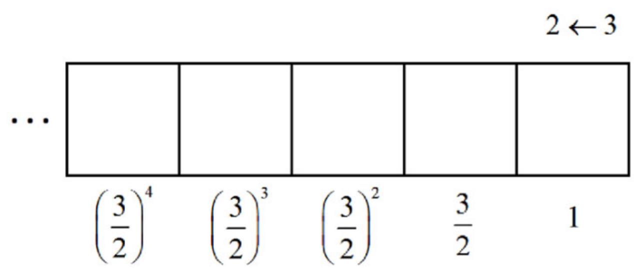
$$\begin{array}{r} 210 \\ +\,211 \\ \hline =421 \end{array} \quad = 2121$$

**Practice 71.4** Compute $10 + 5$ purely by $2 \leftarrow 3$ machine codes. Do you get the code for fifteen?

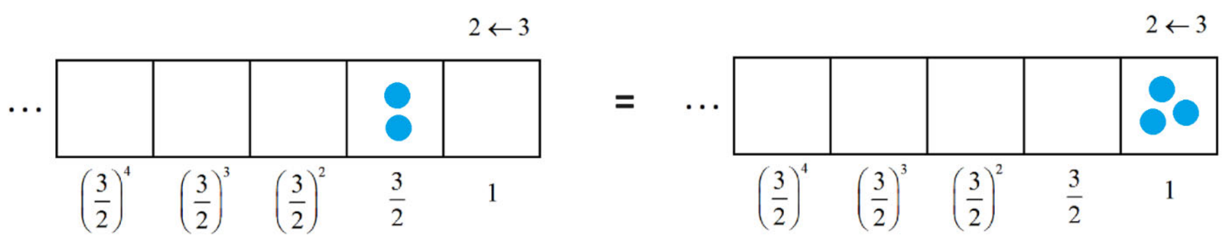
But the real question is:

> What are these codes? Are these codes for place-value in some base?
> If so, which base?

Of course, the title of this Section gives the answer away. So, let's check if the powers of one-and-half, $\frac{3}{2}$, really are the correct place values here.




For starters, three dots in the 1s place should be equivalent to two dots in the $\frac{3}{2}$s place. Are they?

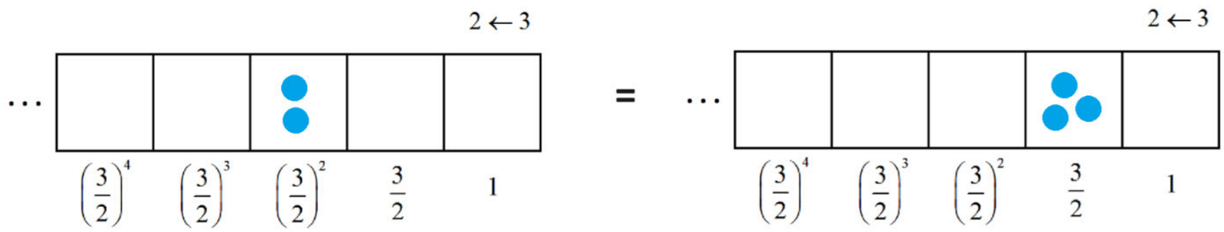



Now

three 1s are: $3 \times 1 = 3$
two $\frac{3}{2}$s are: $2 \times \frac{3}{2} = 3$

Yes! These are the same.

We also need three dots in the $\frac{3}{2}$s place to be equivalent to two dots in the $\left(\frac{3}{2}\right)^2$s place. Let's check.

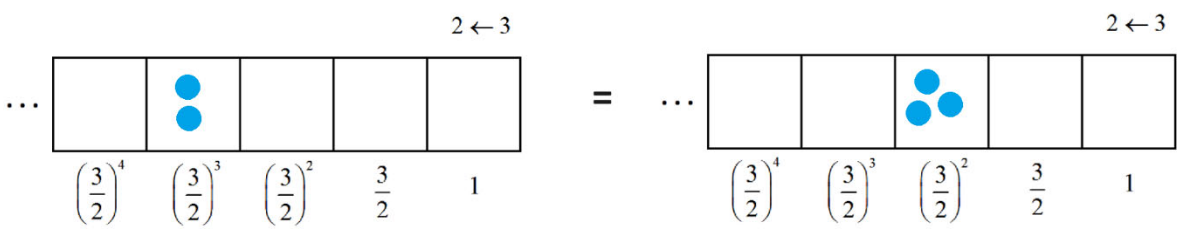


Three $\frac{3}{2}$s are: $\quad 3 \times \frac{3}{2} = 3 \times 3 \times \frac{1}{2}$

Two $\left(\frac{3}{2}\right)^2$s are: $\quad 2 \times \left(\frac{3}{2}\right)^2 = 2 \times \frac{3}{2} \times \frac{3}{2} = 2 \times 3 \times \frac{1}{2} \times 3 \times \frac{1}{2} = 3 \times 3 \times \frac{1}{2}$

These are the same!

Let's check one more.



Three $\left(\frac{3}{2}\right)^2$s are: $\quad 3 \times \left(\frac{3}{2}\right)^2 = 3 \times 3 \times \frac{1}{2} \times 3 \times \frac{1}{2}$

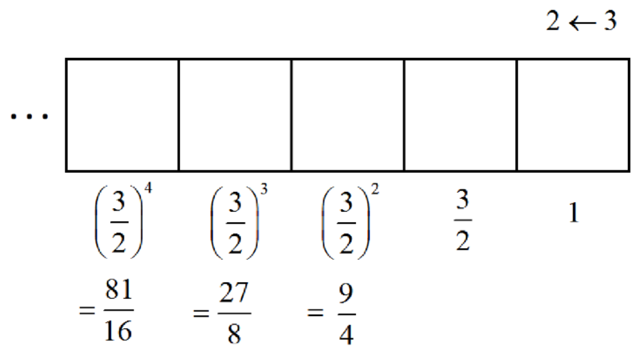
Two $\left(\frac{3}{2}\right)^3$s are: $\quad 2 \times \left(\frac{3}{2}\right)^3 = 2 \times 3 \times \frac{1}{2} \times 3 \times \frac{1}{2} \times 3 \times \frac{1}{2} = 3 \times 3 \times 3 \times \frac{1}{2} \times \frac{1}{2}$

These are the same!

**Practice 71.5 OPTIONAL** If you can, show in general, that three copies of one power of $\frac{3}{2}$ will always match two copies of the next power of $\frac{3}{2}$.

We are indeed working in something that looks like base one-and-a-half!

And if you like we can evaluate the powers of $\frac{3}{2}$ if we want (but they are awkward fractions!)




$\left(\frac{3}{2}\right)^2 = \frac{3}{2} \times \frac{3}{2} = \frac{9}{4}$ $\left(\frac{3}{2}\right)^3 = \frac{3}{2} \times \frac{3}{2} \times \frac{3}{2} = \frac{27}{8}$ $\left(\frac{3}{2}\right)^4 = \frac{3}{2} \times \frac{3}{2} \times \frac{3}{2} \times \frac{3}{2} = \frac{81}{16}$ $\left(\frac{3}{2}\right)^5 = \frac{3}{2} \times \frac{3}{2} \times \frac{3}{2} \times \frac{3}{2} \times \frac{3}{2} = \frac{243}{32}$

This codes from our $2 \leftarrow 3$ machine use the digits $0$, $1$, and $2$, which is a little weird.

> There is no digit ten or bigger in base ten.
> There is no digit two or bigger in base two.
> There is no digit three or bigger in base three.
> And so on.

Yet there is a digit bigger than the base number for these $2 \leftarrow 3$ codes.

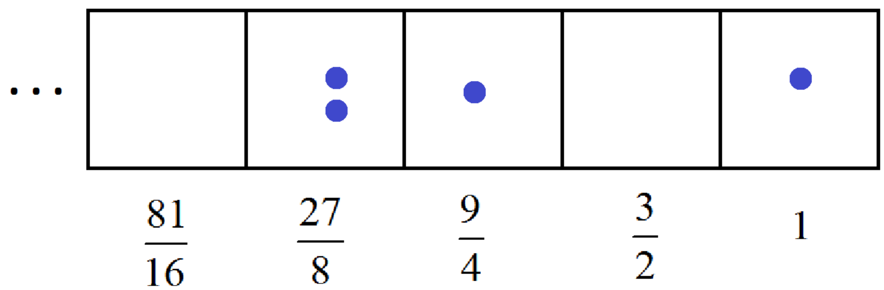
**Comment:** Mathematicians are trying to develop a notion of base one-and-a-half, and other fractional bases, that don't use digits that exceed the base value. (If you are game, look up *beta expansions* and *non-integer representations* on the internet.)

This alternative version of base one-and-a-half was first conceived by mathematician Dr. James Propp. It is called **Propp Base One-and-a-Half**.

I personally find $2 \leftarrow 3$ machine codes intuitively alarming!

We are saying that each and every counting number can be represented as a combination of the ghastly fractions $1, \frac{3}{2}, \frac{9}{4}, \frac{27}{8}, \frac{81}{16}, \ldots$.

For example, we saw that the number ten has code $2101$.




Is it true that this combination of fractions

$$2 \times \frac{27}{8} + 1 \times \frac{9}{4} + 0 \times \frac{3}{2} + 1 \times 1$$

turns out to be the perfect whole number ten?

Check that it does.

(And to that, I say WHOA!)

There are plenty of questions to be asked about the $2 \leftarrow 3$ machine codes of numbers, and many are unsolved to this day.

For reference, here are the first forty numbers in Propp base one-and-a-half (along with zero at the beginning).

| | | | |
|---|---|---|---|
| **0** | | | |
| **1** | **2102** | **21220** | **212021** |
| **2** | **2120** | **21221** | **212022** |
| **20** | **2121** | **21222** | **212210** |
| **21** | **2122** | **210110** | **212211** |
| **22** | **21010** | **210111** | **212212** |
| **210** | **21011** | **210112** | **2101100** |
| **211** | **21012** | **212000** | **2101101** |
| **212** | **21200** | **212001** | **2101102** |
| **2100** | **21201** | **212002** | **2101120** |
| **2101** | **21202** | **212020** | **2101121** |

**Practice 71.6:**

Explain why, after a small initial "hiccup," all the codes begin with the digit 2.

Explain why, after a slightly bigger initial "hiccup," all the codes begin with 21.

(The first three digits of the codes, alas, don't stabilize, even after a big "hiccip.")

**Practice 71.7:** With the exceptions of 0, 1, and 2, explain why if you delete the final digit of any code in this list, what remains is another code in the list. (For example, deleting the final digit of 2101121, the code for forty, leaves 210112, the code for twenty six.)

The Musings to end of this volume (assuming you are reading this optional section) represent a lifetime of work! Musing on them and making progress with them and possibly solving them is likely opening up new original mathematics currently unknown to the world.

Enjoy musing on them. Just play and have fun. And if you happen to make significant headway on a problem, let me know! I'll help the world learn what you've accomplished.

Have fun!

## MUSINGS

**Musing 72.8** **Which combinations of 0s, 1s, and 2s are Propp codes for numbers?**

Look at our list of codes for the first forty numbers. We don't see "201" or 21102."

a) The code 201 represents the mixed number $5\frac{1}{2}$. (Why?)
What mixed number does 21102 represent?

One way to work out whether or not a combination of digits represents a whole number is to simply work out the sums of powers of $\frac{3}{2}$ it represents. But that doesn't seem fun! For instance, how long would it take to determine the number represented by this long code?

210221202021200210202200210101122000221221222021020122011002102010202212

b) **Unsolved Challenge:** Develop a quick and efficient means to look at a sequence of 0s, 1s, and 2s to determine whether or not it corresponds to the code of a whole number. (Of course, how one defines "quick" and "efficient" is up for debate.)

**Musing 72.9** **Divisibility Checks**

Look at the list of the first forty 2 ← 3 machine codes of numbers.

Starting with number zero, every third code ends with a zero, and only every third code (Why?)

| | | | |
|---|---|---|---|
| $\boxed{0}$ | | | |
| 1 | 2102 | $\boxed{21220}$ | 212021 |
| 2 | $\boxed{2120}$ | 21221 | 212022 |
| $\boxed{20}$ | 2121 | 21222 | $\boxed{212210}$ |
| 21 | 2122 | $\boxed{210110}$ | 212211 |
| 22 | $\boxed{21010}$ | 210111 | 212212 |
| $\boxed{210}$ | 21011 | 210112 | $\boxed{2101100}$ |
| 211 | 21012 | $\boxed{212000}$ | 2101101 |
| 212 | $\boxed{21200}$ | 212001 | 2101102 |
| $\boxed{2100}$ | 21201 | 212002 | $\boxed{2101120}$ |
| 2101 | 21202 | $\boxed{212020}$ | 2101121 |

This leads to the following check.

**Divisibility by Three**
A number written in 2 ← 3 code is divisible by three precisely when its final digit is zero.

This makes it easy to tell if a number expressed as a 2 ← 3 machine code is a multiple of three.

a) Find, and explain, a similar divisibility check for 9.

b) Find, and explain, a similar divisibility check for 27 and for 81. (Could you keep going with higher and higher powers of three?)

Every fifth code in our list of 2 ← 3 machine codes of numbers has a curious property too: the alternating sum of its digits is a multiple of five.

| | | | |
|---|---|---|---|
| $\boxed{0}$ | | | |
| 1 | 2102 | 21220 | 212021 |
| 2 | 2120 | 21221 | 212022 |
| 20 | 2121 | 21222 | 212210 |
| 21 | 2122 | 210110 | 212211 |
| $\boxed{22}$ | $\boxed{21010}$ | $\boxed{210111}$ | $\boxed{212212}$ |
| 210 | 21011 | 210112 | 2101100 |
| 211 | 21012 | 212000 | 2101101 |
| 212 | 21200 | 212001 | 2101102 |
| 2100 | 21201 | 212002 | 2101120 |
| $\boxed{2101}$ | $\boxed{21202}$ | $\boxed{212020}$ | $\boxed{2101121}$ |

2-2 = 0

2-1+0-1+1-1 = 0

2-1+2-0+2-0 = 5

c) Can you explain why this is so for every fifth entry, and only the fifth entries?

This then gives a check for divisibility by five.

**Divisibility by Five**
A number is divisible by 5 only if the alternating sum of the digits of its 2 ← 3machine code is a multiple of five.

c) **Unsolved Challenge: How do you tell if a number is even?**

Is there a divisibility check for the number two? Is there an interesting property to the codes of every second number?

As far as I am aware, no one knows how to look at the 2 ← 3 machine code a number and quickly determine if that number is even!

Do you see—and can you prove always hold—something interesting about every second code?

| | | | |
|---|---|---|---|
| 0 | | | |
| 1 | 2102 | 21220 | 212021 |
| 2 | 2120 | 21221 | 212022 |
| 20 | 2121 | 21222 | 212210 |
| 21 | 2122 | 210110 | 212211 |
| 22 | 21010 | 210111 | 212212 |
| 210 | 21011 | 210112 | 2101100 |
| 211 | 21012 | 212000 | 2101101 |
| 212 | 21200 | 212001 | 2101102 |
| 2100 | 21201 | 212002 | 2101120 |
| 2101 | 21202 | 212020 | 2101121 |

**Musing 72.9** **Understanding Code Lengths**

Look at the lengths of the $2 \leftarrow 3$ machine codes for numbers.

| | | | |
|---|---|---|---|
| 0 | | | |
| 1 | 2102 | 21220 | 212021 |
| 2 | 2120 | 21221 | 212022 |
| 20 | 2121 | 21222 | 212210 |
| 21 | 2122 | 210110 | 212211 |
| 22 | 21010 | 210111 | 212212 |
| 210 | 21011 | 210112 | 2101100 |
| 211 | 21012 | 212000 | 2101101 |
| 212 | 21200 | 212001 | 2101102 |
| 2100 | 21201 | 212002 | 2101120 |
| 2101 | 21202 | 212020 | 2101121 |

There are 3 codes one digit long, 3 codes two digits long, 3 codes three digits long, 6 codes four digits long, 9 digits five digits long, 12 codes six digits long, and so on.

If you kept going you will get the following sequence of numbers for how many codes have a given length.

$$3, 3, 3, 6, 9, 12, 18, 27, 42, 63, 93, 141, 210, 315, 474, \ldots$$

Is there a pattern of some kind to these numbers?

**Challenge:** As far as I am aware, no one knows a formula for the numbers in this sequence that would quickly tell you the 500$^{th}$ or the 1200$^{th}$ or millionth number is in this sequence.

Your challenge: Find a direct way to find the $n$th number in this list of numbers for any counting number $n$.

**Musing 72.10** Palindromes

In mathematics, a **palindrome** is a number that reads the same way forwards as it does backwards (in whatever base we happen to be discussing).

Among the first forty $2 \leftarrow 3$ machine codes, there are six palindromes: the codes for $0$, $1$, $2$, $5$, $8$,$17$, and $35$.

| | | | |
|---|---|---|---|
| 0 | | | |
| 1 | 2102 | 21220 | 212021 |
| 2 | 2120 | 21221 | 212022 |
| 20 | 2121 | 21222 | 212210 |
| 21 | 2122 | 210110 | 212211 |
| 22 | 21010 | 210111 | 212212 |
| 210 | 21011 | 210112 | 2101100 |
| 211 | 21012 | 212000 | 2101101 |
| 212 | 21200 | 212001 | 2101102 |
| 2100 | 21201 | 212002 | 2101120 |
| 2101 | 21202 | 212020 | 2101121 |

It turns out that the codes for 170, 278, 422, and 494 are also palindromes in Propp's base one-and-half.

**Challenge:** Is there another number with a palindromic Propp base one-and-a-half code?

(Dr. Propp argues in his essay here that there might not be any more examples. Or, if he is wrong, at most finitely many more examples.)

**Final Comment:**

In these musings I have brought you to an edge of mathematical knowledge.

The school world gives the impression that all in mathematics is “solved” and that mathematicians—most likely—are spending their days just doing calculations on bigger and bigger numbers.

The truth of matters is much more fun and much more interesting. (Calculations, in and of themselves, are boring!) There are so many deep questions and wonderings for mathematicians—and everyone in the world—to explore.

We’ve asked here just a few curious and unsolved questions about the behavior of the number $\frac{3}{2}$. It is such a simple fraction, yet there is clearly so much we don’t yet understand about it!

One of the world’s current foremost mathematicians, Dr. Terrence Tao, has written about the mysteries of this basic fraction here. And other scholars too have looked at the mysteries of Propp’s base one-and-a-half. (For instance, see this work by Ben Chen et al.)

It is astounding to me that we don’t understand the mathematical properties of such a simple fraction. That’s just wonderful! And that’s typical of the source of the adrenalin that pushes mathematicians forward and makes them want to want to strive for more. It’s the pursuit of deeper understanding and deeper clarity on how our intellectual universe works.

I hope this volume has given you a glimpse of how mathematics can be so compelling to so many humans.