ARITHMETIC, ALGEBRA, and
Radical Comprehension of Math

A Refreshingly Joyous, Human, and Accessible approach to
Arithmetic and Algebra
for all those who may have experienced it otherwise

CHAPTERS 9, 10, 11, and 12

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PART 1

Arithmetic: The Gateway to All

Photo: Erick Mathew, Tanzania

Algebra is the practice of avoiding the tedium of doing arithmetic problems one instance at a time, to take a step back and see a general structure to what makes arithmetic work the way it does, and so open one's mind to more than the one view of what arithmetic, and mathematics, can be.

Part 1

Part 2

Introduction: Why Part 2? Where's Part 1?

I remember as a youngster back in Australia in the late 70s eagerly awaiting the release of a new movie that I heard was creating a sensation in America. It was called *Star Wars.* Us Aussies had to wait several months before it was released on our continent, and that waiting was torture for this eleven-year-old lad.

I was, of course, immediately enthralled by the production when I finally got to see it. But I was also perplexed. It was called "STAR WARS: EPISODE IV." Where were episodes I, II, and III? What had I missed?

Well, it turns out much of the world was perplexed too by this in the opening titles and nothing was missed.

I dare not equate these math notes with the magic George Lucas's work (nor try to explain why I think Lucas decided to first share his epic galactic saga to the world midway through its story), but I will settle one similar issue with regard to these notes:

Where are chapters 1 through 8 that supposedly comprise Part 1 of this math story?

The answer is that they are here, in this link, freely available to one and all, including you. Because I put them on a website, you may have missed them.

These first eight chapters are a gift to the world as part of a program I co-found called the *Global Math Project* whose goal is to prove that mathematics—school mathematics even—truly can serve as a portal to intense human connection, meaning, and joy. (This video outlines the history of the project and the impact it has had.)

Part 1 of this series covers the story of grade-school arithmetic. You likely already know the mechanics of what is covered there.

For instance, you no doubt know about the **counting numbers** $0, 1, 2, 3, \ldots$ and how to do arithmetic with them: add them, subtract them, multiply them, and divide them. You likely know the "opposite" numbers $-1, -2, -3, \ldots$, that then bring us to the world of **integers**. And you have probably been trained to do arithmetic with these numbers too. (To recognize that $(-4) \times 5 = -20$ and $(-2) \times (-3) = +6$, for instance.)

And you have no doubt gone further and worked with **fractions** and **decimals** as well.

But notice that I chose to say "mechanics." Doing math versus truly <u>owning</u> and deeply understanding the math you are doing are very different things.

I remember back in grade 5 of my schooling receiving gold star after gold star on my long division worksheets. I was even called the star of the class for apparently truly "getting" long division. But I didn't get it. I had just memorized the algorithm (quite well, apparently) and was performing simply to please my teacher. I was fully cognizant of the fact that I didn't understand what I was doing one whit.

Part 1 invites us to reexamine all we think we know about school arithmetic and see it in new, clear, and sensical light.

However, starting on this second volume without working though Part 1 will likely be fine. Just be willing to refer to sections of Part 1 every now and then as you go along. To get a sense of what I mean by this, have a look at the six Musings at the end of this introduction. They are not prerequisites, but they do illustrate the depth of understanding developed in that first volume.

Each of our educational journeys has bits and bobs that are hazy or are outright missing. That's okay! Try this volume and see how it goes. I'll do my best to direct you to relevant sections of Part 1 as we move along to help out.

Be kind to yourself.

Despite what you might have been trained to believe from school math, there is actually no rush to make sense of mathematics. Don't hurry! Just let a profound beautiful sense of mathematics unfold over whatever amount of time it takes.

So, take your time. Linger. Enjoy!

Mathematics is a gift for you to truly savor.

MUSINGS

Here are some thorny issues covered in detail in Part 1. How do personally feel about attending to these questions?

Musing 1: *People say you can't divide by zero. But why?*
Can you personally explain what goes wrong mathematically if you try to divide by zero?
[See Sections 17 and 38.]

Musing 2: *Why, exactly, is negative times negative positive? Have you a convincing explanation?*
[See Section 25.]

Musing 3: *Every few months the following problem makes the rounds on the internet.*

What is the value of $\mathbf{8 \div 2(2+2)}$?

a) Some people say this expression has value 1*. Do you see why they might say that?*
Other people say that it has value 16*. Do you see how such folk must be thinking?*
b) Can you insert parentheses into the expression to ensure everyone evaluates it as 1*?*
c) Can you insert parentheses into the expression a different way to make sure everyone evaluates it as 16*?*
[See Sections 8 and 9.]

Musing 4: *Are you comfortable evaluating* 17×18 *each of these two ways?*
(The second picture looks like it is a rectangle with negative side lengths and negative areas. Is that allowed?)

	10	8
10	100	80
7	70	56

100 + 70 + 80 + 56 = 306

	20	-2
20	400	-60
-3	-40	+6

400 - 60 - 40 + 6 = 306

[See Sections 11 and 24.]

Musing 5: *Does "*$3n + 1$*" make sense to you as a shorthand way to write "triple an unspecified number and add one to the result"?*

Musing 6: *Here's a piece of math that looks scary:* $6x^2 - 3ax = 3x(2x - a)$.
After a moment and a deep breath, can you start to make some sense of it?
Could you perhaps draw a picture to demonstrate what it is saying?
[See Section 13 and 24.]

Chapter 9

Solving, Graphing, Seeing

72. Math is a Language

People say that “mathematics is a language.” I personally am not sure I know what that means.

Presumably this remark is a curt summary of a statement made by Italian physicist Galileo Galilei (1564-1642) in his work *Opere Il Saggiatore*:

> *[The universe] cannot be read until we have learnt the language and become familiar with the characters in which it is written. It is written in mathematical language, and the letters are triangles, circles and other geometrical figures, without which means it is humanly impossible to comprehend a single word.*

Galileo is making a deep philosophical statement that no doubt has been probed and debated by folk cleverer than me.

Rather than comment on Galileo’s musing myself, I will offer instead a non-erudite interpretation of the curt summary statement.

Mathematics *is* a language, in a very literal way.

Since we are currently communicating in English, the language of math is … English!
(And if I was writing in Hindi or in Korean, then the language of mathematics would be Hindi or Korean.)

The fact is that every mathematical statement is a sentence.

For example, the statement

$$5 \ = 2 + 3$$

has a subject (the quantity “5”), a verb (“equals”), and an object (the quantity “$2 + 3$”).

As such, the sentence when written as a stand-alone statement should come with proper English punctuation: it needs a period at its end.

$$5 \ = 2 + 3.$$

Look up any published mathematics paper and you will see it littered with words and symbols and formulas replete with all the Englich punctuation to go with them—periods, commas, semi-colons, you name it. You’ll see that punctuation even within and throughout the symbols and lines of formulas. (There are occasional instances where the mathematics community has agreed to let go of some punctation for the sake of visual clarity, but it is implied that it is there.)

One can literally read a math paper out loud as though it were an essay from English class.

Let's run with this idea.

Consider, for instance, the statement:

$$7 > 4 + 9.$$

The subject is the number 7, the verb is "is greater than," (well, that's a bit more than just a verb in this case), and the object is the quantity $4 + 9$. (We talked about the *greater than* symbol in section 50.)

> A sentence with *equals* as its verb is called an **equation**.
>
> A sentence that compares quantities or states that two quantities are not equal is called an **inequality**.

Our first sentence, $5 = 2 + 3$, happens to be a true sentence about numbers and our second sentence, $7 > 4 + 9$, a false one. (Seven is not larger than thirteen!)

And although we know in life that sentences made need not be true, mathematics tends to focus on truth and wants to present sentences that are true statements about numbers.

But there is another issue: sentences can also be ambiguous. They may need further context or information before being deemed true or false.

For instance, the sentence

Harold is over six feet tall

cannot be determined as true or false until we are told which particular Harold of the world the speaker of this sentence is referring to.

And any sentence in math class that uses a name for a number without specifically stating what that number is in the author's mind cannot be determined as true or false either. For example,

$$N + 3 = 10$$

is a sentence about a number being called "N." The sentence is currently neither true nor false. Of course, if we are later told that N represents the number 7, then we can deem it a true sentence. If, instead, we are later told that N represents 13, then we can say we have a false sentence.

The sentence

$$x = 2$$

is currently neither true nor false. But we do realize that if x represents the number 2, then it would be a true sentence.

Practice 72.1: The symbol for "does not equal" is $\neq$. (Have you seen this before?)

a) Give a value for a number being called p that makes the sentence

$$p \neq 2$$

a true sentence.

b) Now give value for p that would make it a false sentence.

MUSINGS

Musing 72.2 (We talk about numbers being squared in Section 11.)

Here's a sentence about an unspecified number being called n.

$$n^2 = 25$$

Without knowing anything about n, the sentence is currently neither true nor false.

a) Let me tell you that the unspecified number n is actually 7 in my mind. Knowing that, is the given sentence true or false?

b) Actually, that's not the case: the number n is really 5. Given that, is the sentence true or false?

c) There are two numbers n could be that would make the sentence true. I believe you've just found one of those values. What's the other one?

Musing 72.3 Here is another math sentence about a number being called r. Knowing nothing about r, the sentence is currently neither true nor false.

$$r + 3 > 100$$

Describe all the possible numbers r could represent that would make this a true sentence.

Musing 72.4 Here's a math sentence about two numbers being called a and b.

$$ab = 0$$

Give some examples of numbers that a and b could represent to make this a true sentence.

[The given sentence could also be written $a \times b = 0$ using the multiplication symbol.
See Section 9 for the various ways people indicate two numbers being multiplied together.]

Musing 72.5 Have you noticed spots in my writing where I have <u>not</u> put in the proper punctuation around a math sentence? There is the convention in math writing not to put in punctuation if a math sentence is being "displayed," that is written on its own line, usually centered in the line.

73. Collecting Math Data

When presented with a math sentence that includes the names of numbers not explicitly specified, one feels compelled to think of values for those undeclared numbers that make the sentence true.

For example, in seeing the sentence

$$N + 3 = 10$$

with N the name of some unspecified value, we can't help but think: "N should be 7!"

> **Practice 73.1** Here is a sentence about an unspecified number M.
>
> $$(M - 2) \times (M - 8) \times (M - 100) \times (M - 42\frac{1}{2}) = 0$$
>
> Think of some values for M that would make the sentence true.

People call a symbol, letter, or name that represents a number a **variable**. It's a scary-sounding word. But it comes from the idea that the value the symbol represents could vary if the author of the sentence changes their mind about the number they are actually thinking of. (Or, maybe the math sentence comes from some physical experiment the author is conducting and each run of the experiment produces slightly different values for the symbols representing numbers.)

Those just reading a math sentence without any context might use the word **unknown** instead for the symbol representing an unspecified number.

And if a math sentence contains unknowns, it feels compelling to think of values for the unknowns that make the sentence true.

Collecting Data from Math Sentences

It is natural to collect from a math sentences the values for the unknown(s) in the sentence that bring truth.

For example, for the sentence

$$N + 3 = 10$$

there is just one data value to collect: Only the value 7 for N makes the sentence true.

There is a lot of data we can collect from this equation:

$$a \times b = 12$$

For example, we could have

a is 3 and b is 4 a is 4 and b is 3 a is 2 and b is 6

a is 12 and b is 1 a is $\frac{1}{2}$ and b is 24 (We don't have to stick with whole numbers!)

People usually organize the data they collect in tables.

a	b
3	4
4	3
2	6
12	1
$\frac{1}{2}$	24

This data table is certainly incomplete: there are an infinitude of values to be had that involve fractions and decimals making the statement $ab = 12$ true. And there are also negative data values we can add to the table such as a is -3 and b is -4.

Example: Collect all relevant data for the equation $x = 2$.

Answer: There is only one data value to collect. We need x to be the number 2 for this to be a true sentence.

Example: Describe all relevant data for the inequality $x \neq 2$.

Answer: There is an infinite amount of data to collect for this sentence: having x be any number but 2 will make the sentence true!

Practice 73.2 Back to this sentence about the unknown M.

$$(M-2)\times(M-8)\times(M-100)\times(M-42\frac{1}{2})=0$$

a) Do you have four data values for this sentence?
b) Why isn't 98 one of your data values?
c) Why isn't -2 one of your data values?
d) Convince me that your four data values are the only data values that make this sentence true.

Practice 73.3: Here is a compound sentence:

$$3<q<7$$

It reads "the value 3 is smaller than the value q, which, itself, happens to be smaller than the value 7." (So many words are condensed into those math symbols!)

a) Give six data values that make the sentence true.
b) How many data values are there that make the sentence true?

Here is another math sentence.

$$w^2 = b$$

The subject is an unspecified number w that is squared, the verb is "equals," and the object of the sentence is an unspecified number b.

Again, it is natural to collect values for w and b that make the sentence true. Using trial and error to do so is fine. (But if you have some judicious reasoning to use, go for it!)
Here's my start to a data table.

w	b
2	4
1	1
3	9
-3	9
0	0
$\frac{1}{2}$	$\frac{1}{4}$

We have that w is 3 and b is 9 is a line in the table because $3^2 = 9$ is a true statement.
I didn't put w is 16 and b is 4 in the table because $16^2 = 4$ is a false statement.

Practice 73.4: Start a data table for the inequality

$$s \cdot t < 0$$

Have at least six lines within your table.

MUSINGS

Musing 73.5 Which of these math sentences about two unspecified numbers a and b have an infinitude of data values that make the sentence true?

i) $a + b = 0$ ii) $a - b = 0$

iii) $a^2 + b^2 = 0$ iv) $a^2 - b^2 = 0$

Musing 73.6 Write down a math sentence about an unspecified number x that has no data values that make the sentence true.

Musing 73.7 Consider the math sentence

$$\frac{b}{b} = 1$$

Describe all the data values that make this sentence meaningful and true.

MECHANICS PRACTICE

Practice 73.8 Give all the data values for this math sentence:

$$36 = w^2$$

Practice 73.9 Draw a data table at least six lines long for this math sentence:

$$a \times b = 1$$

Practice 73.10 Find at least one data value for the inequality

$$2x + 3y \neq 5$$

Practice 73.11 Find at least one data value for the equality

$$2x + 3y = 5$$

Practice 73.12 Find all the data values for the math sentence

$$-x = -7$$

74. Visualizing Data: Graphs

Here's a story that is not true.

One time, some students and I conducted an experiment.

We were wondering if eating carrots has any effect on sleeping patterns. So, we decided to each eat some carrots before going to bed one night and note how many hours of sleep we each got for that night.

Here's the (fake) data we collected.

Number carrots eaten	Number hours of sleep
6	9
3	5
4	6
3	6
10	13
1	3
0	0
5	8

You can see that one student forgot to conduct the experiment and then could not sleep a wink out of guilt.

> **Question:** Are there other data values that stand out to you?

It's hard to see—literally see—if this data is indicating anything of note. Is there a connection between carrot-eating and sleep?

Can we make this data visual?

In our early grades, we represented numbers visually on a number line. It has become the convention to draw this line horizontally, with the positive numbers heading off to the right, negative numbers off to the left. We can put dots on the line to highlight certain numbers.

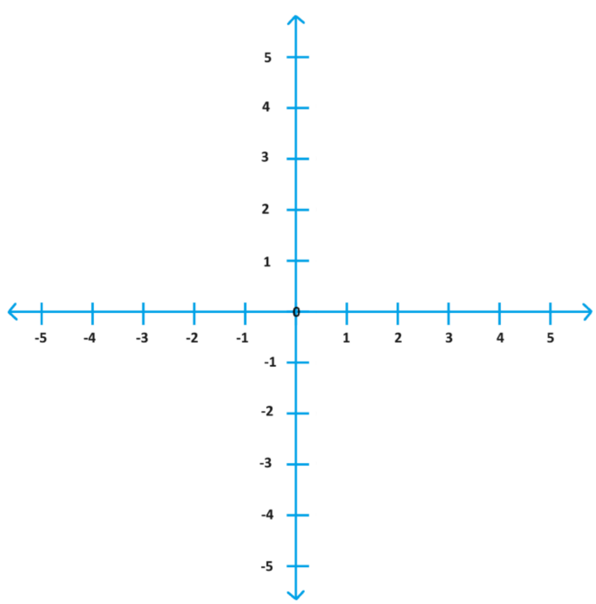
So, we could represent the various counts of carrots eaten with dots on one number line, and the counts of hours slept with dots on a second number line. That's one way to display our data.

But the two number lines should be linked in some way since the data is linked: 6 carrots eaten matches 9 hours of sleep, and so forth.

Hmm.

It took mathematicians a very long time to figure out a way to put two number lines together in a way that would help visualize linked data. In fact, the number line itself wasn't "invented," or seen as useful at least, until the mid-1600s when English mathematician John Wallis suggested using it as way to visualize basic arithmetic. French mathematician René Descartes at that time thought to put dots above and below a number line to start visualizing data from physical and geometric problems.

But it wasn't until a century later in the 1700s that scholars started drawing two numbers explicitly together. They kept one horizontal but made the second line vertical (with positive numbers going upwards). And they had the two lines cross at each of their zeros.



So, let's do this too.

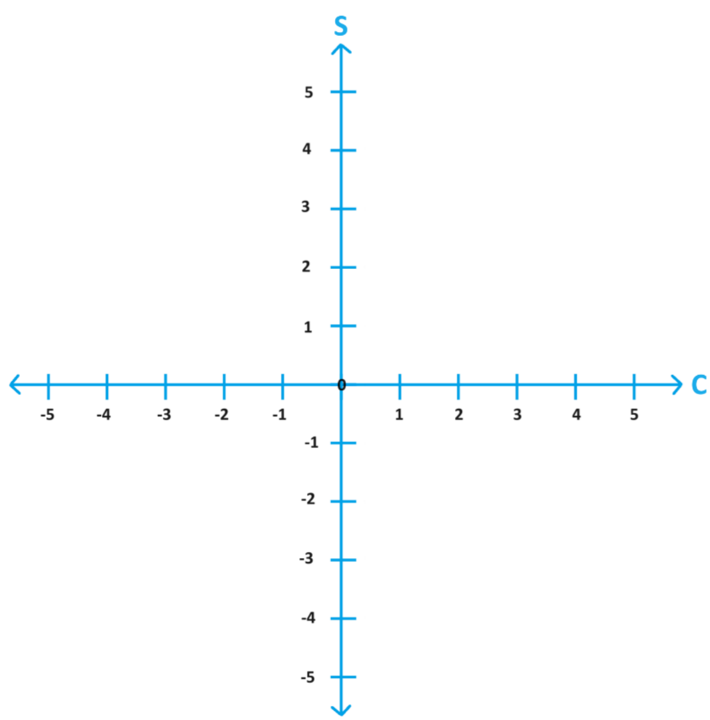
But now there's a question: Which number line should we use for counts of carrots, the horizontal one or the vertical one? And which number line should we use then for the counts of hours of sleep?

Mathematics won't care which one we choose for which, but scientists have come up with a convention:

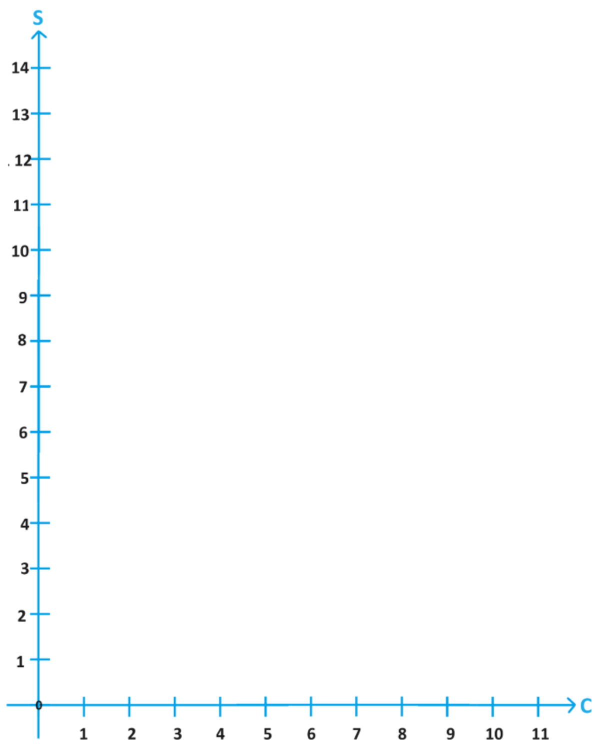
> In an experiment, you are usually in control of one quantity and are looking to see what the response shall be to various control values.
>
> **Convention**: Use the horizontal number line for the control data and the vertical number line for the response data.

In our experiment, students were in control of the number of carrots they ate. So, counts of carrots are our "control data" and we'll consequently use the horizontal number line to represent that data. The number of hours of sleep students got as a result is our "response data" and we'll use the vertical number line for that data.

Let's label our number lines C and S, for "carrots" and "sleep" to show this.



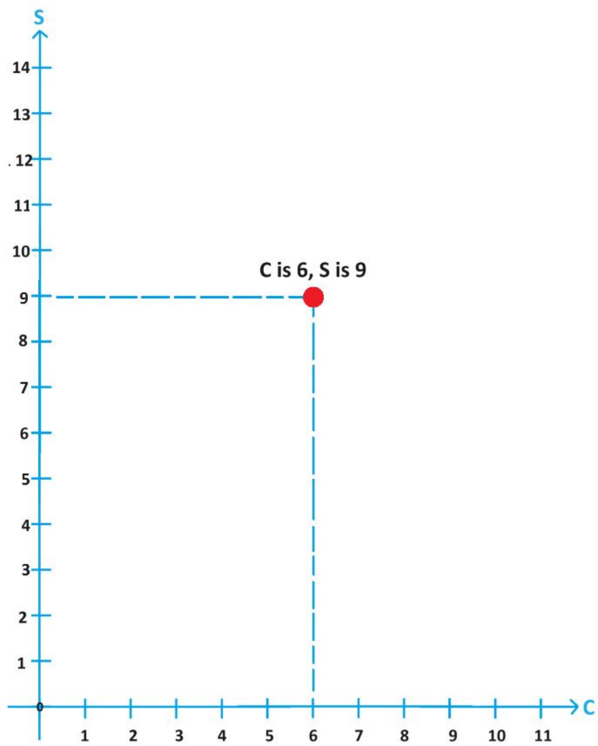
Actually, since our data never gives us negative number results, let's focus on just the positive numbers for each of our number lines.



Number carrots eaten	Number hours of sleep
6	9
3	5
4	6
3	6
10	13
1	3
0	0
5	8

Here’s how folk thought to make linked data visual using these two lines some 300 years ago:

> The first line of the data table has C as 6 and S as 9 (one student ate six carrots and got nine hours of sleep).
>
> To show this piece of data, look for the number 6 on the carrot number line. Then move 9 places vertically up from it and draw a dot at that point. (And notice that you can use the vertical number line for S to help identify that height of 9.)

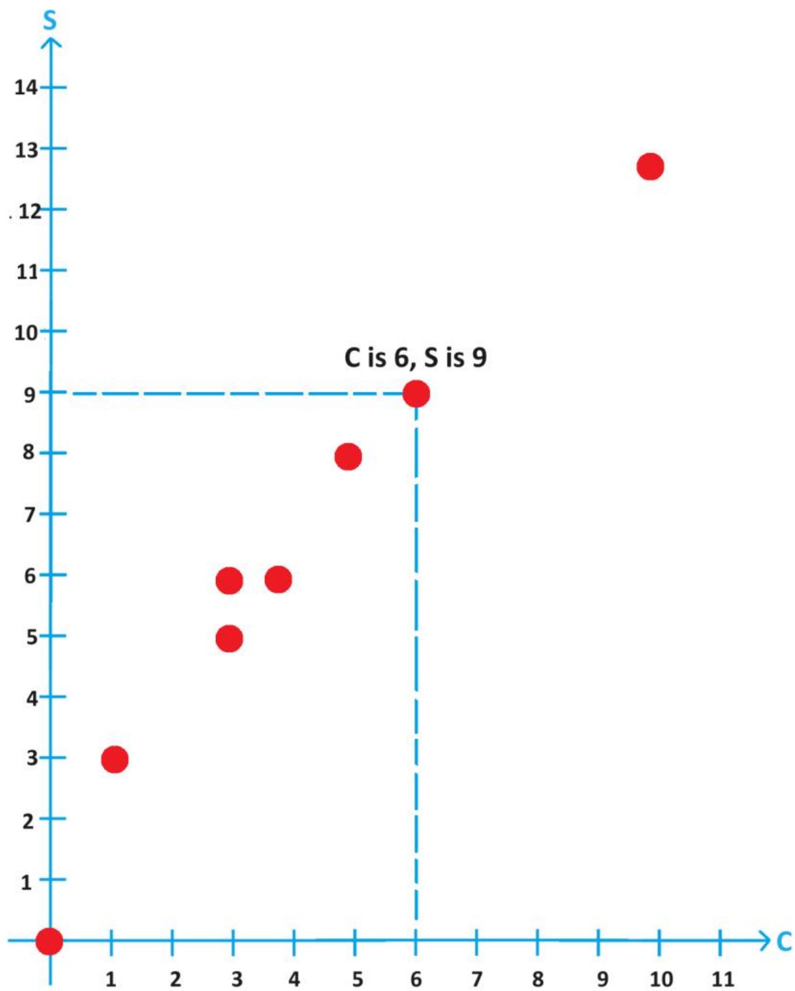


Number carrots eaten	Number hours of sleep
6	9
3	5
4	6
3	6
10	13
1	3
0	0
5	8

> Then do the same for the remaining piece of data.

Plotting the data piece C is 0 and S is 0 is interesting: you have to go “up” a height of zero from the number 0 on the carrot number line.

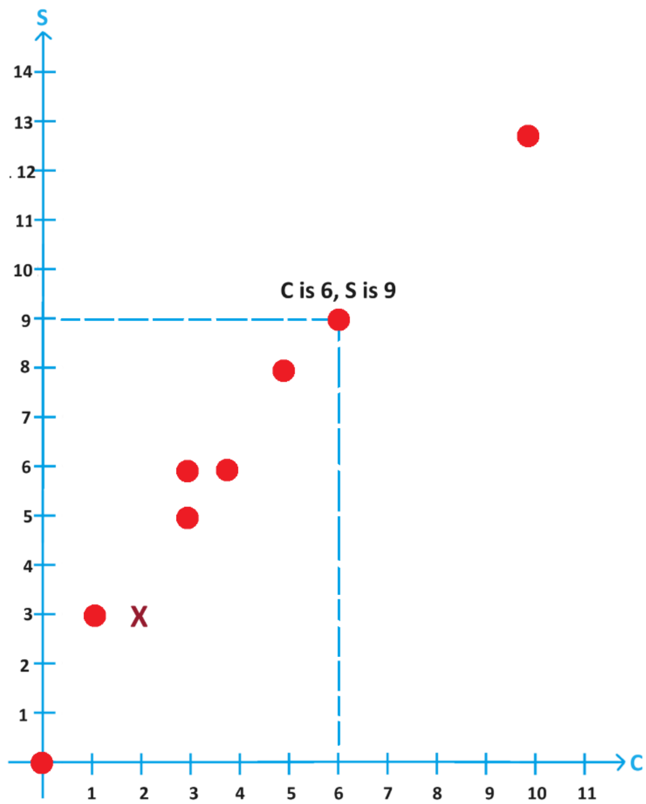
Here's the result of making the data table visual.



Number carrots eaten	Number hours of sleep
6	9
3	5
4	6
3	6
10	13
1	3
0	0
5	8

There does seem to be a trend: the more carrots you eat, the more hours of sleep you get!
The data is showing us that there is something worth investigating about the effect of evening carrot eating for sleep effects (assuming you believe this nonsense story about my students and me eating carrots).

Practice 74.1: A student says we omitted her data point. She marked it on the picture with an X. How many carrots did she eat and how many hours of sleep did she get?



Number carrots eaten	Number hours of sleep
6	9
3	5
4	6
3	6
10	13
1	3
0	0
5	8

We have just made a **graph** of our data. Some people might also call this a **scatter plot**.

Each piece of data is represented as a point, and you may have heard people describe a piece of data as a **data point**.

We had two number lines in our graph and data appears as points in a two-dimensional page.

Some More Jargon and Notation

Each number line in a graph is usually called an **axis**.

And rather label points as "C is 6, S is 9," which is cumbersome, people will write $(6, 9)$ with the first number mentioned in the set of parentheses the value of the control variable, the second the value of the response variable. We call $(6, 9)$ the **coordinates** of that particular data point.

It is unfortunate that mathematicians have settled on using parentheses in this context as they are usually used to represent "groupings" and order of operations as per Section 9. (But maybe folk were thinking that this is appropriate as data values are being grouped together?)

Practice 74.2: Which of the following points are <u>not</u> data points in our graph?

$(3, 5)$ $(5, 3)$ $(5, 6)$ $(5, 8)$ $(0, 0)$ $(4, 5)$ $(6, 4)$ $(4, 8)$

The point where the two axes (number lines) cross has coordinates $(0,\ 0)$.
People call this point the **origin**.

Now to something that sounds like it comes from a high-school algebra textbook.

Example: Graph the equation

$$a^2 = b^2$$

Let's take it slowly to unravel what exactly is being asked of us here.

To start, we see we have a math sentence. The subject of the sentence is a^2, an unspecified number that is squared, the verb is *equals*, and the object b^2, another unspecified number that is squared.

And no doubt we will want to obtain data values, values for a and b, that make this sentence true.

The command of the example is to "graph." The word is being used as a verb, not a noun, but the author of the question must surely mean: "make a graph."

Okay, So, we are to collect data and display it visually with a graph.

Next: What sorts of numbers for a and b make the sentence $a^2 = b^2$ true?

By trial and error, I got these examples.

a	b
3	3
2	2
-2	-2
-2	2
2	-2
0	0
1	1
-1.5	-1.5
-0.7	0.7

Check: Do verify that each of these data points do indeed make the math sentence true.

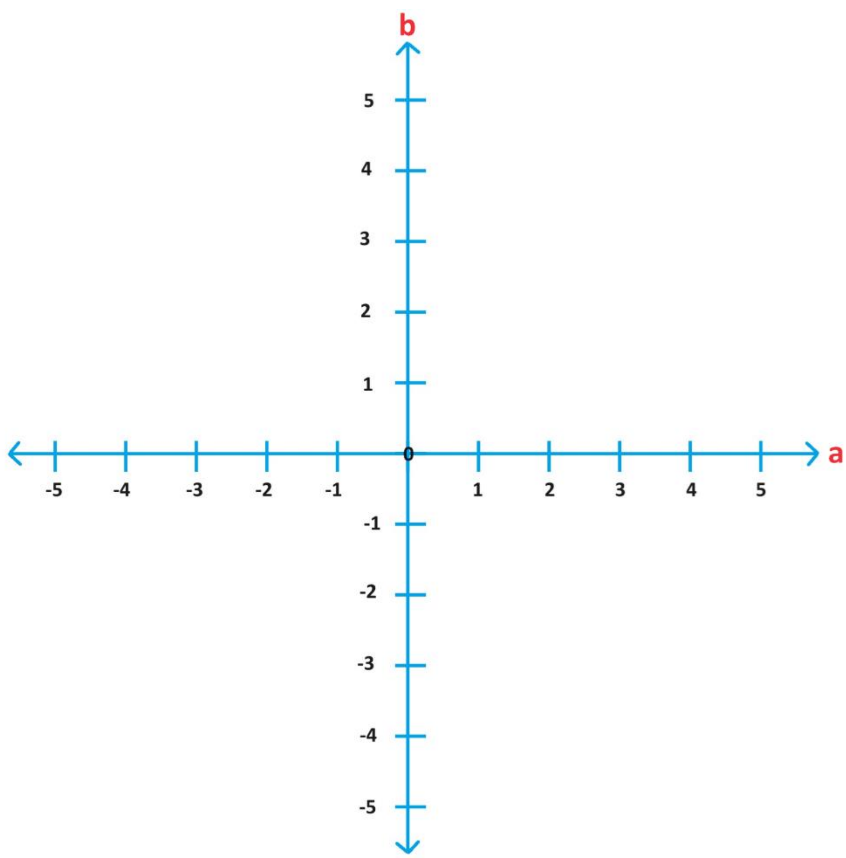
Let’s start creating a graph.

We’ll need one axis (number line) for the data representing a and a second axis for the data representing b. Let’s label the axes “a” and “b.”

But there’s a question: Which axis should be which?
Should we label horizontal axis “a” and the vertical one “b”? Or the other way round?

There is no information in the question to indicate if one of these variables is in “control” and the other is in “response.” So, we can choose to do whatever we wish.

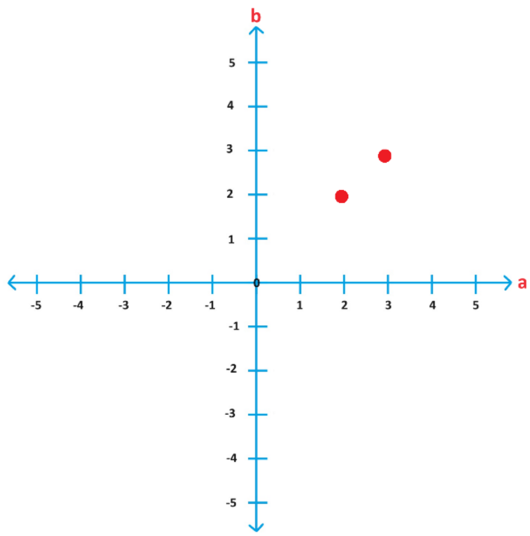
So, let’s label the horizontal axis “a” and the vertical one “b.” (Mathematics does not care.)



a	b
3	3
2	2
-2	-2
-2	2
2	-2
0	0
1	1
-1.5	-1.5
-0.7	0.7

Let’s now plot the data points.

The first two data points $(3,3)$ and $(2,2)$ are manageable, but I want to pause on the third one.

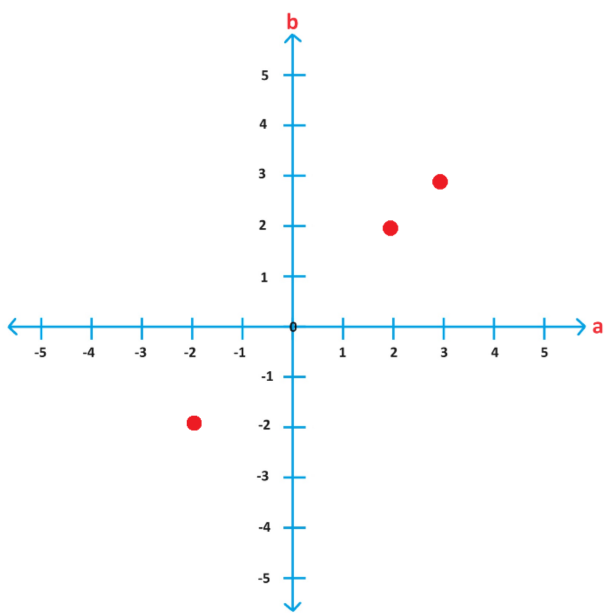


a	b
3	3
2	2
-2	-2
-2	2
2	-2
0	0
1	1
-1.5	-1.5
-0.7	0.7

Plotting $(-2, -2)$ requires finding -2 on the number line for a—not a problem—and then "going up a height of -2" from it. That sounds a bit strange.

But a negative height must be the opposite of a positive height and we go <u>downwards</u> rather than upwards. A point -2 units high, must be 2 units below.

We can plot $(-2, -2)$.

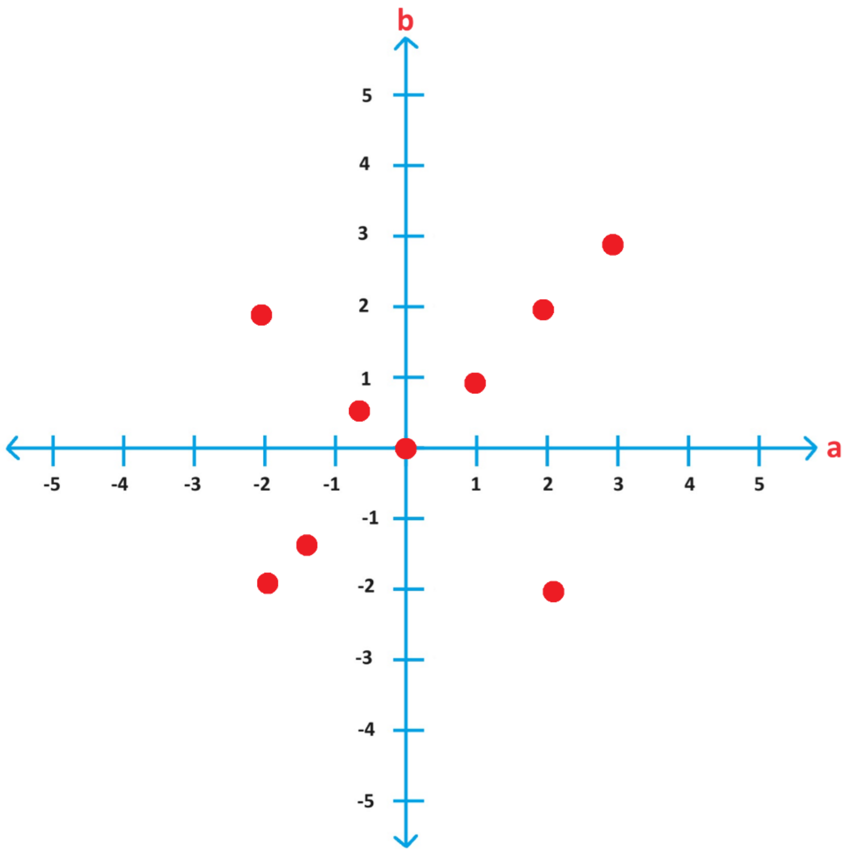


a	b
3	3
2	2
-2	-2
-2	2
2	-2
0	0
1	1
-1.5	-1.5
-0.7	0.7

Practice 74.3: Try plotting the remaining six data points before turning the page.

Here's what I got. Did you get the same picture?

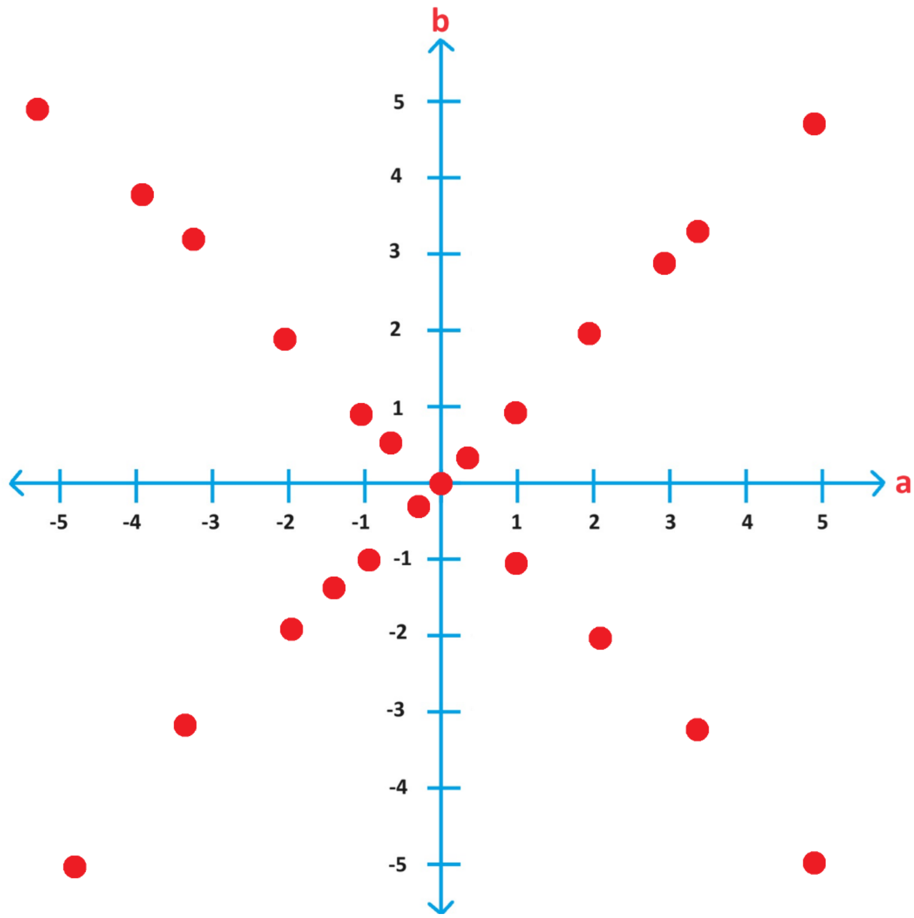
(One should really conduct this work on graph paper. I am currently eye-balling the proper locations of the data points!)



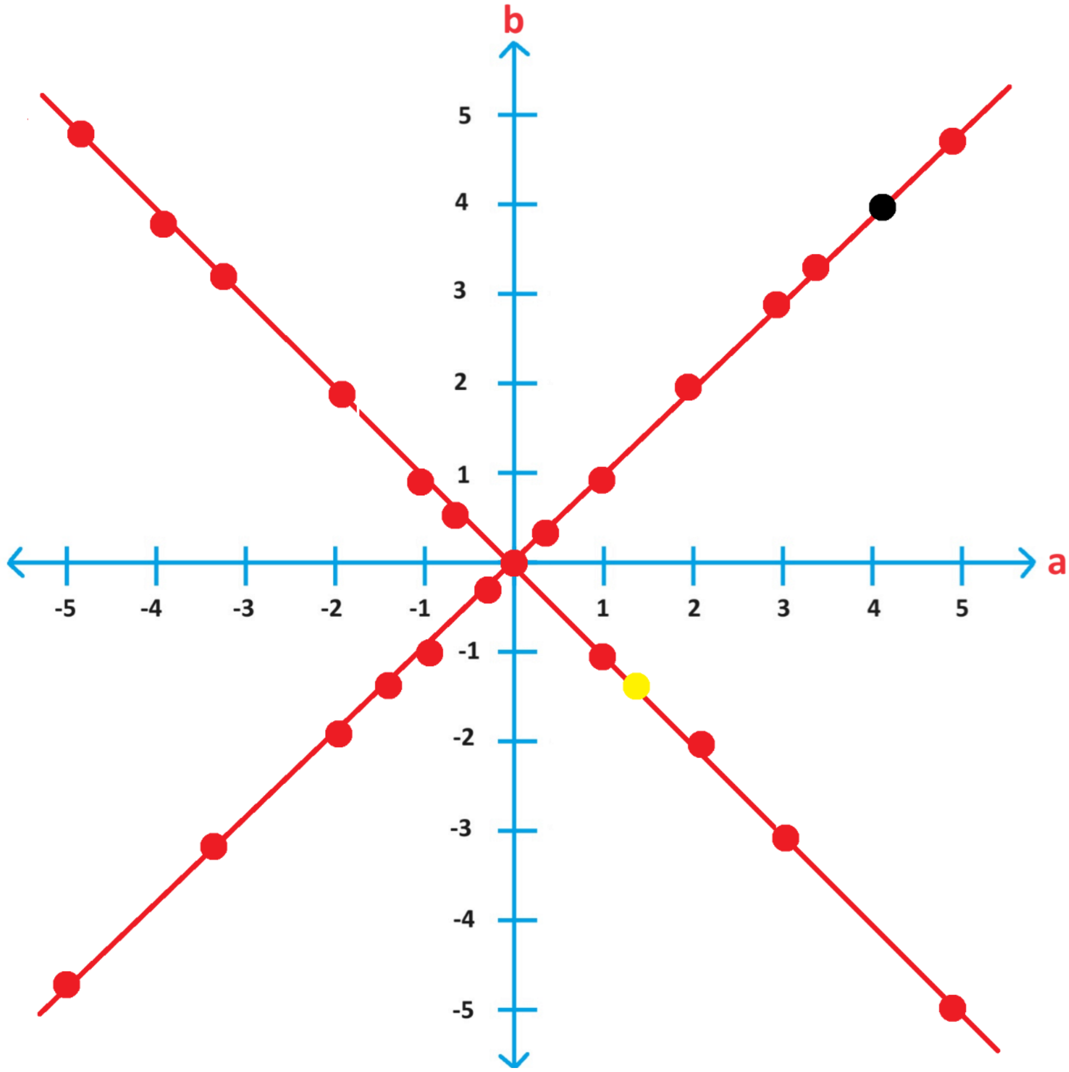
a	b
3	3
2	2
-2	-2
-2	2
2	-2
0	0
1	1
-1.5	-1.5
-0.7	0.7

Practice 74.4: Collect at least ten more data points that make the equation $a^2 = b^2$ a true sentence and plot those points as well on the picture above. Be sure to choose some data points with fractional or decimal values.

As I plot more and more data points, a compelling picture seems to be falling into place.



Actually, if we kept going and going and going with this, I can imagine a whole continuum of dots appearing to make a picture of big X for my graph.



And I can argue that this would be legitimate thing to conclude.

For example, I chose the black dot at random on one part of the X and it looks like it has coordinates $(4.1, 4.1)$. And this is a data value that makes $a^2 = b^2$ a true statement.

In fact, whenever we set a and b to have the same value, the statement $a^2 = b^2$ will be true, and it looks like every point on the north-east diagonal line provides values for a and b that are the same.

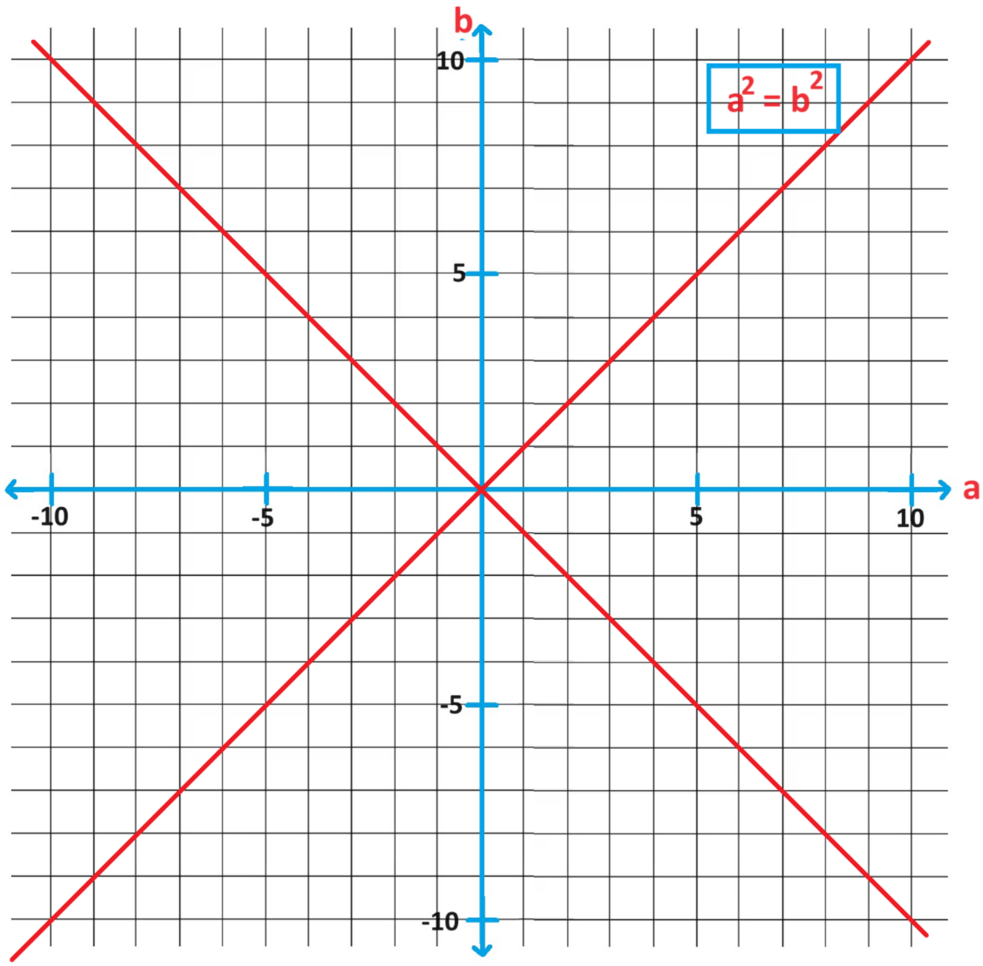
I also chose the yellow dot on the south-east diagonal line at random, it is looks like it has coordinates $(1.3, -1.3)$ and this too is a data value that makes $a^2 = b^2$ a true statement.

In fact, whenever we set a and b to have the same value but make one of the values negative and the other positive, the statement $a^2 = b^2$ will be true. Every point on the south-east diagonal line gives such data values.

The graph of the equation $a^2 = b^2$ really is a pair of lines making an X shape: every point on one of the lines corresponds to a data point that makes the sentence $a^2 = b^2$ true. And every data point that makes the sentence true lies on one of these lines.

The origin happens to lie on both lines.

Here's a picture of the graph properly drawn on graph paper.



Practice 74.5: What would the graph of the equation look like if we had instead labeled the horizontal number line as "b" and vertical one as "a"?

Practice 74.6: Consider this equation:

$$a = b + 6$$

Find some data values for this equation, plot the data points you find, and create the graph of this equation <u>on the same picture</u> as the one above.

What is special about the data point that are on both graphs simultaneously?

Practice 74.7: In each of the following, label the horizontal axis "x" and the vertical axis "y."

a) Sketch a graph of the equality $xy = 0$.

b) Sketch a graph of the inequality $xy \neq 0$.

c) Sketch a graph of the inequality $xy > 0$.

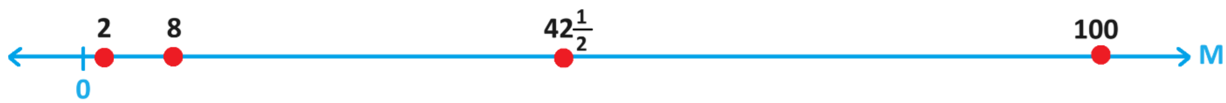
One-Dimensional Graphs

Consider this math sentence from the previous section.

$$(M-2)(M-8)(M-100)(M-42\frac{1}{2})=0$$

There are only four data values that make the sentence true: M must be 2, 8, 100, or $42\frac{1}{2}$.

To display this data visually we need only one number line. We can label it "M"and show these four values as points on it.



This picture is a graph of the equation.

Practice 74.8 Graph the equation $x = 2$.

A graph of the inequality

$$x \neq 2$$

would also be one dimensional and would show a number line labeled "x" that has every point but 2 shaded on it. This is tricky to draw.

Mathematicians have settled on the following drawing conventions:

- To indicate that a particular point is <u>not</u> part of the graph, draw an open dot at that point.
- If it helps to clarify that a particular point <u>is</u> part of the graph, use a solid dot at that point.
- To indicate that a region of points is part of the graph, shade the region. (The idea is to have it look like an infinitude of solid dots drawn throughout.)

Here's the graph of $x \neq 2$.

Practice 75.9 Graph the inequality $2 < x$.

The symbol $\leq$ means "less than, or possibly equal to."

Example: Graph the statement $-4 < p \leq 3$.

Answer:

Here's a tricky example.

Example: Graph the inequality $w^2 \geq 4$.

Answer: What sorts of values for w make this statement true? Certainly w could be 3 or 10 or 10.067, for instance. It could also be 2.

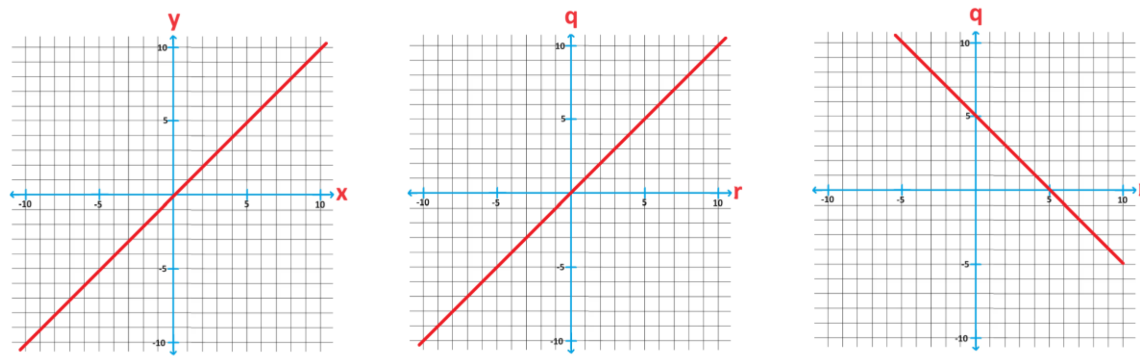
Actually, it could also be -2 or -3 or -10 or -10.067. (Do you see why?)

We have the following graph (with filled-in dots).

Practice 74.10 Graph $r^2 < 1$.

MUSINGS

Musing 74.11 For each of the following, give a mathematics statement whose graph could be as shown.



a) b) c)

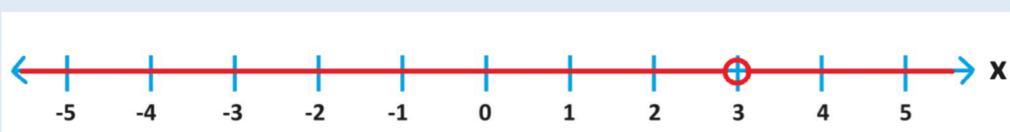
Musing 74.12 Let's look at some one-dimensional graphs.
a) Graph the equation $a = a$.
b) Graph the inequality $a \neq a$.
c) Graph the equation $\frac{a}{a} = 1$.

Musing 74.13 Sketch a graph of

$$x \; + \; 0 \times y \; = \; 3$$

(Most people would just write this sentence as "$x = 3$," but I want to point out that it is really a statement about two unknowns x and y to thus obtain a two-dimensional graph.)

Musing 74.14 Challenge
Give TWO math statements whose graph would match this picture.

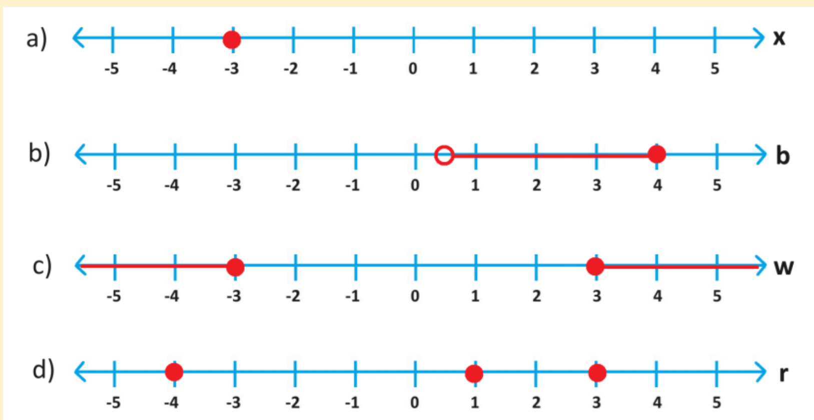


MECHANICS PRACTICE

Practice 74.15 Sketch (one-dimensional) graphs for each of these statements.

a) $x(x-3)(x+4)=0$
b) $5>p\geq 3$
c) $w^2=0$

Practice 74.16 Give a math statement whose graph would match each of these pictures.



Practice 74.17 Graph

$$2x+y=9$$

with the horizontal axis labeled "x."

Start by collecting data points in a table and plotting those. (Is there a systematic way to do this? Perhaps ask "If x is 1, then y would have to be …?" for instance. Don't forget fractional values too.)

Then imagine collecting more and more and more data points.
Would you have a continuum of data points in the graph?

75. What it means to Solve an Equation or Inequality

One hears all the time in algebra class, "Please solve …" (but often without the "Please"!)

Simply put, to **solve** a sentence about numbers is to identify all the data points that make the sentence true. We've been doing this already.

For example, the equation

$$w + 4 = 10$$

has only one value for w that would make this a true sentence, namely 6. We say that this equation has one only **solution**, namely that w must be 6.

The inequality

$$2 < x$$

has a whole range of solutions: all values larger than 2.

The equation

$$c^2 = 16$$

has two solutions, namely, c can be 4 or -4 to make this a true sentence.

We can express the solutions to a sentence about numbers in any way that successfully communicates to the reader what they are. For example, the inequality

$$ab > 0$$

has solutions:

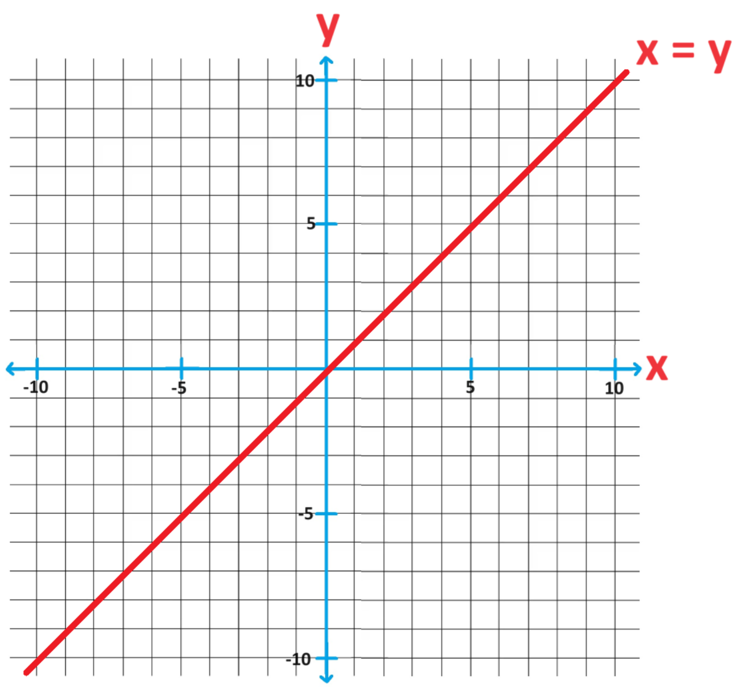
> *Any two values for a and b that are either both positive or both negative*.

There! We have just solved the inequality $ab > 0$.

> **Practice 75.1:** Solve the equation $x = y$.

A graph is a visual representation of all the solutions to a sentence about numbers. So, one could communicate the solutions of an equation or an inequality simply by presenting a graph of the equation or inequality.

This graph answers Practice 75.1. It's a picture of all the data points that have x and y the same value.



Here's an example that confuses many people, including math teachers.

> **Example:** Solve $\mathrm{x} = 3$.
>
> **Answer:** There is only one possible value for x that makes this a true sentence, namely, x must represent the number 3.

This is confusing to math folk because they have forgotten that a statement like $x = 3$ is a sentence that is technically neither true nor false as it stands. It is only when someone tells you the value for x they have in mind do we know whether or not the sentence is true.

And why have people forgotten this?

Because people do something that is a tad lazy and use an equal sign to express the solutions to a number sentence.

As an example of that I mean, consider the very first equation I presented in this section

$$w + 4 = 10$$

People might write this response to this

If $w + 4 = 10$, then $w = 6$

and stop there.

Actually, people will usually write even less

$$w + 4 = 10$$
$$w = 6$$

and skip all the in-between words.

And what they mean in these curt writings is the following:

> *The solutions to $w + 4 = 10$ are the same as the solutions to $w = 6$, and everyone knows what the solutions to this second sentence are.*

As another example, people might write

$$\textit{If } c^2 = 16, \textit{ then } c = 4 \textit{ or } -4$$

or just

$$c^2 = 16$$

$$c = 4 \text{ or } -4$$

These both stands for:

> *The solutions to $c^2 = 16$, are the same as the solutions to the sentence " $c = 4$ or -4," and it is clear what the solutions to this second sentence are.*

People have gotten into the habit of not bothering to write "and the solutions to this second sentence are clear," assuming it is understood. As a result, many have forgotten that sentences like "$w = 6$" and "$c = 4 \text{ or } -4$" are sentences that have solutions that need to be described too.

But you can see a process here.

The **art of solving a sentence about numbers** is to

> Take the given math sentence and turn into a new sentence which you are confident has exactly the same solutions as the original sentence, and whose solutions are blatantly clear and obvious to anyone who reads the new sentence.

And how do we do conduct such a transformation?

The process we use to do so was first developed by the Persian scholar al-Khwarizmi (ca. 780 – ca. 850) in his book "The Science of Restoring and Balancing." He described methods for converting one sentence about numbers into a new sentence without changing the solutions it has. He called his method *al-jabr* from the Arabic term for a "reunion of broken parts." From that term came the word **algebra** we use today.

(In addition to this, the word **algorithm**, which we use for any set of instructions for carrying out a task, is derived from his name!)

MECHANICS PRACTICE

Practice 75.2 Here's a (curt) piece of written mathematics.

$$(x-2)(x-3)(x+5)=0$$

$$x = 2\,,3\,,\text{or} -5$$

Write out in full what is being said (and understood to be said) here.

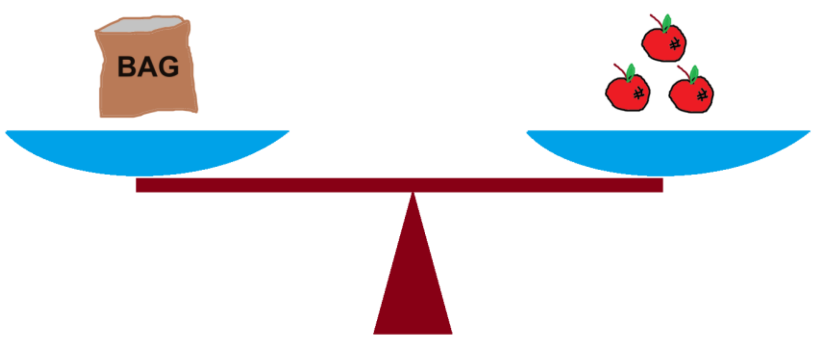
76. The Art of Balancing: What We Say is True about Equality

al-Khwarizmi evoked the idea of two quantities in a math sentence being in “balance” if the two quantities are deemed equal.

This provides lovely imagery that has been adopted by many math curriculums: two quantities are shown as “equal” if they balance on a simple two-pan balance.

For example, here’s a picture of a bag of apples balancing perfectly with three apples. What can we conclude?

[Let’s assume henceforth that all apples we discuss are of identical weight, that bags only contain apples, and that the weight of the material of any bag is immaterial compared to the weight of apples.]



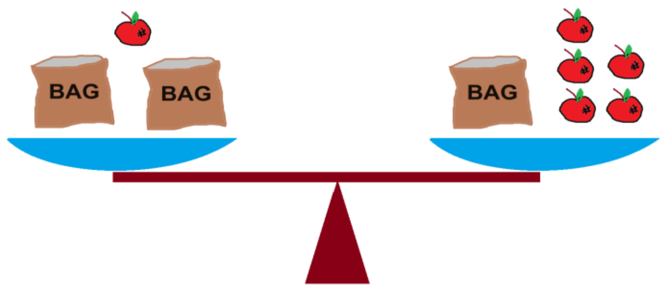
The picture represents the equation

$$BAG = 3$$

and the solution must be that “BAG” represents the number 3. That is, there must be three apples in the bag.

Let's use the imagery of "balance" to identify what we believe is true about equality.
And let's do that by playing with this more complicated balance picture.

(Assume all bags in this picture contain the same count of apples.)

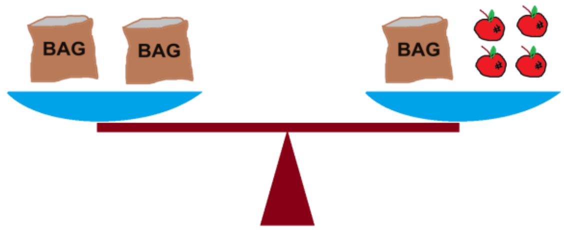


You can no doubt see that each bag must contain four apples for this picture to be true.
But that solution is less easy to see if we just look at the math sentence that describes the picture.

$$2\ BAG + 1 = BAG + 5$$

Let's make seeing that easy via al-Khwarizmi's method of *al-jabr*.

Our everyday experience suggests removing one apple from each side of the scale won't affect the balance of the scales and thus won't affect the truth of the situation. The number of apples in each bag that make our original picture true is precisely the same as the number of apples in each bag to make this slightly less complicated picture true.



In terms of math sentences, we've just converted the statement

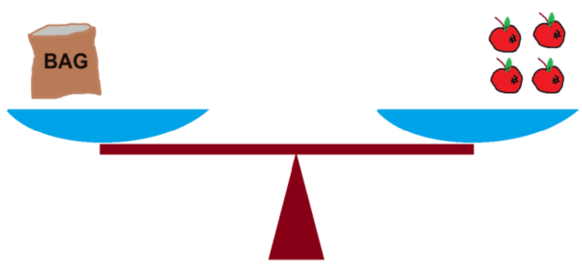
$$2\ BAG + 1 = BAG + 5$$

to the statement

$$2\ BAG = BAG + 4$$

believing that we haven't affected the truth of the statement and hence its solutions.

Since all bags are identical, we can also take a bag off of each side of the scale and not affect the truth of the picture.



This means we have just converted the statement

$$2\ BAG = BAG + 4$$

to

$$BAG = 4$$

without affecting the truth and the solutions.

From

$$BAG = 4$$

it is clear that BAG represents the number 4. There are four apples in the bag.

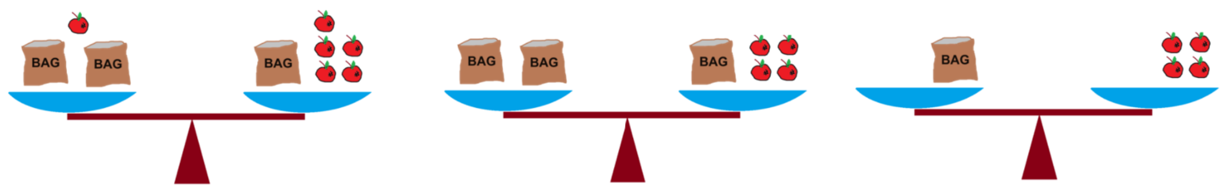
In summary: We turned the statement

$$2\ BAG + 1 = BAG + 5$$

into much simpler statement

$$BAG = 4$$

without affecting the truth along the way. They are equivalent statements. Any value for BAG that makes the first statement true, makes the second statement true too, and vice versa.



Jargon: Two math sentences about the same unknowns are **equivalent** if they have exactly the same solutions (that is, the same set of values that make each sentence true).

Algebra is the art of turning one math sentence into an equivalent sentence whose solutions are more readily seen.

As we worked through this example, we employed a feature we like to believe is true of equality:

> Suppose $A = B$ is a statement about equality of quantities.
>
> For any number k, the statements
>
> $$A = B$$
>
> and
>
> $$A + k = B + k$$
>
> are equivalent.

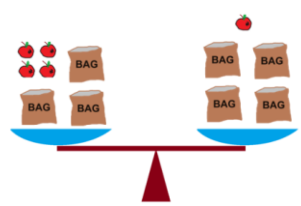
(If k is a negative number, then we're really making a statement about subtraction here.)

Adding an apple to each side of balance pan ($k = 1$) does not affect truth.
Subtracting an apple to each side of a balance pan ($k = -1$) does not affect truth.
Adding a bag to each side of a balance pan ($k = BAG$) does not affect truth.
And so on.

Example: Solve

$$4 + 3BAG = 4BAG + 1$$

Answer: The left side of the math sentence is $4 + 3 \times BAG$ and the right side is $4 \times BAG + 1$. We have this picture:



4 + 3BAG = 4BAG +1

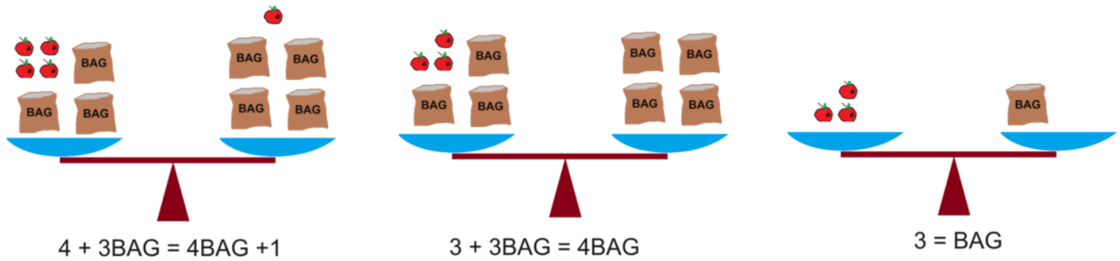
Subtracting 1 from each side of the equation gives

$$3 + 3BAG = 4BAG$$

Subtracting $3BAG$ from each side of the equation gives

$$3 = BAG$$

(Are you comfortable with "$4BAG - 3BAG$" giving "BAG"? See Sections 13 and 25.)



We have converted the sentence $4 + 3BAG = 4BAG + 1$ into the equivalent sentence $3 = BAG$.

It must be that BAG represents the number 3.

Here's the curt way to present this reasoning.

$$4 + 3BAG = 4BAG + 1$$

$$3 + 3BAG = 4BAG$$

$$3 = BAG$$

thus

$$BAG = 3$$

Students are usually required to annotate their work

$$4 + 3BAG = 4BAG + 1 \quad \text{(subtract 1 from both sides)}$$

$$3 + 3BAG = 4BAG \quad \text{(subtract 3BAG from both sides)}$$

$$3 = BAG$$

thus

$$BAG = 3$$

I need to point out two more things.

1. In the previous solution, I converted the sentence $3 = BAG$ into the sentence $BAG = 3$ without comment. This presumed another belief about equality.

> If $A = B$, then $B = A$, and vice versa.

Not a big deal, but I thought I should be explicit about this.

2. If a math expression, like $4 + 3BAG$, involves expressions sitting between $+$ and – signs and some of these pieces are just numbers and others involve unknowns, then it has become the convention to write the pieces that involve unknowns first and the pieces that are just numbers second.

Consequently, people prefer to $4 + 3BAG$ as

$$3BAG + 4$$

There is a seemingly contradictory convention that if a number and an unknown are multiplied together, then one should write the number first and the unknown second. (So, write $3BAG$ and not $BAG3$.)

Got that?

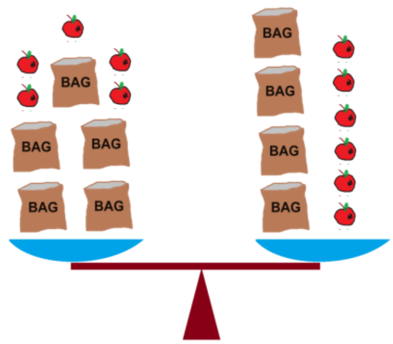
Mathematics doesn't care about any of this. It is just societal style thing.

Practice 76.1: For each picture below, write a math sentence that matches the picture.

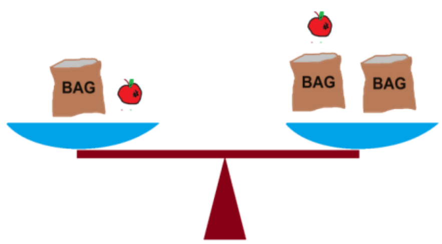
Then perform the steps of algebra to convert the sentence to an equivalent one that makes it blatantly clear how many apples must be in each bag. (Giving a curt-style presentation is fine.)

Imagine how the balance picture is changing as you perform each of your steps.

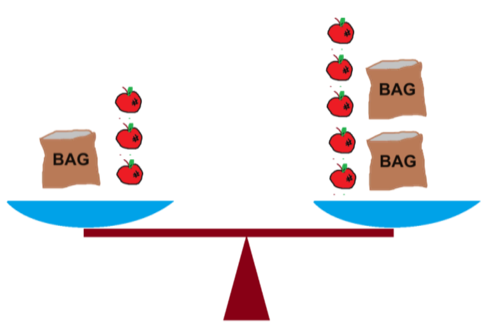
a)



b)



c)

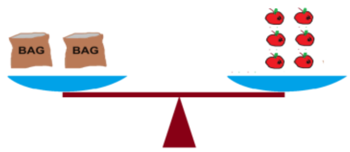


(Anti-apples?)

Here's a picture and its matching sentence.

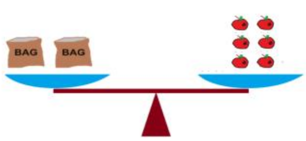
BAG = 3

Let's double the quantities on each side of the pan-balance scale. Commonsense tells us that this won't change the truth of the picture.

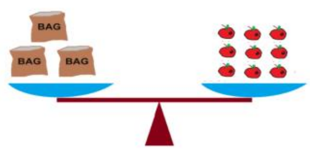


2BAG = 6

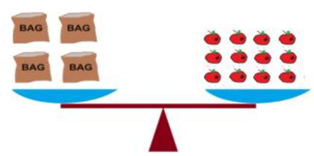
We could triple or quadruple the quantities on each side and still have no effect on the truth of matters.



2BAG = 6



3BAG= 9



4BAG = 12

We can even multiply quantities on each side of the scale by fractions and preserve truth. For example, multiplying each side of "$4BAG = 12$" (the right picture) by $\frac{1}{2}$ gives "$2BAG = 6$" (the left picture.)

We have a second belief about sentences that are equations.

> Suppose $A = B$ is a statement about equality of quantities.
>
> For any non-zero number k, the statements
>
> $$A = B$$
>
> and
>
> $$k \cdot A = k \cdot B$$
>
> are equivalent.

(See chapters 4 and 5 to see that multiplying a quantity by a fraction of the form $\frac{1}{n}$ is the same as dividing that quantity by n. There really is no such thing as division. It's just multiplication by a fraction.)

Practice 76.2: Why shouldn't k be zero in this belief?
Could two statements of the form $A = B$ and $0 \cdot A = 0 \cdot B$ have different solution sets?

Actually, let me answer this practice question.

Consider, for instance, the math sentence

$$x = 7$$

This sentence has just one solution: x must be 7 for the sentence to be true.

Now consider the sentence

$$0 \cdot x = 0 \cdot 7$$

This sentence is true no matter what number x represents: multiplying any number by zero gives zero in all cases.

The sentence $x = 7$ has solution set: the single number 7.
The sentence $0 \cdot x = 0 \cdot 7$ has solution set: the set of all numbers.

Multiplying each side of math sentence by zero gives a sentence that is suddenly true for all values for the unknowns.

Another issue: Do you believe in anti-apples?

I snuck some in for practice problem 76.1, but I am not sure if you liked that.

The solution to 76.1 c) requires each bag to contain two anti-apples. If each (proper) apple exerts a downward force due to gravity, each anti-apple does the opposite and wants to float upward by that same amount of force (so that an apple and anti-apple together have zero combined effect).

But, of course, making such "factual" claims is pointing to the absurdity of trying to make mathematics apply, in full, to any one physical model. As we saw in Part 1, physical models can inspire mathematics, motivate mathematics, and provide intuition for some aspects of mathematics—but not all of it. Each model starts to become "absurd" when pushed beyond its natural parameters.

Nonetheless, given that we do have negative numbers in our mathematical universe, it is natural to test our stated belief and decide if it should extend to negative numbers as well.

For example, taking $k = -1$, does it seem right that $A = B$ and $-A = -B$ are equivalent statements?

Practice 76.3
a) What are the solutions to $r = 3$? What are the solutions to $-r = -3$?
b) What are the solutions to $x = y$? What are the solutions to $-x = -y$?

The solution sets for each pair of equations in this problem simply must be the same. And here's why.

Example: Explain, in general, why $A = B$ and $-A = -B$ are equivalent statements. (Here A and B each represent expressions that involve numbers and unknowns.)

Answer: Start with $A = B$ and add the number $-A$ to each side.

$$A = B$$

$$A + -A = B + -A$$

This is

$$0 = B + -A$$

Now add $-B$ to each side.

$$0 + -B = B + -A + -B$$

$$-B = -A$$

We can rewrite this as $-A = -B$.

So, believing that adding the same number to each side of an equation does not alter truth forces us to conclude that $A = B$ and $-A = -B$ are two equivalent equations.

So, it does not matter whether or not you believe in and want to play with apples and anti-apples. Mathematics offers a way to move forward and find solutions to equations that involve negative quantities nonetheless.

Okay, let's practice some *al-jabr*.

Example: Solve

$$5b + 2 = 2b + 14$$

(I am getting tired of writing BAG. I am just writing b now.)

Answer: Let's add -2 to both sides of the equation. (That is, if I am picturing bags and apples on a balance scale, let's remove two apples from each side.)

$$5b + 2 = 2b + 14$$

$$5b + 2 + -2 = 2b + 14 + -2$$

$$5b = 2b + 12$$

Let's now add $-2b$ to each side of the equation. (In my mind and removing two bags from each side of the balance scale.)

$$5b + -2b = 2b + 12 + -2b$$

$$3b = 12$$

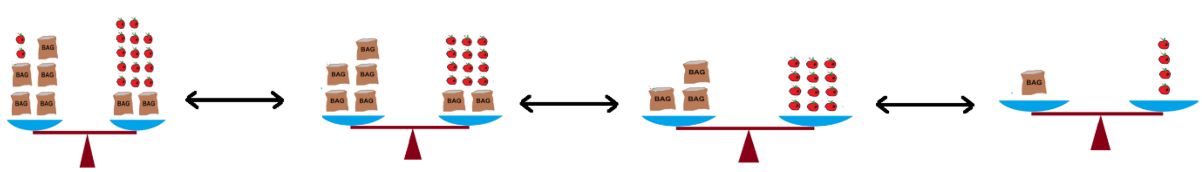
Let's now multiply each side of the equation by $\frac{1}{3}$. (That is, let's scale down each side of the balance scale by a factor of three.)

$$\frac{1}{3} \times 3b = \frac{1}{3} \times 12$$

$$\frac{1}{3} \times 3 \times b = \frac{1}{3} \times 3 \times 4$$

$$b = 4$$

Thus, the equation $5b + 2 = 2b + 14$ is equivalent to the equation $b = 4$.



For truth, b must be the number 4.

Curt, Wordless, Makes-Math-Seem-Inhuman-and-Scary Presentation:

$5b + 2 = 2b + 14$

$5b + 2 - 2 = 2b + 14 - 2$

$5b = 2b + 12$

$5b - 2b = 2b + 12 - 2b$

$3b = 12$

$\frac{1}{3} \times 3b = \frac{1}{3} \times 12$

$b = 4$

Remember that technically the final line here is not a solution – it's just another equation with the same solution set as the original equation. One should write a final sentence along the lines "And so b having the value 4 is the solution to the original equation," but the practice is to omit such a final sentence and assume it is understood.

Example: Solve

$$19z + 2 = 17z + 3$$

(If I am still thinking about bags, I guess I am labeling them "z" now.)

Answer: You've probably sensed a general strategy:

1. Add or subtract a number to each side of the equation to reduce the number of pieces in the math sentence that are numbers.

2. Add or subtract some number of the unknown to each side of the equation to reduce the number of pieces in the math sentence that involve the unknown.

3. Follow your nose from there.

Here goes:

$$19z + 2 + -2 = 17z + 3 + -2$$

$$19z = 17z + 1$$

Now let's work the pieces involving the unknown:

$$19z + -17z = 17z + 1 + -17z$$

$$2z = 1$$

I can see that for this equation to be true, z better be $\frac{1}{2}$, and I can stop here having said that.

But … if you are giving a curt, wordless presentation, then you might want to go a little further and add these two lines

$$\frac{1}{2} \times 2z = \frac{1}{2} \times 1$$

$$z = \frac{1}{2}$$

Example: Solve

$$2 - 2w = w + 6$$

Answer: Here's how my brain went with this:

$$2 - 2w + -2 = w + 6 + -2$$
$$-2w = w + 4$$
$$-2w + 2w = w + 4 + 2w$$
$$0 = 3w + 4$$
$$0 + -4 = 3w + 4 + -4$$
$$-4 = 3w$$
$$\frac{1}{3} \times (-4) = \frac{1}{3} \times 3 \times w$$
$$-\frac{4}{3} = w$$

so

$$w = -\frac{4}{3}$$

(See Chapters 5 and 6 to be clear on this fractions work.)

MUSINGS

Musing 76.4 What do you believe about "not equal to"?

What would you say is/are the solution(s) to the following inequality?

$$5x + 7 \neq x + 5$$

As you work through this, imagine a pan balance with apples and bags of apples (each labeled "x") on each side but <u>not</u> in balance. As you add and remove apples and bags, does it feel right to say that the pan-balance is still out of kilter? Is this still the case if you scale the contents of each pan by a non-zero number k?

MECHANICS PRACTICE

Practice 76.5 Solve each of the following equations. (Giving curt presentations of your work is fine.)

a) $2w = -4$

b) $19z + 2 = 17z - 3$

c) $8x + 7 = 5x + 31$

d) $2p + 1 = 12p$

e) $3R + 5 + 2R + 9 = 4R + 22$

77. Another Algebraic Move

In Part 1 we established ten basic rules of arithmetic—and their logical consequences. (See the **Appendix**.)

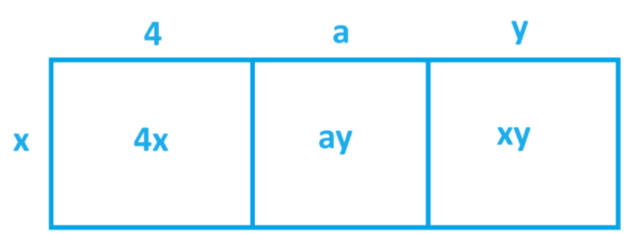
We understand these rules to speak truth in, and of, themselves.

For example, we believe we can change the order of the sum of two numbers without contradiction (**Rule 1**):

> $a + b = b + a$ *is a true sentence no matter which two numbers* a *and* b *represent.*

and we believe the distributive rule (a.k.a. "chopping up rectangles," **Rule 8**). For instance:

> $x(4 + a + y) = 4x + ax + xy$
>
> *is a true sentence no matter which numbers* x, y *and* a *represent.*



It seems right to believe that applying a fundamental rule of arithmetic to any part of a math sentence does not change the truth, and hence the solutions, of that sentence.

For example, the sentences

$$3 + 2w + 5 = w - 7$$

and

$$2w + 8 = w - 7$$

are equivalent math sentences.

Why? Because Rule 3 shows that we can add a string of addition in any order we like, so $3 + 2w + 8$ can be deemed no different than $2w + 3 + 5$, which in turn, is no different than $2w + 8$.

Example: Kindly solve

$$3d + 2 - d = 4(d - 1)$$

Answer: Let's try applying some basic rules of arithmetic first to portions of this sentence.

For starters, the expression $3d + 2 - d$ is no different than $d + d + d + 2 + -d$, which is just $2d + 2$.

Also, $4(d - 1)$ is the same as $4(d + -1)$, which is $4d - 4$.

So, our given math sentence is equivalent to the sentence

$$2d + 2 = 4d - 4$$

This looks like the type of example we solved in the last section.
Adding 4 to each side of the equal sign gives the equivalent sentence

$$2d + 6 = 4d$$

Adding $-2d$ to each side of the equal sign then gives

$$6 = 2d$$

Multiplying each side of the sentence by $\frac{1}{2}$ gives

$$3 = d$$

The solution to the original math sentence is that d must be the number 3.

Curt Presentation:

$3d + 2 - d = 4(d - 1)$

$2d + 2 = 4d - 4$ (arithmetic)

$2d + 2 + 4 = 4d - 4 + 4$

$2d + 6 = 4d$ (arithmetic)

$2d + 6 + -2d = 4d + -2d$

$6 = 2d$ (arithmetic)

$\frac{1}{2} \times 6 = \frac{1}{2} \times 2d$
$3 = d$ (arithmetic)

So,

$d = 3$

Example: Please solve

$$2(x - 3) + 2 = 2(x + 3)$$

Answer:

$2(x - 3) + 2 = 2(x + 3)$
$2x - 6 + 2 = 2x + 6$ (arithmetic)
$2x - 4 = 2x + 6$ (arithmetic)
$2x - 4 + -2x = 2x + 6 + -2x$
$-4 = 6$ (arithmetic)

The original math sentence is equivalent to a math sentence that is never true.
There are no solutions to the given equation. That is, there are no values for x that could make the sentence true.

Some people will say "The solution set is empty."

Example: Please solve

$$2(x - 3) + 12 = 2(x + 3)$$

Answer:

$2(x - 3) + 12 = 2(x + 3)$
$2x - 6 + 12 = 2x + 6$ (arithmetic)
$2x + 6 = 2x + 6$ (arithmetic)
$2x - 4 + -2x = 2x + 6 + -2x$
$6 = 6$ (arithmetic)

The original math sentence is equivalent to a math sentence that is always true, irrelevant to whatever value x may be.
Every number is a solution to the given equation.

Some people will phrase this as "The solution set is the set of all numbers."

MUSINGS

Musing 77.1 Are you a logic purist? Did you realize that we have already been using the idea of this section in the previous section? (This means I should have presented this section of text first!)

For example, in Section 76, I presented this example:

> **Example:** Solve $5b + 2 = 2b + 14$
> **Answer:**
> $5b + 2 = 2b + 14$
> $5b + 2 - 2 = 2b + 14 - 2$
> $5b = 2b + 12$ (arithmetic)
> $5b - 2b = 2b + 12 - 2b$
> $3b = 12$ (arithmetic)
> $\frac{1}{3} \times 3b = \frac{1}{3} \times 12$
> $b = 4$ (arithmetic)

Do you see that in going from the second line to the third we used the idea that $5b + 2 - 2$ is no different than $5b$ (by the rules of arithmetic) and that $2b + 14 - 2$ is no different than $2b + 12$. And so on throughout this answer, and throughout the entire Section.

Musing 77.2 Make up a complicated, scary-looking math sentence using the unknown y, whose solution is that y must be 1.

MECHANICS PRACTICE

Practice 77.3 Kindly solve each of these equations.

a) $6m + 5 - 3m + 4 + 7m = 4 + 3m + 2 + 8m + 2$

b) $2(t + 2) - 3t = 2(t + 1)$

c) $4(w - 5) - 5(w - 4) = w - 6$

d) $p(p + 5) = 5p + 36$

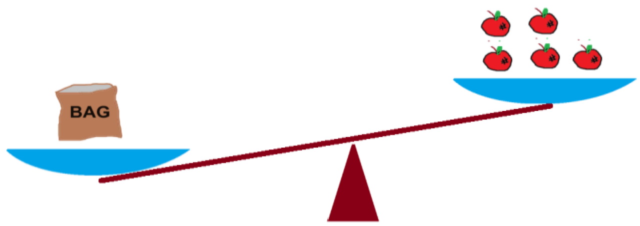
e) $3(x - 3) + 9 = 3x + 1$

f) $z = z$

78. What We Say is True about Inequality

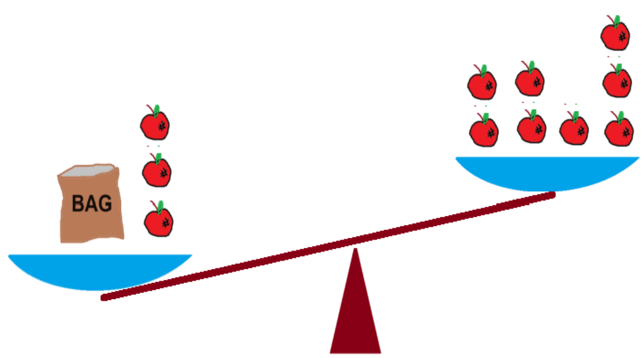
Here is a picture of a balance-scale not in balance.

$$BAG > 5$$



For the picture to be true, the bag must contain more than five apples. That is, the solution set the inequality $BAG > 5$ is the set of all numbers greater than 5.

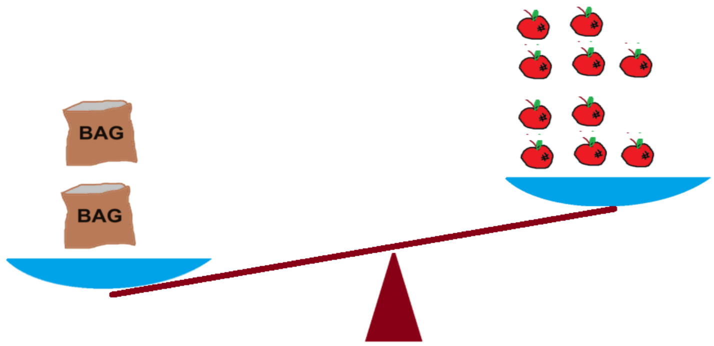
What happens if we add, say, three apples to each pan on each side of the balance scale? Common sense tells us that the scale will again be unbalanced and tilted in the same direction as before.



It seems that the sentences $BAG > 5$ and $BAG + 3 > 5 + 3$ are equivalent statements. They have the same solutions, namely, that the unknown in each case must be a number greater than 5.

What if, instead, we doubled the quantities on each side of the balance scale?

Common sense tells us that the balance pan will again be titled and in the same direction as before.



It seems that the sentences $BAG > 5$ and $2 \times BAG > 2 \times 5$ are equivalent statements too, each having the same solutions of requiring the unknown to represent a number larger than five.

The same would be the case if we tripled, quadrupled, or centupled the quantities on each side of the balance scale: the scale will remain tipped in the same direction. We can even scale each quantity by a fractional amount, the tilt of the scale will not change.

It is unclear, however, if we change the quantities on each side of the balance scale by a negative factor. This will require thinking through the meaning of “anti-apples” and “anti-bags.” and it is not clear if they are even meaningful!

At the very least, we have:

Suppose $A < B$ is a statement about the inequality of two quantities.

For any number k, the statements

$$\boldsymbol{A < B}$$

and

$$\boldsymbol{A + k < B + k}$$

are equivalent.

If k is <u>positive</u>, then we have a third equivalent statement.

$$\boldsymbol{k \cdot A < k \cdot B}$$

This observation holds too for the inequalities $\leq$, $>$, and $\geq$.

Question: Does it feel right to you to say that $A \geq B$ and $A + k \geq B + k$ are equivalent statements for a number k?

If k is a positive number, does it feel right to you that we actually have three equivalent statements?

$$A \geq B$$

$$A + k \geq B + k$$

$$k \cdot A \geq k \cdot B$$

(Remember, $A \geq B$ means that the quantity A is larger, or possibly equal to, the quantity B.)

Practice 78.1 Draw lines in this picture to connect pairs of statements that are equivalent.

$\boxed{B < A}$ $\boxed{B > A}$

$\boxed{A > B}$

$\boxed{A < B}$

$\boxed{A \geq B}$ $\boxed{A \leq B}$

$\boxed{B \leq A}$ $\boxed{B \geq A}$

Example: Please solve

$$3w - 2 \leq 16$$

Answer:

$$3w - 2 \leq 16$$

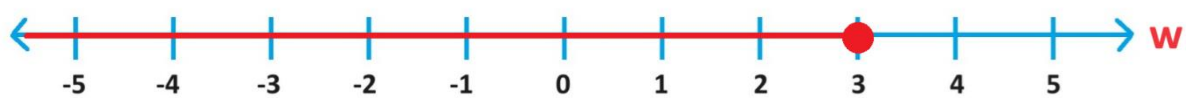
$$3w - 2 + 2 \leq 16 + 2$$

$$3w \leq 18$$

$$\frac{1}{3} \times 3w \leq \frac{1}{3} \times 18$$

$$w \leq 3$$

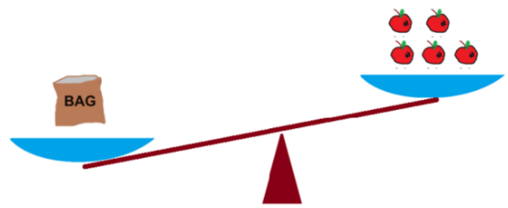
Solution set: All numbers less than, or possibly equal to, 3



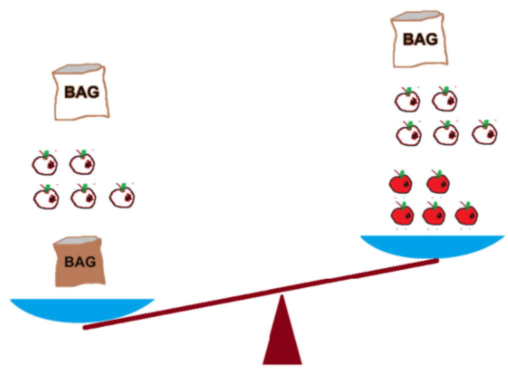
Practice 78.2: Solve $4s + 7 > 2s + 5$ and present the solution set as a graph.

Practice 78.3: Let's try playing with anti-apples and anti-bags.

Here's a picture of $BAG > 5$.



Let's now add five anti-apples and one anti-bag to each side of the pan balance.



Is there something to deduce from this picture?

The approach of Practice problem 78.3 suggests a means to understand inequalities that involve negative quantities (without resorting to “anti” shenanigans).

We know that

$$\boldsymbol{A < B} \qquad \text{and} \qquad \boldsymbol{A + k < B + k}$$

are equivalent statements for any number k. Well, the expressions A and B themselves, even if they contain unknowns, do represent numbers—we just might not be privy to what numbers they actually are.

Let’s choose k to be the number $-A + -B$.

Consequently, the statement

$$A < B$$

is equivalent to

$$A + -A + -B < B + -A + -B$$

Tidying this up, it reads

$$-B < -A$$

We could rewrite this as

$$-A > -B$$

if we like.

Suppose $A < B$ is a statement about the inequality of two quantities.

Then

$$\boldsymbol{A < B}$$

and

$$\boldsymbol{-A > -B}$$

are equivalent statements.

Similar observations hold too the inequalities $\leq$, $>$, and $\geq$.

Practice 78.4: Does this feel right to you?
For instance, we know that

$$3 < 5$$

From what we've just established, it must be that

$$-5 < -3$$

Do you agree that -5 is "less than" -3?

a) Draw a number line and show the location of the points 3 and 5, and the location of the points -3 and -5 on it.

If a number to the left is always considered "less than" a number its right, is $3 < 5$ and is $-5 < -3$?

b) Recall from Section 50 that we say $a < b$ (read as "less than") if there is a positive number n so that

$$a + n = b$$

(that is, we need to "adding something to a to get up to the number b").

For example, $3 < 5$ because $3 + 2 = 5$.

i) Establish that $-5 < -3$ according to this definition.

ii) Show that $-13 < 100$ according to this definition.

iii) Establish that every negative number is "less than" zero.

iv) Suppose $a < b$. Then there is a positive number n so that $a + n = b$.

What is the value of $-b + n$?
Explain why we have $-b < -a$.

Practice 78.5: Establish that $A \geq B$ and $-2A \leq -2B$ are equivalent sentences.

Did you learn the following rule in school?

If you multiply an inequality through by a negative number, then you must flip the sign of the inequality.

Hopefully now you can see that no flipping is actually involved.

For example, to answer the practice problem, we have

$$A \geq B$$

$$A + -A + -B \geq B + -A + -B$$

$$-B \geq -A$$

Now multiply through by the positive number 2

$$-2B \geq -2A$$

Notice that the inequality sign has **not** "flipped."

It is only when we choose to rewrite $-2B \geq -2A$ as

$$-2A \leq -2B$$

does "flipping" seem to occur.

I often find it much clearer when working with an inequality to not multiply through by negative number: I just add quantities to each side of the inequality. That way I can keep the direction of the inequality straight.

Example: Kindly solve

$$3 - 2r \geq 3 - 2s$$

and graph the solutions (with "r" as the horizontal axis in the graph).

Answer: Let's start by subtracting 3 from each side of the inequality.

$$3 - 2r + -3 \geq 3 - 2s + -3$$

$$-2r \geq -2s$$

Let's now add $2r$ throughout

$$-2r + 2r \geq -2s + 2r$$

$$0 \geq -2s + 2r$$

and add $2s$ throughout.

$$0 + 2s \geq -2s + 2r + 2s$$

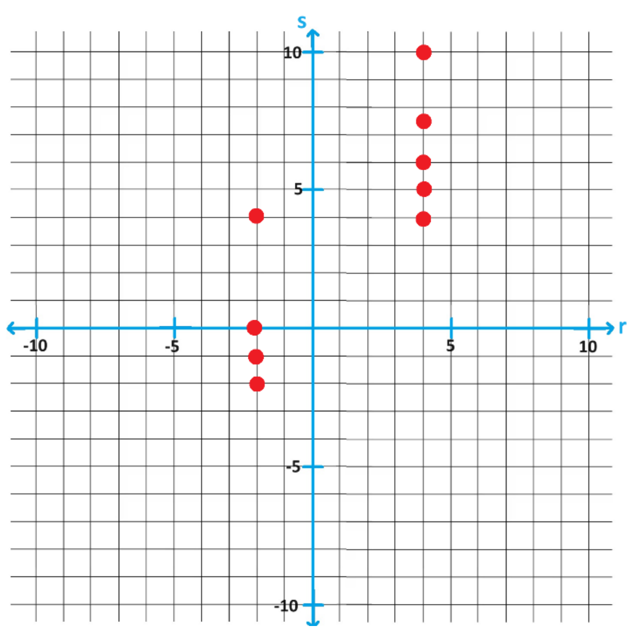
$$2s \geq 2r$$

Now multiply though by $\frac{1}{2}$ to obtain

$$s \geq r$$

The solution is the set of all values for s and r with s having a value larger than or equal to r.

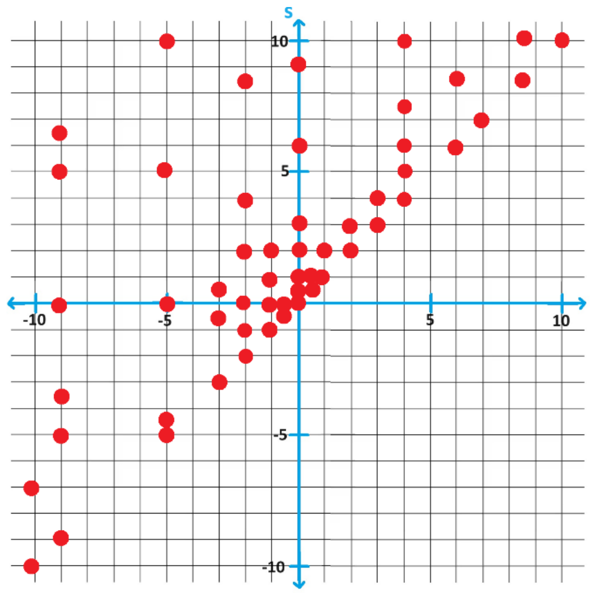
To graph this, let's collect some data values.



r	s
4	4
4	5
4	6
4	$7\frac{1}{2}$
4	10
−2	−2
−2	−1
−2	0
−2	4

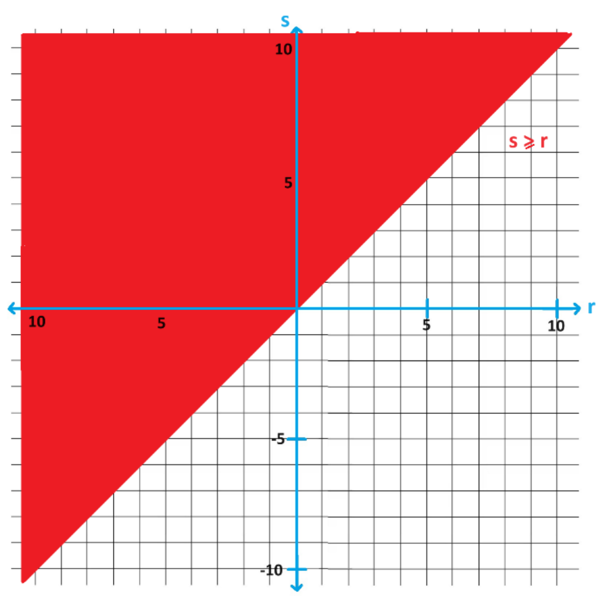
This is not enough data to see a meaningful picture.

Here's some more data points.



Every point along the northeast diagonal line has r and s equal in value, which is a valid data point. And every point vertically above a point on this line represents a point with s larger than r.

The graph of the solutions is a diagonal half of the entire plane.



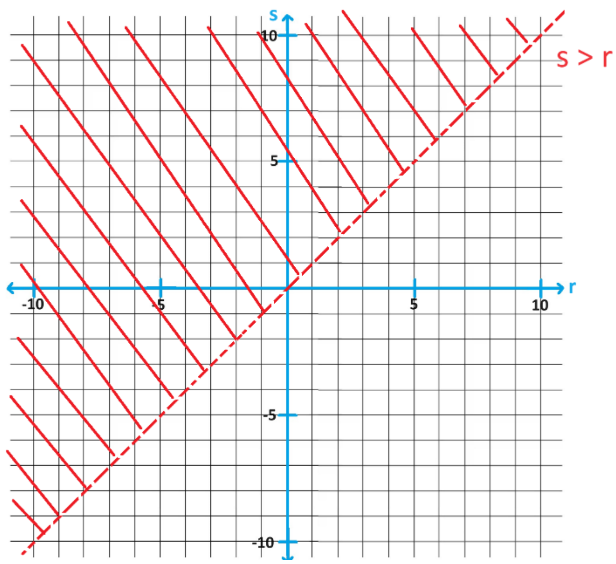
Practice 78.6

a) Would the graph of $3 - 2r \geq 3 - 3s$ be the same as the graph of $s \geq r$?

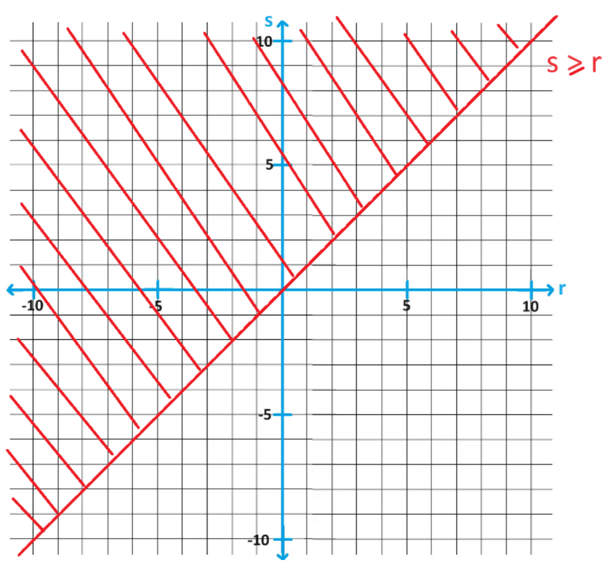
b) In general, would two equivalent statements have the same graph?

The graph of $s > r$ would be similar to the graph of $s \geq r$. It just has the points along the northeast diagonal omitted.

The way people indicate that is to draw a dashed line for the line of omitted points and not use a solid block of color for a region of points, using instead "line strokes" to indicate that a region is filled in.



We could draw our previous graph similarly using a solid line for the included points along the boundary.



Practice 78.7 Graph

$$2(b-a)+2<6-2a$$

with the horizontal axis of your graph labeled "a."

Practice 78.8

a) Graph $x+y=0$.

b) Graph $x+y\geq 0$

c) Graph $x+y>0$

d) Graph $x+y\neq 0$

Label the horizontal axis of your graph "x" in each case.

MUSINGS

Musing 78.9
a) Describe the graph of the solutions to $(a - a) \cdot b = 0$.
b) Describe the graph of the solutions to $(a - a) \cdot b \geq 0$.
c) Describe the graph of the solutions to $(a - a) \cdot b > 0$.

Musing 78.10
a) Create an inequality in one unknown using the symbol $>$ whose solution set is all numbers different from zero.
b) Create an inequality in one unknown using the symbol $\geq$ whose solution set is just the number zero.

MECHANICS PRACTICE

Practice 78.11 Solve each of these inequalities and graph its solutions. (Always choose to label the horizontal axis "x.")

a) $-3x + 9 \leq 3x + 9$

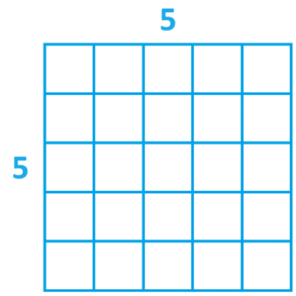
b) $x(x - 2) + 2x < 4$

c) $3(y - 2) - (x - 2) \neq 2(y - x + 2) + 2$

d) $2x + y + 3 \leq 3(y + 1)$

79. Squares and Square Roots

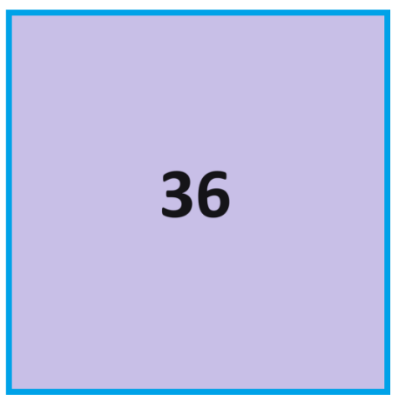
A square with side length 5 units has area $5 \times 5 = 25$ unit squares.



As we saw in Sections 11 and 60, we write 5^2 for 5×5 and call it five **squared**. The connection to geometry in this language is deliberate.

And we can go in reverse.

Suppose I gave you the area of a square first. Let's say we have a square with area 36 squared units. It is natural to wonder what the base or "root" feature of this square must be.



Of course, we are all thinking that the side length of the square must be 6 units and that I am using strange language to ask this.

But this is the language al-Khwarizmi used to when thinking of an equation of the form $s^2 = 36$. He used the Arabic word for *root*, which when transcribed into Latin by western Scholar became **radix**.

The symbol for "the root feature of a square," that is, for **square root** is √, which we call a **radix**. But the symbol usually comes attached with a vinculum (as was discussed in Section 11).

We write

$$\sqrt{36} = 6$$

In the same way, we have $\sqrt{25} = 5$.

Practice 79.1 Compute the following square root values.
(Remember, we are talking about quantities related to geometric square figures.)

$\sqrt{100}$ $\sqrt{49}$ $\sqrt{196}$ $\sqrt{1024}$ $\sqrt{1.21}$ $\sqrt{\frac{81}{4}}$

It is understood that the radix (and its vinculum) $\sqrt{\blacksquare}$ is a symbol from geometry, in which case all quantities being discussed when using the symbol are assumed to be from geometry. As such, they must be positive numbers. (All measurements of lengths and areas are positive numbers.)

Writing

$$\sqrt{-9}$$

is meaningless, as there is no square of area -9 in geometry.

And writing

$$\sqrt{9} = 3 \quad \text{or} \quad -3$$

is also meaningless as a square cannot have a side length of -3.

This latter point is important. The equation

$$x^2 = 9$$

has solutions

$$x = 3 \quad \text{or} \quad -3$$

and writing this is good and correct. No radix was used here and so there is no implicit command keep this piece of work in the context of geometry.

Practice 79.2
a) Is writing $\sqrt{11 + \ -2}$ technically meaningful? If so, what is its value?
b) Is writing $\sqrt{(-3)^2}$ technically meaningful? If so, what is its value?

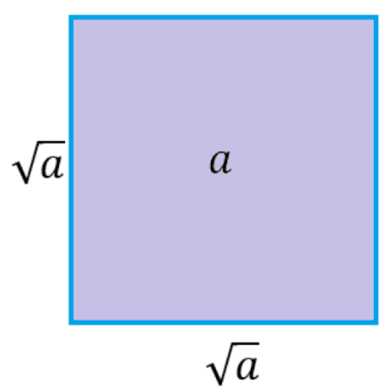
Practice 79.3 a) Describe the solution set to $s^2 = -3$.
b) Describe the solution set to $s^2 = 0$.

There is one exception to this "geometry rule." People will consider squares of zero area. Such squares have a side length of zero units. It is accepted to observe and write:

$$\sqrt{0} = 0$$

Here's the formal definition of the square root of an allowed number.

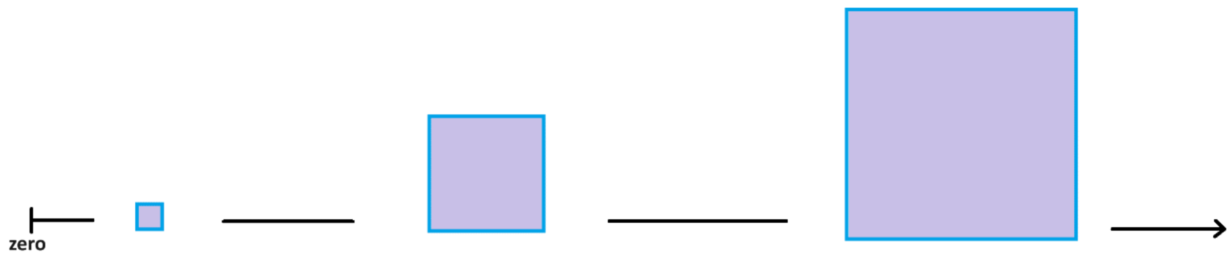
If a is a positive number, or possibly zero, then the **square root** of a is a number $\sqrt{a}$ with the property that $\sqrt{a} \times \sqrt{a} = a$.



We can check that $\sqrt{16} = 4$, for instance, by noting that $4 \times 4 = 16$.
(Picturing an actual square is always a good move.)

Also, $\sqrt{\frac{81}{4}} = \frac{9}{4}$ is not correct because $\frac{9}{4} \times \frac{9}{4} = \frac{81}{16}$, which is not $\frac{81}{4}$.

As squares can shrink and grow to any size we want, every positive number (and zero) has a square root.

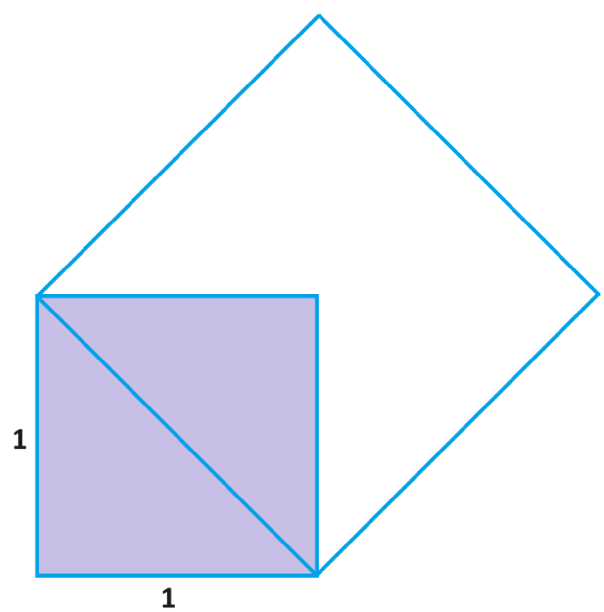


Practice 79.4 *Does the square root of two exist?*

One way to answer this question is to exhibit a square of area 2. The side length of that square is then, by definition, the square root of 2.

Draw a square with a side length 1 and draw a tilted square with side the diagonal of that unit square.

Can you see that the titled square has area 2.

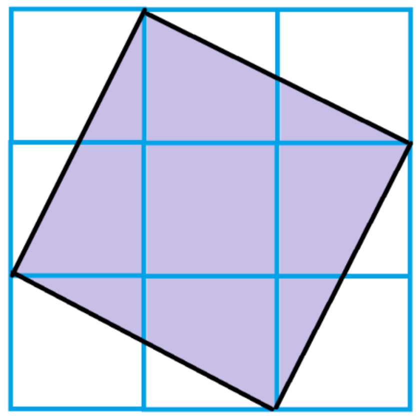


Comment: We mentioned that the diagonal of the unit square has length $\sqrt{2}$ back in Section 59, but this picture gives a more natural way to see that this must be so.

We proved that $\sqrt{2}$ is an irrational number in Section 59, and we tried to write $\sqrt{2}$ as a decimal in Practice Problem 56.17.

Practice 79.5 On a calculator what is $\sqrt{2}$ as a decimal rounded to some large number of decimal places?

Practice 79.6 Use this picture to demonstrate that $\sqrt{5}$ exists.



Example: Kindly solve $2\sqrt{w} + 4 = 0$.

Answer: The use of the radix indicates that we must be thinking of positive (or zero) values with regard to squares and square roots.

We have

$$2\sqrt{w} + 4 = 0$$

$$2\sqrt{w} = -4$$

$$\sqrt{w} = -2$$

The original equation is equivalent to an equation which can never be true. (A square root value cannot be negative.)

The original equation has no solutions.

MUSINGS

Musing 79.7 Collect some data for the equation $b = \sqrt{a}$. Graph the data with the horizontal axis labeled "a."

MECHANICS PRACTICE

Practice 79.8 Make the expression $\sqrt{\frac{\sqrt{3}}{2.1}} \times \sqrt{\frac{\sqrt{3}}{2.1}}$ a tinier bit friendlier.

Practice 79.9 What is the value of $\sqrt{\sqrt{\sqrt{256}}}$?

Practice 79.10 Show that $\frac{3\sqrt{20}}{2}$ is the same number as $\frac{30}{\sqrt{20}}$.

Practice 79.11 Please solve each of these math sentences.

a) $4\left(\sqrt{r} + 3\right) = 3\sqrt{r} + 5$

b) $x(x + 3) - 2x + 2 \leq x - 5$

c) $\sqrt{a} - 3 \neq 3 - \sqrt{a}$

80. Going Rogue

Two mathematical sentences (mentioning the same unknowns) are equivalent if they have exactly the same solutions. And the process of solving an equation or an inequality is to carefully transform the given sentence into an equivalent one whose solutions we readily recognize.

Thanks to the al-Khwarizmi, we have a number of algebraic steps we can take that won't alter the truth, and hence solutions, of a given math sentence.

- **Applying a fundamental rule of arithmetic to one piece of a math sentence won't affect the truth of the sentence.**

For example,

$$3(x+6)=2x \quad \text{and} \quad 3x+18=2x$$

are equivalent sentences because a rule of arithmetic allows us to expand brackets and rewrite the piece of the sentence $3(x+6)$ as $3x+18$.

- **Adding (or subtracting) identical quantities to each side of a math sentence does not affect the truth of the sentence.**

For example, the adding $-a$ to each side of this inequality

$$3a+2 \geq a-1$$

gives the equivalent sentence

$$3a+2+\ -a \geq a-1+\ -a$$

which, by the first point, is equivalent to

$$2a+2 \geq -1$$

- **Multiplying each side of a math sentence by a quantity known to be a <u>positive</u> number won't affect the truth of the sentence.**

For example, scaling each side of the sentence $2a + 2 \geq -1$ by a factor of $\frac{1}{2}$ gives the equivalent sentence

$$a + 1 \geq -\frac{1}{2}$$

And that's it! These are the three algebraic operations that preserve the truth of math sentences.

Practice 80.1 Are

$$b > 2$$

and

$$(b^2 + 1)b > 2(b^2 + 1)$$

equivalent sentences?

One need not fuss about what happens when we multiply through by a negative value: the addition and subtraction of quantities will handle that.

Practice 80.2
a) Show that

$$-A = -B \quad \text{and} \quad A = B$$

are equivalent sentences by adding $A + B$ to each side of the first equation.

b) Show that

$$-3 \leq x \quad \text{and} \quad -x \leq 3$$

are equivalent sentences.

But we did see "rules" we can follow if you prefer the shortcut they offer.

- Multiplying each side of an equality ($=$) by a negative number does not affect the truth of the equality.
- In multiplying each side of an inequality ($>$ or $\geq$ or $<$ or $\leq$) by a negative number, one most also "flip" the direction of the inequality sign in order to preserve truth.

Practice 80.3 Is there a special "rule" to be deduced for multiplying an inequality of the from $A \neq B$ through by a negative number?

One is, of course, welcome to transform a math sentence in any way one desires.

But if you deviate from the bulleted items listed on the previous two pages, then all bets are off as to what remains true about the new math sentence you create. It is up to you to decide if the solutions to your new sentence bear any relevance to the solutions of the original sentence.

For example, here's an equation.

$$x = 2$$

It has just one solution, namely that x must have the value 2 to make it a true sentence.

Now let's square each side of the equation. (This is a rogue move: it's not one of the allowed moves of algebra.)

We get the sentence

$$x^2 = 4$$

This new sentence has two solutions: x must be 2 or -2 to make the sentence true.

The sentences $x = 2$ and $x^2 = 4$ are not equivalent sentences.

Solution sets will likely change if you deviate from the standard steps!

However, there is some potential value here.

We can observe that if two numbers are equal, then the squares of those two numbers will also be equal. So, if we have some values that make a math sentence

$$A = B$$

true, then those values will also make the sentence

$$A^2 = B^2$$

true.

The solutions to $A = B$ will appear among the solutions to $A^2 = B^2$.

And we saw that: the solution to $x = 2$ is among the solutions to $x^2 = 4$.

So, if you feel like squaring each side of an equation, go for it. But you now know that your equation will likely have more solutions than the original equation, but the solutions to the original equation you seek will be among them.

You can then just check each potential solution in turn to see which ones actually work.

Comment: School curricula call the appearance of additional potential solutions a phenomenon of **extraneous solutions**. They require students to "check all your solutions" to weed out which ones don't apply to the original equation. And that's appropriate if a student has taken a step that's broken away from the standard steps of algebra (our previous bullet points).

But if a student has not deviated from the standard steps of algebra, then they can be assured that all the solutions obtained are all the solution of the original equation. There is no need to "check all your solutions" (except to catch arithmetic errors, perhaps).

Practice 80.4

a) Describe the set of solutions to

$$\sqrt{w} = -3$$

b) Squaring each side of this equation gives

$$w = 9$$

What are the solutions to this equation? Are any of the solutions "extraneous"?

Going rogue is dangerous.

For example, consider this equation

$$(b-5)^2 = 36$$

Students are often encouraged to take the square root of each side of the equation and to even draw in the radix.

$$\sqrt{(b-5)^2} = \sqrt{36}$$

This leads students to then write

$$b - 5 = 6$$

$$b = 11$$

I have no idea what taking the square root of each side of math sentence typically does to the set of solutions of the original sentence. I am not sure what to say about concluding "b must have value 11."

Practice 80.5

a) Does b having the value 11 make the sentence $(b-5)^2 = 36$ true?

b) Have we fully solved $(b-5)^2 = 36$? Are we missing solutions?

Like I said, if you choose to go rogue, you are completely on your own!

Example: Solve

$$\frac{x}{x} = 1$$

Answer Attempt 1: Multiply each side of the equation by the number x to obtain

$$x \times \frac{x}{x} = x \times 1$$

$$x = x$$

Every possible value for x makes this final sentence true.

Solution set: all numbers.

The trouble with this answer is that we've subtly gone rogue.

We are welcome to multiply an equation through by a number, as long as that number is positive or negative—but not zero!

So, we have to be careful with our thinking.

Answer Attempt 2: If x represents a number that is not zero, then we can multiply each side of our equation x to obtain

$$x \times \frac{x}{x} = x \times 1$$

$$x = x$$

Every possible value for x makes this final sentence true—but remember, we are only considering non-zero numbers at present.

<ins>Solutions so far</ins>: The set of all non-zero numbers.

So, what if we do consider x to be the number zero?

Well, in that case our math sentence $\frac{x}{x} = 1$ is not meaningful, yet alone true. (We can't have a denominator of zero in a fraction.) So, zero is not a solution after all.

<ins>Final answer</ins>: The solutions are all non-zero numbers.

Practice 80.6 Godspeed was asked to solve this equation:

$$a \times a = a$$

He presented this solution:

$$a \times a = a$$

$$\frac{1}{a} \times a \times a = \frac{1}{a} \times a$$

$$a = 1$$

I conclude that there is one solution: a is 1

Any commentary?

Example: Solve

$$\frac{1}{x-3} = 2$$

Answer: The equation does not make sense, yet alone be true, if x is 3. We have to keep the value 3 out of our considerations.

But if x is not 3, then $x - 3$ is not zero and we are welcome to multiply both sides of the equation by $x - 3$ to get an equivalent equation (but still under the proviso that x is not 3).

$$(x-3) \times \frac{1}{x-3} = (x-3) \times 2$$

$$1 = 2x - 6$$

We can keep going

$$7 = 2x$$

$$\frac{1}{2} \times 7 = \frac{1}{2} \times 2x$$

$$3\frac{1}{2} = x$$

We have the solution that x is $3\frac{1}{2}$ (and the fits the proviso of not being 3!

MUSINGS

Musing 80.7 Consider the equation

$$x < \frac{1}{x}$$

a) Is 2 a solution to this equation? Is -2 a solution?

b) Is $\frac{1}{2}$ a solution to this equation? Is $-\frac{1}{2}$ a solution?

c) Why can't zero be considered as a potential solution to this equation?

d) If we restrict our minds to consider only positive values for x, show that the equation is then equivalent to $x^2 < 1$. What then are the solutions to the equation (within this restricted mindset)?

e) If we restrict our minds to consider only negative values for x, show that the equation is then equivalent to $x^2 > 1$. What then are the solutions to the equation (within this restricted mindset)?

f) Describe the full set of solutions to the original equation.

MECHANICS PRACTICE

Practice 80.8 Fully solve

$$\frac{x+3}{x-3} = 0$$

explaining your reasoning with care as you go along.

Practice 80.9 Give two solutions to the equation

$$(a+3)^2 = 100$$

Practice 80.10 Kindly solve

$$\sqrt{w+6} = 9$$

Chapter 10

Lines

81. Do Straight Lines Exist?

Suppose I asked you to go outside and walk in a straight line directly east. Barring obstacles such as fences, buildings, lakes, mountains, oceans, and such, do you feel you could do that—at least for some distance?

Would the path feel "straight" to you?

Most people would say it does feel as though you are walking in a straight line when heading directly east, but also acknowledge that if they kept going (again barring obstacles) they would eventually return to start having followed a line of latitude circling a portion of the Earth. This not the expected behavior of a straight line!

Practice 81.1: If you stand at the North Pole, which direction is east?

Hence my question. We feel like we can walk a straight path on the surface of the Earth, but because the Earth is curved, we know such paths are not straight.

So, do straight lines exist?

Of course, we can say that the edge of a door, for instance, is meant to be a straight line but, if examined under a magnifying glass, it is likely not. (And certainly, at the atomic level that allegedly straight edge is bumpy.) So, we can argue that perfectly straight lines don't exist in our practical world.

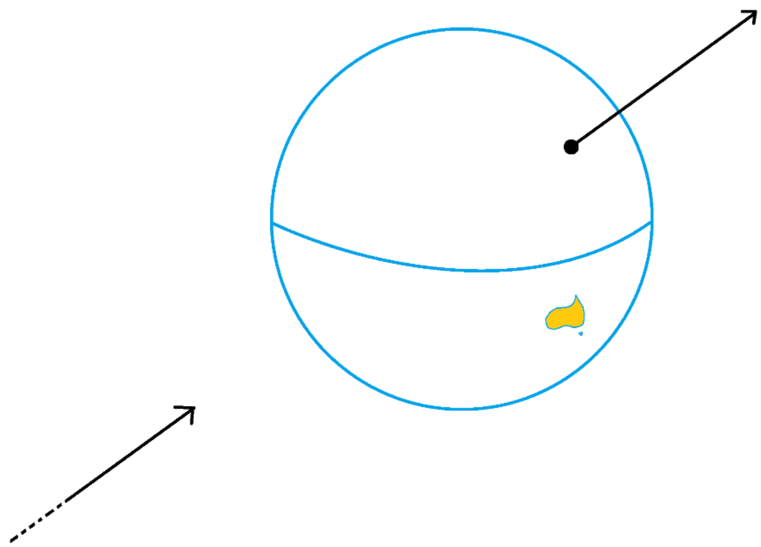
What about in our intellectual world?

We've stepped back from the Earth and seen that it is a sphere. There are no straight lines on the surface of the Earth. But what about a line through the Earth or out from it? Are there straight lines in the universe?

Imagine a brave astronaut having agreed to conduct an experiment. She is going to fly away from the Earth in a rocket ship in a straight line and never to deviate from a straight path.

And suppose some 300 years later she reappears from the opposite direction, fervently claiming that she did not deviate from a straight path.

What would we conclude?



We can't step back from the universe to see what shape it is, but we would have to conclude that the universe is curved in some way analogous to the surface of the Earth being curved.

Maybe straight lines don't exist and can't exist at all?

Nonetheless, we feel in our minds that we know what a "straight line" is and that such things do exist, perhaps just intellectually.

Can we pin down what exactly is on our heads on this matter?

What Makes a Line "Straight"

We feel like the ground is "flat," at least in our very immediate surrounds.
And we feel like we know what it means to have a perfectly flat, "level" line.

Here's a picture of something we probably agree represents one (or, at least, a section of a flat, level line).

What feels flat about this line is that no point on it is "higher" than any other point.

Any how do we measure height? By measuring along another type of line we feel also exists, a vertical line.

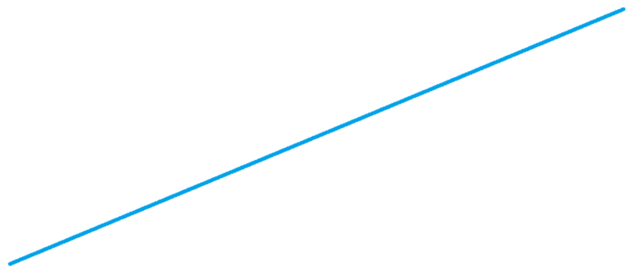
We can go down a rabbit hole trying to make sense of all the interconnected words we keep using—flat, level, horizontal, vertical, high, and so on.

But rather than delve into the realm of mathematical philosophers, let's just claim, for now at least, that two basic examples of "straight lines" exist.

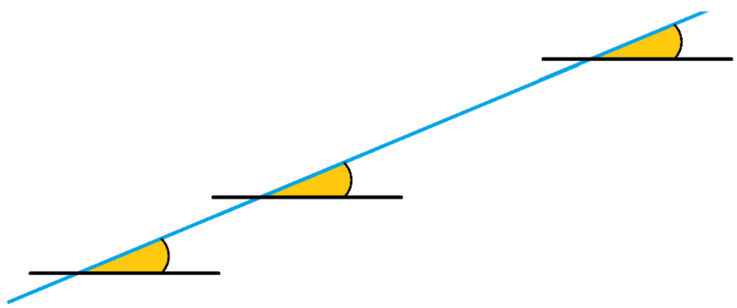
> Horizontal and vertical lines exist and are examples of straight lines.

Let's next follow our intuition about what makes other lines "straight" having these two basic examples to bounce off of. (And if down the road we find a way to firm up this shaky and questionable start, we will!)

Okay. Given we've got horizontal and vertical lines to play with, what makes us think that a line like this diagonal one also deserves to be called "straight"? (Again, this is just a section of the full picture we have in our minds.)

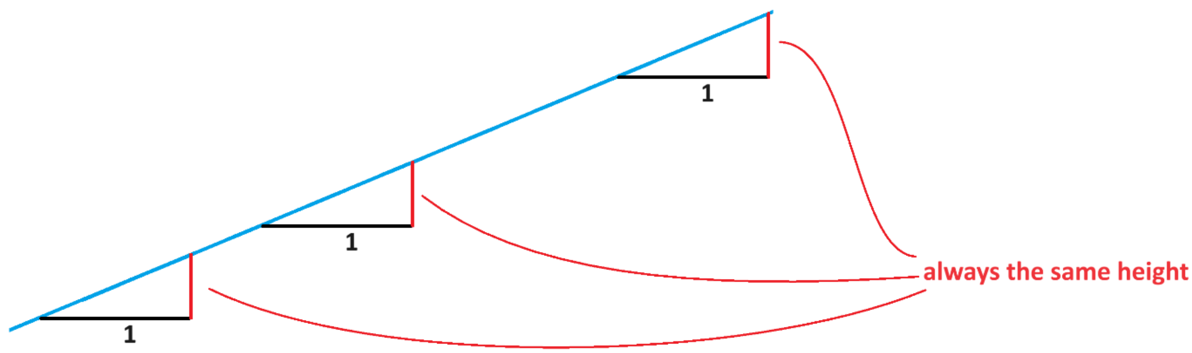


One thing to note is that if we were to draw horizontal segments along the line, we'd expect to see the line rising at the same angle from the horizontal at all places along the line.

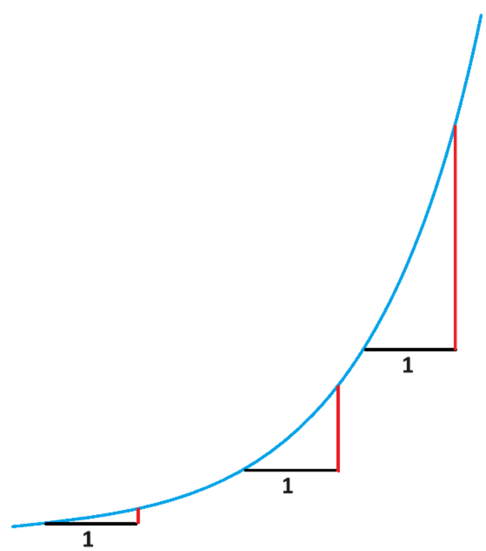


Rather than try to pin down what we mean exactly by "angle," let's get at this notion of upness by making use of vertical lines.

For line to be straight, we expect for each horizontal segment say, 1 unit long, drawn from a point on the line, the same vertical distance is needed to return to the line from the endpoint of the segment.



Certainly not all curves drawn in the plane have this "constant height" property. So, it looks we're capturing the notion of "straightness."



Comment: I have fallen into the convention of assuming horizontal lines, like the horizontal axis of a graph, are drawn with positive measurements made rightward, and vertical segments are drawn with positive measurements made upwards.

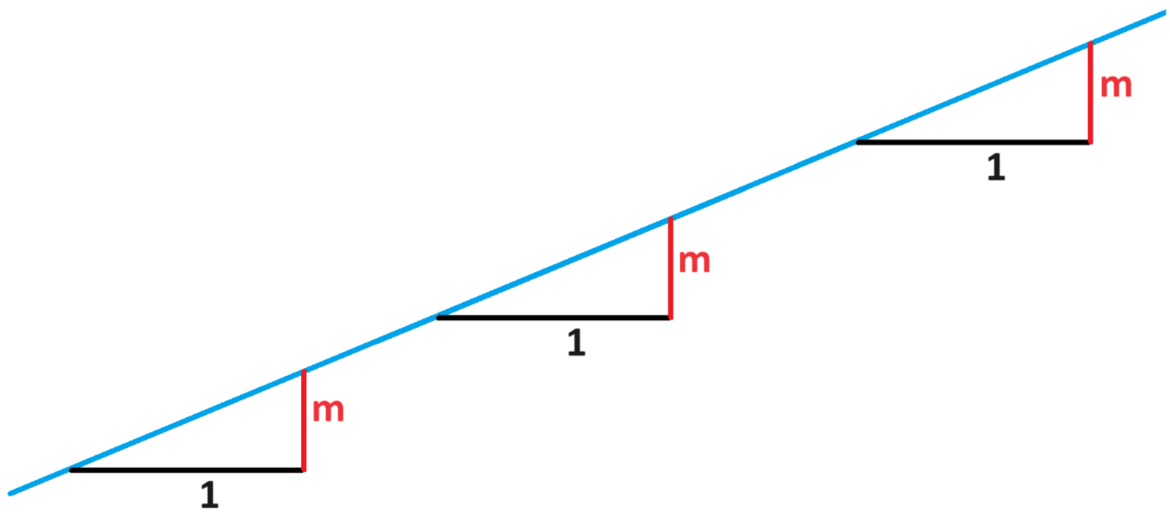
Let's follow that convention just to be consistent with our convention for graphing.

We can make a definition of a what it means for a curve drawn in the plane to be straight.

There are two fundamental examples of straight lines: horizontal lines and vertical lines.

For any other curve drawn in the plane, conduct this experiment: For each point on the curve draw a horizontal line segment of length 1 (measured rightward) and determine the vertical distance required (measured upwards) to return to a point on the curve.

If these vertical distances are consistently the same value, call it m, then the curve is called a ***straight line*** *and we say that the line has* ***slope*** *m.*

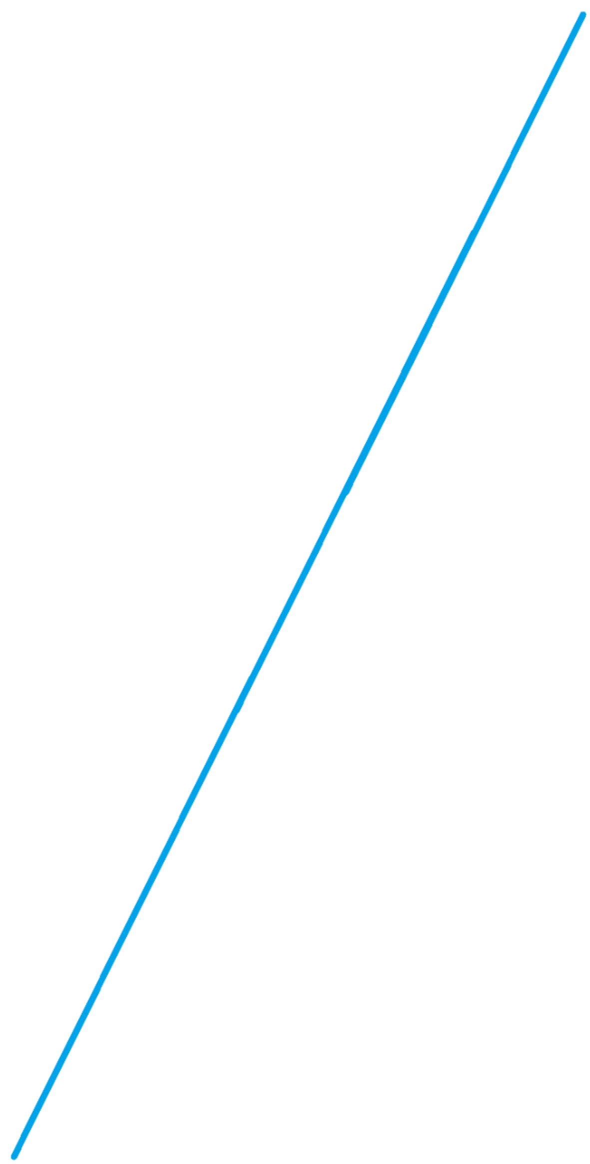


A straight line of slope *m*

For some reason it has become the custom to use the letter m to denote the common height value of a given straight line. It seems no one really knows how this came to be so (though a common speculation espoused on the internet is that it comes from the French word *monter* meaning “to climb”).

However, folk in some counties use the letter k or the letter a for slope. They might also use the word **gradient** instead of the word slope.

Practice 81.2 Pull out a ruler and determine the slope of this line.
(Physically measure and draw horizontal sections one unit long and determine the length of the matching vertical segments that bring one back up to the line.)



A PROBLEM

The previous practice example brings up a concern.

> Jennie is Australian, and the ruler she used to answer the previous question was marked in centimeters. Thus, she drew horizontal segments 1 cm long and noticed that 2 cm of vertical length brought her back up to the line each time.
>
> She concluded:
>
> > *The slope of the line is* 2*.*
>
> (Actually, being Australian, she wrote: "The gradient of the line is 2.")

On the other hand,

> Jared is American and answered this question using a ruler marked in inches. He drew horizontal segments 1 inch long and saw that vertical segments 2 inches long aways brought him back up to the line.
>
> He concluded:
>
> > *The slope of the line is* 2*.*

Our previous definition of "straight" never specified what unit of length we should use to determine the slope of the line. Luckily, Jennie and Jared got the same value for slope using different units. But will that always be the case?

This could be a serious issue.

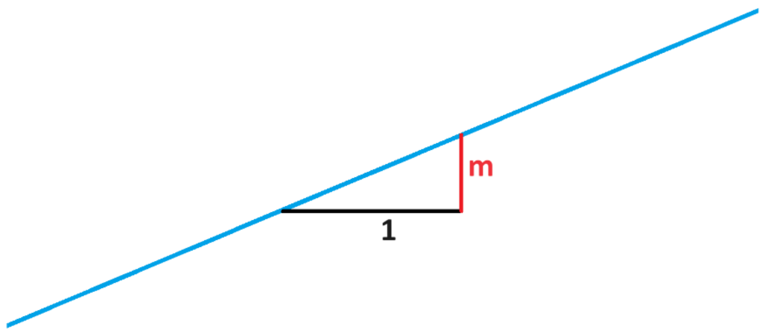
For instance, when Jennie looked at Jared's work, she thought it was strange that he chose to draw a horizontal segment 2.54 cm long. She agrees that he then measured a matching vertical segment 5.08 cm long but doesn't understand why Jared then also concluded that the slope of the line is 2.

Jared's work is peculiar from her perspective.

> **Question:** Would Jennie's work likely be equally perplexing to Jared?

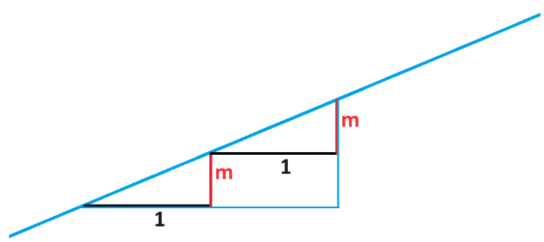
RESOLVING THE ISSUE

Let's consider a line of slope m.

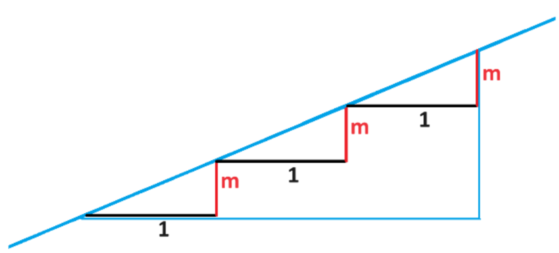


When determining the slope of the line, it seems we need to be more flexible about what the length of the horizontal segment can be, just in case someone else has a different notion of "one unit of length." So, let's examine what happens with different horizontal segment lengths.

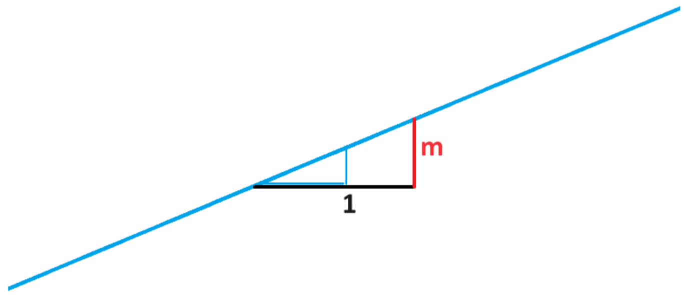
For starters, this picture shows that if we draw a horizontal length 2 units long, it will take a vertical length $2m$ units long to return to the line.



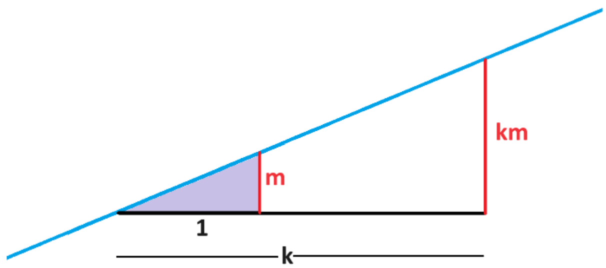
This picture shows that if we draw a horizontal length 3 units long, it will take a vertical length $3m$ units long to return to the line.



And this picture shows that if we draw a horizontal length $\frac{1}{2}$ units long, it will take a vertical length $\frac{1}{2}m$ units long to return to the line.



We're being led to believe that if we scale the horizontal length we draw by a factor k , then the matching vertical line segment we draw is also scaled by the same factor k.



Jargon: When given a straight line and drawing a horizontal segment from a point on the line:

> *The length of the horizontal segment is called the* **run**.
> *The length of the matching vertical segment to reach back to the line is called the* **rise**.
> *The quantity*
>
> $$\frac{rise}{run}$$
>
> *is called the* **slope** *(or* **gradient***) of the line.*

With a horizontal segment of length 1 unit and vertical segment of length m units, the slope of the line is $\frac{m}{1}$, which equals m, just as we had before.

If the picture is scaled by a factor k, then the slope of the line is computed as $\frac{km}{k}$, which still gives the value m.

"Rise over run" is sure to equal the slope of the line, no matter how you choose to scale the picture.

Phew!

Having thought this through, Jennie can now say:

Jared's work is fine!

As there are 2.54 *cm in an inch, his picture is just a scaled version of my picture. He has* $k = 2.54$.

When I work out slope, I get $\frac{2}{1}$ *(two centimeters over one centimeter), which is* 2.

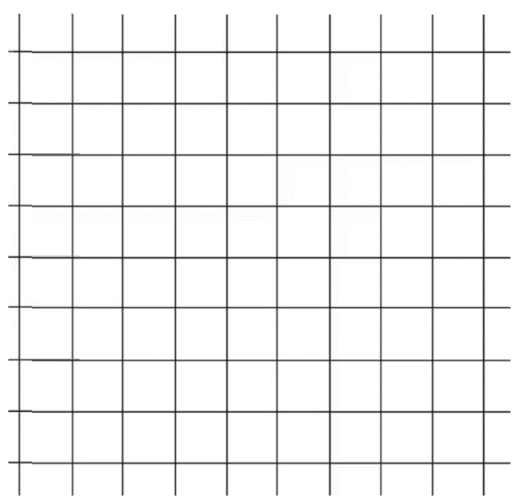
Whan Jared works out slope, he gets, in my opinion, $\frac{2.54\times 2}{2.54\times 1}$*, which, of course, gives the same value* 2.

Our computed values for slope were sure to agree.

Practice 81.3 On a sheet of graph paper, draw examples of lines of the following slopes.

a) Slope 3 b) slope $\frac{1}{3}$ c) slope $\frac{4}{5}$ d) slope -2 e) slope -1 f) slope 0

(For part f), what does it mean to measure -2 units upwards?)

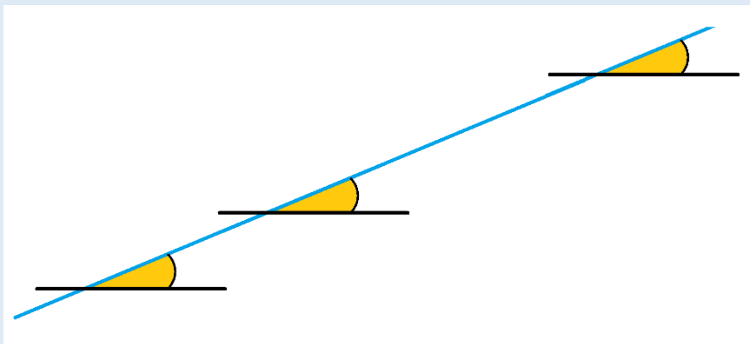


Practice 81.4 Jennie and Jared each draw an example of a line of slope $-\frac{1}{2}$ on graph paper, but the size of the squares on their sheets differ. Does that matter?

If they put their papers on top of one another, would they see that they've drawn the same line?

MUSINGS

Musing 81.5 A straight line makes an angle of $45°$ to each horizontal line segment. What is the slope of the line?

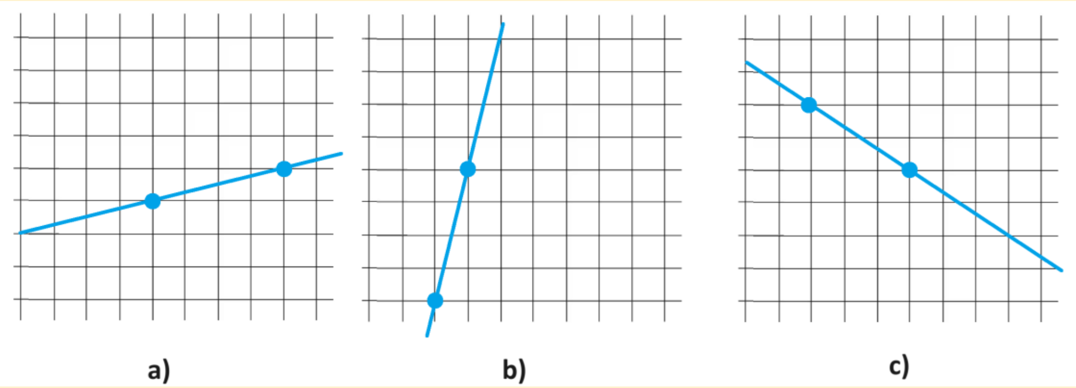


Musing 81.6 What does this sign mean?

Musing 81.7
a) Does a horizontal line have a value for slope?
b) Does a vertical line have a value for slope?

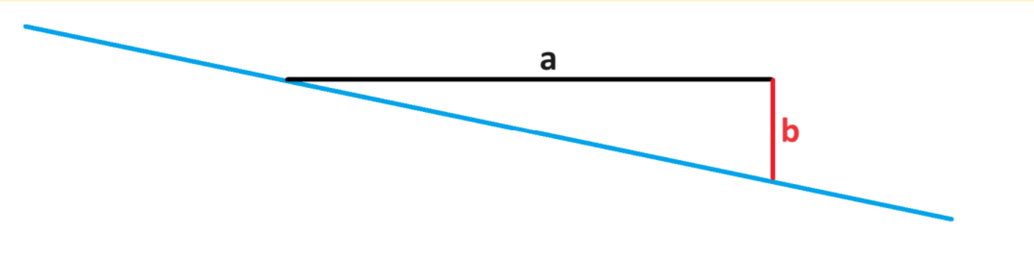
MECHANICS PRACTICE

Practice 81.8 Find the slope of each of these straight lines.



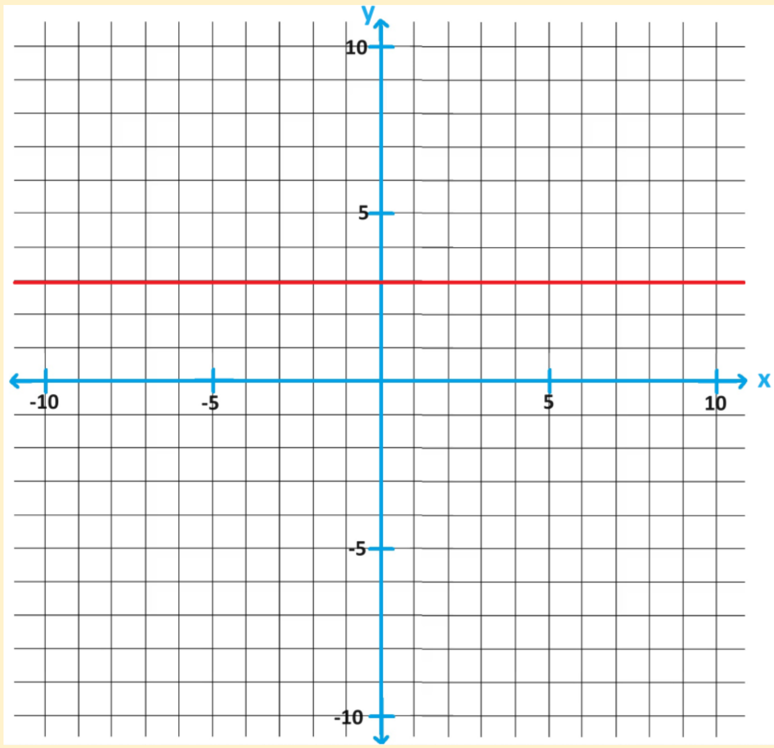
Practice 81.9 The slope of the line shown is:

a) $\frac{a}{b}$ b) $\frac{b}{a}$ c) $-\frac{a}{b}$ d) $-\frac{b}{a}$



Practice 81.10 Here's a horizontal line drawn on some graph paper with axes labeled x and y.

What's an equation for this line?



To answer the question we can collect some data values we see from the graph.

x	y
3	3
2	3
−6	3
0	3
10	3
1	3
−4	3

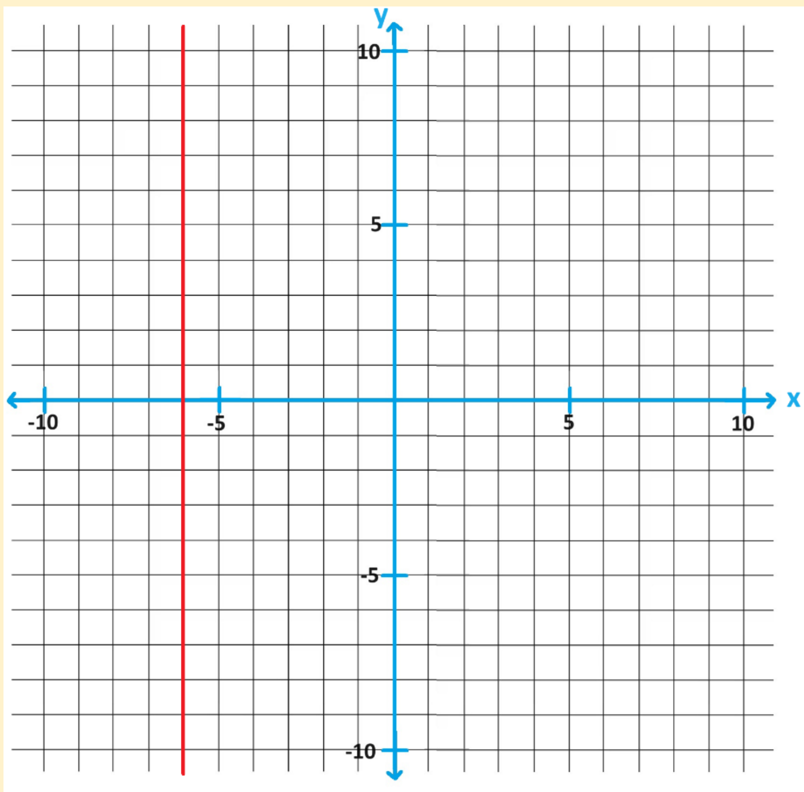
It is evident that for a data point (x, y) to be on this line, it must have y equal to 3, and conversely, any data point with second coordinate 3 will be on this line.

The equation of the horizontal line is thus

$$y = 3$$

(All the values for x and y that make this equation true give a point on this line.)

Your Turn: Find the equation of this vertical line.



82. The Equation of a Line

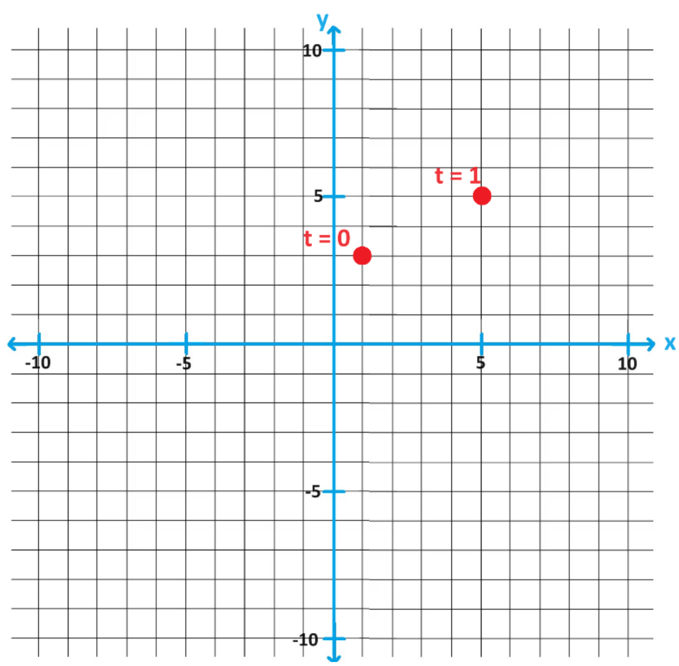
Let's start with an exercise. It's a meaty one!

Do your best to work through the ideas here on your own before reading on.

Example: A robot has been programmed to walk in a straight line at a uniform speed.

To keep track its location, a grid like graph paper was drawn on the floor of the lab and two axes, labeled x and y were marked.

At time $t = 0$ minutes the robot was at position $(1,3)$. After one minute, at time $t = 1$ minute, the robot is at position $(5,5)$.



a) Where will the robot be at time $t = 2$ minutes? Plot this location on the grid.

b) Where will the robot be at time $t = 3$ minutes? At time $t = 10$ minutes?
(Warning: The robot will be off the grid as shown at these times. Imagine that the grid extends beyond what is shown.)

c) Where was the robot at time $t = 30$ seconds? Plot this location on the grid.

d) Where was the robot at time $t = 45$ seconds? Plot this location on the grid.

e) Where was the robot a minute before these observations started, at time $t = -1$ seconds? Plot this location on the grid.

f) At what time was the robot at position $(7,6)$?

g) At what time will the robot be at position $(21, 13)$?

h) Will the robot ever be at position $(13, 12)$? If so, at what time? If not, how do you know?

i) Was the robot ever at position $(-9, -2)$? If so, at what time? If not, how do you know?

BONUS CHALLENGE:

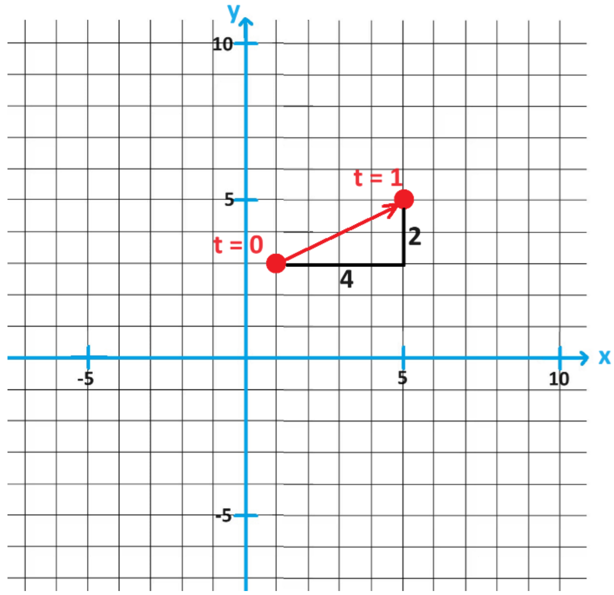
Describe the location of the robot, as a point in the plane, at a general time t.

Check that what you write down gives the location $(1,3)$ for $t = 0$, the location $(5,5)$ for $t = 1$, and matches all your answers to the various parts of this exercise.

Okay, my turn to go through the work I just asked you to do.

a) and b): Where will the robot be at times $t = 2$ and $t = 3$ minutes?

We see that the robot, in its diagonal motion, is shifting 4 units rightward and 2 units upward each minute.

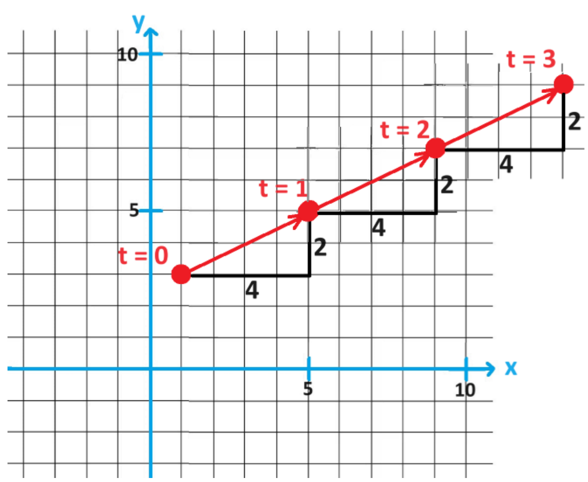


So, after one more minute, it will shift another 4 units to the right and 2 units upwards.

a) At time $t = 2$ minutes the robot will be at position $(\mathbf{5} + 4, \mathbf{5} + 2) = (9,7)$.

After another minute, it will shift again the same amount from $(9,7)$

b) At time $t = 3$ minutes the robot will be at position $(\mathbf{9} + 4, \mathbf{7} + 2) = (13,9)$.



We can keep this line of thinking and build up to understanding where the robot will be at time $t = 10$ minutes, one minute at a time. But let's see if we can start being swifter in our work.

At the start, time $t = 0$ minutes, the robot is at position $(1,3)$. After ten minutes, the robot will shift rightward 4 units, ten times (that's 40 units rightward in total) and shift upwards 2 units, ten times (that's 20 units upwards). Thus

b) Continued: At time $t = 10$ minutes the robot will be at position $(\mathbf{1} + 40, \mathbf{3} + 20) = (41,23)$.

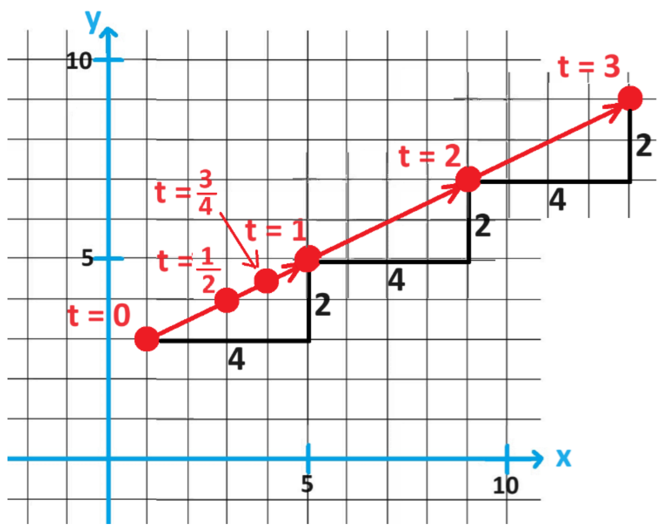
c) and d): Where was the robot at times $t = 30$ and $t = 45$ seconds?

At $t = \frac{1}{2}$ minutes (30 seconds), the robot would have moved only half as far as it did in one minute: only $\frac{1}{2} \times 4 = 2$ units rightward and $\frac{1}{2} \times 2 = 1$ unit upwards.

c) At $t = \frac{1}{2}$ minutes, the robot was at position $(\mathbf{1} + 2, \mathbf{3} + 1) = (3,4)$.

At $t = \frac{3}{4}$ minutes (45 seconds), the robot would have moved only three-quarters as far as it did in one minute: namely, $\frac{3}{4} \times 4 = 3$ units rightward and $\frac{3}{4} \times 2 = \frac{3}{2}$ units upwards.

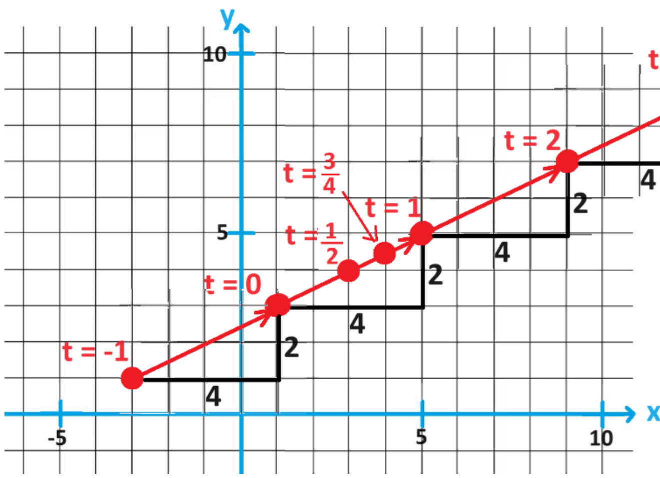
d) At $t = \frac{3}{4}$ minutes, the robot was at position $\left(\mathbf{1} + 3, \mathbf{3} + \frac{3}{2}\right) = (3,4\frac{1}{2})$.



e) Where was the robot at time $t = -1$ minutes?

One minute before these observations, the robot must have been 4 units leftward and 2 downward from the starting location $(1,3)$.

e) At time $t = -1$ minutes, the robot was at position $(\mathbf{1} - 4, \mathbf{3} - 2) = (-3, 1)$.

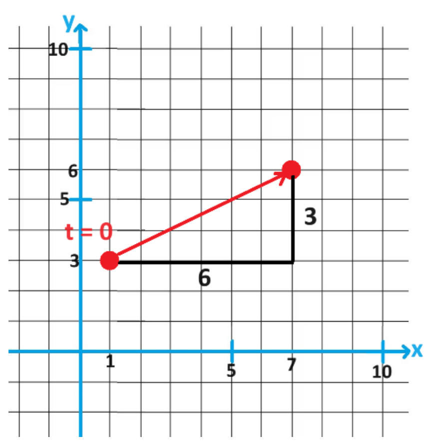


f) and g): When was the robot at $(7,6)$ and $(21,13)$?

In order to move from the start $(1,3)$ to the point $(7,6)$ the robot needs to shift

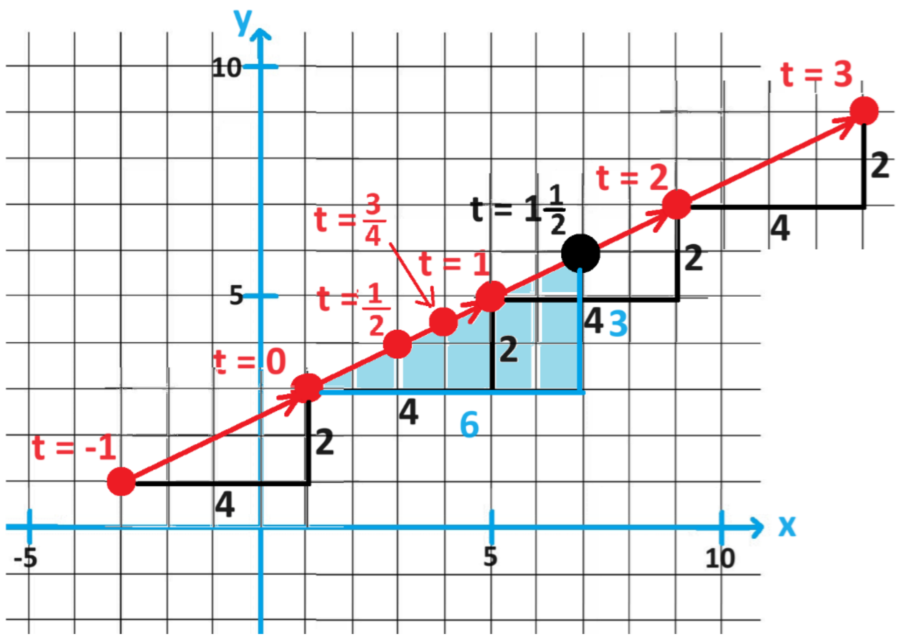
$7 - 1 = 6$ units rightwards

$6 - 3 = 3$ units upwards



In one minute, it shifts 4 and 2 units in these directions, so another half a minute will do the trick.

f) At time $t = 1\frac{1}{2}$ minutes the robot will be at position $(7,6)$.



(My picture is getting messy!)

To move from the starting position $(1,3)$ to $(21,13)$ the robot needs to shift

$21 - 1 = 20$ units rightwards

$13 - 3 = 10$ units upwards

In each minute it moves 4 and 2 units in these directions, so 4 minutes of motion will do the trick.

g) The robot will be at position $(21,13)$ at time $t = 4$ minutes.

h) and i): Was the robot at $(13,12)$ and $(-9,-2)$? If so, when?

To move from the starting position $(1,3)$ to $(13,12)$ the robot needs to shift

$13 - 1 = 12$ units rightward

$12 - 3 = 9$ units upwards

In each minute it moves 4 and 2 units in these directions.

So, in three minutes the robot will shift 12 units to the right—as we want—but it will shift only shift 6 units upwards, not 9.

h) The robot does not pass through the point $(13,12)$.

If the robot was, at some point of time at location $(-9,-2)$, then it moved to $(1,3)$ by shifting

$9 + 1 = 10$ units rightwards

$2 + 3 = 5$ units upwards

It can shift 10 units rightward in $2\frac{1}{2}$ minutes (since $10 = 2\frac{1}{2} \times 4$).
In that time it would shift $2\frac{1}{2} \times 2 = 5$ upwards, which is perfect!

i) The robot was indeed at position $(-9,-2)$ at time $t = -2\frac{1}{2}$ minutes.

BONUS CHALLENGE:
During one minute of motion the robot shifts 4 units rightward and 2 units upward.
So, during t minutes of motion, the robot shifts

$4t$ units rightwards
$2t$ units upwards

It's location at time t minutes will thus be

$$(\mathbf{1} + 4t, \mathbf{3} + 2t)$$

(and this agrees with all the answers just presented).

What Are We Learning?

The robot's motion was in a straight line, and we just saw how useful it is to think in terms of

shifts rightward and **shifts upward**

to understand and answer questions about its motion.

We recognize that these shifts were called **run** and **rise**, respectively, in the previous section.

Also, during the course of the exercise, you may have noticed that as the robot moves from location to location, **its vertical shift is always half its horizontal shift**.

> **Question:** Can you see this in my answers to parts f), g), and i)?
> And can you see the problem with part h)? The desired vertical shift there was not half the horizontal shift.

Now that I reflect on this, I shouldn't be surprised by this. We are just noticing that robot is following the path of a straight line of slope one half: for each horizontal segment of length 4, the matching vertical segment to return to the line is 2 units long.

$$\frac{rise}{run} = \frac{\text{vertical shift}}{\text{horizontal shift}} = \frac{2}{4} = \frac{1}{2}$$

But I like this idea of explicitly thinking about horizontal and vertical shifts.

Leaning into Shifts

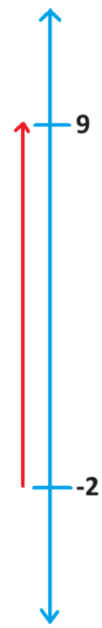
Here are some practice problems worth leaning into.

Practice 82.1

a) What value must I add to 3 on the number line to reach the number 7?

b) What value must I add to -2 on the number line to reach the number 9?

c) What value must I add to -4 on the number line to reach the number -9?



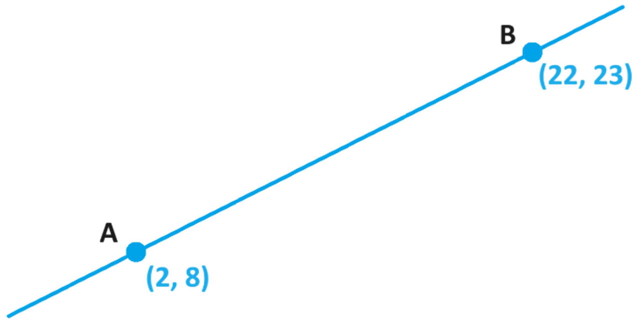
d) What number must I add to 100 to reach the number 120?

e) What number must I add to 100 to reach the number -120?

f) What number must I add to a number a on the number line to reach the number b? Does your answer here jibe with your answers to the previous five questions (especially part b), c) and e)?

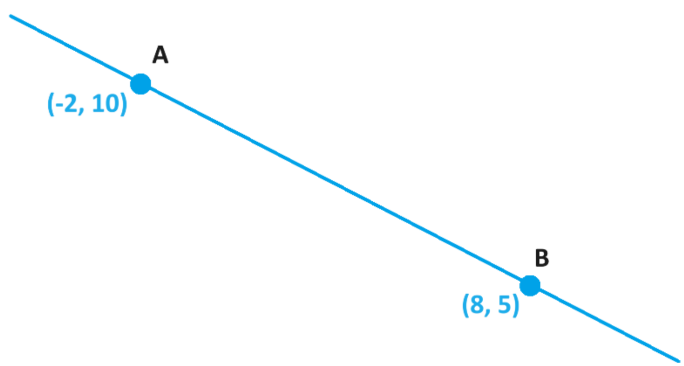
Practice 82.2

What is the horizontal shift from point A to point B?
What is the vertical shift from point A to point B?
What is the slope of the line on which these two points sit?



Practice 82.3

What is the horizontal shift from point A to point B?
What is the vertical shift from point A to point B?
What is the slope of the line on which these two points sit?

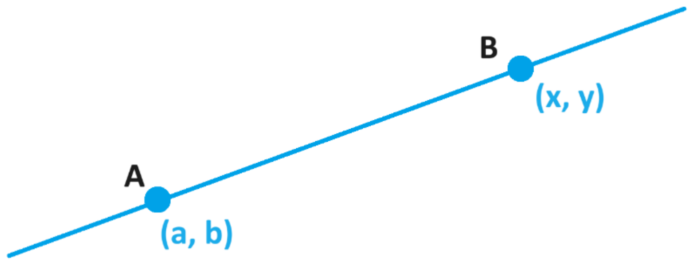


Now to be mind-bendy …

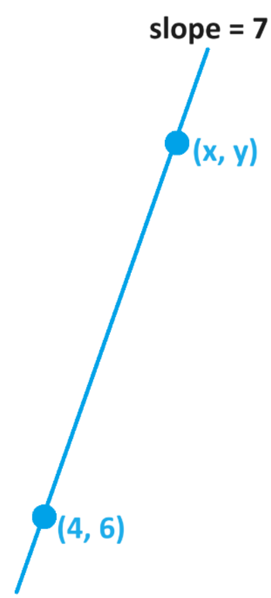
What is the horizontal shift from point B to point A?
What is the vertical shift from point B to point A?
Why is the value of $\frac{\text{vertical shift}}{\text{horizontal shift}}$ computed this backwards way the same value as you computed just above for the slope of the line?

Practice 82.4 *Being abstract …*

If (a, b) and (x, y) are two points on a line, what is a formula for the slope of the line on which they sit?



I'm liking this shift thinking!



Example: What must be true about two values x and y for the point (x, y) to lie on the line of slope 7 that passes thought the point $(4, 6)$?

Answer: We are told that for any two points on the line

$$\frac{\text{vertical shift}}{\text{horizontal shift}} = 7$$

In particular, for the two points mentioned, we must have

$$\frac{y-6}{x-4} = 7$$

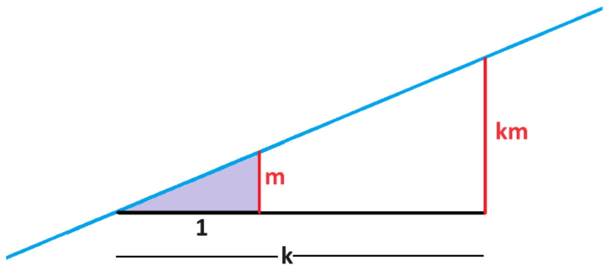
This is an equation that must be true for the point (x, y) to lie on the line described … almost!

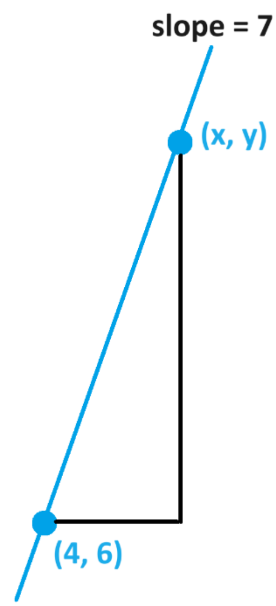
There's a problem with what we have: the point with $(4,6)$ is on the line but the equation $\frac{y-6}{x-4} = 7$ is meaningless if x is the number 4 and y is the number 6.

We need an equation that works for all potential points (x, y) on the line.

Hmm.

Let's go back to the multiplicative thinking of the last section. If a line has slope m, then a horizontal segment (shift) of a certain length will be matched with a vertical segment (shift) of length m times as long.





In our example, this means that our vertical shift $y - 6$ is seven times as long as our horizontal shift $x - 4$. This gives the equation

$$y - 6 = 7(x - 4)$$

And this equation holds true even for the data point $(4, 6)$.

Wonderful!

Practice 82.5

a) Verify that setting $x = 4$ and $y = 6$ in the equation $y - 6 = 7(x - 4)$ does indeed give a true math sentence.

b) If $x = 5$, what value for y then makes the equation a true?

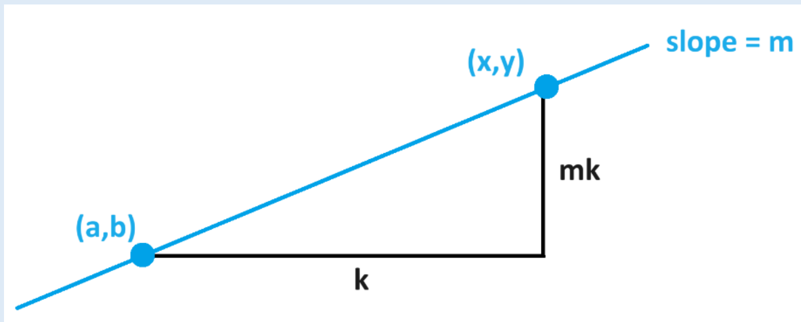
c) If $y = 20$, what value for x makes the equation true?

d) Is the point $(0, -22)$ on the line?

e) What is the point on the line that has second coordinate equal to zero?

We've discovered the **equation of a line**.

If a line of slope m passes through a point (a, b), then for another point (x, y) to be on the line, we need the vertical shift between the two points to be m times the horizontal shift between the points.



This leads to the equation

$$y - b = m(x - a)$$

Any data point (x, y) that makes this equation a true sentence is a point on the line, and vice versa, any point on the line represents a data point that makes the equation true.

Example: Find the equation of the line that passes through the points $(-1,2)$ and $(3, 10)$.

Answer: These two points have

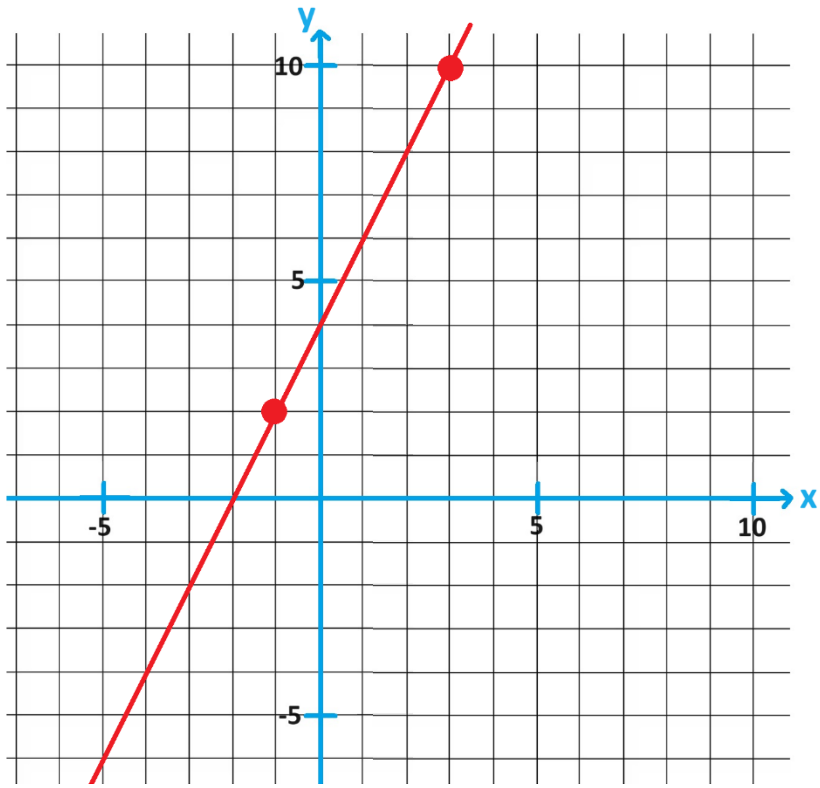
horizontal shift $= 3 - (-1) = 4$

vertical shift $= 10 - 2 = 8$

So, the slope of the line is

$$\frac{rise}{run} = \frac{\text{vertical shift}}{\text{horizontal shift}} = \frac{8}{4} = 2$$

So, we seek the equation of the line of slope 2 that passes through the point $(-1,2)$ and the point $(3,10)$.



Let's work with the point $(-1,2)$. To shift to another point (x, y) we need the vertical shift to be double the horizontal shift. This gives the equation

$$y - 2 = 2\big(x - (-1)\big)$$

that is,

$$\boldsymbol{y - 2 = 2(x + 1)}$$

That's the equation of the line.
Done!

But I am worried! What if we used the point $(3,10)$ instead? That seems to give a different equation for the line, namely.

$$\boldsymbol{y - 10 = 2(x - 3)}$$

Which equation is correct?

Well, let's play with each of the equations.

First:

$$y - 2 = 2(x + 1)$$

$$y - 2 = 2x + 2$$

$$y - 2 + 2 = 2x + 2 + 2$$

$$y = 2x + 4$$

Our first equation is equivalent to the equation $y = 2x + 4$.

Now the second equation:

$$y - 10 = 2(x - 3)$$

$$y - 10 = 2x - 6$$

$$y - 10 + 10 = 2x - 6 + 10$$

$$y = 2x + 4$$

Our second equation is also equivalent to $y = 2x + 4$.

So, we have three equivalent equations for this line:

$y - 2 = 2(x + 1)$ and $y - 10 = 2(x + 3)$ and $y = 2x + 4$

Practice 82.6

a) The point $(100, 204)$ also happens to be on the line. Show that the equation $y - 204 = 2(x - 100)$ is also equivalent to $y = 2x + 4$.

b) **OPTIONAL CHALLENGE:** Show, in general, that if (p, q) is any other point on the line, then the equation $y - q = 2(x - p)$ is sure to be equivalent the equation $y = 2x + 4$ as well!

Schoolbooks typically use only the letters x and y to represent the unknowns in an equation and, when drawing graphs of such equations, always set the horizontal axis as the x-axis and the vertical one as the y-axis. I've been following that practice too in this section.

They also prefer to rewrite equations, if it is possible, in the form

$$y = \text{something involving } x$$

Such a form of an equation is often called the **standard form** of the equation.

We've seen several different, but equivalent, equations for the line in our worked example. But they all led to the same standard form equation.

$$y = 2x + 4$$

So that is it clear we all recognize and can talk about the same equation, schoolbooks will insist that the equation of a line be presented in standard form, if possible.

The **standard form** of a line is the equation for the line written as

$$y = mx + b$$

for some numbers m and b
(assuming that the equation can be rewritten to follow this form).

Practice 82.7 Look back at practice problem 81.10.
Can the equation of a horizontal line be written in standard form?
Can the equation of a vertical line be written in standard form?

School books go to great lengths to make sure students have a strong grasp of the "meaning" of the numbers m and b that appear in the standard form $y = mx + b$ of the equation of a line.

Practice 82.8 Consider the equation $y = 5x + 3$ of a line. The graph of this equation is a line.

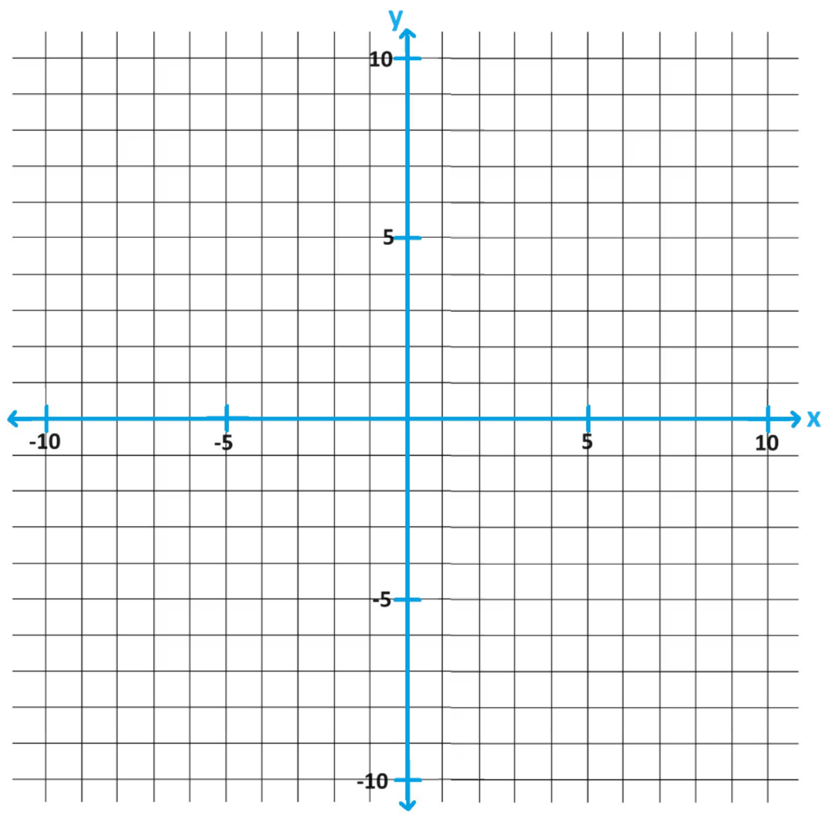
a) When $x = 0$, what is the appropriate value for y that makes the equation true?
Plot this point on the graph paper.

b) When $x = 1$ (a horizontal shift of one unit from part a), what is the appropriate value for y that makes the equation true? Plot this point on the graph paper.

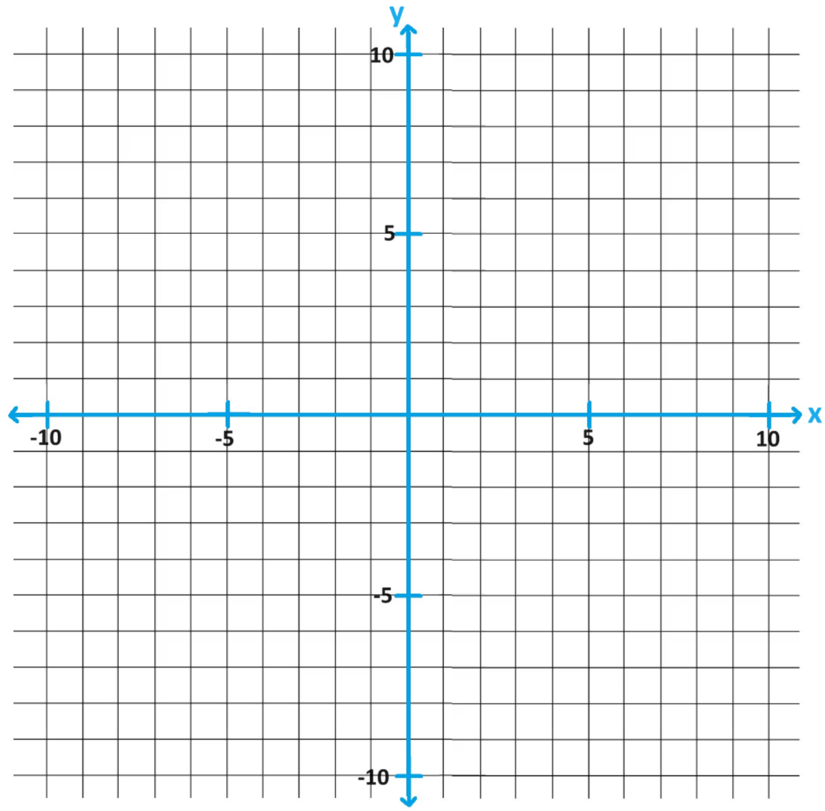
What vertical shift does your answer here represent compared to your answer to part a)?

c) Draw in the line that passes through these two points you found in parts a) and b).
This line is the graph of $y = 5x + 3$.

d) What is the slope of the line? Through which number does the line pass on the vertical axis?



Practice 82.9 Repeat the previous question for the line with equation $y = -2x + 4$.



Practice 82.10 Consider the equation of a line $y = mx + b$ where m and b are fixed numbers.

a) Show that the point $(0, b)$ is on the line.

b) Show that the point $(1, m + b)$ is on the line.

c) What is the horizontal shift in going from $(0, b)$ to $(1, m + b)$? What is the vertical shift?

d) What is the slope of the line?

e) Through which number on the vertical axis does the graph of $y = mx + b$ pass?

We've just learned that for an equation of a line presented in standard form,

$$y = mx + b$$

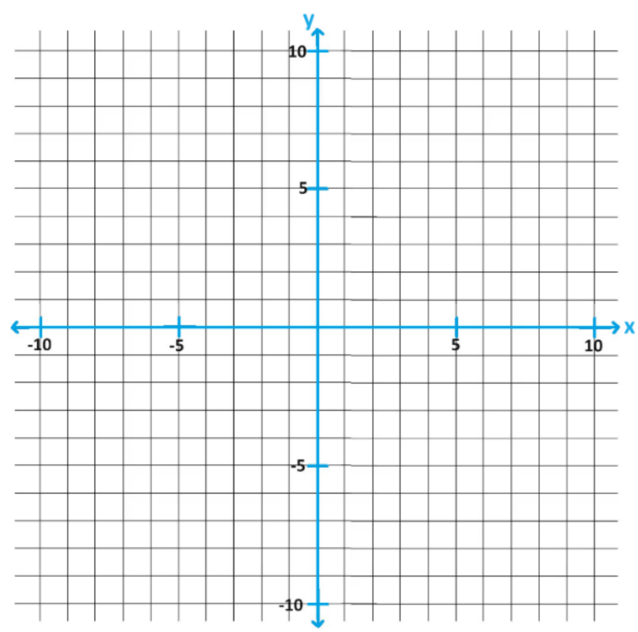
we have

m corresponds the slope of the line
b is the number at which the line crosses the vertical axis (the y-axis)

Practice 82.11
a) Give an example of a line that has an equation that cannot be rewritten in standard form.
b) A horizontal line passes through the point $(7,8)$. Write an equation for that line that is in standard form.

Let's bring in all the school jargon. There is a lot of if!

Recall that schoolbooks have students draw graphs with the horizontal axis labeled x, the vertical axis labeled y, which means that equations are always presented with the letters x and y as the unknowns.



The horizontal axis is thus called the **x-axis** and the vertical one the **y-axis**.

Labeling the axes in this order gives the sense that x is the "control variable" and the that y is the "response variable." This idea is reinforced by writing equations in standard form.

$$y = \text{something involving } x$$

Such an equation feels as though it is saying “when given a value for x, here’s what you need to do to compute the matching value of y.”

With this setup, schoolbooks tend to call x the **independent variable** and y the **dependent variable** (its value seems to depend what value is given to x).

For a point (a, b) plotted on the grid paper, the first coordinate a is often called the **x-coordinate** of the point and the second coordinate b the **y-coordinate** of the point.

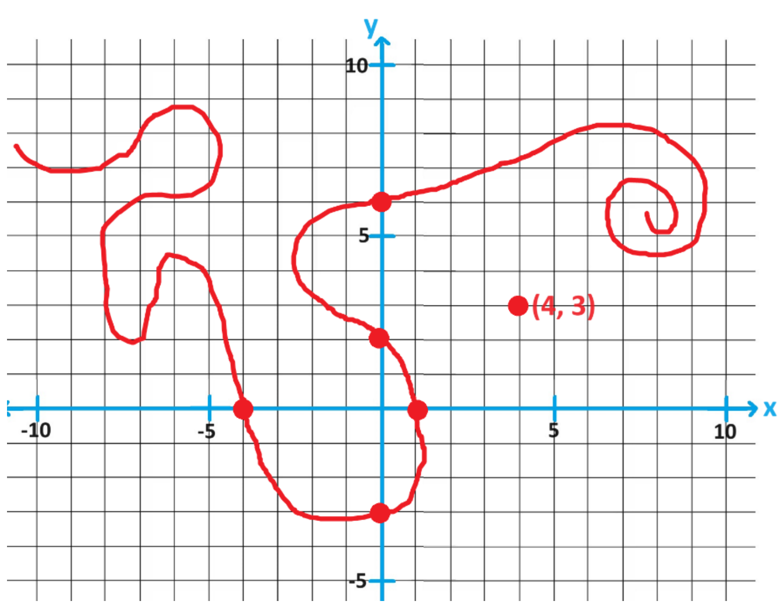
The set of all points that represent data making the given equation true is, of course, the graph of the equation. When given a picture of a line drawn on grid paper, to “find the equation of the line shown” means to find an equation whose graph matches the picture given.

If the graph of an equation crosses the vertical axis, the y-axis, at some number b, then that number is called a **y-intercept** of the graph. People write: “$(0, b)$ is a y-intercept” or “a y-intercept is $y = b$” or just “a y-intercept is b.”

If the graph crosses the horizontal axis, the x-axis, at some number a, then that number is called an **x-intercept** of the graph. People write: “$(a, 0)$ is an x-intercept” or “an x-intercept is $x = a$” or just “an x-intercept is a.”

Example: In this picture, the graph has x-intercepts -4 and 1, and y-intercepts -3, 2, and 6.

The point with x-coordinate 4 and y-coordinate 3 is not on the graph.



Students are often taught a "quick" way to sketch the graph of a line.

Example: Sketch a graph of the line $y = 3x - 5$.

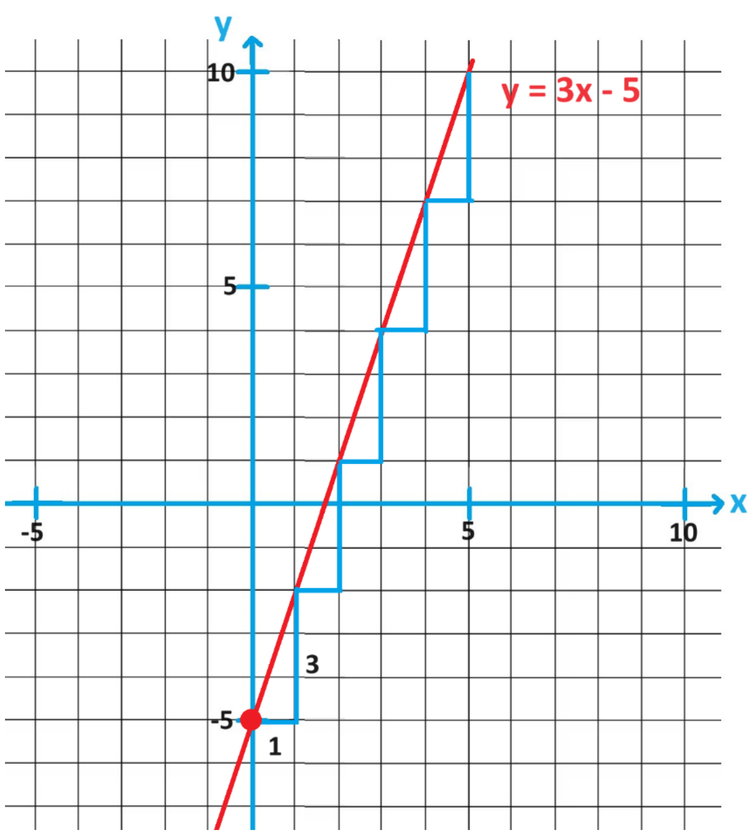
Answer: Thinking of this equation as

$$y = 3x + (-5)$$

we see we are dealing with a line of slope 3 and y-intercept -5.

So, we know the line crosses the vertical axis at the number -5, and to each 1 step rightward, we need to step 3 units upwards to return to the line.

We're set to sketch!



On another note …

I find the schoolbook insistence on always using the letters x and y for the names of the unknowns somewhat unnatural.

Example: I can purchase one gumball for 10 cents, two for 20 cents, fifty for 500 cents (that is, for \$5), seven thousand for $70{,}000$ cents (that is, \$700), and so on. There is no discount for purchasing large quantities of gumballs, alas.

There is natural data to collect in this scenario: the number of gumballs I could purchase and the matching cost of that purchase.

i) If I were to graph this data, why, philosophically, must the data lie on a straight line?

ii) What is the slope of the line on which the data lies?

iii) What are the intercepts of the line?

iv) What is the equation of the line on which the data sits?

Answer: I would personally like to call the number of gumballs purchased N and the cost of that purchase C, but let's follow schoolbook practice and use the letters x and y, respectively, instead. (In this scenario, I am in control of the number of gumballs I purchase, so it does seem appropriate to view that as the control variable here.)

x	y
1	10
2	20
50	500
7,000	70,000
0	0

i) and ii): Let's imagine shifting from one data point to another.

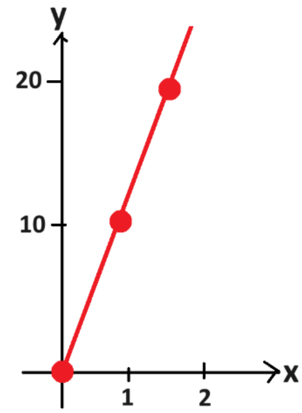
If I were to purchase one more gumball, then the amount I pay increases 10 cents.

That is, each horizontal shift of 1 unit in the data induces a vertical shift of 10 units.

This is precisely our understanding of what makes a line straight.
The data sits on a straight line of slope 10.

iii) If I purchase 0 gumballs, I owe 0 cents. I added this data point to the table.

We can see that the line has x-intercept 0 and y-intercept 0.



iv) The standard form of a (non-vertical) line is

$$y = mx + b$$

with m the slope of the line and b its y-intercept.

Thus, the equation of the line in this example is $y = 10x + 0$, which can be rewritten:

$$y = 10x$$

By the way, economists call the additional cost induced by purchasing just one more item the **marginal cost** in the scenario. This provides additional internet theory as to what the letter m is used as the symbol for slope.

Your turn:

Practice 82.12: I can rent a jeep for \$100 down and then \$30 per day for each day of use. (Just to be clear, if I enter into this agreement, as soon as I sign, I pay \$100. Even if I decide not to rent after all after this, that is, I rent the jeep for 0 days, it will still cost me \$100.)

There is natural data to collect in this scenario: the number of days I rent the jeep, and the cost of doing so.

i) If I were to graph this data, why, philosophically, must the data lie on a straight line?

ii) What is the slope of the line on which the data lies?

iii) What is the equation of the line on which the data sits?

iv) What are the intercepts of the line? (Are they relevant and meaningful for this scenario?)

MUSINGS

Musing 82.13 In this section we looked for the equation of the line of slope 7 that passes through the point $(4,6)$.

We first wrote down this equation

$$\frac{y-6}{x-4} = 7$$

but then settled on this equation

$$y - 6 = 7(x - 4)$$

Sketch a graph of each of the two equations. How do they differ?

Musing 82.14 The point A has coordinates $(-4, 7)$ and the point B coordinates $(2, -5)$.
What are the coordinates of the one-third of the way along the line segment that connects A and B, closer to A than it is to B?

MECHANICS PRACTICE

Practice 82.15 On grid paper draw an example of a line of …

a) positive slope b) negative slope c) slope zero d) slope 1 e) slope $-1{,}000{,}000$

Practice 82.16 Consider the line with the equation $y - 6 = 7(x - 4)$.

a) Is the point $(5,11)$ on the line?
b) Is the point $(3, -1)$ on the line?
c) What is the line's x-intercept?
d) What is the line's y-intercept?
e) What is the equation of the line in standard form?
f) Is there a point on the line with x- and y- coordinates the same value? If so, what is that point?

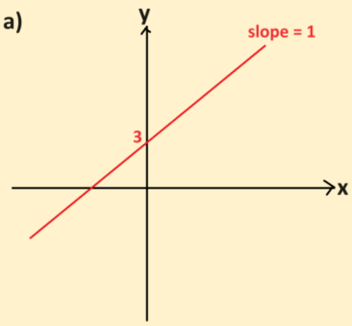
Practice 82.17 Find the equation of the line that passes through the points $(-1, 2)$ and $(4,6)$.

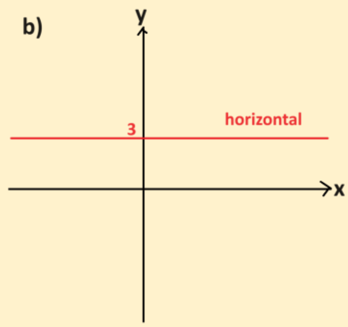
Practice 82.18 A line has equation $y = -2x + 2$.

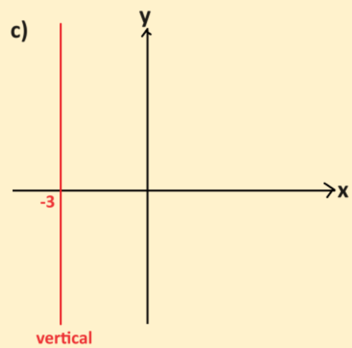
a) What is the slope of the line?
b) What is the y-intercept of the line?
c) Make a quick sketch of the graph of the line.
d) What is the x-intercept of the line?

Practice 82.19 A line has equation $y - 5 = 3(x - 2)$.
What is the slope of the line and what is a point on the line?

Practice 82.20 Write down an equation for each of these lines.

a)


b)


c)


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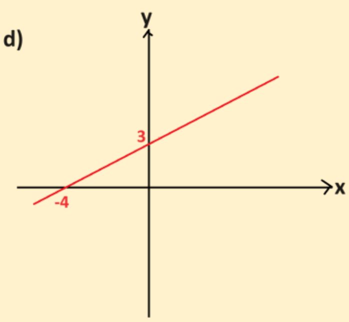

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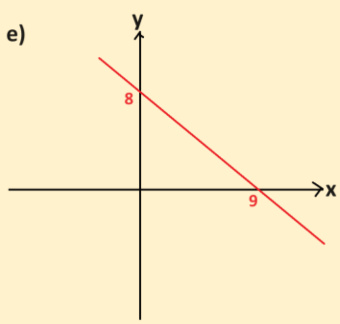

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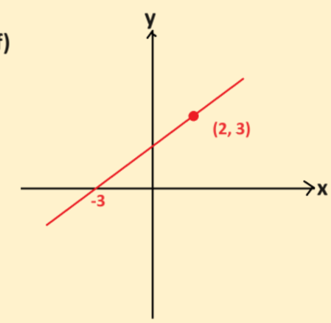


Practice 82.21 A line has x-intercept 5 and y-intercept -3. What is the equation of the line?

Practice 82.22 A line of slope -3 passes through the points $(2,5)$ and $(4, k)$. What is the value of k?

Practice 82.23 Does $2x - 3y = 6$ represent the equation of a line?
If so, what are the x- and y- intercepts of the line, and how does the graph of the line appear?

[You can guess that the answer to the opening question is YES!]

Practice 82.24 I currently have \$102 in my piggy bank, and starting today, the first of the month, I am going to spend \$3 from it each day.

So, day 1 starts with \$102 in the bank, day 2 starts with \$99 in the bank, day 3 with \$96 in the bank, and so on.

a) On which day will I first see \$0 in the bank?

b) There is natural data to collect in this scenario: $(1,102)$, $(2,99)$, $(3,96)$, and so on.

Why, philosophically, are these data points sure to lie on a line?
What is the slope of that line?

c) What are the two intercepts of that line?

d) Find an equation for the line.

83. One Type of Equation for all Lines

If a line has a slope m and passes through a point (p, q), then

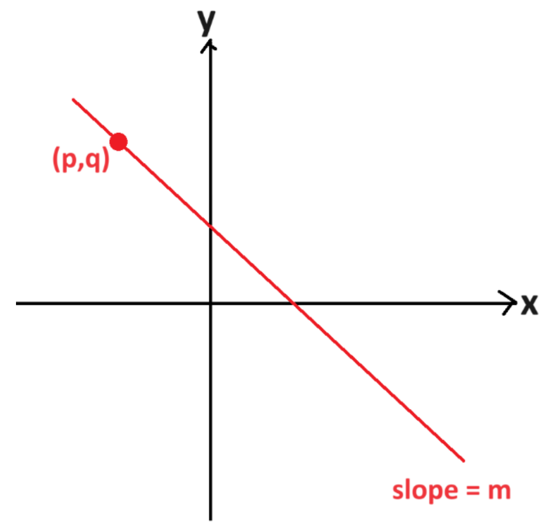
$$y - q = m(x - p)$$

is an equation whose graph matches the line.

We've seen that we can rewrite such an equation into an equivalent "standard form"

$$y = mx + b$$

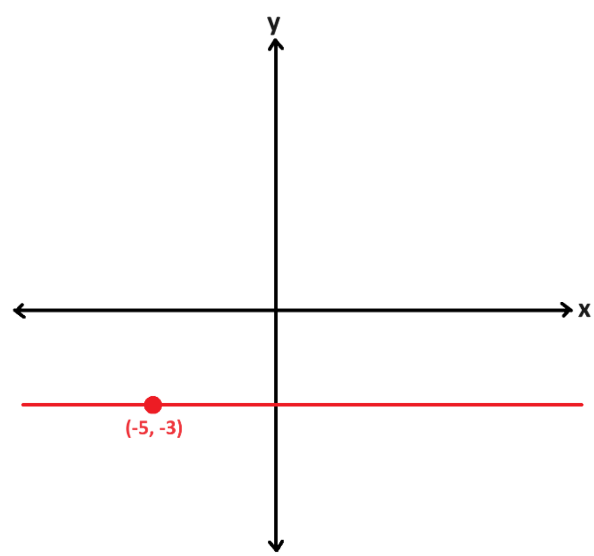
where m is still the slope of the line and b is the y-intercept of the line.



Practice 83.1 Write the equation of the line of slope -1.2 that passes through the point $(-3.4, 7.5)$ in standard form.

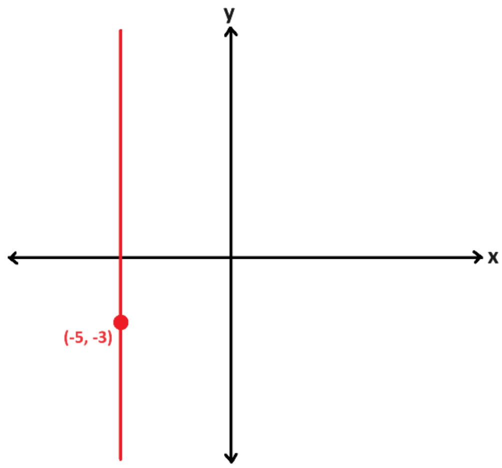
Horizontal lines have slope 0: shifting one unit horizontally from a point on a horizontal line requires shifting zero units vertically to return to the line.

Practice 83.2 Write the equation of the horizontal line through the point $(-5, -3)$ in standard form.



But we cannot write the equation of a vertical line in standard form.

Practice 83.3 Write the equation of the vertical line through the point $(-5, -3)$. What prevents you from writing the equation in standard form?



But people have noticed there is something in common to each type of equation we've just discussed: they can each be written in the form

$$Ax + By = C$$

for some numbers A, B, and C.

For example, the horizontal line of Problem 83.2 has equation $y = -3$, which can be expressed as

$$0 \cdot x + 1 \cdot y = -3$$

The vertical line of Problem 83.3 has equation $x = -5$, which can be expressed as

$$1 \cdot x + 0 \cdot y = -5$$

The line of Problem 83.1 has equation $y = -1.2x + 3.42$, which can be expressed as

$$1.2x + 1 \cdot y = 3.42$$

Every line—horizontal, vertical, and diagonal— can be expressed via an equation of **general form**

$$Ax + By = C$$

for some numbers A, B, and C.

No fuss is needed to distinguish between vertical and non-vertical lines.

Practice 83.4 Consider the line with equation given by

$$-4x + 5y = 60$$

a) What is the x-intercept of this line? What is its y-intercept?

b) Sketch a graph of the line.

c) What is the slope of the line?

d) What is the equation of the line in standard form?

When writing the equation of a line in general form $Ax + By = C$, people expect the numbers A and B to not both be zero (but it is fine for C to be zero).

Practice 83.5

a) Describe the graph of $0 \cdot x + 0 \cdot y = 6$.
b) Describe the graph of $0 \cdot x + 0 \cdot y = 0$.
c) Describe the graph of $x + y = 0$.

MUSINGS

Musing 83.6 Consider the equation of a line given by

$$x + y = 1$$

a) What are the two intercepts of this line?
b) Sketch a graph of this line.

Now …

c) Sketch a graph of $\frac{1}{x} + \frac{1}{y} = \frac{1}{xy}$.

Musing 83.7 Consider the general equation $Ax + By = C$ for some numbers A, B, and C.

a) If A is zero and B is non-zero, what can you say about the line?
b) If, instead, B is zero and A is non-zero, what can you say about the line?
c) What can you say about the line if both A and B are non-zero, but C is zero?

Musing 83.8 COMING FULL CIRCLE?

We started this chapter wondering if straight lines actually exist, which then led us to enquire what we mean by "straight" in the first place!

Have we come full circle?

Could we now say that "a curve drawn in the plane is called a **straight line** if it matches the graph of an equation of the form $Ax + By = C$ for three numbers A, B, and C with A and B not both zero" ?

Do we now, finally, have a solid definition of "straight"?

What do you think?

MECHANICS PRACTICE

Practice 83.9 Rewrite each of the equations in general form.

a) $y = 2x + 3$
b) $x = 9$
c) $x = y - 1$
d) $y = 0$
e) $y - 7 = -2(x - 5)$

Practice 83.10 Consider the equation of a line given by

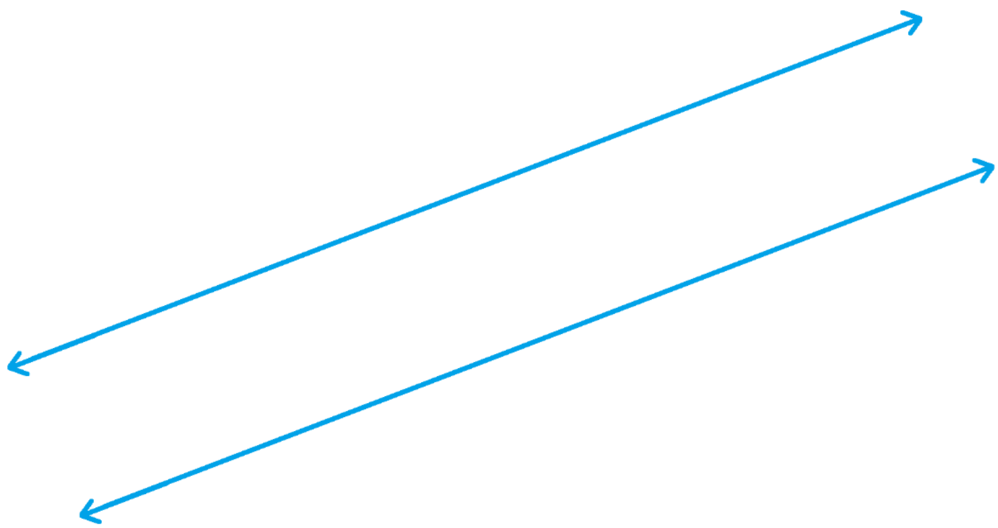
$$\frac{x}{2} + \frac{y}{3} = 1$$

a) What are its two intercepts?
b) Sketch a graph of the line.

Practice 83.11
Sketch a graph of the line with equation $\frac{x}{a} - \frac{y}{b} = 1$ for a and b are each a positive number.

84. Parallel Lines

In geometry class, one defines two straight lines drawn in an infinite plane to be **parallel** if they are sure to never meet.



My question: How can we ever be sure if two lines fail to meet? Do we have to check infinitely far in each of two directions to make sure they never cross?

That seems beyond human!

But maybe with some algebra under our belts, we can make sense of parallelism without having to do conduct work for an infinite amount of time.

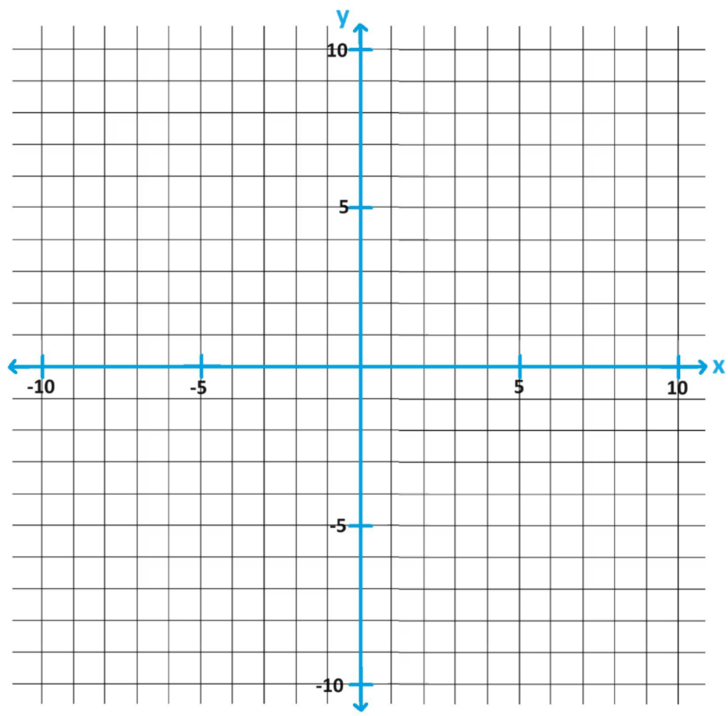
To explore what it means for two lines not to meet, let's make sure we understand what it means for when they do.

Practice 84.1 On the one grid below graph each of these two lines:

i) the line of slope 2 through the origin: $y = 2x$
ii) the line of slope 3 with y-intercept -2: $y = 3x - 2$

a) At which point do these two lines intersect?

b) Give a value for x and a value for y that make the math sentences $y = 2x$ and $y = 3x - 2$ simultaneously true.



If you conduct this exercise, you will see that the two lines intersect at the point $(2,4)$.

Thus $(2,4)$ is a data point that makes the equation $y = 2x$ true since it lies on the graph of this equation, and, simultaneously, makes the equation $y = 3x - 2$ true since it also lies on the graph of this second equation.

We have that $x = 2$ and $y = 4$ are values that make both sentences true at the same time.

Practice 84.2 Show that there is no pair of values, one for x and one for y, that make these two equations simultaneously true.

i) $y = 4x + 3$
ii) $y = 4x + 2$

Actually, let me answer this question right now.

Suppose there is a number p for x and a number q for y that makes each equation true.

We have

$$q = 2p + 3$$

and

$$q = 2p + 2$$

Let's add $-2p$ to both sides of each equation.

$$q - 2p = 3$$

$$q - 2p = 2$$

Whatever "$q - 2p$" is, it supposedly equals 3 and equals 2 at the same time.

That is impossible! There can be no values p and q that make both equations true at the same time.

We've just concluded that the two lines described in problem 84.2 cannot intesect.
The lines must be parallel!

Practice 84.3 Show that if two different lines have the same slope m, they must be parallel.

Hint: Suppose the lines have equations $y = mx + b$ and $y = mx + c$, with b different from c (otherwise, they would be the same line). Follow the approach above.

Practice 84.4 Explain why two different vertical lines cannot intersect.

Hint: What is the form of the equation for a vertical line?

We have just established:

If two different lines have the same slope, or are both vertical, then they are parallel.

Practice 84.5 Show that the lines given by these equations are parallel.

$$y + 2 = -3(x + 1)$$

$$3x + y = 7$$

Practice 84.6 Are the lines given by these two equations parallel?

$$5x - y = 10$$

$$y = 5(x - 2)$$

MUSINGS

Musing 84.7 (TOUGH!)
We showed that if two different lines have the same slope, they never meet and so must be parallel.

Let's now show that if two lines have different slopes, then they are sure to meet and so not be parallel. Do this as follows:

Consider lines with equations

$$y = mx + b$$
$$y = nx + c$$

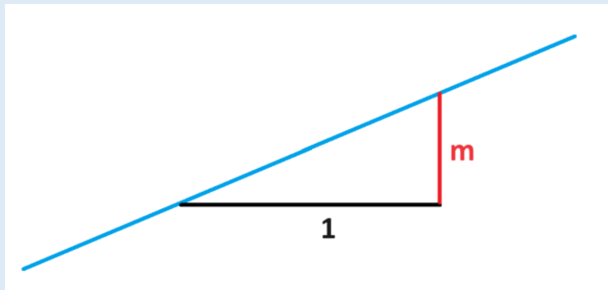
with slopes m and n different values.

We can be sure that $m - n$ is not zero.

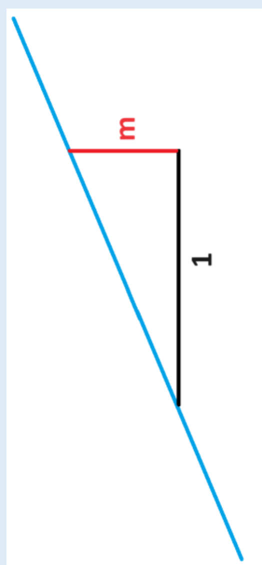
Show that setting x to have value $\frac{c-b}{m-n}$ and y to have value $c - b$ makes each equation true.

This means that the two lines intersect at the point $(\frac{c-b}{m-n}, c - b)$ and are thus not parallel.

Musing 84.8 Here's a picture of a line of slope m: shifting one unit to the right horizontally requires a vertical shift of m units to return to the line.



Let's now rotate the diagram $90°$ to give a picture of a line **perpendicular** to our original line. (I was a bit silly and rotated the letter m and the digit 1 as well!)



Geometry textbooks say that if a line has slope m, then a line perpendicular to it has slope $-\frac{1}{m}$. Do you see that this is the case in the picture?

MECHANICS PRACTICE

Practice 84.9 Do the lines given by the equations $y - 8 = 3 - x$ and $x + y = 10$ meet at a point?

Practice 84.10 Find a "simultaneous solution" to the equations:

$$x = y$$

$$y = 2x - 3$$

(Can you guess what this question is asking?)

85. Solving Equations Simultaneously via Graphing

Algebra books are chock full of problems of the following ilk.

Example: Two rental car companies, *Cars R Us* and *Drive-o-Rama*, offer the following rates.

Cars R Us: Deposit of \$100 and \$30 for each day of rental.
Drive-o-Rama: Deposit of \$200 and \$30 for each day of rental.

a) If you intend to rent a car for 7 days, which company is cheaper for you?

b) If, on a later trip, you intend to rent a car for 14 days, which company is cheaper for you?

c) Is there a "cross-over" point – a certain number of days where the total rental charge will be the same from each company?

Answer: Let C represent the total rental charge for renting a car from *Cars R Us* for n days, and D the matching rental cost from *Drive-o-Rama.*

We have

$$C = 100 + 30n \text{ dollars}$$

$$D = 200 + 20n \text{ dollars}$$

(Do you agree? Make sure you really do! Do each of these formulas make sense for renting a car for $n = 0$ days? For $n = 1$ day? For $n = 2$ days?)

a) For $n = 7$ days of rental,

$$C = 100 + 30 \times 7 = 310 \text{ dollars}$$

$$D = 200 + 20 \times 7 = 340 \text{ dollars}$$

Going with *Cars R Us* is a better deal.

b) For $n = 14$ days of rental,

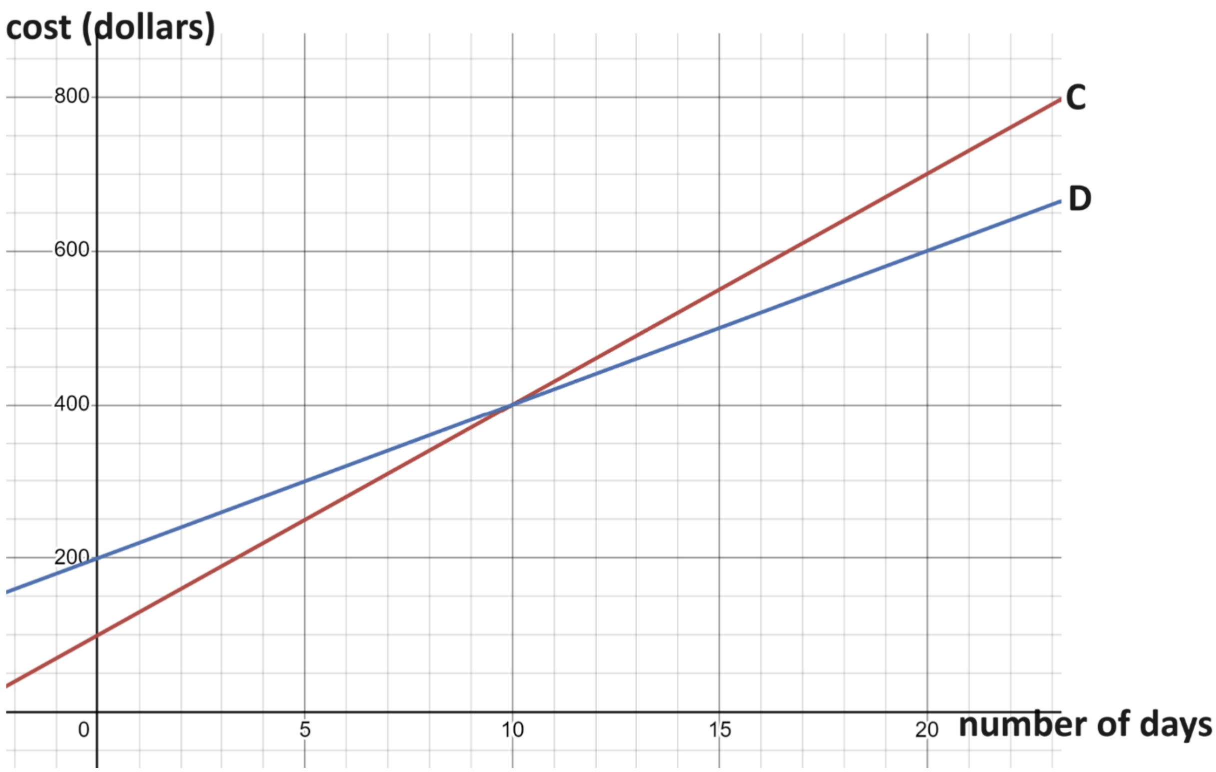
$$C = 100 + 30 \times 14 = 520 \text{ dollars}$$

$$D = 200 + 20 \times 14 = 480 \text{ dollars}$$

Going with *Drive-o-Rama* is a better deal.

c) If you graph these two equations (they are both equations of lines) we see something interesting.

(I used an online graphing tool this time, and then labeled the axes appropriately.)



The two lines seem to meet at the point $(10{,}400)$.

To check, for day $n = 10$ we have

$$C = 100 + 30 \times 10 = 400 \text{ dollars}$$

$$D = 200 + 20 \times 10 = 400 \text{ dollars}$$

Yep, 10 days of rentals is a cross-over point: If renting for less than ten days, go with *Cars R Us*, they are cheaper. If renting for more than ten days, going with *Drive-o-Rama* is cheaper. If renting for exactly ten days, flip a coin—the cost will be the same for the two.

Practice 85.1 I am looking for two numbers that sum to 100 with one number double the other. (I don't know why, but I am.)

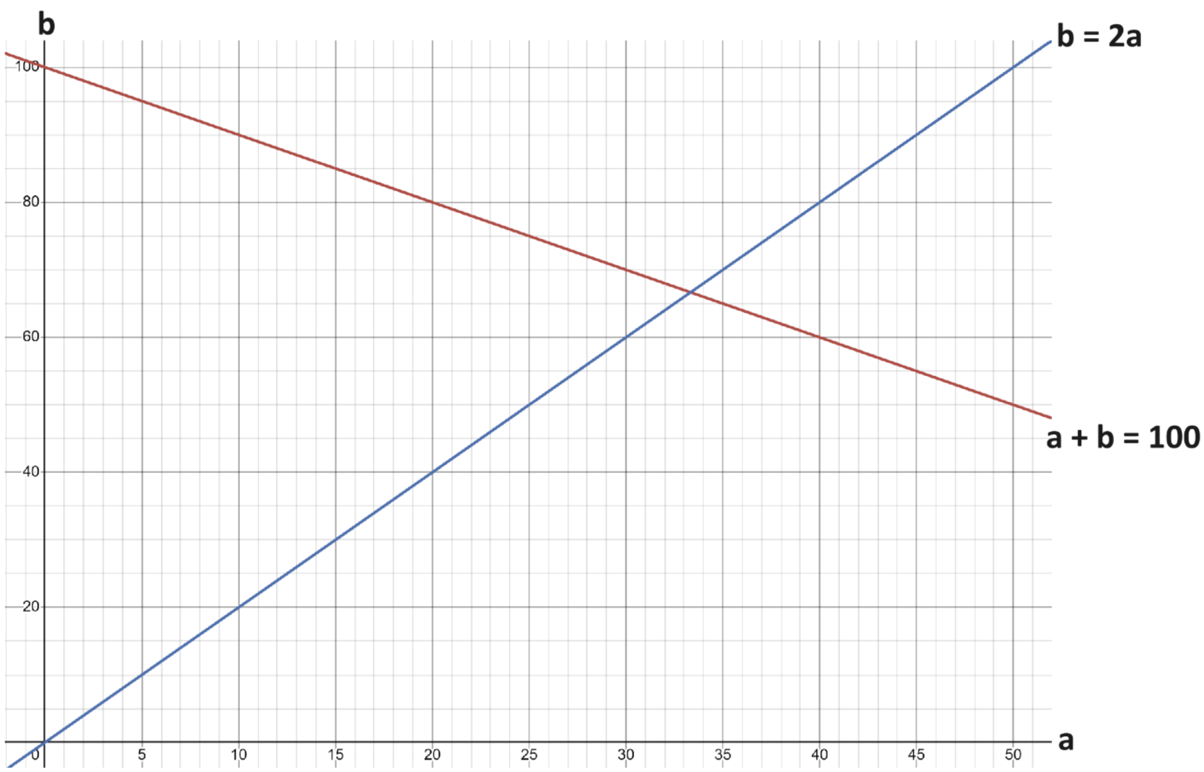
I decided to call the two numbers a and b and write down two equations I want to be true about these numbers:

$$a + b = 100$$

$$b = 2a$$

I then used technology to graph each of these equations.

Is the graph revealing the two numbers I seek?



Here's the sort of test question that gives math a bad name. (Who ever speaks this way?)

Example: In a certain household there are some humans and some cats.

Each human has two legs and two ears.
Each cat has four legs and two ears.
Each human has one nose, as does each cat.

There are a total of 20 ears and 32 legs in the household.

How many noses are there in total? How many of those are cat noses?

Let me answer this the common-sense way.

Each being has two ears and one nose. That there are 20 ears in the household then tells me that there are 10 beings in the household, and so 10 noses altogether.

Each being has two legs emanating from its hips. With a total of 10 beings, this accounts for 20 legs. The extra 12 legs must be front legs of cats and so we conclude that there are 6 cats.

Answer: There are 10 noses altogether: 6 are cat noses and 4 are human noses.

Practice 85.2 Let's now answer this questions the schoolbook algebra way.

Let h represent the number of humans in the household and c the number of cats.

a) Write an equation that represents the statement "There are a total of 20 ears in the household."

b) Write an equation that represents the statement "There are a total of 32 legs in the household."

c) Use technology (or by hand) graph both of these equations on grid paper.

d) Does your picture reveal the solution to the puzzle?

MUSINGS

Musing 85.3 Are there two numbers that sum to 100 with one number being the square of the other?

MECHANICS PRACTICE

Practice 85.4 Following the opening example of this section …

A third car rental company, *Just Drive*, has opened up business. It charges $40 a day for rental with no deposit.

a) You intend to rent a car for 7 days. Which company, *Cars R Us*, *Drive-o-Rama*, or *Just Drive* offers the best deal for you?

b) You next intend to rent a car for 14 days. Which company, *Cars R Us*, *Drive-o-Rama*, or *Just Drive* offers the best deal for you?

c) Write equations that give the cost incurred for renting a car for n days from each of the three companies and graph each of the equations on the same grid paper. What do you notice?

Practice 85.5 Can you think of two numbers that sum to 20 and differ by 13?

Even if you can, also solve this question by writing down two relevant equations, graphing on the same grid paper, and examining what the picture reveals for you.

Practice 85.6 Priyanka is walking along a straight road at a constant rate of 3.5 miles per hour. Gordon is 1 mile ahead of her along the road and is walking at a constant rate of 3 miles per hour.

Will Priyanka catch up to Gordon? If so, how do you know and how long will it take her to catch up?

a) Is there a "common sense" way to think through and answer this problem?
b) The schoolbook approach would want you to write two relevant equations, graph them, and determine the answer to the question from the graph. Conduct that approach too!

Practice 85.7 Yesterday, Albert stuffed and addressed $1{,}200$ envelopes. He'll continue the job today at a constant rate of attending to 120 envelopes per hour, nonstop.

Bilbert is new to the job today. He hasn't yet stuffed and addressed any envelopes but will work at this job today at double the rate Albert will, also nonstop.

Is there a "crossover point," a moment when Bilbert will have stuffed and addressed just as many envelopes as Albert up to that point? If so, when?

86. Solving Equations Simultaneously without Graphing

An equation with two unknowns is called **linear** if its graph is a straight line.

For example, any equation that can be written or rewritten in the form $Ax + By = C$ (with x and y the unknowns) is linear.

But people actually use the word *linear* for <u>any</u> equation that can be written in this form, even if there are more than two unknowns. For example,

$$2x + 6y - 8z = 19$$

and

$$76a - \frac{1}{2}b + 44c + 98d - 0.09e + 11f + \sqrt{2}g = 119\frac{3}{4}$$

are linear equations. (Good luck making sense of what it means to graph these!)

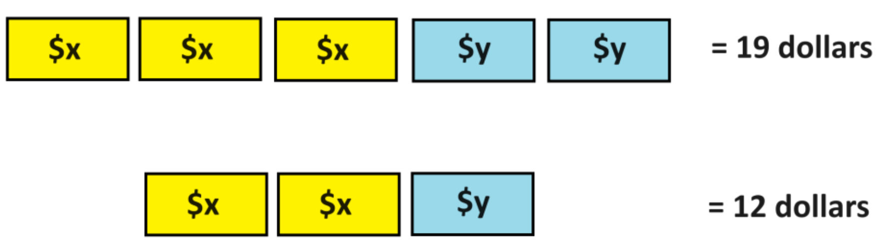
We've used graphing to find simultaneous solutions to pairs of linear equations in two unknowns. That was grand and good, but it is tedious: it's hard to draw careful graphs, especially if no technology is close at hand.

Fortunately, there is an easier way to look for simultaneous solutions to sets of linear equations.

Let's start with a purely visual approach.

Try this problem on your own before turning the page.

Example: We have \$x dollar bills and \$y dollar bills and the following information about them.

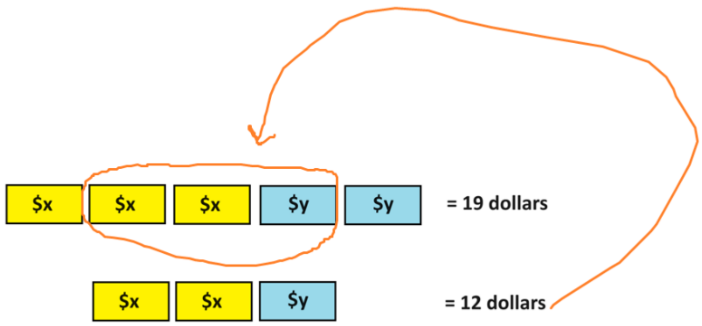


Just staring at the picture …

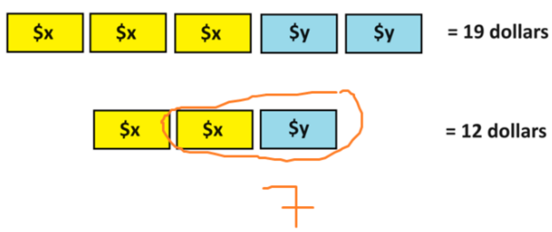
i) What's the value of $x + y$?
ii) What then is the value of x?
iii) What then is the value of y?

Here's what I saw.

First, $x + y$ must by 7.



Then, knowing that, I saw that x must be 5.



And that then forces y to be 2.

I think it is kinda cool we just solve a system of two linear equation in two unknowns as though it was a logic puzzle.

$$3x + 2y = 19$$

$$2x + y = 12$$

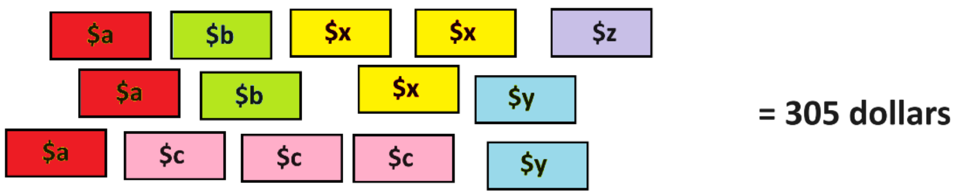
Practice 86.1 Graph these two lines together. Do they indeed intersect at the point $(5,2)$?

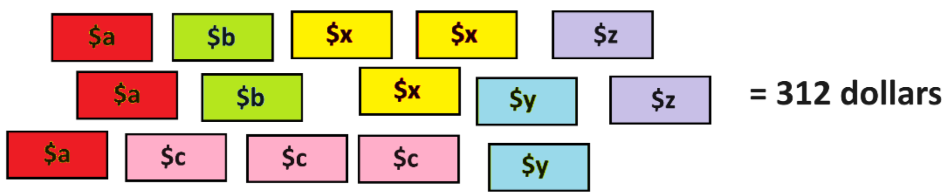
Let's now have some fun honing our logical and visual thinking skills to get on top of simultaneous linear equations and make sense of the algebra that this visual thinking suggests.

In what follows, the values of x-dollar and y-dollar bills will vary from problem to problem.

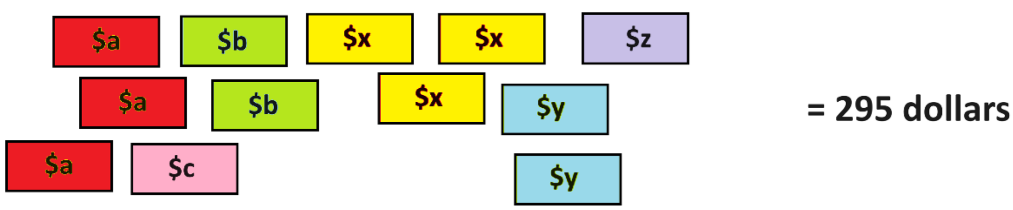
Practice 86.2 A certain country has six different types of bills. For some reason citizens call them a, b, c, x, y, and z dollar bills and mark them with just those letters, not by the number of dollars each letter represents. (Weird!)

a) Here are two pieces of information. Can you deduce any useful about this nation's currency?





b) Here's an additional piece of information. What can you deduce now?



Practice 86.3 What must be the value of h if these two linear equations are simultaneously true?

$$2a + 5b + 18c + 7d + e + 92f + 100g + 86h = 987$$

$$2a + 5b + 18c + 7d + e + 92f + 100g + 87h = 997$$

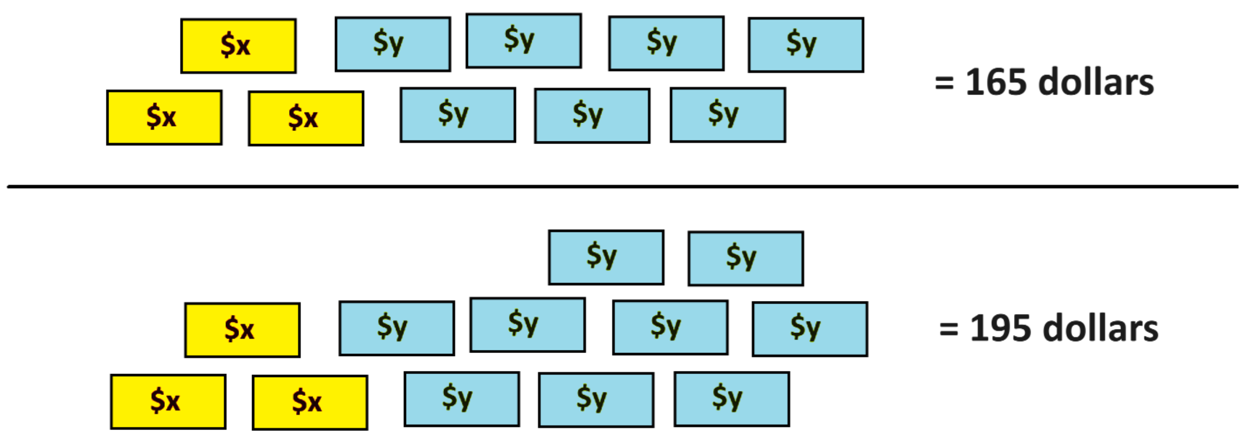
Practice 86.4 What must be the value of w if these two linear equations are simultaneously true?

$$98p + 4w + 8z = 17$$

$$98p + 9w + 8z = 57$$

Schoolbooks usually have students work only with equations containing two unknowns.

Practice 86.5 Following the currency of the previous page, find the value of x and y given these two pieces of information.



We can present this example in a schoolbook way.

Example: Kindly solve this system of linear equations

$$3x + 7y = 165$$

$$3x + 9y = 195$$

Your answer to Problem 86.5 should show that $y = 15$ and $x = 20$.

Let's keep going.

Practice 86.6 We don't yet know the values of a and b in our currency example.

Here's two pieces of information about them.

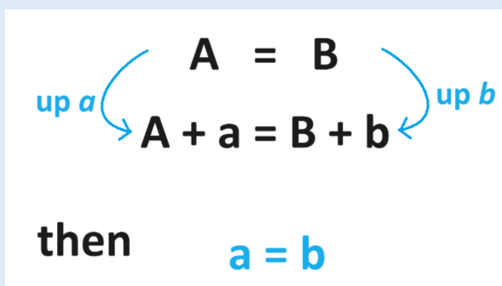
$$2a + 7b = 221$$

$$12a + 7b = 1221$$

a) Can you imagine the picture that goes with each of these equations?
b) What is the value of a and what is the value of b?

We're using the following idea about equality.

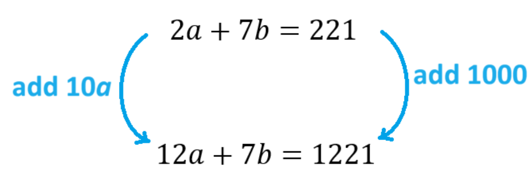
**If we have equality between two quantities,
and if we increase each by a certain amount and equality remains true,
then it must be that the increments match.**



For instance, in Problem 86.6, we have the equality

$$2a + 7b = 221$$

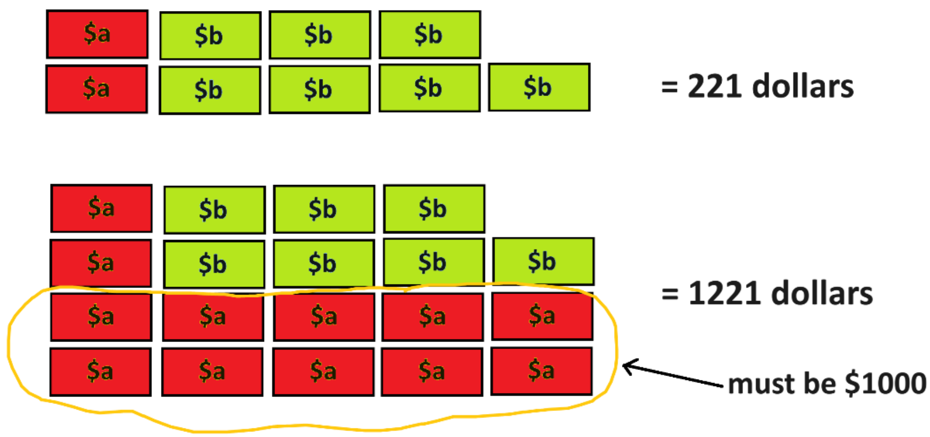
And we are told that increasing the left side by $10a$ and the right side by 1000 again yields equality:



$$12a + 7b = 1221$$

We deduce then that $10a = 1000$ and so $a = 10$. (And knowing a represents the number 100, we then see that $200 + 7b = 221$, giving that b must be 3.)

This new observation about equality makes sense with our dollar bill thinking.



But we can also deduce this observation as a logical consequence of what we already believe about equality as outlined in Section 76. (Seeing this removes us from relying on a single real-world model to "explain" mathematics. As we have learned, mathematics is sufficiently robust to justify itself!)

Explaining the Observation Mathematically (if you are curious):
If we have an equality $A = B$, then we know from Section 76 that for any number a, the equality

$$A + a = B + a$$

holds too.

But we are being told that $A + a = B + b$ is true as well, for some number b.

Hmm.

So, $A + a$ equals $B + a$ and it also equals $B + b$. We must have

$$B + a = B + b$$

Adding $-B$ to each side of this equality gives us

$$a = b$$

Practice 86.7 When Aparna was asked to solve this system of equations,

$$3x + y = 5$$
$$3x + 2y = 7$$

she immediately deduced that y must be 2. (Do you agree?)

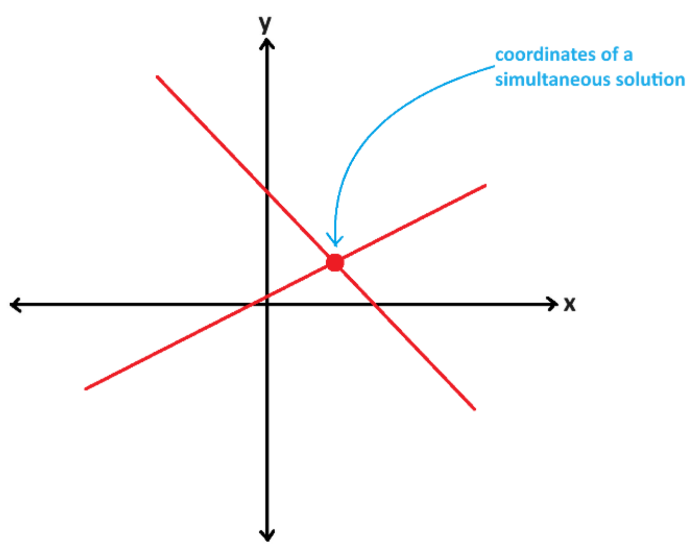
Then she looked at the first equation and deduced that x must have the value 1 (do you again agree?) yielding $x = 1$ and $y = 2$ being the values that make the two equations simultaneously true.

But what if she looked at the second equation instead to find the appropriate value of x? Would she still have deduced that x is required to have the value 1?

This problem brings up an important point:

Suppose we are given a system of two linear equations in two unknowns, and we seek a simultaneous solution to them.

Our work on graphing showed that a simultaneous solution corresponds to the coordinates of the point where the graphs of the lines intersect. (So, we better hope we don't have two parallel lines!)



Thus, if we know one of the coordinates of this special point, then it doesn't matter which of the two lines we work with to find the other coordinate: that point of intersection lies on both lines, so just pick whichever one you feel like working with! You'll get the coordinates of that <u>one</u> point of intersection either way.

Okay. We're now set to solve all the usual schoolbook examples thrown students' ways.

Try this series of problems. You'll still need your wits—but that's what actually makes these (otherwise tedious) problems fun!

Practice 86.8 Please find values for x and y that make these two linear equations simultaneously true.

$$3x + 5y = 5$$
$$3x + 10y = 25$$

Practice 86.9: Solve the following system of equations.

$$3x + 5y = 60$$
$$3x - 2y = 18$$

Practice 86.10: Solve the following system of equations.

$$x + y = 3$$
$$3x + 4y = 11$$

Remember: If there is something in life you want, make it happen! (And deal with consequences.)

Would you like a "$3x$" to be part of the first equation? If so, make it happen!

Practice 86.11: Kindly solve the following system of equations.

$$x + y = 3$$
$$3x + 4y = 8$$

Practice 86.12: Please solve the following system of equations.

$$a + 4b = -29$$
$$2a - 3b = 8$$

Practice 86.13: Consider the following system of equations.

HOUSTON: We have a problem!

$$3x - 4y = 13$$
$$5x + 2y = 13$$

a) To solve the system, try by multiplying the first equation through by 5 and the second equation through by 3. Can you see what that does for us?

b) Solve the system again but by doing something enlightened that makes the "y-terms" in each equation match.

Practice 86.14: Kindly solve the following system of equations.

$$5m - 0.4n = 1.0$$
$$0.5m + 0.1n = 1.5$$

Practice 86.15: I most humbly invite you to solve the following system of equations.

$$x = 3y - 2$$
$$y = 5 - 4x$$

Practice 86.16: Please find a common solution to

HOUSTON: We have a problem!

$$6x + 2y = 5$$
$$3x + y = 0$$

Practice 86.17: Is it possible for two linear equations (in two unknowns) to have two or more simultaneous solutions?

Allow me to close a logical gap.

We've been playing with systems of two linear equations in two unknowns. For instance:

$$3x + 2y = 21$$
$$x + y = 8$$

As long as the two lines represented by the equations are not parallel, we know that the two lines have a unique point of intersection: an x value and a y value that make each equation true simultaneously.

To find that point of intersection, we've been conducting logical maneuvers of the following type:

> *If we are working with numbers that make the two equations true,*
>
> *then this must also be true … and this must also be true .. and this must also be true …*

until we get to statements that tell us a value for x that must be true and a value for y that must be true.

But this is a whole sequence of logical **ifs.**
Are our final conclusions sure to be **actually true**?

The answer is yes.

And that is because we do happen to know *a priori* that there is a solution to be had (assuming the lines are not parallel) and there only one solution to be had, so there will be no mix-up of which x values for truth match up with which y values.

To be painstakingly clear, the logic we are following in our work is this:

> *The point of intersection of the two lines given by the equations*

$$3x + 2y = 21$$
$$x + y = 8$$

> *gives x and y values that make both of these equations true.*
>
> *Which means they make both of these equations too.*

$$3x + 2y = 21$$
$$3x + 3y = 24$$

> *Which means they also make the statement $y = 3$ true.*

Which means they also make

$$3x + 6 = 21$$

true.

Which means they also make $x = 5$ *a true statement.*

Well, if the point of intersection makes the sentences

$$x = 5$$

$$y = 3$$

both true, then it must actually be the point $(5,3)$*!*

Of course, no one writes out all the words in this reasoning: it is assumed understood. (But, because it is never actually written out, I am not sure it properly is understood!)

Also, notice I've been presenting all our pairs of linear equations in the form:

$$ax + by = c$$

$$dx + ey = f$$

Then we can multiple one or both of the equations through by numbers to make either the "x terms" or the "y terms" match.

Of course, one can always rewrite a given pair of linear equations to follow this structure.

Practice 86.18 Solve the following system of two linear equations.

$$y + x + x = 4 - y + x$$

$$3 - x = 8 - y$$

“Real-World” Problems

Algebra textbooks often try to give the impression that all aspects of algebra is real-world relevant and is actively being used—just as presented—by engineers, actuaries, chemists, sociologists, and the like. And, moreover, when comparing rental car plans, or doing carpentry, or trying to count the number of nickels and dimes in your piggy bank, you too might well set up a system of two linear equations to solve.

You, no doubt, see through this.

For instance, here is a textbook “real world” example.

> **Example**: A carpenter cut a ten-foot-long plank into two pieces, one two feet longer than the other. How long were the two pieces?
>
> **Answer:** Call the length of the two pieces a feet and b feet. Then we have
>
> $$a + b = 10 \text{ feet}$$
>
> with
>
> $$b = a + 2$$
>
> We need to solve this system
>
> $$\begin{aligned} a + b &= 10 \\ -a + b &= 2 \end{aligned}$$
>
> A decrease of $2a$ on the left matches a decrease of 8 on the right.
> Consequently, $a = 4$ feet and so $b = 6$ feet.

But what carpenter expresses a problem this way? When is a carpenter in a position of needing one board two feet longer than the other with their combined length previously specified?

Despite the hokey-ness of the these “real world” examples, I will always advocate for practicing thinking, engaging in common sense reasoning, and exercising mathematics. I cannot predict how such intellectual work will end up being relevant to you and your life’s doing and work, but having intellectual techniques and prowess under your belt just can’t be a bad thing.

Allow me to now present some typical schoolbook applications of “systems of equations.” But please understand, I am not trying to convince you that these examples demonstrate practical importance of this work. (But I hope they do illustrate the power of general intellectual might.)

Practice 86.19 Joi has 2,400 coins on a table. Some are pennies (1 cent coins) and the rest of nickels (5 cent coins).

Joi doesn't know how many coins of each type she has, but she somehow knows that their total value is \$58.80.

She is curious about the total value of the pennies alone.

Rather than count pennies, she writes the following on a napkin.

> *Let p be the number of pennies and n the number of nickels on the table.*
> *We have*
> $$p + n = 2400$$
> $$p + 5n = 5880$$

Continue Joi's work and determine the total value of the pennies please.

Here's a "common sense" way to think though Joi's problem.

> *If all 2,400 of Joi's coins were pennies, their total value would be \$24.*
> *But she has a total value of total value of \$58.80, which is \$34.80 more.*
>
> *This extra value must be coming from the nickels present, each contributing an extra 4 cents compared to each penny. So, the number of nickels must be*
> $$3480 \div 4 = 870$$
> *This means that there are*
> $$2400 - 870 = 1530$$
> *pennies making \$15.30 in cash.*

Did you get this answer too for Problem 86.19?

Practice 86.20 Two pieces of Mouthstick gum and five pieces of Chew-a-Lot gum together cost 40 cents, whereas eight pieces of MouthStick gum and three pieces of Chew-a-Lot gum together cost 58 cents.

For some reason you never looked at the actual prices of each piece of gum.

What is the cost of one piece of MouthStick gum?
What is the cost of one piece of Chew-a-Lot gum?

Okay, here is an application that might actually be used in the real-world.

Example: In a chemistry lab there are two acid solutions: Container A of 20% acid solution and Container B of 30% acid solution.

You need a supply of 24% acid solution.

What proportion of the two solutions should you combine to create this?

(I am not a chemist. Forgive me if my setup and wording here is not quite right—but I am not far off in how algebra textbooks usually present such problems.)

Solution (Ha!): The question doesn't specify how much of this 24% acid solution you need, but let's pick an amount of 100 milliliters, say. (Choosing the number "100" will likely make expressing proportions as easier.)

Let a represent the number of milliliters to take from Container A and b the number of milliliters from Container B.

We need

$$a + b = 100$$

$$0.20 \times a + 0.30 \times b = 0.24 \times 100$$

(Do you understand the second statement? It's about the total among of acid you have present in the solutions.)

Let's multiply the second equation through by 10.

$$a + b = 100$$

$$2a + 3b = 240$$

Let’s multiply the first equation through by 2 to make the “a terms” match.

$$2a + 2b = 200$$

$$2a + 3b = 240$$

Now we see that we need $b = 40$ milliliters of liquid from Container B and consequently 60 milliliters of liquid from container A.

Thus, we need to combine liquids from containers A and B in a $60:40$, that is, 3:2, ratio. (Three parts from Container A and two parts from Container B.)

Practice 86.21: In the same chemistry lab with Container A of 20% acid solution and Container B of 30% acid solution, I now need

a) a supply of 25% acid solution

b) a supply of 29% acid solution

c) a supply of $33\frac{1}{3}\%$ acid solution

Find the proportions of the two acid solutions needed to create each of these supplies.

Reflection:
What does common sense tell you is the answer to part a)?
Does your mathematics show this too?

What does common sense tell you is the answer to part c)?
Does your mathematics show this too?

Practice 86.22: A dinner-cruise boat travels along a river.
It travels 6 miles downstream in one hour and then returns upstream in one-and-a-half hours.

What is the speed of the river current in miles per hour, and what would be the speed of the boat in still water?

Actually, problem 86.22 might be a "real-world" problem for curious diners. They are not privy to the speed of the boat set by the captain, nor do they know the speed of the current. But they do know how long they traveled in each direction and perhaps how far downstream they went.

Practice 86.23: Ibrahim invested money into two accounts: some in one account that earned 9% interest on his deposit by the end of the year and one that earned him 12% interest by the end of the year.

He deposited a total of $10,000 at the beginning of the year and now has $11,200.

He cannot recall how much he put into each account and all records have been lost.
(You know how that goes.)

Please help him out.

Practice 86.24: Anni-Frid found some $2 and $5 bills.
She counted 28 bills in all adding to $116 in cash but was incapable of counting the number of $2 bills alone and the number of $5 bills alone.

How many bills of each type did she find?

Practice 86.25: Concert organizers sold two types of tickets: student tickets at $12 each and general admission tickets at $850 each.

They lost track of how many tickets of each type sold, but they do know they sold a total of 101 tickets to bring in a total of $2,888.

Can you help them out?

(Surely the organizers would notice how many $850 tickets they managed to sell!)

MUSINGS

Musing 86.26 Apart from the basic properties of equality we outlined back in Section 76, we have been using another belief of equality without mention. It's this:

> *If a quantity A equals a quantity B and it also equals a quantity C, then it must be that B and C are equal.*

That is, if

$$A = B \text{ and } A = C$$

then

$$B = C$$

We've used this idea multiple times throughout these notes (and in this section during our brief discussion of equality – can you spot its use?)

a) Do you think this idea seems reasonable to believe?
b) Do you think I should I have explicitly mentioned this additional belief of equality earlier on?

MECHANICS PRACTICE

Practice 86.27 This was an exercise-rich section! Did you try all 25 problems that appeared?
(I won't suggest you keep doing more!)

Practice 86.28 Actually, a great way to test if you really "got" an idea is to see if you can create your own exercises on the topic for others to try.

a) Write variations of problems 86.19 to 86.25 that are manageable to solve and have answers involving numbers that are not too icky.

b) Create a system of two linear equations in two unknowns with no solutions.

c) Also create a system of two linear equations in two unknowns with infinitely many solutions.

87. OPTIONAL ASIDE: The Expected Schoolbook Approach(es)

Our algebraic approach to solving systems two linear equations in two unknowns is probably surprising to most textbook authors.

Example: Kindly find a common solution to this system of equations:

$$3m - 2q = 10$$

$$5m - 2q = 14$$

Answer: This system is set up for us to readily compare increases.

$$\textbf{up } 2m \quad \begin{array}{c} 3m - 2q = 10 \\ 5m - 2q = 14 \end{array} \quad \textbf{up } 4$$

We have that $2m$ matches 4 and so we need $m = 2$ for the two equations to be true.

Looking then at the first equation, we now see it reads as

$$6 - 2q = 10$$

Applying the usual techniques of algebra gives

$$6 = 10 + 2q$$

$$-4 = 2q$$

$$-2 = q$$

So, we need $m = 2$ and $q = -2$ for the two equations to be simultaneously true.

Practice 87.1 Here's a start to finding the common solution to:

$$3p - 2q = 22$$

$$4p + 5q = 37$$

1. Make the "p terms" the same by multiplying the first equation through by 4 and the second equation through by 3.

$$12p - 8q = 88$$

$$12p + 15q = 111$$

2. Now compare increases.

$$\begin{array}{ccc} & 12p - 8q = 88 & \\ \textbf{up } \boldsymbol{23q} & & \textbf{up } \mathbf{23} \\ & 12p + 15q = 111 & \end{array}$$

Please complete the work.

Rather than "compare increases," algebra books typically have students "subtract equations."

Let me explain what is meant by this.

The principle we've been following is that if equality holds, then "increases must match."

$$A = B$$

up a $\quad A + a = B + b \quad$ up b

then $\quad a = b$

But people say that it looks like we took our two equations and subtracted one from the other to get to this conclusion.

$$\begin{array}{rc}
 & A + a = B + b \\
\text{subract} & A \;=\; B \\ \hline
 & a \;=\; b
\end{array}$$

Sure! Subtraction is the process of making increments explicit. So, the thinking of subtracting equations as suggested is a valid move (and is just a rephrasing of our approach). We can adopt this practice directly too if we want.

Example Revisited: Kindly find a common solution to this system of equations:

$$3m - 2q = 10$$

$$5m - 2q = 14$$

Answer: Subtracting the first equation from the second gives

$$2m = 4$$

$$\begin{array}{rl}
 & 5m - 2q = 14 \\
\text{subtract} & 3m - 2q = 10 \\ \hline
 & 2m \qquad = \; 4
\end{array}$$

So, $m = 2$ and, as before, $q = -2$.

Practice 87.2 Show how subtraction in Problem 87.1 yields $23q = 23$.

Since subtraction is really addition of the opposite, we can interpret this subtraction practice as first multiplying one equation through by -1 and then adding two equations.

$$\begin{array}{lccc} & A + a & = & B + b \\ \text{subract} & A & = & B \\ \hline & a & = & b \end{array} \qquad \begin{array}{lccc} & A + a & = & B + b \\ \text{add} & -A & = & -B \\ \hline & a & = & b \end{array}$$

And sometimes we might want to work with the addition of two equations right off the bat.

For example, for this system of two equations

$$3x - y = 7$$

$$2x + y = 8$$

adding the two together gives

$$5x = 15$$

yielding that we need $x = 3$ and consequently $y = 2$ for a simultaneous solution.

This action is just the same subtraction process, if we first multiply the second equation through by -1 to make the "y-terms" match.

$$\begin{array}{lrcl} & 3x - y & = & 7 \\ \text{add} & 2x + y & = & 8 \\ \hline & 5x & = & 15 \end{array} \qquad \begin{array}{lrcl} & 3x - y & = & 7 \\ \text{subract} & -2x - y & = & -8 \\ \hline & 5x & = & 15 \end{array}$$

So, feel free to add to equations or subtract two equations. Both approaches are just our "increments must match" thinking in disguise.

Practice 87.3 Practice these subtraction and addition techniques to solve the following systems of equations.

a)

$$4x + 7y = 16$$

$$4x - y = 0$$

b)

$$4x + 7y = 16$$

$$4x - 7y = 64$$

c)

$$m - 0.2n = -5$$

$$4m - n = 10$$

d)

$$3r - 2s = 6$$

$$2r + 3s = -4$$

The Substitution Method

Consider the following system of equations

$$y = -2x + 1$$

$$y = 3x - 1$$

$$\begin{array}{r} y = -2x + 1 \\ \text{subract} \quad \underline{y = 3x - 1} \\ 0 = -5x + 2 \end{array}$$

Since the y-terms are matching, it makes sense to subtract the two equations to obtain

$$0 = -5x + 2$$

and we can work from here.

But we could follow the reasoning outlined in Musing 86.26:

> *If we have x and y values that makes the equations true, then it looks like the number for y equals two different values.*
>
> $$y = -2x + 1$$
>
> $$y = 3x - 1$$
>
> *It must be the case then that*
>
> $$-2x + 1 = 3x + 1$$

This can be rewritten as $5x = 2$ and we can go from there.

Schoolbooks call this approach **solving by substitution**. It looks like we took one stated value of y from one equation and replaced the appearance of y in the second equation with that stated value.

Practice 87.4 Solve this system of equations by the substitution method.

$$y = 100x$$

$$y = 99x + 3000$$

Actually, schoolbooks are usually much bolder with this substitution approach.

Consider the following system of equations

$$y = 5x - 1$$

$$3y - 13x = 9$$

If there are x and y values that make the equations simultaneously true, then the first equation is telling us that the value of y matches the value of $5x - 1$. So, we might as well use that for the value of y in the second equation. We must have:

$$3(5x - 1) - 13x = 9$$

Now we can work with this equation to see that x must have value 6 and consequenctly y has value $5 \times 6 - 1 = 29$. (Check this!)

This reasoning feels okay. But does the math justify what we just did?

It does. And here's how.

We can rewrite the second equation using the standard techniques of algebra

$$3y - 13x = 9$$

$$3y - 13x + 13x = 9 + 13x$$

$$3x = 13x + 9$$

$$\frac{1}{3} \times 3y = \frac{1}{3} \times (13x + 9)$$

$$y = \frac{13}{3}x + 3$$

So, our system of equations can alternative be presented as

$$y = 5x - 1$$

$$y = \frac{13}{3}x + 3$$

The standard substitution technique then gives

$$5x - 1 = \frac{13}{3}x + 3$$

Now, let's undo our algebra steps on this. (We added $13x$ and multiplied by $\frac{1}{3}$. Backwards, we multiply by 3 and subtract $13x$).

$$5x - 1 = \frac{13}{3}x + 3$$

$$3 \times (5x - 1) = 3 \times (\frac{13}{3}x + 3)$$

$$3(5x - 1) = 13x + 9$$

$$3(5x - 1) - 13x = 13x + 9 - 13x$$

$$3(5x - 1) - 13x = 9$$

which does indeed match writing "$5x - 1$" for y directly into our second equation $3y - 13x = 9$.

Phew!

Here's the general schoolbook substitution method:

To solve a system of equations of the form

$$y = mx + b$$

$$Ax + By = C$$

work with

$$Ax + B(mx + b) = C$$

and go from there.

Practice 87.5 Solve these systems of equations any way you like (and that includes via graphing, or better, finding some online computer algebra system to do everything for you).

a)

$$4x + 7y = 16$$

$$y = 2x$$

b)

$$y = -2x + 1$$

$$y = x - 11$$

c)

$$x + x + y + x + y + x = -x - x - y$$

$$y = x$$

d)

$$y = 5 - 7x$$

$$2y = 8 - 7x$$

MUSINGS

Musing 87.6

a) Create a system of two linear equations in the unknows x and y whose solution is $x = -2$ and $y = 1$.

b) When Sunil answered this question he gave the system

$$x = -2$$
$$y = 1$$

Do you agree that this is a system of two linear equations? Does the system have solution $x = -2$ and $y = 1$?

c) Solve Sunil's system by graphing, Does the intersection point $(-2,1)$ appear? (It should!)

MECHANICS PRACTICE

Practice 87.7 Solve the following system of equations

$$3x - 2y + z = 48$$
$$3x - 2y + 2z = 61$$
$$y = 23 - x - z$$

Chapter 11

Proportional Reasoning

(Figuring Out when Common Sense Does the Trick)

88. Just Do It!

The middle-school curriculum presents a line of thinking called "proportional reasoning"- an idea that manifests itself in a large variety of situations and scenarios.

It is the practice of mathematics to recognize the same underlying structure in multiple scenarios, and thus be able to "see through" the scenarios with expert prowess. Studying proportional reasoning provides an excellent opportunity to practice just this.

But the typical school presentation of this topic, in my opinion, fails to keep this message clear and often gets lost in clutter and loses sight of the common-sense thinking behind it all.

The underlying logic of proportional reasoning is actually quite natural and intuitive—one doesn't even need to know what "proportional reasoning" is in order to engage in it.

To illustrate what I mean, try this proportional reasoning problem right now!
(This problem is not as tedious as it first appears.)

Example: With 10 emu eggs I can make 35 omelets.

10 eggs ⟷ 35 omelets

Assuming all emu eggs are identical and all omelets I make are identical too …

a) How many omelets can I make with 20 eggs?
b) How many omelets can I make with 200 eggs?
c) How many omelets can I make with 2 eggs?
d) How many omelets can I make with 16 eggs?
e) How many omelets can I make with 1 egg?
f) How many omelets can I make with 122 eggs?

g) Here's a YES/NO question: If I were to ask you how many omelets I could make with 6 eggs, 19 eggs, 847 eggs, could you figure it out?

h) I want to make 350 omelets. How many eggs will I need?
i) I want to make 70 omelets. How many eggs will I need?
j) What fraction of an egg is needed to make 1 omelet?

k) Here's a YES/NO question: If I were to ask you how many eggs are required to make 10 omelets, 870 omelets, 922 omelets, could you figure it out?

We are dealing with **two quantities** in this omelet-making scenario: a count of eggs and a count of omelets.

The number of eggs we have determines the number of omelets we can make, and vice versa, the number of omelets we wish to make determines how many eggs we'll need.

Also, common sense tells us that:

> Doubling the number of eggs doubles the number of omelets we can make.
>
> Tripling the number of eggs triples the number of omelets we can make.
>
> Halving the number of eggs halves the number of omelets we can make.
>
> And so on.

Changing the count of eggs by some factor changes the number of omelets we can make by the same factor, and vice versa.

And that's the key insight. In fact, it is the very insight that solves everything!
Here's how:

With 10 emu eggs I can make 35 omelets.

$\boxed{10 \text{ eggs} \leftrightarrow 35 \text{ omelets}}$

Assuming all emu eggs are identical and all omelets I make are identical too …

a) How many omelets can I make with 20 eggs?
b) How many omelets can I make with 200 eggs?
c) How many omelets can I make with 2 eggs?
d) How many omelets can I make with 16 eggs?
e) How many omelets can I make with 1 egg?
f) How many omelets can I make with 122 eggs?

g) Here's a YES/NO question: If I were to ask you how many omelets I could make with 6 eggs, 19 eggs, 847 eggs, could you figure it out?

h) I want to make 350 omelets. How many eggs will I need?
i) I want to make 70 omelets. How many eggs will I need?
j) What fraction of an egg is needed to make 1 omelet?

k) Here's a YES/NO question: If I were to ask you how many eggs are required to make 10 omelets, 870 omelets, 922 omelets, could you figure it out?

We are told that 10 emus eggs make 35 omelets.

> 10 eggs ⟷ 35 omelets

Let's double the number of eggs, and hence the number of omelets.

> 20 eggs ⟷ 70 omelets

Let's scale up the number of eggs by a factor of ten (and hence do the same to the number of omelets).

> 200 eggs ⟷ 700 omelets

Let's now scale down by a factor of one hundredth (that is, scale both sides by 0.01).

> 2 eggs ⟷ 7 omelets

Let's scaling by eight then gives:

> 16 eggs ⟷ 56 omelets

Can you see that that I am scaling each expression to give the right number of eggs to answer parts a), b), c), d), in turn?

But what part e) is suggesting to do seems helpful. Let's find out how many omelets 1 egg will make.

Let's go back to

2 eggs $\leftrightarrow$ 7 omelets

Halving now shows

1 egg $\leftrightarrow$ $3\frac{1}{2}$ omelets

This is powerful because I can scale this statement by any desired number to see how many omelets I can make with that number of eggs.

Let's be general. Let's scale by a number N.

N eggs $\leftrightarrow$ $3.5 \times N$ omelets

I can now see that with N eggs, I can make $3.5 \times N$ omelets.

I am now set to answer part f) and YES to part g).

The remainder of the question focuses on the count of omelets.

We can start with

10 eggs $\leftrightarrow$ 35 omelets

and scale to answer parts h) and i).

100 eggs $\leftrightarrow$ 350 omelets (scaled by ten)

20 eggs $\leftrightarrow$ 70 omelets (scaled by a fifth)

But as part i) is leading us, it seems that knowing the number eggs needed for 1 omelet will unlock everything.

Going back to

$10 \text{ eggs} \leftrightarrow 35 \text{ omelets}$

and scaling each side by $\frac{1}{35}$ gives

$\frac{10}{35} = \frac{2}{7} \text{ eggs} \leftrightarrow 1 \text{ omelet}$

(Can you see that we could have gone back to <u>any</u> line of our previous work to deduce the same result?)

We can now scale this by any number we want, say k, to see how many eggs are needed to make k omelets.

$\frac{2}{7} \times k \text{ eggs} \leftrightarrow k \text{ omelets}$

We are now set to answer YES to part k).

Here's what we have learned:

As soon as we recognize in a scenario

two quantities that naturally vary in value

(like a count of eggs and a count of omelets) for which real-world common sense tells us that

scaling the value of one quantity in that scenario by a given factor
forces the other quantity to change by that same factor too

(doubling one quantity forces the other quantity to double as well; reducing one quantity by a tenth forces the other quantity to reduce by a tenth as well, and so on), then we are all set to answer complicated questions about the scenario using just common sense!

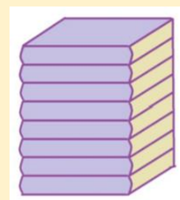
I personally like to use the term **scale in tandem** for two quantities A and B in a scenario that scale together in the way just described.

I also like to use a double arrow to indicate I have two quantities scaling in tandem and to do mathematics using this notation (just like I did in solving the previous problem).

amount of quantity A $\longleftrightarrow$ **amount of quantity B**

MECHANICS PRACTICE

Practice 88.1 Yvette likes to stack books.



She has many copies of the same book and likes to make stacks of different heights using them.

a) Can you identify two quantities in this scenario whose values can vary?

b) If so, do these two quantities scale in tandem?

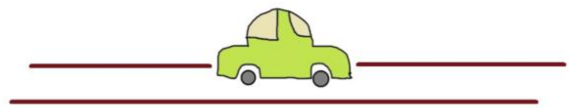
If the answer to b) is YES:

c) Yvette notices 4 books make a stack 5 inches tall. How tall would a stack of 17 books be?

d) Yvette's ceiling is 108 inches (9 feet) high. What is the maximum number of books she could stack starting with a book on the floor?

89. Exercising our Judgment

Consider these two problems before turning the page.

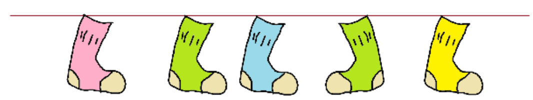


Example 1: Lisa drives along a straight and very lengthy stretch of road at a constant speed.

a) Can you identify two quantities in this scenario whose values can vary?
b) If so, do these two quantities scale in tandem?

And if so again …

c) Lisa is traveling at 65 miles per hour. How much distance along road will she cover in 40 minutes?
d) Lisa has 100 miles of road still to go. How long will it take her to cover that final stretch?



Example 2: Bernard has some socks on the line to dry.

a) Can you identify two quantities in this scenario whose values can vary?
b) If so, do these two quantities scale in tandem?

And if so again …

c) Today it takes 24 minutes for 5 socks on the line to dry. If he hung out 8 socks instead, how long would it take for them to dry?
d) He is hoping to have a pair of dry socks within 10 minutes. Could that happen for him?

Here are my responses.

Answer 1: a) and b): We do have two quantities whose values can vary:

time spent driving along the road

the distance covered during that time

And common sense tells me that if we double, or triple, or halve, say, the amount of time driving, because Lisa's speed is constant, the amount of distance she covers will change the same way.

c) We have

1 hour of driving $\leftrightarrow$ 65 miles of road covered

Thus

$\frac{2}{3}$ hour of driving $\leftrightarrow \frac{2}{3} \times 65 = 43\frac{1}{3}$ miles of road covered

d) From

1 hour of driving $\leftrightarrow$ 65 miles of road covered

we have

$\frac{1}{65}$ hour of driving $\leftrightarrow$ 1 mile of road covered

and so

$\frac{100}{65}$ hour of driving $\leftrightarrow$ 100 miles of road covered

Now $\frac{100}{65} = 1\frac{7}{13}$ hour is one hour and $\frac{7}{13} \times 60 \approx 32$ minutes.

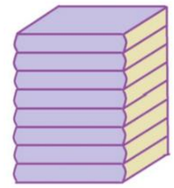
Answer 2 a) and b): I can identify one quantity whose value can vary in this scenario.

the count of socks on a line

But common sense tells me that the length of drying time does not vary. Whether one has 1 sock, or 5 socks, or 17 socks on the line, they will all take the same amount of time to dry.

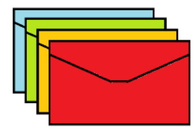
Parts c) and d) of this question are moot! (Though the answer to part c) is 24 minutes. And the answer to part d) is "NO. Two socks will also take 24 minutes to dry.")

Here are three more musings for you to try.



Example 3: Yvette from Practice 86.1 continues to stack her (identical) books. She is now making these stacks on her kitchen table but is measuring the height of the stack as the distance of the top book in her stack to the floor.

a) Identify two quantities in this scenario whose values can vary.
b) Do these two quantities scale in tandem?



Example 4: Jaspreet is stuffing envelopes. She started yesterday and stuffed 1200 of them. She is continuing the job today, working at the constant rate of stuffing 200 envelopes per hour.

We have two quantities in this scenario whose values can vary:

- The total number envelopes Jaspreet has stuffed (yesterday and today)
- The number of hours she'll spend today working on the job

Do these two quantities scale in tandem?

Example 5: Lisa drives along that straight road again at a constant speed of 65 mph. The start of that road is 52 miles from her home and the road takes her directly further away as she travels along it.

We have two quantities in this scenario whose values can vary:

- The time Lisa spends driving along that road
- The number of miles she is from home after driving along that road for that given time

Do these two quantities scale in tandem?

None of these three scenarios have quantities that scale in tandem.

To see why, try doubling the value of one quantity and see if the other doubles as well. (Or triple, or ten-tuple!)

For Yvette in Practice 86.1 and Example 3 we have:

4 books $\longleftrightarrow$ 5 inches + height of table

Doubling the number of books gives

8 books $\longleftrightarrow$ 10 inches + height of table

The height of the stack, as it is measured now, has not doubled!

For Jaspreet in example 4 we have

1 hour of work today $\longleftrightarrow 200 + 1200$ envelopes stuffed

Doubling the number of hours of work today gives

2 hours of work today $\longleftrightarrow 400 + 1200$ envelopes stuffed

The total number of envelopes stuffed has not doubled.

For Lisa we have

1 hour of travel along the road $\longleftrightarrow 65 + 52$ miles from home

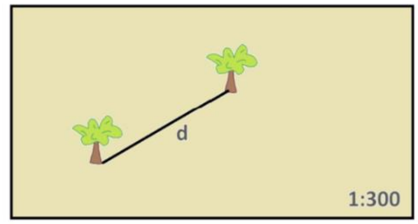
Doubling the number of hours driven gives

2 hours of travel along the road $\longleftrightarrow 130 + 52$ miles from home

The distance from home has not doubled.

Practice 89.1 What makes the answers to Example 1 and Example 5 so different? The first had two quantities that do scale in tandem and the second did not.

And now just two more examples for you to mull on. These ones come up in real life, for sure.



Example 6: A map is a "scaled" drawing of a real-world location.

Each map comes with a "key." For example, the map shown has the key 1:300.

a) Identify two quantities associated with a map whose values can vary and naturally scale in tandem.
b) What does the key 1:300 mean?
c) The pictures of the two trees on the map are 1 inch apart. How far apart are the corresponding trees in real life?

Example 7: "There are approximately 2.54 centimeters in an inch."

a) Does this statement imply a scenario with two quantities whose values can vary and do so in tandem?
b) Approximately how tall am I in centimeters given than I am 72 inches (6 feet) tall?

Answer 6: We have two quantities whose values can vary.

- distances between objects drawn on the map
- the matching real-life distances between the objects these pictures represent

And for the map to be a "scaled drawing" means precisely that the map is drawn so that these quantities scale in tandem.

The key tells us that

$$1 \text{ unit of length on the map} \leftrightarrow 300 \text{ units of length in the real world}$$

So, 1 cm of length on the map matches 300 cm of length in the real world; 1 inch of length on the map matches 300 inches of length in the real world; 1 hand width of length on the map matches 300 hand widths of length in the real world, and so on.

As just stated, the answer to part c) is 300 inches, which is 25 feet.

Answer 7: We have two quantities whose values can vary

- the numerical value of a length measured in centimeters
- the numerical value of the same length measured in inches

Commonsense tells us that these scale in tandem: if the length of an object measures as a centimeters and as b inches, then an object twice as long is sure to measure as $2a$ centimeters and as $2b$ inches, for example.

We have (up to the approximation)

$$2.54 \text{ centimeters of length } \longleftrightarrow 1 \text{ inch of length}$$

Thus,

$$72 \times 2.54 \approx 183 \text{ centimeters of length } \longleftrightarrow 72 \text{ inches of length}$$

I am approximately 183 centimeters tall.

We have just seen that scenarios involving

- **Constant rates** (driving at a constant speed, stuffing envelopes at a fixed rate, increasing a stack height one book-width at a time)
- **Unit conversion**
- **Scaled drawings**

often yield pairs of quantities that scale in tandem. (But one must still think through matters and use commonsense to make sure.)

MECHANICS PRACTICE

Practice 89.2 At present, 1 Australian dollar is worth 67 U.S. cents.

a) Does this statement imply a scenario with two quantities whose values can vary and do so in tandem?

b) I have $500 U.S. dollars saved up for spending money in Australia. How many Australian dollars is that amount?

Practice 89.3 Write a version of Jaspreet's envelope-stuffing scenario (Example 4) that does lead to two quantities that scale in tandem.

90. Schoolbook Examples

Let's get a piece of language out of the way.

Imagine a scenario that has two quantities whose values do, or can, or can be imagined to vary and that they do so by scaling in tandem. (Tripling the value of one quantity triples the value of the other; halving the value of one quantity halves the value of the other, and so on.)

Then we say that the two quantities are in a **proportional relationship**.

Example 8: Here is a data table of some quantities labeled x and y.

Each row of the table shows a value for the quantity called x and its matching quantity called y. Could this data have come from a proportional relationship?

x	y
6	9
10	15
4	6
0	0
20	30
7	10.5
100	150

What do you think?

Answer 8:
If we do have a proportional relationship at play, then the first line of the table gives

6 for value $x \leftrightarrow$ 9 for value y

Scaling by a factor of $\frac{1}{6}$ gives

1 for value $x \leftrightarrow \frac{3}{2}$ for value y

Then scaling by an arbitrary number n gives

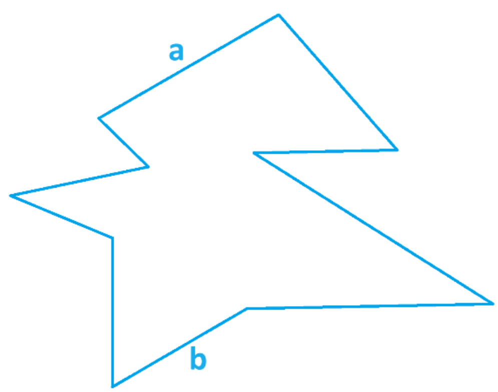
n for value $x \leftrightarrow \frac{3}{2}n$ for value y.

Every row of this table does conform to this pattern: each y value is one-and-a-half as large as its corresponding x-value.

This data could come from a proportional relationship.
(But who really knows? Maybe a next data pair might disobey this pattern?)

Here's an example from geometry class.

Example 9: Consider the set of all figures **similar** to this shape.



Suppose for each of these shapes we record the length of the side that matches side a shown in the diagram and the length of the side that matches side b shown.

Will the data we collect be in a proportional relationship?

Answer 9:

We learn in a geometry class that two figures are similar if

- all corresponding angles have the same measure
- all corresponding side lengths are scaled by the same factor (call it k).

In this scenario we are looking at many scaled copies of the polygon shown, matching the length of particular top edge of each polygon with the length of a particular bottom edge of the same polygon.

In the figure shown we have, for the given figure:

The top edge under consideration has length a ⟷ The bottom edge under consideration has length b

For a similar figure with scale factor k geometry tells us:

The top edge under consideration has length ka ⟷ The bottom edge under consideration has length kb

This precisely fits the definition of a proportional relationship.

The data is in a proportional relationship.

What do you think of this next example?
Is there a hidden proportional relationship afoot?

Example 10: We have that 120% of a quantity has value 300. What is the value of quantity itself?

Answer 10:

A percentage is simply a fraction with denominator 100. So, if a quantity has value Q, then 120% of the quantity is the fraction $\frac{120}{100}$ of the quantity, namely, $\frac{120}{100} \times Q$.

We are told that this has value 300.

$$\frac{120}{100} \times Q = 300$$

We deduce that $Q = \frac{100}{120} \times 300 = \frac{5}{6} \times 300 = 250$.

But we can also answer this question by thinking of a proportional relationship.

> *The stated percentage of a given quantity is in a proportional relationship with the corresponding portion of the quantity.*

(Halving the stated percentage that describes a portion corresponds to halving the portion, for instance.)

We have:

120% of the quantity ⟷ the value 300	
20% of the quantity ⟷ the value 50	(Scale by one sixth)
100% of the quantity ⟷ the value 250	(Scale by five)

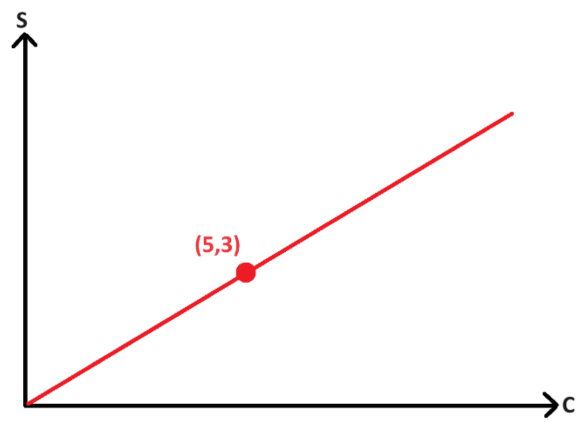
Question: Which of these two approaches felt more natural to you?

Now let's get very schoolbook-y!

Example 11: The following is the graph of some data showing that there seems to be a direct correlation between the count C of carrots one eats in an evening and the number of hours sleep S one obtains that night from doing so.

Does this fictitious (and absurd!) data represent a proportional relationship?

(Assume you really are seeing a straight line passing through the origin in this picture.)

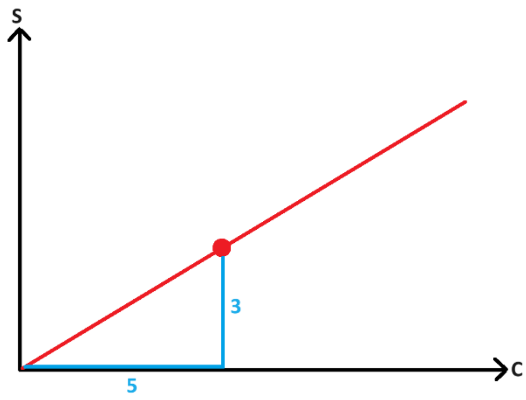


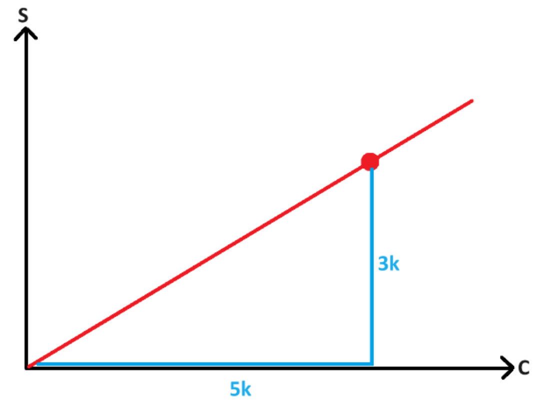
Answer 11: We see one data point from the graph:

eating 5 carrots ↔ 3 hours of sleep

But does the data depicted in the graph suggest matters are scaling in tandem?

We have a straight line of slope $\frac{3}{5}$.



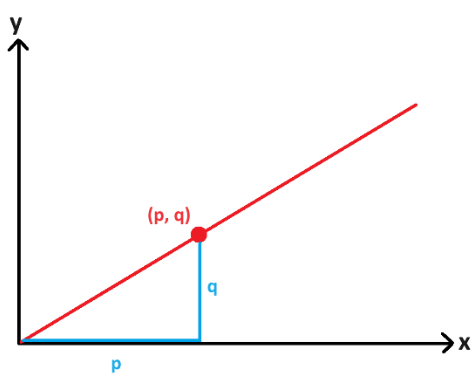


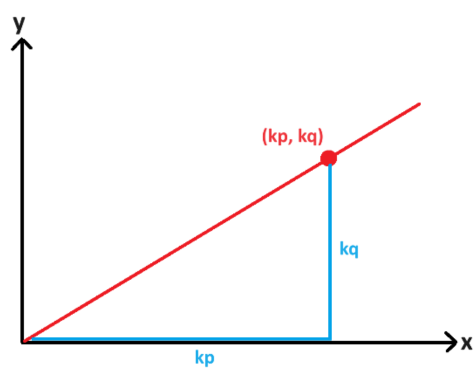
But we know from Chapter 10 that when computing slope of a line we can adjust the horizontal and vertical steps by the same scale and not affect matters. Thus, since $(5,3)$ is a point on the line, so is $(5k, 3k)$ for any scale factor k. And having $(5k, 3k)$ a data point is saying that

eating $5k$ carrots $\leftrightarrow$ $3k$ hours of sleep

We do have data scaling in tandem!

This example shows that if data comes from a graph that is a straight line through the origin, then we can be sure that the data is in a proportional relationship.





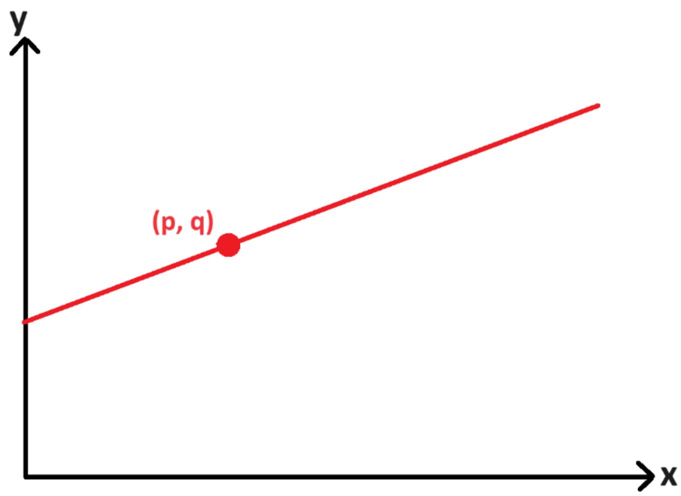
first coordinate p $\leftrightarrow$ second coordinate q

first coordinate kp $\leftrightarrow$ second coordinate kq

Practice 90.1 Is data coming from a straight-line graph that does not pass through the origin in a proportional relationship?

To be specific:

If p and q represent data that comes from a point on a straight line graph $y = mx + b$,
will kp and kq also represent data that comes from a point on the line?
Assume here that the y-intercept b is not zero.



MUSINGS

Musing 90.2 Given a square, we can measure two quantities associated with it:

- its area
- its perimeter

Different squares give different values for these quantities.

Is the area and perimeter data of squares in a proportional relationship?

Musing 90.3

Given a circle we can measure its diameter and its circumference. Different circles give different values for these quantities.

Is the diameter and circumference data of circles in a proportional relationship?

Musing 90.4

a) Each student in a certain class an equal number of black pens and blue pens in their pencil case.

The students collected data by asking each other how many black pens and how many blue pens they each have.

Will the data be seen as coming from a proportional relationship?

b) Each student in another class has twice as many black pens as blue pens in their pencil case.

The students collected data by asking each other how many black pens and how many blue pens they each have.

Will the data be seen as coming from a proportional relationship?

MECHANICS PRACTICE

Practice 90.5
a) Does this data look like it could come from a proportional relationship?

x	y
12	10
3	$2\frac{1}{2}$
9	$7\frac{1}{2}$
0	0
15	$12\frac{1}{2}$
-12	-10

b) Trusting that the data does come from a proportional relationship, make a prediction for the value of y if x has value 100, and make a prediction for the value of x if y has value 100.

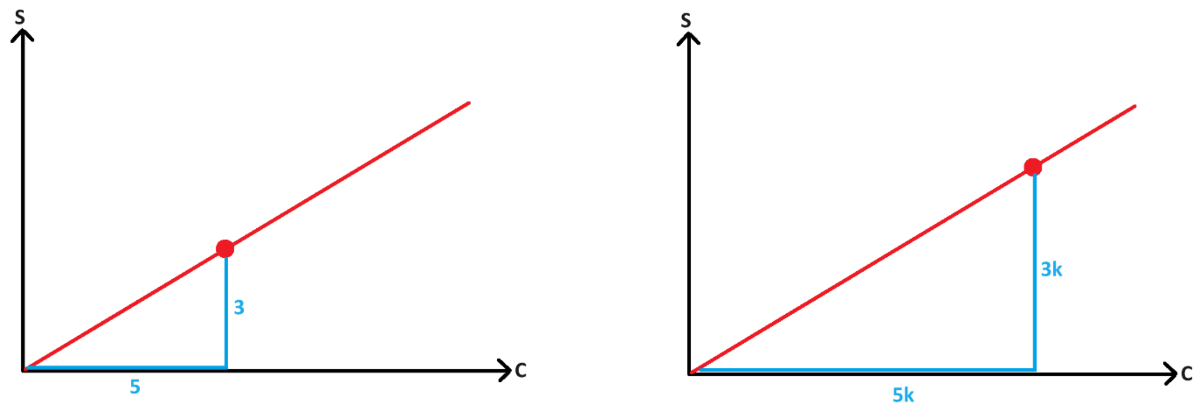
Practice 90.6 If 210% of a number is 700, what is the number?

Practice 90.7 Could this data be coming from a proportional relationship?

x	y
100	105
110	110
120	115

91. A Prime Example of a Mathematical Model

We have just seen that if the graph of data coming from a scenario in which two quantities can vary in value sits on a straight line passing through the origin, then that data is in a proportional relationship.



5 carrots eaten $\leftrightarrow$ 3 hours of sleep

$5k$ carrots eaten $\leftrightarrow$ $3k$ hours of sleep

Is the converse true?

If we graph data that we first know to be in a proportional relationship, will it yield graph with data points sitting in a straight line though the origin?

Here is the start of a table of data points for two quantities A and B known to be in a proportional relationship.

A	B
5	8

Even though we have only one data point, knowing that we have a proportional relationship at hand allows us to deduce what every possible data point shall be!

We have

$$5 \text{ of quantity } A \leftrightarrow 8 \text{ of quantity } B.$$

Scaling by a fifth, we obtain

$$1 \text{ of quantity } A \leftrightarrow \frac{8}{5} = 1.6 \text{ of quantity } B.$$

Scaling by a value x yields

$$x \text{ of quantity } A \leftrightarrow 1.6x \text{ of quantity } B.$$

A	B
5	8
x	$1.6x$

Here we have focused on quantity A as the "primary" variable (the independent variable) and followed the practice to subsequently denote such a variable by the letter x. If we then use y to denote the matching value of quantity B, then we have just shown that

$$y = 1.6x$$

The data is indeed following an equation whose graph we know is a line through the origin. (It is a line with slope 1.6 and y-intercept 0.)

Practice 91.1 Following this work, show that for a number y we also have

$$0.625y \text{ of quantity } A \;\longleftrightarrow\; y \text{ of quantity } B$$

What equation is this suggestion for the data if we do the unconventional thing of regarding y as the independent variable and x the dependent variable?

A **mathematical model** of a real-world scenario is a piece of mathematics (an equation, a collection of equations, a graph, for instance) that describes the real-world scenario and allows you to make predictions about the scenario.

Mathematicians try to create models that match the real-world scenario as closely as possible, but recognize that there will always be "noise" and human measurement errors and outside influencing parameters, and so on. Models are usually seen as close approximations.

But we have just seen an example of a perfect mathematical fit with real-world phenomenon.

We have shown:

If a real-world phenomenon is known to be in a proportional relationship, then the data from that scenario is sure to be perfectly described by the equation of a line that passes through the origin.

$$y = mx$$

And conversely, data that comes from points on a line passing through the origin is sure to be in a proportional relationship.

And we saw this in our very first example of the chapter.

We had

With $x = N$ eggs you can make $y = 3.5 \times x$ omelets

And if we want to make the count of omelets the focus:

To make $x = k$ omelets, you will need $y = \frac{2}{7}x$ eggs.

These are both equations of straight lines through the origin.

Comment: As I said earlier, I often find this insistence on naming one's unknowns as "x" and "y" unnatural. In this example, I'd rather write:

Let N be the number of eggs required to make k omelets.
Then we have

$$k = 3.5 \times N$$

and

$$N = \frac{2}{7} \times k$$

as we did in the opening example (though I don't know why I chose "k" for the count of omelets).

For each scenario with data in a proportional relationship, one has a choice as to which of the two quantities should be deemed the control (independent) variable. The choice made, of course, will depend on context.

Example: In problem 89.2 we stated "At present, 1 Australian dollar is worth 67 U.S. cents."

In international markets, an Australian dollar is denoted AUD and U.S. dollars as USD.

a) Write an equation that relates AUD to USD with the independent variable being USD.
b) Write an equation that relates AUD to USD with the independent variable being AUD.

c) Lulu, an American, is about to travel to Australia with a wallet full of USD. Which of these two equations is likely more useful to her?

d) After returning from Australia, Lulu has some Australian notes left in her wallet. Which of the two equations above is likely to be most useful to her now?

Answer: Common sense tells us that we have a proportional relationship here with

$$0.67 \text{ AUD} \leftrightarrow 1 \text{ USD}$$

a) Let's scale by a number x. Then we obtain

$$0.67x \text{ AUD} \leftrightarrow x \text{ USD}$$

This tells me that x USD are worth $0.67x$ AUD. ("Just multiply by 0.67.")

Rather than try to write this statement as a formal schoolbook equation of a line, I think the world will better understand me expressing this as:

$$AUD = 0.67 \times USD$$

b) Let's now scale our opening data point by $\frac{1}{0.67} \approx 1.49$ and then by a number x. Then we obtain

$$0.67x \text{ AUD} \leftrightarrow x \text{ USD}$$

$$1 \text{ AUD} \leftrightarrow 1.49 \text{ USD}$$

$$x \text{ AUD} \leftrightarrow 1.49x \text{ USD}$$

This tells me that x AUD are worth $1.49x$ USD. ("Just multiply by 1.49.")

Again, I think the world will best understand this if I write:

$$USD = 1.49 \times AUD$$

c) Heading to Australia with USD in her wallet, the equation in part a) is probably most useful to Lulu.

d) Upon her return, with AUD in her wallet, the equation in part b) is probably most useful to her.

Of course, we've gone deep into the study of data in a proportional relationship and related it to schoolbook mathematics.

But, as the opening example of this chapter shows, we actually conducted all this work without being explicit about it: the thinking behind proportional reasoning is somewhat innate.

Just follow your common sense thinking as best you can!

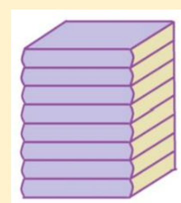
MUSINGS

Musing 91.2 When collecting data from a scenario with two quantities in a proportional relationship, why must $(0,0)$ be one the data points you could collect?

Musing 91.3 Would you like to look up the term **unit rate** from a textbook chapter on proportional reasoning and try to make sense of its connection to the slope of a line and a data point of the form $(1, r)$ and whatever else is thrown your way on this? (Please say NO. All this is unnecessary clutter.)

MECHANICS PRACTICE

Practice 91.4 Recall from Problem 86.1 that Yvette likes to stack identical books. She had



4 books ↔ a stack 5 inches tall

Write a formula for the height of a stack with N books in the stack.

Practice 91.5 Recall from Example 1 that Lisa is driving in a straight road a constant speed of 65 miles per hour.

We have

1 hour of time passing ↔ 65 miles of road traversed.

Write a formula for the distance along the road she covers in t hours.

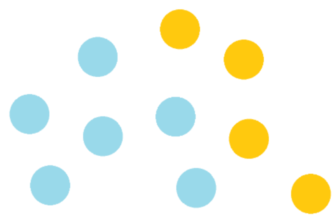
Practice 91.6 Recall from Example 7 that there are 2.54 centimeters in an inch. Write a formula for the number of inches in N centimeters.

92. Ratios

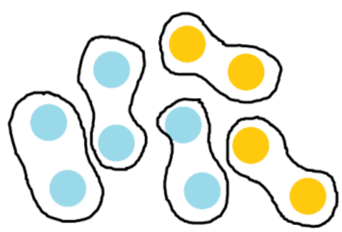
Two quantities A and B that can be deemed of the same type in some way (for example, A could represent a collection of apples and B a collection of bananas, but apples and bananas could all be deemed "fruit") are said to be in an $\boldsymbol{a}$**:** $\boldsymbol{b}$ **ratio** ("a to b ratio") if it is possible to divide quantity A into a parts of equal size and quantity B into b parts of the same equal size.

Some examples make this clear.

Example: The blue dots and the orange dots in the picture are in a 6-to-4 ratio. We can see the blue dots as 6 groups of one dot and the orange dots as 4 groups of one dot.



We can also say the dots come in the ration 3:2. This time see the blue dots as 3 groups of two dots and the orange dots as 2 groups of two dots.



We could also say they come in a 12:8 ratio: view the blue dots as 12 sets of half dots and the yellow dots as 8 sets of half dots.

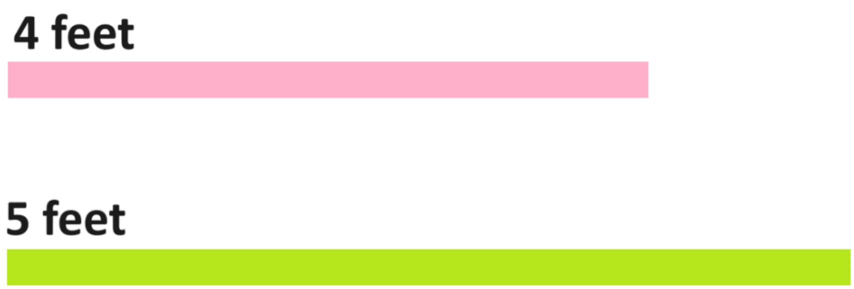
And we could keep going!

Note: The order we present words and numbers is important. In the above, it is clear that we were talking about blue dots first, orange dots second. Thus, in writing a ratio 6:4 or 3:2 or 12:8, it is understood that the first number mentioned refers to the blue dots and the second number to the orange dots.

This is the practice followed when talking about ratios.

Practice 92.1 Come up with three other ways to describe the ration of blue dots to orange dots in the previous example. (Give the group size you are thinking of for each ratio you present.)

Example: The lengths of these two sticks come in a 4:5 ratio.



Here it is clear we must be referring to the pink stick first and the green stick second and that we're thinking of dividing each length into the equal-sized part of 1 foot.

If we think of equal-sized parts of inches, we could say that these sticks come in a 48:60 ratio.
If we think yards, we would say they come in a $1\frac{1}{3}:1\frac{2}{3}$ ratio. (Though that is awkward!)

Example: Some Americans and Australians attend a party.
The ratio of Americans to Australians is 3:2.

a) If 18 Australians are present, how many Americans are present too?
b) If instead 60 Americans are present, how many Australians are present too?
c) The Americans can be split into three equal-sized groups of N people. How many Australians are there (in terms of N)?

Answer:
Part c) encompasses the thinking needed to answer all the questions.

The statement that we have a 3:2 ratio of Americans to Australians tells us that we the Americans can be split into three groups of equal size and the Australians into two groups of the same size. We must have $3N$ Americans and $2N$ Australians for some number N. (This answers part c).)

For part a), we see that $2N = 18$ and so N must be 9. Consequently, there are $3N = 27$ Americans.
For part b), we see that $3N = 60$ and so Nmust be 20. Consequently, there are $2N = 40$ Aussies.

We have:

Two quantities A and B come in an $a\!:b$ ratio, precisely if there is a common part size N so that

Amount of quantity A $= a \times N$
Amount of quantity B $= b \times N$

But as we saw in our first two examples, a $6\!:4$ ratio could also be seen as a $3\!:2$ ratio or as a $12\!:8$ ratio, and a $4\!:5$ ration can be seen as a $48\!:60$ ratio or as a $1\frac{1}{3}\!:1\frac{2}{3}$ ratio.

Practice 92.2 Some pink and purple dots come in a $5\!:3$ ratio.

I got this value by looking at groups of size twelve and saw that there were $5 \times 12 = 60$ pink dots and $3 \times 12 = 36$ purple dots.

a) When Cecile looked at the dots, she said they came in a $10\!:6$ ratio. What sized groups must she have been noticing?

b) When CeCe looked at the dots, she said they came in a $30\!:18$ ratio. What sized groups must she have been noticing?

c) When Cecelia looked at the dots, she said they came in a $60\!:36$ ratio. What sized groups must she have been noticing?

d) When Celine looked at the dots, she said they came in a $180\!:108$ ratio. What sized groups must she have been noticing?

Here's a general result with a mathy proof.

Result: If two quantities come in an $a\!:b$ ratio, then they can also be said to come in a $ka\!:kb$ ratio for any positive number k.

As you can guess, this must come from changing one's perspective of what the equal-sized groups are in a given picture.

Proof:

Suppose quantities A and B are in an $a : b$ ratio. Then there is a number N (the "group size") so that

Amount of quantity A = $a \times N$
Amount of quantity B = $b \times N$

We have a groups of size N for quantity A and we have b groups of size N for quantity B.

If we start thinking "half the group size," then we can write

Amount of quantity A = $a \times 2 \times \left(\frac{N}{2}\right)$
Amount of quantity B = $b \times 2 \times \left(\frac{N}{2}\right)$

We have $2a$ groups of size $\frac{N}{2}$ for quantity A and we have $2b$ groups of size $\frac{N}{2}$ for quantity B, and so the quantities can be said to also come in a $2a : 2b$ ratio.

If we think "third the group size," then we can write

Amount of quantity A = $a \times 3 \times \left(\frac{N}{3}\right)$
Amount of quantity B = $b \times 3 \times \left(\frac{N}{3}\right)$

We have $3a$ groups of size $\frac{N}{3}$ for quantity A and we have $3b$ groups of size $\frac{N}{3}$ for quantity B, and so the quantities can be said to also come in a $3a : 3b$ ratio.

If we think "quadruple the group size," then we can write

Amount of quantity A = $a \times \frac{1}{4} \times (4N)$
Amount of quantity B = $b \times \frac{1}{4} \times (4N)$

We have $\frac{1}{4} \times a$ groups of size $4N$ for quantity A and we have $\frac{1}{4} \times b$ groups of size $4N$ for quantity B, and so the quantities can be said to also come in a $\frac{1}{4} \times a : \frac{1}{4} \times b$ ratio.

In general:

If we think "$\frac{1}{k}$ of group size," then we can write

Amount of quantity A = $a \times k \times \left(\frac{N}{k}\right)$
Amount of quantity B = $b \times k \times \left(\frac{N}{k}\right)$

We have ka groups of size $\frac{N}{k}$ for quantity A and we have kb groups of size $\frac{N}{k}$ for quantity B, and so the quantities can be said to also come in a ka: kb ratio.

This result has the feel of a proportional relationship of some kind, but it is not really one. We're just finding different ways of expressing one static situation: a fixed picture of blue and orange dots; a fixed picture of two sticks.

Our third example of Americans and Australians at a party, however, gave us a way to think dynamically about a static example. There was a fixed party, but we were not told the actual number of attendees. This allowed us to imagine different scenarios that could be true for the situation described and thus to engage in dynamic thinking.

Example: Every night for a year Americans and Australians congregated at a residence for a party. Although the number of attendees varied from night to night, matters were such that ratio of Americans to Australians was always 3:2.

a) Explain why the number of Americans that attended each night and the matching number of Australians is data that scales in tandem.

b) One night there were 500 attendees at a party! How many of those attendees were Australian?

Answer

a) Because there is a 3:2 ratio at play, we know on any given night we can split the Americans into 3 groups of equal size and the Australians into 2 groups of that same size.

   Suppose a later party has double the number of Americans and double the number Australians. We can still split attendees into 3 and 2 equal-sized groups—just the equal-sized groups will now be doubled in size. We'll still have a 3:2 ratio and a valid party.

   Suppose another instead has half the number of Americans and half the number Australians. We can still split attendees into 3 and 2 equal-sized groups—just the equal-sized groups will be half the size they were before in this case. We'll still have a 3:2 ratio and a valid party.

   In general, if we scale the number of Americans and Australians each by a factor k, we can still split attendees into 3 and 2 equalsized groups—the equal-sized groups will be k times as big. We still have a 3:2 ratio and a valid party.

   We are seeing data scaling in tandem: if (p, q) is a valid data point in this scenario (p Americans and q Australians), then (kp, kq) is a valid data point too.

a) Following the notation above, we know

$$p = 3N$$
$$q = 2N$$

   for some value N.

   We are told that $p + q = 500$ and so $5N = 500$ giving $N = 100$.
   The number of Australians present is $2N = 200$.

This example shows how statements about ratios can be turned into dynamic situations that then lead to data in a proportional relationship.

MUSINGS

Musing 92.3 Ratios are not limited to a discussion with just two quantities. For instance, can you imagine a picture of blue, pink, yellow, orange, and red dots in a $3: 6: 2: 7: 1$ ratio?

a) In a picture of colored dots, these five colors with counts in the above ratio, there are five red dots. How many dots are there of each other color?

b) If instead in the picture there are 18 pink dots, how many dots are there of each color this time?

Musing 92.4 Here's an annoying puzzle-book problem. Can you solve it using just common-sense thinking?

If 4 cats can catch 9 rats in 2 days, to the nearest hour, how long does it take 1 cat to catch 1 rat?

MECHANICS PRACTICE

Practice 92.5 The ratio of blue pens to black pens in my pencil case is $7: 9$. I have 64 blue and black pens in total. How many of those pens are blue?

(And why am I carrying so many pens? And why am I speaking about them in such a cryptic way?)

Practice 92.6 What does it mean to say that two quantities are in a "one-to-one ratio"?

Practice 92.7 Two quantities are in a $1\frac{3}{4} : \frac{2}{3}$ ratio. Give a much simpler way to express this ratio.

Practice 92.8 One in five Americans like the taste of Vegemite. What is the ratio of Americans who like the taste of Vegemite to the number who don't?

93. Visualizing Ratio Problems

A discussion of a "ratio" fundamentally refers to quantities coming in sets of equal-sized groups.

If we represent equal-sized groups visually as equal-sized blocks, then we might be able to "see through" a problem about rations with some ease.

All the problems presented in this section are somewhat artificial. So, just enjoy them as intellectual logic puzzles to sharpen one's mind in some general sense.

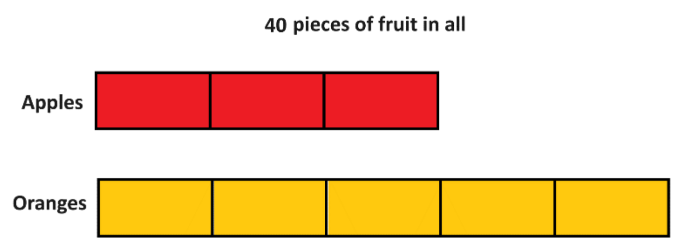
(Also, the problems, like ratio problems tend to be, are "static." They are each about just one instance of a particular scenario and do not involve proportional reasoning *per se*.)

Example: There are apples and oranges in my fruit bowl making a total of 40 pieces of fruit in all. The ratio of apples to oranges is $3:5$.

How many more oranges are there than apples in the bowl?

Answer: We can divide the fruit into equal-sized groups with three of those groups making for the apples and five the oranges.

Here's a picture showing this.



We see 40 pieces of fruit divided into 8 groups. There must be 5 items per group.

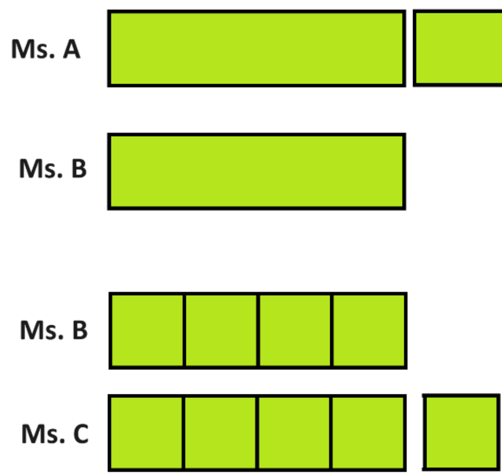
We also see that are two more groups of oranges than apples, and so there are $2 \times 5 = 10$ more oranges than apples.

Practice 93.1 On a tray sit glasses of milk and glasses of soda. There are 35 drinks in all with the ratio of milk to soda drinks $2:5$. How many drinks of each kind are there?

Example: Ms. A has twice as much money in her savings account as Ms. B does. Ms. B has four fifths the amount of money in her account as Ms. C does.

What is the ratio of Ms. A's account balance to Ms. C's?

Answer: Does this picture do it for you? Do you see that the ratio we seek must be $8:5$?



Practice 93.2 Mr. D has triple the amount money in his savings account as Mr. E does. And Mr. E has 150% the amount of money as Mr. F does.

What is the ratio of Mr. F's account balance to Mr. D's?

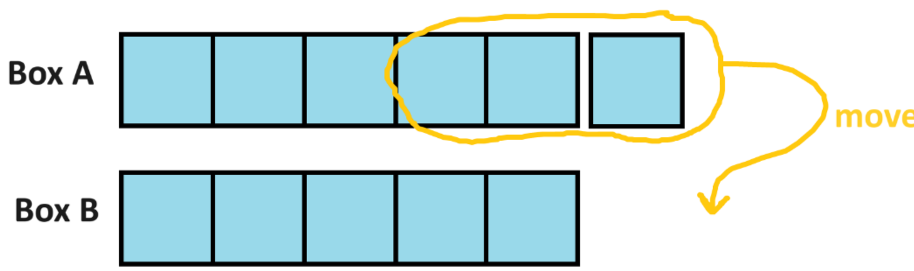
Example: Box A contains a 120% more books than box B does.

If half the books are taken out of box A and moved to box B, what now is the ratio of the number of books in box A to the number of books in box B?

Answer: Start with a picture showing box A with 20% (a fifth) more books than box B.

Can you see then see that after the transfer of books, box A will have just $6 - 3 = 3$ units of books and box B will have $5 + 3 = 8$ units of books?

The ratio we seek is thus $3{:}\,8$.



Practice 93.3 Crate A of kiwi fruit has $33\frac{1}{3}\%$ more fruit than crate B. Unfortunately, upon inspection, $33\frac{1}{3}\%$ of the fruit in crate B had to be discarded.

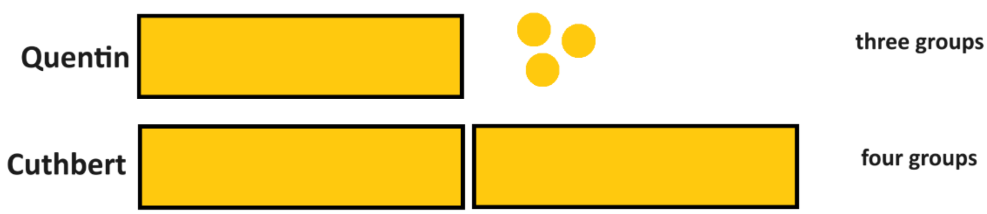
What now is the ratio of the count of fruit in crate A to the count in crate B?

Example: Quentin has some gumdrops. Cuthbert has twice as many.

When Quentin was later given 3 more gumdrops the ratio of the count of his gumdrops to the count of Cuthbert's changed to $3:4$.

How many gumdrops do the fellows each now have?

Answer: We start with a picture showing a $1:2$ ration of gumdrops, but add 3 gumdrops of Quentin's count is meant to show 7 equal-sized groups of gumdrops with three of those groups belonging to Quenton and four to Cuthbert.



It must be that those three extra gumdrops represent one group.

Thus, Quentin has 9 gumdrops and Cuthbert 12.

Practice 93.4 Allistaire has two-thirds the number of parking tickets as Poindexter has.

Allistaire just got two more tickets. He still had less tickets than Poindexter, but the ratio of tickets between them has changed to 7:9.

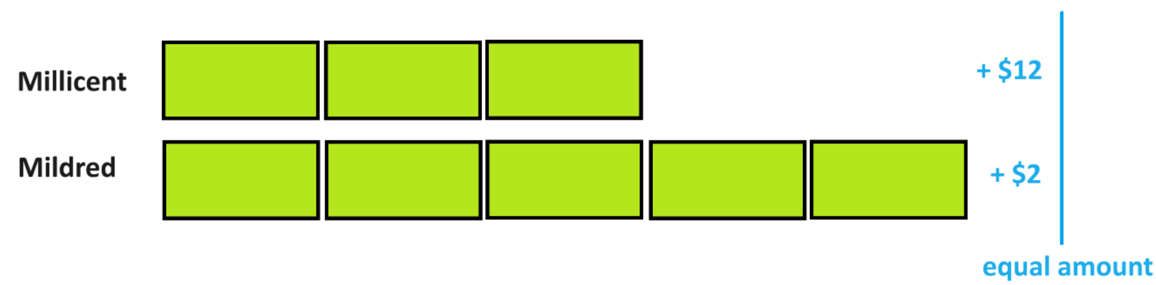
How many tickets do they each now have?

Example: Millicent and Mildred each started with equal amounts of cash. After Millicent spent \$12 and Mildred \$2, the cash they each had remaining was in a 3: 5 ratio.

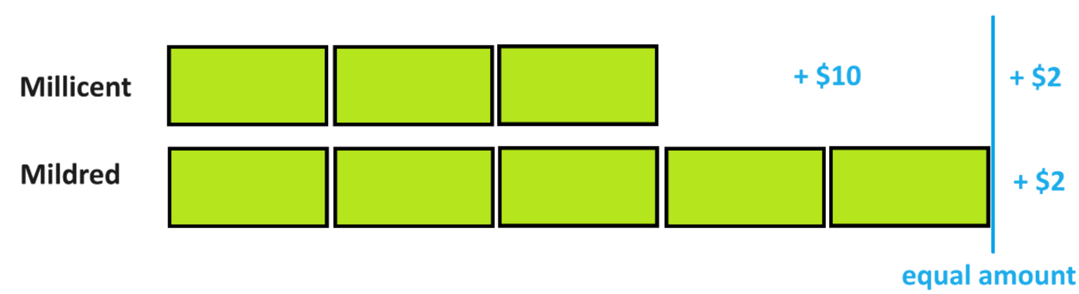
How much cash did they each start with?

Answer: This picture shows blocks of cash illustrating a 3: 5 ratio.

We have that three blocks of cash for Millicent and \$12 more matches five blocks of cash for Mildred and \$2 more.



Actually, let's tweak this picture.



We see that each block of cash must be \$5.

They thus each started with \$27.

Practice 93.5 Hal and Hank each started with equal amounts of cash. Hal spent \$29 and Hank \$26 and as a result Hal was left with half as much cash as Hank.

How much cash did they each start with?

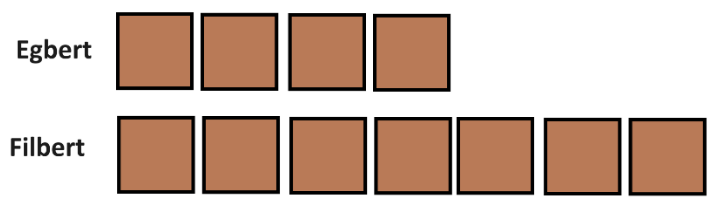
Example: Egbert and Figbert each have some cookies. The count of their cookies come in a $4:7$ ratio (with Figbert having the most cookies).

When Egbert eats half of his cookies and Figbert east 20 of his, the ratio of their cookie count changes to $1:3$.

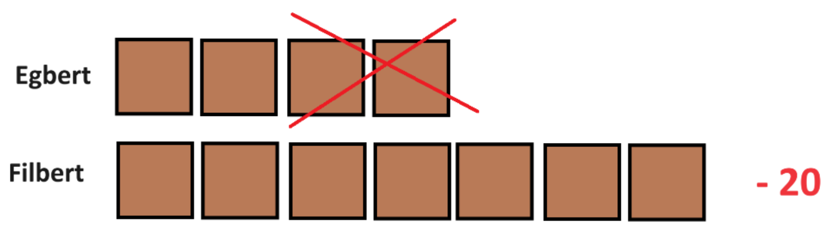
How many cookies did Egbert eat?

Answer: Look at this picture showing a $4:7$ ratio of cookie counts.

BEFORE



AFTER



For Filbert to end up with triple the number cookies than Egbert, he must have six blocks of cookies remaining. Thus "eating 20 cookies" must be equivalent to "eating one block of cookies."

Thus each block represents 20 cookies.

As Egbert ate two blocks, he ate 40 cookies.

Practice 93.6 I have a supply of lime candies and cherry candies, currently in a $3:2$ ratio. (There are more lime candies than cherry ones.) I eat 10 lime candies and the ratio changes to $2:3$.

How many candies of each type did I have to begin with?

MUSINGS

Musing 93.7 *Here is a tricky puzzle.*

There are 33 people in a room. Of all the men in the room, a third of them are Australian. Of all the women in the room, three-sevenths of them are Australian. There are 13 Australians altogether.

How many Australian men are there? How many non-Australian men?
How many Australian women are there? How many non-Australian women?

MECHANICS PRACTICE

Practice 93.8 Did you try the six problems in the section?

Practice 93.9 Make up two more problems that can be solved in the style of this section.

94. Summary

The goal of this chapter is to make a simple point: **Don't forget to use common sense thinking!**

The fact that we could answer the emu-egg omelet proportional reasoning problem without any preparation in "proportional reasoning" says it all. (We even wrote down the equations of lines through the origin for that problem without realizing it.) It's too easy to get bogged down with fancy mathematics and lose sight of uncluttered and natural thinking.

So, when faced with a challenge about two quantities whose values can or do vary—or at least be imagined to vary—take a step back and ask:

Is the data here scaling in tandem?

If the answer is YES, then you are golden. You can answer any reasonable question about the scenario just by following common sense.

You can even answer any schoolbook question about it too if you recall that the such data, when graphed, is sure to lie on a straight line that passes through the origin. (And, conversely, any data that comes from such a graph is sure scale in tandem.)

Data that scales in tandem often arises in these situations:

- **Constant rate problems** (walking or driving at a constant speed, stuffing envelopes at fixed rate, for example)
- **Unit conversion problems** (converting lengths measured in inches to lengths measured in centimeters, converting from USD to AUD, for example)
- **Percentage problems**
- **Scaled maps and scaled drawings**
- **Ratio problems**

Ratio problems tend to be "static problems" (about the ratio of Americans and Australians at a certain party) but can be turned into "scaling in tandem" problems if the static situation is repeated (Australians and Americans party every night together for a year, always in the same ratio of nationalities, for instance.)

And that's it. If you recognize scaling in tandem, then common sense is your path to success!

Chapter 12

Some Algebra Tricks and Hacks

95. Mind Reading Tricks

You are probably familiar with tricks of the following ilk:

> **TRICK 1:**
> *Think of a number.*
> *Add 3 and multiply the result by five.*
> *Add 5 and double the result.*
> *Divide by ten and subtract the number you first thought of.*
>
> *Barring arithmetic mistakes, you are now thinking the number* 4.

Wow!

Some tricks like this are easy to see through.

> **TRICK 2:**
> *Think of a number.*
> *Add* 17 *and subtract the number you first thought of.*
>
> *You are now thinking* 17.

Not so wow!

But some are more mysterious.

> **TRICK 3:**
> *Roll three dice.*
> *Multiply one of the rolls you see by* 5 *and add* 1.
> *Double the result and add a second roll you see.*
> *Multiply the result by* 5 *and add* 1.
> *Double the result and add the third roll you see.*
> *You have a three-digit number.*
>
> *Subtract* 22 *and get a new three-digit number.*
> *That number has as digits your three dice rolls!*

(As a magic trick, I would ask you tell me the three-digit number you have from line six, secretly subtract 22 from it myself, and then announce to you what your three dice rolls were—to your astonishment!)

Algebra is the way to explain these tricks.

In fact, when someone performs one of these tricks on me, I always choose in response to "think of a number" the symbol N—for some (unknown to me) number a person might think of.

TRICK 2 is the unexciting trick, so let's examine what happens with that one first.

Think of a number. N

Add 17. $N + 17$

Subtract the number first thought of:

$$N + 17 - N$$

TRICK 2:
Think of a number.
Add 17 *and subtract the number you first thought of.*

You are now thinking 17.

And of course, we are thinking 17 because algebra tells us that $N + 17 - N$ is 17.

Now to TRICK 1:

Think of a number. N
Add 3. $N + 3$
Multiply the result by five.

$$5(N + 3) = 5N + 15$$

Add five: $5N + 20$

Double the result:

$$2(5N + 20) = 10N + 40$$

Divide by ten:

$$\frac{1}{10} \times (10N + 40) = N + 4$$

Subtract the number you first thought of:

$$N + 4 - N = 4$$

TRICK 1:
Think of a number.
Add 3 and multiply the result by five.
Add 5 and double the result.
Divide by ten and subtract the number you first thought of.

You are now thinking the number 4.

Yep! Everyone is sure to end up with the number 4.

Practice 95.1: Make up your own "think of a number" trick that you are sure is going to work!

Practice 95.2 Open up a month of a calendar.

a) Without you looking, have a friend circle (or just mentally select) a 2-by-2 block of dates, sum the four dates selected, and tell you the sum.

For instance, in the example shown, she tells you that her sum is 60.

S	M	T	W	T	F	S
					1	2
3	4	5	6	7	8	9
10	11	12	13	14	15	16
17	18	19	20	21	22	23
24	25	26	27	28	29	30
31						

Divide the answer you hear by 4 (halve it twice) and subtract 4.

The answer you get will be the top left number in the bock of numbers which means you can now tell your friend the four dates she selected.

b) This time have a friend select a 3-by-3 block of dates and share with you the middle number in her block.

For instance, in the example shown, she will tell you that the middle number is 22.

S	M	T	W	T	F	S
					1	2
3	4	5	6	7	8	9
10	11	12	13	14	15	16
17	18	19	20	21	22	23
24	25	26	27	28	29	30
31						

Now race your friend to sum all nine selected numbers. You have the disadvantage as you know only one of the numbers.

But you will win this race simply by multiplying middle number told to you by 9.

$$22 \times 9 = 22 \times 10 - 22 = 220 - 22 = 198$$

and lo and behold, $14 + 15 + 16 + 21 + 22 + 23 + 28 + 29 + 30 = 198$!

Why is the sum of the nine numbers sure to be 9 times the middle number?

Let's now attend to TRICK 3. We start with an observation.

Every positive integer can be written in the form $10n + a$ where a is a single digit that matches the final digit of the original number.

For example, 623 has final digit 3 and we have $623 = 620 + 3 = 10 \times 62 + 3$.

In the same way,

$$4047 = 4040 + 7 = 10 \times 404 + 7$$

$$30 = 10 \times 3 + 0$$

$$8 = 10 \times 0 + 8$$

Practice 95.3 Convince me that every integer of two or more digits can be written in the form

$$100n + ab$$

where "ab" represents a one- or two-digit number given by the final two digits of the original number.

Along this line of thinking …

Every three-digit number can be written in the form $100a + 10b + c$ where a, b, and c are the digits of the original number.

For example, $623 = 600 + 20 + 3 = 100 \times 6 + 10 \times 2 + 3$.

In the same way,

$$500 = 100 \times 5 + 10 \times 0 + 0$$

We're now ready to make sense of TRICK 3.

Call the three dice rolls a, b, and c.

Multiple the first roll by five and add one: $5a + 1$

Double the result and add the second roll:

$$2(5a + 1) + b = 10a + 2 + b$$

Multiply by five and add one:

$$5(10a + 2 + b) + 1 \quad = 50a + 10 + 5b + 1$$

$$= 50a + 5b + 11$$

Double result and add the third roll:

$$2(50a + 5b + 11) + c = 100a + 10b + 22 + c$$

Subtract 22:

$$100a + 10b + c$$

Yep! The final result is sure to be a three-digit number with digits the rolls of the dice.

TRICK 3:
Roll three dice.
Multiply one of the rolls you see by 5 *and add* 1.
Double the result and add a second roll you see.
Multiply the result by 5 *and add* 1.
Double the result and add the third roll you see.
You have a three-digit number.

Subtract 22 *and get a new three-digit number.*
That number has digits of your three dice rolls!

MECHANICS PRACTICE

Practice 95.4 Explain this mind-reading trick.

> *Think of a number.*
> *Add* 3 *and multiply the result by* 3*.*
> *Add 6 and divide the result by* 3*.*
> *Subtract the number you first thought of.*
>
> *You are now thinking of the number* 5*.*

Practice 95.5 Please explain this mind-reading trick too.

> *Roll two dice.*
>
> *Double the value of one of the rolls and add* 4*.*
> *Multiply the result by* 5 *and* 1*.*
> *Add the value of the second roll.*
> *Subtract* 21 *from your total.*
>
> *You are now thinking of a two-digit number whose digits are the values of your rolls.*

Practice 95.6 Choose a four-by-five block of twenty numbers from a calendar page. Explain why the sum of all twenty numbers is sure to be ten times the sum of the smallest and largest numbers in the block. (For instance, in the example shown the sum of all the selected dates is $10 \times (4 + 29) = 330$.)

S	M	T	W	T	F	S
					1	2
3	4	5	6	7	8	9
10	11	12	13	14	15	16
17	18	19	20	21	22	23
24	25	26	27	28	29	30
31						

Practice 95.7

a) Write down a three-digit number and multiply it by 1,001. What do you notice? Explain what you notice.

b) Write down a three-digit number and multiply it b 7 and then by 11 and then 13. What do you notice? Explain what you notice.

Practice 95.8 The 1089 Trick

Write down three digits not all the same and make the biggest three-digit number you can with them. Also make the smallest three-digit number you can. (If you one or two of your digits are zero, you will technically have a two- or one-digit number here.)

Subtract the small number from the large one.

Reverse the digits of your answer and add that reversed answer to your answer.

Without me seeing your work, I know you just got 1089.

$$\begin{array}{r} 541 \\ -\ 145 \\ \hline =396 \end{array} \qquad \begin{array}{r} 220 \\ -\ 022 \\ \hline =198 \end{array}$$

$$\begin{array}{r} 396 \\ +\ 693 \\ \hline =\boxed{1089} \end{array} \qquad \begin{array}{r} 198 \\ +\ 891 \\ \hline =\boxed{1089} \end{array}$$

Why is 1089 sure to always result?

96. Arithmetic Hacks

I remember learning in school a multiplication trick for multiplying two-digit numbers by 11.

> *To multiply a two-digit number by* 11*, split apart the two digits and write their sum between them to produce a three-digit number. That number is the product you seek.*

$$11 \times 23 = 253$$
$$11 \times 45 = 495$$
$$11 \times 10 = 110$$

If you are willing to adopt some *Exploding Dots* thinking, this trick seems to work even if the sum obtained is a two-digit number.

$$11 \times 75 = 7|12|5 = 825$$
$$11 \times 86 = 8|14|6 = 946$$
$$11 \times 99 = 9|18|9 = 1089$$

Of course, the interesting question is: Why does this trick work?

Explanation: We can see this result if we use the long multiplication algorithm on a general two-digit number.

$$\begin{array}{rccc} & & a & b \\ \times & & 1 & 1 \\ \hline & & a & b \\ & a & b & 0 \\ \hline = & a & a+b & b \end{array}$$

For a more algebraic approach, we can observe as we saw last section that every two-digit number can be written in the form $10a + b$ where a and b are the digits of the number.

So, we're looking at the product $11 \times (10a + b)$.

This equals $110a + 11b$, which can be rewritten as

$$100a + 10a + 10b + b$$

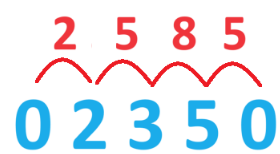
That is, as

$$100a + 10(a + b) + b$$

We do indeed have a hundreds, $a + b$ tens, and b ones as the answer to our product.

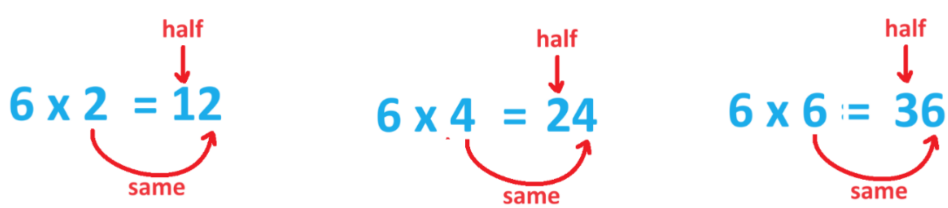
Practice 96.1 Can you explain this strange method for multiplying by 11?

To multiply a multi-digit number by 11, *write a* 0 *at the beginning and at the end of the number. Reading left to right, sum the pairs of digits you see. They spell out the answer to the product.*



For example, 235×11 is 2585

Practice 96.2 Have you ever noticed that if you multiply an even digit by 6 the result is half that digit and the digit placed together to make a two-digit number?



This holds for all even numbers if you are willing to adopt the Exploding Dots mentality for thinking about and writing numbers.

6 x 10 = 5|10 = 60

6 x 14 = 7|14 = 84

6 x 32 = 16|32 = 192

Explain why this pattern is true.

In this section we'll go through some arithmetic patterns and hacks. These hacks aren't worth memorizing and using. But it is fun to figure out why they work.

The only "hacks" I do recommend are ones that promote a sense of common-sense and clever thinking that help avoid hard work when doing a numerical calculation.

For instance, when faced with computing

$$1637 - 498$$

I naturally think "subtract 500 and then add 2 to compensate."
I am realizing in my mind that $1637 - 498 = 1637 - 500 + 2$. The answer 1139 is apparent.

Example: Compute $137 \div 5$.

My Answer: I think to double the number and dividing that number by 10.

$$\frac{1}{5} \times 137 = \frac{1}{10} \times (2 \times 137)$$

And I'll double 137 the *Exploding Dots* way of Section 33.

$$2 \times 137 = 2|6|14 = 274$$

So

$$137 \div 5 = 27.4$$

Example: Compute 96×32.

My Answer: Well, $100 \times 32 = 3200$. But this is too large by "four copies of 32."

Doubling and doubling 32 gives 128.

Consequently

$$96 \times 32 = 3200 - 100 - 20 - 8 = 3072$$

Are you comfortable with each of these ideas?

- To add 42 to a number, add 40 and then add 2.

$$N + 42 = N + 40 + 2$$

- To subtract 97 from a number, subtract 100 and add 3.

$$N - 97 = N - 100 + 3$$

- To multiply a number by 11, multiply the number by 10 instead and add the original number to the result.

$$(10 + 1)N = 10N + N$$

- To multiply a number by 9, multiply the number by 10 instead and subtract the original number from the result.

$$(10 - 1)N = 10N - N$$

- To multiply a number by 98, multiply it by 100 instead and subtract two copies of the original number.

$$(100 - 2)N = 100N - 2N$$

- To divide a number by 4, halve the number and then halve again.

$$\frac{1}{4} \times N = \frac{1}{2} \times \frac{1}{2} \times N$$

- To multiply a number by 5, multiply the number by 10 instead and halve the result.

$$5 \times N = \frac{1}{2} \times 10 \times N$$

Practice 96.3 Compute each of these quantities in your head!

a) $1152 - 98$ b) $978 + 39$ c) $36{,}487 + 5{,}490$

d) $872 \div 4$ e) $2{,}688 \times 5$ f) 38×9

g) 55×62

Okay, now to some hacks that are interesting only because they present mysteries to crack. Why do they work?

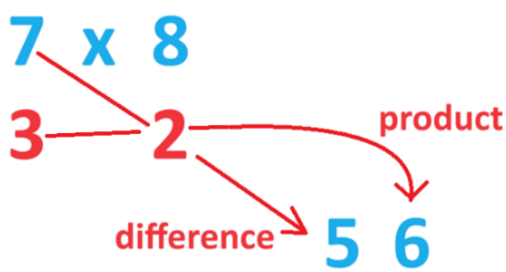
Multiplying Two Numbers Each Close to 10

To multiply two numbers that are each close to the number ten, write down the product and below each number the value you need to add to it to get 10.

For example, in computing 7×8 I'd write a 3 under the 7 and a 2 under the 8.

Write down the difference of the two digits along the southeast diagonal—the first number of the product and the number you wrote down below the second number of the product. That difference is the first digit of your answer.

The second digit is the product of the two smaller numbers you wrote down.

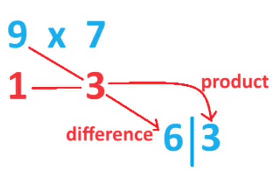


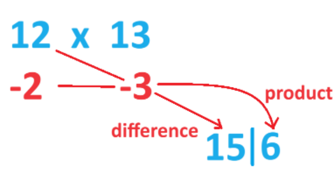
Here are some more examples. They show us that

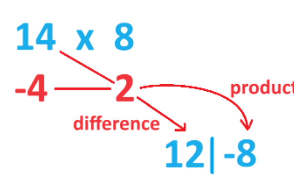
9×7 is 6 tens and 3 ones: 63

12×13 is 15 tens and 6 ones: 156

14×8 is 12 tens and -8 ones: 112







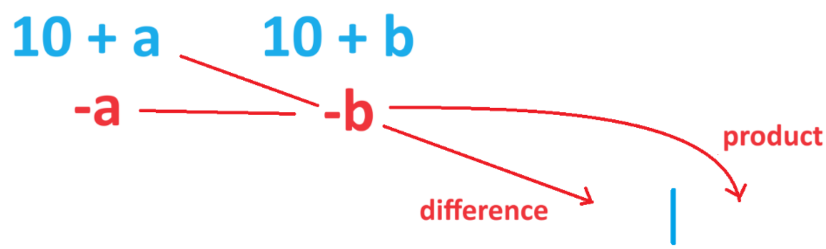
What's going on?

Let's start by naming our two numbers that are each "close to ten" in a way that makes it clear how far off from ten each are: $10 + a$ and $10 + b$.

So, we are trying to work out the product

$$(\mathbf{10} + \boldsymbol{a})(\mathbf{10} + \boldsymbol{b})$$

Now let's follow the procedure of the hack with these two abstract numbers.



The difference we need is

$$10 + a - (-b) = 10 + a + b$$

The product is:

$$(-a) \times (-b) = ab$$

So, the two-digit number the method gives is

$$\mathbf{10} \times (\mathbf{10} + \boldsymbol{a} + \boldsymbol{b}) + \boldsymbol{ab}$$

Is this the product of the two original numbers in disguise?

We can rewrite our product:

$$(10 + a)(10 + b) = 100 + 10a + 10b + ab$$

And we can rewrite the expression from the hack:

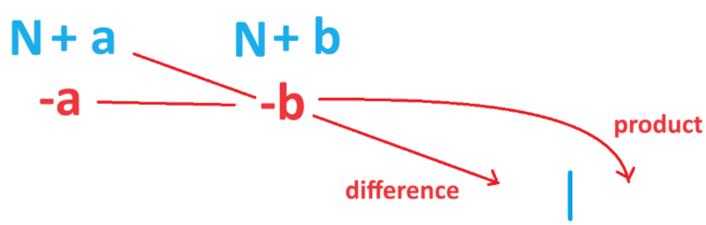
$$10(10 + a + b) + ab = 100 + 10a + 10b + ab$$

Ahh! They are indeed the same quantity in disguise! The hack is sure to always work.

What I love about algebra is that it frees us up from working with specific numbers and a feeling of "specialness" to certain numbers.

Practice 96.4 Suppose we wish to multiply two numbers that are each close to N (where N could be 10, as before).

What does the technique of this hack say is the value of $(N+a) \times (N+b)$?



It is tempting the say in this problem that the hack suggests that the product is a two-digit answer with first digit $N+a+b$ and second digit ab, that the answer is

$$10 \times (N+a+b) + ab$$

But this is suspicious! The number 10 just appeared and there should be no ten-ness in this problem!

Okay. Let's rethink this.

We're looking for a disguised version of the product

$$(\boldsymbol{N}+\boldsymbol{a})(\boldsymbol{N}+\boldsymbol{b})$$

So, let's play with this expression and see if we can make the difference $N+a+b$ and product ab appear in what we seem.

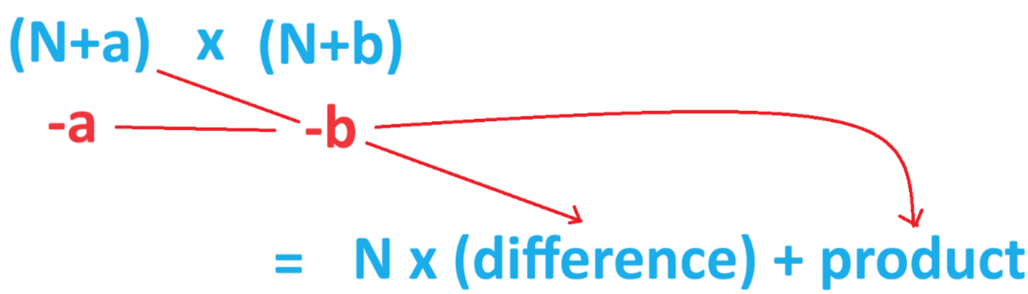
It seems natural to expand $(N+a)(N+b)$ and write is as

$$N^2 + aN + bN + ab$$

I see the product ab.

And I see the difference $N + a + b$ too if I factor out N from the first three terms!

$$N \times (N + a + b) + ab$$

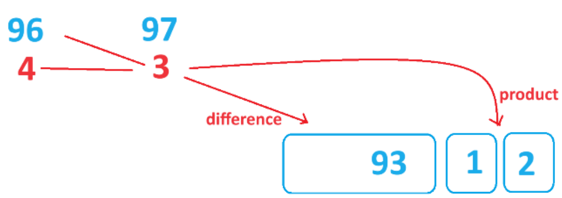


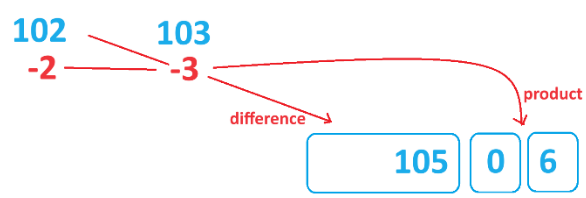
The answer is a two-digit number, but a two-digit number in base N!

Multiplying Two Numbers Each Close to 100

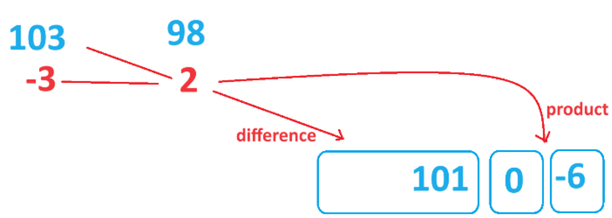
Let's take N to be 100 in the previous hack.

> *To compute the product of two numbers each close to* 100*, write down the product, and then below each term in the product the value you need to add to it to get* 100*.*
>
> *Write down the difference of the two numbers along the southeast diagonal, and to the right of that difference, the product of the two smaller numbers you wrote down as a two-digit answer.*
>
> *Now read off the product you seek.*





Practice 96.5 How does one make sense of the result of this method for 103×98?



Multiplying Two Two-digit Numbers with the Same First Digit and Second Digits that Add to 10

How’s that for an absurdly specific situation!

But if you find yourself in a position of trying to compute a product like one of the following, and don’t want to use a calculator, and don’t want to use the area-model or standard multiplication, then here’s the trick for you!

$$\begin{array}{r} 4\ 7 \\ \times\ 4\ 3 \end{array} \qquad \begin{array}{r} 8\ 2 \\ \times\ 8\ 8 \end{array} \qquad \begin{array}{r} 5\ 1 \\ \times\ 5\ 9 \end{array}$$

To multiply two two-digit numbers with the same first digit N *compute* $N \times (N+1)$ *and compute the product of the two second digits as a two-digit answer and write those two answers in juxtaposition.* `

$$\begin{array}{r} 4\ 7 \\ \times\ 4\ 3 \\ \hline \underbrace{20}_{4 \times 5}\underbrace{21}_{7 \times 3} \end{array} \qquad \begin{array}{r} 8\ 2 \\ \times\ 8\ 8 \\ \hline \underbrace{72}_{8 \times 9}\underbrace{16}_{2 \times 8} \end{array} \qquad \begin{array}{r} 5\ 1 \\ \times\ 5\ 9 \\ \hline \underbrace{30}_{5 \times 6}\underbrace{09}_{1 \times 9} \end{array}$$

Practice 96.6 Does this method seem to method work for the product of any two integers that are identical except for their final digits, which happen to sum to 10?

$$\begin{array}{r} 1\ 2\ 3 \\ \times\ 1\ 2\ 7 \\ \hline \underbrace{156}_{12 \times 13}\underbrace{21}_{3 \times 7} \end{array}$$

What's going on here?

We have two numbers identical except for their final digits which add to 10. If the final digit of one is a, then the final digit of the is $10 - a$.

$47 = 10 \times 4 + 7$ and $43 = 10 \times 4 + (10 - 7)$
$82 = 10 \times 8 + 2$ and $88 = 10 \times 8 + (10 - 2)$.
$51 = 10 \times 5 + 2$ and $59 = 10 \times 5 + (10 - 1)$.
$123 = 10 \times 12 + 3$ and $127 = 10 \times 12 + (10 - 3)$.

So, we have a number $10N + a$ with final digit a and a second number that is identical, but with final digit is $10 - a$. It's $10N + (10 - a)$.

We are trying to work out their product:

$$(\mathbf{10N} + \boldsymbol{a}) \times (\mathbf{10N} + \mathbf{10} - \boldsymbol{a})$$

	10N	10	-a
10N	$100N^2$	100N	$\cancel{-10Na}$
a	$\cancel{10Na}$	10a	$-a^2$

This is the same as

$$100N^2 + 100N + 10a - a^2$$

Let's look $100N^2 + 100N$ and look at $10a - a^2$.

In $100N^2 + 100N$ there is a common factor of 100 and a common factor of N and so we can rewrite it as $100N(N + 1)$.

In $10a - a^2$ there is a common factor of a and so we can rewrite it as $a(10 - a)$.

So, our product is

$$100 \times N(N + 1) + a \times (10 - a)$$

Look! We computed $N \times (N + 1)$, multiplied it by 100, which "pushes" its digits two places to the left leaving space to write the product $a \times (10 - a)$, the product of the two final digits.

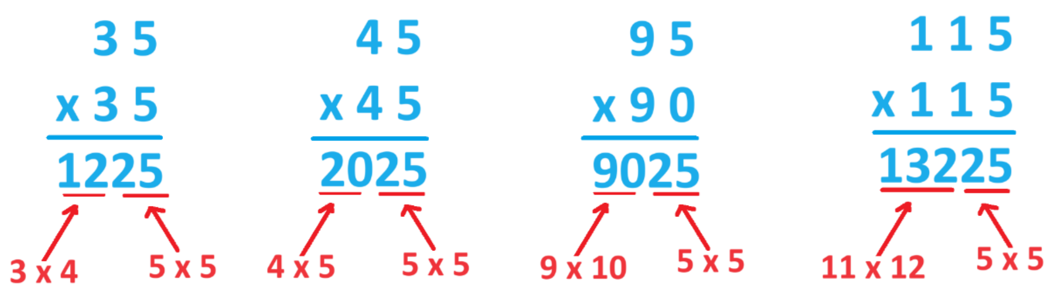
This trick is always sure to work, even if N itself has more than one digit.

Whoa!

Squaring a Number that Ends with a 5

To square a number is to multiply the number by itself.

If that number ends has final digit 5, then we are multiplying two identical numbers whose final digits add to 10. We can use the previous hack!



To square a number that ends with a five, delete the final digit 5 to leave a smaller number. Call it N.
Compute $N \times (N + 1)$ and then tack on the digits 2 and 5 to that answer.

(Do you see how this is an appropriate rephrasing of the previous hack for this special case?)

There seem to be plenty of hacks on the internet for squaring numbers. (It is not always clear how practice they are!)

Here are a few for your amusement.

Practice 96.6 Squaring a Number that ends with a 1

Can you detect a pattern? Can you put the pattern into words?
Can you then explain why the pattern holds?

$$\mathbf{21^2} = \mathbf{20} \times \mathbf{21} + \mathbf{21} = 420 + 11 = \mathbf{441}$$

$$\mathbf{31^2} = \mathbf{30} \times \mathbf{31} + \mathbf{31} = 930 + 31 = \mathbf{961}$$

$$\mathbf{41^2} = \mathbf{40} \times \mathbf{41} + \mathbf{41} = 1640 + 41 = \mathbf{1681}$$

$$\mathbf{51^2} = \mathbf{50} \times \mathbf{51} + \mathbf{51} = 2550 + 51 = \mathbf{2601}$$

Practice 96.7 Squaring a Number that ends with a 9

Can you detect a pattern? Can you put the pattern into words?
Can you then explain why the pattern holds?

$$\mathbf{19^2} = \mathbf{20^2} - (\mathbf{19} + \mathbf{20}) = 400 - 40 + 1 = \mathbf{361}$$

$$\mathbf{29^2} = \mathbf{30^2} - (\mathbf{29} + \mathbf{30}) = 900 - 60 + 1 = \mathbf{841}$$

$$\mathbf{39^2} = \mathbf{40^2} - (\mathbf{39} + \mathbf{40}) = 1600 - 80 + 1 = \mathbf{1521}$$

$$\mathbf{49^2} = \mathbf{50^2} - (\mathbf{49} + \mathbf{50}) = 2500 - 100 + 1 = \mathbf{2401}$$

Practice 96.8 Squaring a Number near 50

Can you detect a pattern? Can you put the pattern into words?
Can you then explain why the pattern holds?

$$\mathbf{47^2 = 2500 - 300 + 3^2 = 2209}$$

$$\mathbf{45^2 = 2500 - 500 + 5^2 = 2025}$$

$$\mathbf{46^2 = 2500 - 400 + 4^2 = 2116}$$

$$\mathbf{40^2 = 2500 - 1000 + 10^2 = 1600}$$

Does the pattern persist for numbers “under 50 by a negative amount”?

$$\mathbf{52^2 = 2500 + 200 + 2^2 = 2704}$$

$$\mathbf{53^2 = 2500 + 300 + 3^2 = 2809}$$

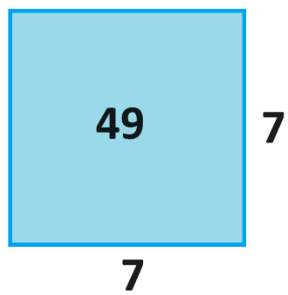
$$\mathbf{59^2 = 2500 + 900 + 9^2 = 3481}$$

Finding Square Roots of Three- and Four-digit Perfect Squares

Recall from section 79 that the square root of a positive number N is the side length of a square of area N.

Consequently, it is a positive number that multiplies by itself to give the value N.

For example, $\sqrt{49} = 7$.



Example: Between which two consecutive integers does $\sqrt{105}$ lie?

Answer: We recognize that 105 lies between 100 (which is 10×10) and 121 (which is 11×11). Consequently,

$$\sqrt{100} < \sqrt{105} < \sqrt{121}$$
$$10 < \sqrt{105} < 11$$

People call an integer a **perfect square** if it equals an integer squared. (So, the term "perfect square" is just another name for a square number, as per Chapter 2.) For example, 100 and 121 are perfect squares.

A perfect square has a square root that is an integer. So, if you recognize between which two perfect squares an integer lies, then you have an estimate of the value of its square root.

Example: Show that $\sqrt{1893}$ has a value between 40 and 50.

Answer: We have $40 \times 40 = 1600$ and $50 \times 50 = 2500$.
So,

$$\sqrt{1600} < \sqrt{1893} < \sqrt{2500}$$
$$40 < \sqrt{1893} < 50$$

Practice 96.9: The number 4761 is a perfect square.
Knowing this, deduce that its square root is a two-digit number with first digit 6.

Practice 96.10: The number 7569 is a perfect square.
What is the first digit of its square root?

These practice exercises are suggesting how to determine the first digit of the square root of a four-digit number known to be a perfect square.

Knowing the basic square numbers seems to be important.

$$1^2 = 1$$

$$2^2 = 4$$

$$3^2 = 9$$

$$4^2 = 16$$

$$5^2 = 25$$

$$6^2 = 36$$

$$7^2 = 49$$

$$8^2 = 64$$

$$9^2 = 81$$

The we can deduce that $40 \times 40 = 1600$, for example. So,

$$\sqrt{1600} = 40$$

We also see

$$\sqrt{8100} = 90$$

$$\sqrt{400} = 20$$

$$\sqrt{2500} = 50$$

$$\sqrt{3600} = 60$$

and so on.

You know the square roots of $\mathbf{100,\ 400,\ 900,\ 1600,\ 2500,\ 3600,\ 4900,\ 6400,}$ and $\mathbf{8100.}$

But how do we determine the second digit of the square root of a four-digit perfect square?

Well, the natural answer is to pull out a calculator and use the square root button!

But since the algebra hacks one finds on the internet presume we don't live in the 21^{st} century, here's a deeply mysterious procedure that allows you to figure this out with pencil and paper. It works for three-digit perfect squares too.

To find the square root of a four- or three-digit number known to be a perfect square.

1. *Find the first digit of the square root by identifying between which two of these perfect squares it lies:* $\mathbf{100, 400, 900, 1600, 2500, 3600, 4900, 6400, 8100.}$
   Call that first digit N.

2. *Look at the final digit of the number we're working with.*

   If it is a 0, *then the square root is the two-digit number* $10N$.
   If it is a 5, *then the square root is the two-digit number* $10N + 5$.

   If it is a 1, *then the square root is either* $10N + 1$ *or* $10N + 9$.
   If it is a 4, *then the square root is either* $10N + 2$ *or* $10N + 8$.
   If it is a 9, *then the square root is either* $10N + 3$ *or* $10N + 7$.
   If it is a 6, *then the square root is either* $10N + 4$ *or* $10N + 6$.

3. *To figure out which of the two options it is, work out* $N \times (N + 1) \times 100$.
   If the original number is bigger than this value, choose the larger option for the square root.
   If the original number is smaller than this value, choose the smaller option for the square root.

Whoa!

Here's a worked example.

Example: The number 1444 is a perfect square. Kindy find its square root without a calculator.

Answer: Here goes.

1. We have

$$\sqrt{900} < \sqrt{1444} < \sqrt{1600}$$

$$30 < \sqrt{1444} < 40$$

The square root is a number in the 30s. The first digit is $\boldsymbol{N = 3}$.

2. The number 1444 ends with a 4, so, allegedly its square root is either 32 or 38.

Rather than quickly square each of these to find out which, we continue the mysterious ways.

3. We have $N \times (N+1) \times 100 = 3 \times 4 \times 100 = 1200$.

Our number 1444 is bigger than 1200, so we choose the bigger option.

$$\boldsymbol{\sqrt{1444} = 38}$$

Let's do another one.

Example: The number 529 is a perfect square. Please find its square root without a calculator.

Answer:

1. We have

$$\sqrt{400} < \sqrt{529} < \sqrt{900}$$

$$20 < \sqrt{529} < 30$$

The square root is a number in the 20s. It's first digit is $\boldsymbol{N = 2.}$

2. Since 289 ends with a 9, the square root is either 23 or 29.

3. We have $N \times (N+1) \times 100 = 2 \times 3 \times 100 = 600$.

Our number 529 is smaller than this, so we choose the smaller option.

$$\boldsymbol{\sqrt{529} = 23}$$

Practice 96.11 If this procedure delights you, use it to work out the square roots of these perfect squares. Check your answers with a calculator.

a) 7569 b) 5329 c) 2025 d) 961 e) 2916

We'll leave explaining this bizarre method for computing the square roots of perfect squares to the Musings of this section. All will be revealed there if you want to know.

MUSINGS

Musing 96.12 Is there a pattern to be discovered and proved about multiplying two-digit numbers by 99?

$$16 \times 99 = \underbrace{15}_{\text{one less than 16 is 15}}\underbrace{84}_{99-15\,=\,84}$$

Musing 96.13 Watch out! Not every arithmetic trick you see on the internet is real.

For example, I found this "trick" (in the real sense of the word!) as a suggested shortcut for finding the square roots of numbers.

To find the square root of a number, just sum its digits and subtract 2.

Here are some examples. Each statement is mathematically correct.

$$\sqrt{25} = 2 + 5 - 2 = 5$$
$$\sqrt{64} = 6 + 4 - 2 = 8$$
$$\sqrt{4} = 4 - 2 = 2$$
$$\sqrt{289} = 2 + 8 + 9 - 2 = 17$$

Can you find another example where this trick, by luck, happens to work?

MECHANICS PRACTICE

As I said earlier, the only arithmetic "tricks" one needs are those that arise from common-sense thinking about using the structure of numbers to avoid hard work. For instance, if I needed to add 98 *to a number, I'd personally add* 100 *and subtract* 2.

Practice 96.14 Kindly work out each of these quantities with flexible and efficient thinking.

a) $173 + 298$ b) $1000 - 273$ c) A quarter of 72 d) 15% of $\$10.20$.
e) 78×99 f) 5×484 g) 25×484 h) $17 + 29 + 83 + 71$

OPTIONAL MUSING: Explaining the Square Root Hack

Let's now explain that mysterious method for determining the square roots of three- and four-digit numbers known to be perfect squares. (Perhaps review that method now to remind yourself of it.)

We'll do this using a series of observations.

Observation 1:
$1^2 = 1$ and $9^2 = 81$ both end with a 1, and $1 + 9 = 10$.
$2^2 = 4$ and $8^2 = 64$ both end with a 4, and $2 + 8 = 10$.
$3^2 = 9$ and $7^2 = 49$ both end with a 9, and $3 + 7 = 10$.
$4^2 = 16$ and $6^2 = 36$ both end with a 6, and $4 + 6 = 10$.

And $5^2 = 25$ "matches with itself" and ends with a 5.
(Technically $0^2 = 0$ also "matches with itself" and ends with a 0.)

Some exercises next to get to our second observation.

Musing 96.15

a) Show that multiplying a number that ends with a 1 by itself is sure to give an answer that ends with a 1 (the same as 1^2 does).

b) Show that multiplying a number that ends with a 2 by itself is sure to give an answer that ends with a 4 (the same as 2^2 does).

c) Show that multiplying a number that ends with a 4 by itself is sure to give an answer that ends with a 6 (the same as 6^2 does).

d) Show that multiplying a number that ends with a 9 by itself is sure to give an answer that ends with a 1 (the same as 9^2 does).

If you had the patience to check every possibility, you'd see

Observation 2:
If a number ends with the digit a, then its square has the same final digit as a^2 does.

This observation has logical consequences.

If a perfect square ends with a 1*, then its square root ends with a* 1 *or a* 9*.*

Reason: We've just seen that only numbers that end with a 1 or 9 will multiply by themselves to give an answer that ends with a 1.

If a perfect square ends with a 5, then its square root ends with a 5.

Reason: We've just seen that only numbers that end with a 5 will multiply by themselves to give an answer that ends with a 5.

If a perfect square ends with a 6, then its square root ends with a 4 or a 6.

Reason: We've just seen that only numbers that end with a 4 or 6 will multiply by themselves to give an answer that ends with a 6.

And so on!

Here's what we have so far:

Summary:
If a perfect square ends with 0, its square root also ends with 0.
If a number ends with 5, its square root also ends with 5.

If a perfect square ends with 1, its square root also ends with either 1 or 9.
If a perfect square ends with 4, its square root also ends with either 2 or 8.
If a perfect square ends with 9, its square root also ends with either 3 or 7.
If a perfect square ends with 6, its square root also ends with either 4 or 6.

Musing 96.16 Explain why no perfect square can end with a 2, 3, 7, or 8.

Let's now work with a specific example to see how these observations apply.

Let's work out the square root of 7744, which happens to be a perfect square.

Step 1: We observe

$$6400 < 7744 < 8100$$

so

$$80 < \sqrt{7744} < 90$$

Thus, $\sqrt{7744}$ is a number in the 80s. It's first digit is sure to be $N = 8$.

Step 2: Our number 7744 ends with a 4. Its square root ends either with a 2 or an 8.

We deduce that the square root of 7744 is thus either 82 or 88.

Step 3: One of these options is larger than 85 and one is smaller.

We can determine which to choose by working out 85^2 and seeing if 7744 is larger than this (in which case 7744 is 88^2) or smaller (in which case 7744 is 82^2). And we have a hack for squaring a number than ends with a 5!

$$85^2 = "8 \times 9" \mid 25 = 7225$$

Our number is larger than this so

$$\sqrt{7744} = 88$$

The key parts to step 3 are:

i) Whenever we have a choice between two possible square roots, one option will have second digit smaller than 5 and the other larger than 5. (We see this in the Summary statement.)

ii) We have a hack for squaring a two-digit number with fist digit N and second digit 5. Its square is

$$N \times (N+1) \times 100 + 25$$

So, we just need to compare our given number with this value to determine whether we have the larger or smaller potential square root.

But as we’re working with large numbers, the “$+25$” part of this value will be immaterial. We can compare our given number just with

$$N \times (N+1) \times 100$$

Whew! That ties things up.

But what an involved and ingenious hack!
(But still .. I’d rather just ask Siri or Alexa or Google to find the square root of a given number.)

97. An Antiquated Hack: Rationalizing the Denominator

Back in my high-school days in Australia (the early 80s) we did not have calculators.

Instead, we were given booklets with tables of values. For example, the “square roots pages” listed the decimal values of the square roots of the first one-hundred or so integers, rounded to three decimal places.

$$
\begin{aligned}
\sqrt{1} &= 1.000\\
\sqrt{2} &= 1.414\\
\sqrt{3} &= 1.732\\
\sqrt{4} &= 2.000\\
\sqrt{5} &= 2.236\\
\sqrt{6} &= 2.449\\
\sqrt{7} &= 2.646\\
\sqrt{8} &= 2.282\\
\sqrt{9} &= 3.000\\
\sqrt{10} &= 3.162\\
\\
\sqrt{11} &= 3.317\\
\sqrt{12} &= 3.464\\
&\vdots
\end{aligned}
$$

Sometimes when working on a mathematics problem, we’d obtain answers that involved the square roots of numbers. Such answers are good and fine mathematically. But if you are working on a realistic problem, knowing that a bridge is $12\sqrt{11}$ meters long, say, is not that helpful. A decimal value would be of better practical use.

So, in this situation we were required to look at the value of $\sqrt{11}$ in our tables and compute the appropriate arithmetic by hand.

We’d see that our answer is close to 39.8 meters.

$$
\begin{array}{rr}
 & 3.317\\
\times & 12\\
\hline
 & 6.634\\
 & 33.170\\
\hline
= & 39.804
\end{array}
$$

This work is tedious, but doable.

But there are some forms of quantities involving square roots whose decimal values are truly miserable to compute by hand.

Example: What is approximate value of $\dfrac{1}{\sqrt{2}}$ as a decimal?

Answer Attempt 1: According to our table, $\sqrt{2} = 1.414$ up to three decimal places.

To compute $\frac{1}{\sqrt{2}}$ we need to compute $\frac{1}{1.414}$.

To avoid decimals within fractions, let's multiply top and bottom each by 1000 and work with $\frac{1000}{1414}$. (According to Chapters 5 and 6, this does not change the value of the fraction.)

This is a long division problem.

$$1414\overline{)1000}$$

Ick!

Fractional quantities with square roots of values in the denominator are always tough to covert to decimal approximations by hand.

But there is a hack.

If wish find a the decimal value of a fractional quantity with denominator the square root of an awkward number, try multiplying the top and bottom of that quantity by that square root number.

Back to our problem.

Answer Attempt 2: Rather than work with $\frac{1}{\sqrt{2}}$, let's try working with

$$\frac{1 \times \sqrt{2}}{\sqrt{2} \times \sqrt{2}} = \frac{\sqrt{2}}{2}$$

We know from our work with fractions this has not changed the value of our quantity.

Now, the decimal approximation to $\frac{\sqrt{2}}{2}$ is

$$\frac{1.414}{2}$$

I see this as "half of fourteen and half of fourteen."

We have

$$\frac{1}{\sqrt{2}} = 0.707$$

at least to some degree of accuracy.

Practice 97.1 What is the approximate decimal value of $\frac{1}{\sqrt{10}}$?

Rewriting

$$\frac{\boldsymbol{a}}{\sqrt{\boldsymbol{b}}}$$

as

$$\frac{a \times \sqrt{b}}{\sqrt{b} \times \sqrt{b}} = \frac{\boldsymbol{a}\sqrt{\boldsymbol{b}}}{\boldsymbol{b}}$$

is a fabulous aid **IF** you are required to work out the decimal value of quantity by hand, using tables of given values for square roots.

This process became known as **rationalizing the denominator** and it was commonly taught in schools before calculators were invented.

And it remained commonly taught in schools all through the 1980s, the 1990s, the 2000s, and you will even see it still taught to this day. In fact, many school curricula regard leaving square root numbers in the denominator of a fractional quantity as mathematically wrong and insist students always rationalize the denominator. This is an absurd thing to insist upon.

Practice 97.2 a) Compute $\frac{1}{\sqrt{10}}$ on a calculator by pressing 1 then ÷ then √ then 10 and then enter.

b) Compute $\frac{\sqrt{10}}{10}$ on a calculator by pressing √ then 10 then ÷ then 10 and then enter.

c) Was significant time saved by working with $\frac{\sqrt{10}}{10}$ rather than $\frac{1}{\sqrt{10}}$?

Practice 97.3 Please be naughty and **rationalize the numerator** of

$$\frac{\sqrt{30}}{5}$$

MECHANICS PRACTICE

Practice 97.4 *Match time!*

Match each quantity on the left with a quantity of the same value on the right.

$\frac{\sqrt{10}}{2}$	$\frac{1}{10\sqrt{10}}$
$\frac{\sqrt{20}}{2}$	$\frac{1}{10}$
$\frac{\sqrt{10}}{100}$	$\frac{10}{\sqrt{20}}$
$\frac{5}{2\sqrt{10}}$	$\frac{\sqrt{10}}{4}$
$\frac{1}{\sqrt{100}}$	$\frac{5}{\sqrt{10}}$

98. Expected Schoolbook Work on Square Roots

While we are here, we might as well detail everything a typical algebra curriculum expects students to know about the arithmetic of the square roots of numbers.

The square root of a positive number N is a number $\sqrt{N}$ that multiplies by itself to give N.

$$\boldsymbol{\sqrt{N} \times \sqrt{N} = N}$$

Recall from section 79 when using the radix symbol $\sqrt{\ }$ from geometry all the quantities discussed must be positive (or zero, that is allowed too).

Practice 98.1 Write down the values of each of these quantities.

a) $\sqrt{225}$ b) $\sqrt{1}$ c) $\sqrt{0}$ d) $\sqrt{\frac{1}{225}}$ e) $\sqrt{\frac{169}{225}}$

The fifth example here illustrates a schoolbook rule.

We know that $\sqrt{169} = 13$ and $\sqrt{225} = 15$ and so you likely answered that

$$\sqrt{\frac{169}{225}} = \frac{13}{15}$$

We do indeed have $\frac{13}{15} \times \frac{13}{15} = \frac{169}{225}$.

It looks like

$$\boldsymbol{\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}}$$

for non-negative numbers a and b with b not zero.

We can prove that this is so in general by checking if $\frac{\sqrt{a}}{\sqrt{b}}$ multiplies by itself to give the answer $\frac{a}{b}$.

Check: It does!

$$\frac{\sqrt{a}}{\sqrt{b}} \times \frac{\sqrt{a}}{\sqrt{b}} = \frac{\sqrt{a} \times \sqrt{a}}{\sqrt{b} \times \sqrt{b}} = \frac{a}{b}$$

Practice 98.2 Make each of these quantities look friendlier.

a) $\frac{\sqrt{40}}{\sqrt{4}}$ b) $\frac{\sqrt{125}}{\sqrt{5}}$ c) $\frac{\sqrt{30}}{\sqrt{3000}}$ d) $\sqrt{80} \div \sqrt{5}$ e) $\frac{\sqrt{0.01}}{\sqrt{100}}$

Practice 98.3 Find four values for b that make $\dfrac{\sqrt{36}}{\sqrt{b}}$ an integer.

The answer to Problem 98.2 d) suggests something interesting. We have

$$\sqrt{80} \div \sqrt{5} = \sqrt{80 \div 5}$$

(After all, the left side is just $\frac{\sqrt{80}}{\sqrt{5}}$ and the right side is $\sqrt{\frac{80}{5}}$, and we know these are the same.)

The schoolbook rule could be rewritten

$$\sqrt{\boldsymbol{a} \div \boldsymbol{b}} = \sqrt{\boldsymbol{a}} \div \sqrt{\boldsymbol{b}}$$

showing that square roots and division "play nice" together.

This makes me wonder: Do square roots play nice with multiplication? With addition? With subtraction?

Question: Are any of the following always true?

$$\sqrt{\boldsymbol{a} \times \boldsymbol{b}} = \sqrt{\boldsymbol{a}} \times \sqrt{\boldsymbol{b}}$$

$$\sqrt{\boldsymbol{a} + \boldsymbol{b}} = \sqrt{\boldsymbol{a}} + \sqrt{\boldsymbol{b}}$$

$$\sqrt{\boldsymbol{a} - \boldsymbol{b}} = \sqrt{\boldsymbol{a}} - \sqrt{\boldsymbol{b}}$$

Practice 98.4 On a calculator, work out $\sqrt{5}$ and $\sqrt{2}$ and test if any of these statements are true.

a) $\sqrt{5} \times \sqrt{2}$ equals $\sqrt{5 \times 2}$
b) $\sqrt{5} + \sqrt{2}$ equals $\sqrt{5 + 2}$
c) $\sqrt{5} - \sqrt{2}$ equals $\sqrt{5 - 2}$

Practice 98.5 Can you find any value for a and any value for b for which

$$\sqrt{a+b}$$

happens to equal

$$\sqrt{a} + \sqrt{b} \text{ ?}$$

Practice 98.6 Can you find any value for a and any value for b for which

$$\sqrt{a-b}$$

happens to equal

$$\sqrt{a} - \sqrt{b} \text{ ?}$$

I am sure you have a conjecture.

Square roots and multiplication play nicely together: $\sqrt{\boldsymbol{a} \times \boldsymbol{b}} = \sqrt{\boldsymbol{a}} \times \sqrt{\boldsymbol{b}}$

Square roots and addition DO NOT play nicely together.

Square roots and subtraction DO NOT play nicely together.

Actually, Problem 98.4 parts b) and c) have established the last two claims: square roots are not friends with addition and not with subtraction.

But we can prove the first claim is true.

Statement: For two positive numbers a and b (or possibly zero) we have that $\sqrt{a} \times \sqrt{b}$ is $\sqrt{a \times b}$, the square root of $a \times b$.

Reason: Let's check. Does multiplying $\sqrt{a} \times \sqrt{b}$ by itself give $a \times b$?

$$\sqrt{a} \times \sqrt{b} \times \sqrt{a} \times \sqrt{b} = \sqrt{a} \times \sqrt{a} \times \sqrt{b} \times \sqrt{b} = \mathrm{a} \times b$$

It does!

We now have two schoolbook rules about square roots.

$$\sqrt{\frac{\boldsymbol{a}}{\boldsymbol{b}}} = \frac{\sqrt{\boldsymbol{a}}}{\sqrt{\boldsymbol{b}}}$$

$$\sqrt{\boldsymbol{a} \times \boldsymbol{b}} = \sqrt{\boldsymbol{a}} \times \sqrt{\boldsymbol{b}}$$

Example: Make $\sqrt{9x^2}$ look friendlier.

Answer: Right away I can see that $3x \times 3x = 9x^2$ and so

$$\sqrt{9x^2} = 3x$$

(We are assuming x is not a negative number. We can't have a negative answer.)

If I didn't happen to see this right off the bat, we could use the fact that square roots and multiplication play nice.

$$\sqrt{9x^2} = \sqrt{9} \times \sqrt{x^2} = 3 \times x = 3x$$

(Again, we are assuming that x is not a negative number.)

Practice 98.7

a) Give a value for x for which $\sqrt{x^2}$ equals x.

b) Give a value for x for which $\sqrt{x^2}$ does not equal x.

Example: Between which two consecutive integers does $3\sqrt{7}$ lie?

Answer: We could answer this by using a calculator or looking at the table of square root values in the previous section.

But here is another approach:

Think of 3 as $\sqrt{9}$. Then we have

$$3\sqrt{7} = \sqrt{9} \times \sqrt{7} = \sqrt{63}$$

This is tad shy of $\sqrt{64} = 8$. So, $3\sqrt{7}$ is between 7 and 8, quite close to 8.

A Comment on "Simplify"

Algebra books say that $3\sqrt{7}$ is the "simplified form" of $\sqrt{63}$.

I personally think that $\sqrt{63}$ looks much friendlier than $3\sqrt{7}$, and certainly this form of number was much more useful to us in the previous problem.

The command "simplify" is dangerous. It is dependent on what one wants to do with the number and is often meaningless as a stand-alone command.

Example: Show that $\sqrt{180}$ is the same as $6\sqrt{5}$.

Answer: We have that $180 = 18 \times 10 = 2 \times 9 \times 2 \times 5 = 36 \times 5$.

So

$$\sqrt{180} = \sqrt{36 \times 5} = \sqrt{36} \times \sqrt{5} = 6\sqrt{5}$$

Practice 98.8 Write $\sqrt{8400}$ in the form $a\sqrt{b}$ with a and b integers and a as large as possible.

There are some special instances for which we can add and subtract quantities expressed as square roots.

Example: Show that $\sqrt{5}+\sqrt{20}$ equals $\sqrt{45}$.

Answer: The only thing I can think to do with the sum is to try rewriting $\sqrt{20}$ within it. We have

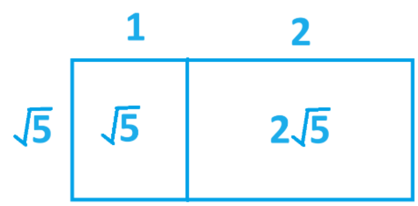
$$\sqrt{20}=\sqrt{4\times 5}=2\sqrt{5}$$

Then $\sqrt{5}+\sqrt{20}$ is

$$\sqrt{5}+2\sqrt{5}$$

There is a common factor of $\sqrt{5}$ to play with here.

$$\sqrt{5}+2\sqrt{5}=\sqrt{5}(1+2)=3\sqrt{5}$$



Can we turn this into $\sqrt{45}$?
Yes!

$$3\sqrt{5}=\sqrt{9}\times\sqrt{5}=\sqrt{45}$$

Practice 98.9 Write each of these quantities as the square root of a single number.

a) $5\sqrt{3}+2\sqrt{3}$
b) $5\sqrt{3}-2\sqrt{3}$
c) $\sqrt{160}+\sqrt{90}$
d) $\sqrt{2}+\sqrt{8}$
e) $\sqrt{63}-\sqrt{28}-\sqrt{7}$

There is a "cheat" way to answer each of these questions. We do have that

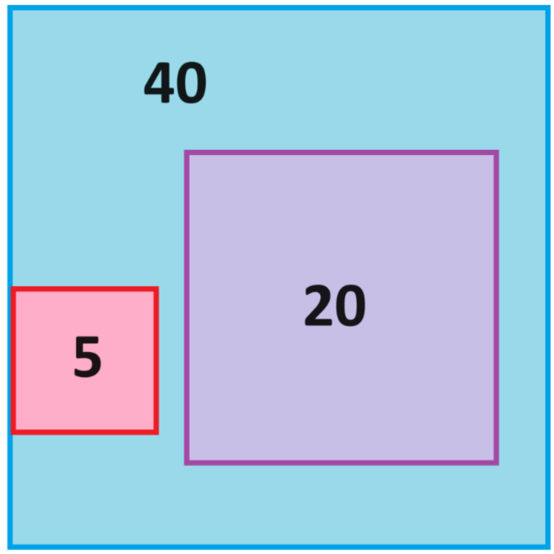
$$5\sqrt{3}+2\sqrt{3}=\sqrt{(5\sqrt{3}+2\sqrt{3})^2}$$

for instance!

Practice 98.10 Why won't this "cheat" work for $\sqrt{18}-\sqrt{32}$?

Practice 98.11 This picture shows a square of area 5 and a square of area 20 sitting inside a square of area 40. No square is tilted.

Is this picture physically possible?



MUSINGS

Musing 98.12 We saw that $\sqrt{a+b}$ usually does not equal $\sqrt{a}+\sqrt{b}$.
For example,

$$\sqrt{9+16}=5 \quad \text{but} \quad \sqrt{9}+\sqrt{16}=7$$

Only when one of numbers is zero do we have equality. For instance,

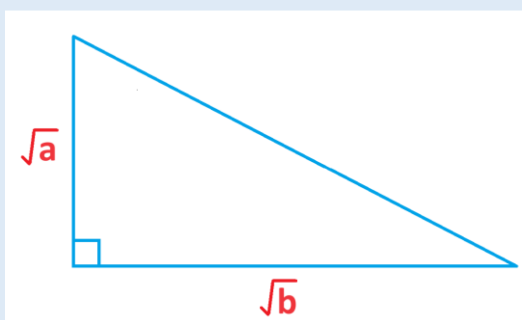
$$\sqrt{3+0}=\sqrt{3}+\sqrt{0}$$

In what follows, assume that neither a nor b is zero.

Using a calculator …

a) Is $\sqrt{3+5}$ larger or smaller than $\sqrt{3}+\sqrt{5}$?
b) Is $\sqrt{10+7}$ larger or smaller than $\sqrt{10}+\sqrt{7}$?
c) Can you find two positive numbers a and b with $\sqrt{a+b}$ larger than $\sqrt{a}+\sqrt{b}$?

d) **OPTIONAL**: Do you know the Pythagorean Theorem? If so, what is a formula for the length of the hypotenuse of this right triangle?



What does this picture say about the value of $\sqrt{a}+\sqrt{b}$ compared to the value of $\sqrt{a+b}$?

MECHANICS PRACTICE

Practice 98.13 Write each of these quantities in the form $a\sqrt{b}$ with a and b integers and a as large as possible.

a) $\sqrt{7200}$ b) $\sqrt{1000}$ c) $\sqrt{6125}$

Practice 98.14 Between which two consecutive integers do each of the following quantities lie?

a) $2\sqrt{3}$ b) $4\sqrt{5}$ c) $3\sqrt{8}$

Practice 98.15 Which of the following equals an integer?

a) $\sqrt{250} - 5\sqrt{10}$ b) $\sqrt{\frac{25}{0.04}}$ c) $\sqrt{8} \div \sqrt{2}$

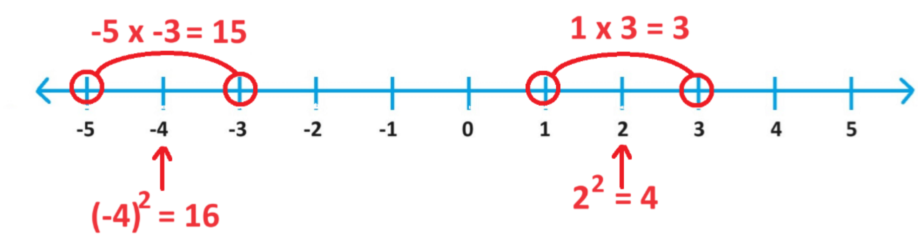
Practice 98.16 Find all the integers b that make the following quantity an integer.

$$\frac{\sqrt{48}}{\sqrt{b}}$$

99. The Difference of Two Squares

Here's a cute pattern.

Choose an integer on the number line.
Multiple the two numbers either side of it.
That product is always one less than the square of your chosen number.



Some more examples:

$$100 \times 102 = \mathbf{10{,}200} \quad \text{and} \quad 101^2 = \mathbf{10{,}201}$$

$$17 \times 19 = \mathbf{323} \quad \text{and} \quad 18^2 = \mathbf{324}$$

$$774{,}092 \times 774{,}094 = \mathbf{599{,}219{,}972{,}648} \quad \text{and} \quad 774093^2 = \mathbf{599{,}219{,}972{,}649}$$

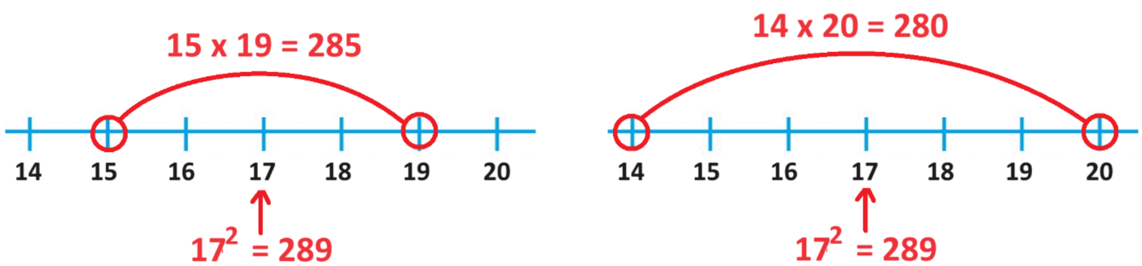
We seem to have

$$N^2 = (N-1)(N+1) + 1$$

"The square of a number is one more than the product of the number one below it and one above it."

Practice 99.1 Make a conjecture about the square of a number and the product of the number 2 below it and the number 2 above it. Experiment with some different values.

Also make a conjecture about the square of a number and the product of the number 3 below it and the number 3 above it.



If you play with this, you might come to think:

The square of a number N is k^2 more than the product of the number k below it and the number k above it.

$$N^2 = (N-1)(N+1)+1$$

$$N^2 = (N-2)(N+2)+4$$

$$N^2 = (N-3)(N+3)+9$$

$$N^2 = (N-4)(N+4)+16$$

$$\vdots$$

$$N^2 = (N-k)(N+k)+k^2$$

If this is true, it leads to another arithmetic hack.

Squaring a number that is near 100

Suppose you wish to compute 97^2 in your head for some reason.

Now 97 is three less than 100, which is an easy number to work with. Let's multiply the numbers three less and three more than 97 and add 9.

$$97^2 = (97-3)(97+3)+9$$

$$97^2 = 94 \times 100 + 9$$

$$97^2 = 9409$$

In the same way, 102 is two away from 100. So, we have

$$102^2 = (102-2)(102+2)+4$$

$$102^2 = 100 \times 104 + 4$$

$$102^2 = 10{,}404$$

Practice 99.2 Use this trick to compute

a) 99^2 b) 104^2 c) 92^2 d) 995^2

Of course, we need to demonstrate that what we are observing and concluding is true.

Practice 99.3 a) Draw and chop up a rectangle to show that $(N-k)(N+k)$ equals N^2-k^2.
b) What then is the value of $(N-k)(N+k)+k^2$?

Practice 99.4 Is

$$90{,}564{,}387^2 - 4$$

a prime number? How do you know?

In problem 99.3 you showed

$$\boldsymbol{N^2 - k^2 = (N-k)(N+k)}$$

This is called the **difference of two squares formula**. It shows that the difference of two square numbers can be written as the product of two factors.

School books have questions of the following ilk:

Please factorise $4x^2 - y^2$

or

Please factor $4x^2 - y^2$.

(In the U.S. the word "factor" is used both as a noun and a verb.)

Students are expected to recognize that the expression is a difference of two squares and so apply the difference of two squares equation.

$$\begin{aligned} 4x^2 - y^2 &= (2x)^2 - y^2 \\ &= (2x-y)(2x+y) \end{aligned}$$

Practice 99.5
a) Explain why $9x^2$ equals $(3x)^2$.
b) Explain why $a^2b^2c^2d^2 = (abcd)^2$.
c) Explain why $y^6 = (y^3)^2$.

Practice 99.6 Kindly factor each of the following expressions.

a) $16m^2 - 25n^2$

b) $4 - a^2$

c) $49x^2y^2 - z^2w^2r^2$

d) $1 - y^6$

e) $7a^2 - 28b^2$

MUSINGS

Musing 99.7
Explain why $97^2 - 169$ is divisibly by 84.
Explain why it is also divisible by 11.

Musing 99.8
If you are not wary of square roots, does $x^2 - 5$ factor?

100. Expected Schoolbook Work: Factoring $ax^2 + bx + c$

We can factor expressions of the form

$$x^2 - k^2$$

by recognizing them as the difference of two squares.

$$x^2 - k^2 = (x - k)(x + k)$$

But schoolbooks generally don't want students to stop there. They want them to practice factoring all sorts of expressions that involve an x^2 and possibly and x and some numbers. (Or the equivalent using a different letter for the unknown.)

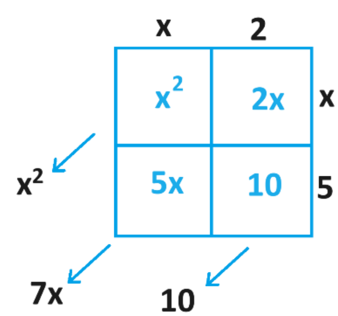
I am not sure why. Perhaps it is just to practice and sharpen one's general algebraic skills and wits? (I know high school teachers reading this are saying: "But James! Students will need this factoring for the upcoming chapter on quadratic equations." I respond, "Actually, we don't" and I argue that factoring actually gets in the way of properly understanding of quadratics.)

To get us going on this factoring work, let's come at it backwards.

Here's an expression that is already factored.

$$(x + 2)(x + 5)$$

We can "unfactor" it by using the area model.



We see

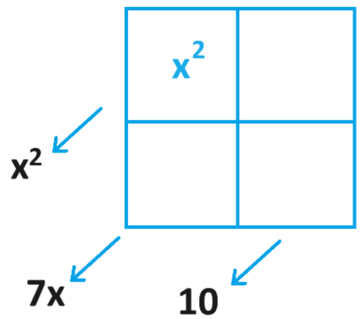
$$(x + 2)(x + 5) = x^2 + 7x + 10$$

Now, forwards. Imagine we were given this answer first:

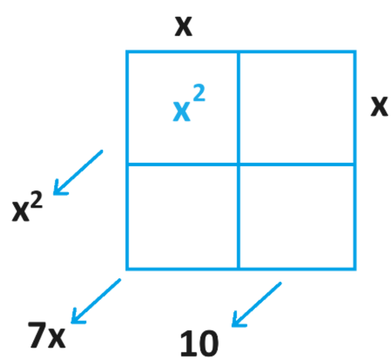
$$x^2 + 7x + 10$$

Could we figure out the rectangle that gives this answer, and so find the two factors that made this expression?

It seems reasonable to guess that $x^2 + 7x + 10$ came from a rectangle with four pieces (two of which combined to make $7x$) with the top left piece have x^2 in it.

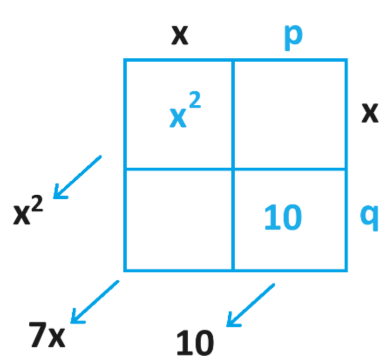


That x^2 likely came from "x times x." (It could have come from $10x \times \frac{1}{10}x$ or from $0.4x \times 2.5x$ or something else obnoxious, but let's make the guess that the schoolbook authors are not going to be that sneaky in these problems.)

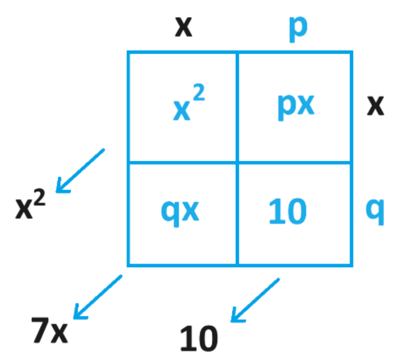


We can also see that the "10" must have come from the bottom right box. But did it appear as 2×5 or as 10×1 or something else?

Let's put in some abstract numbers p and q and see if we can guess what they need to be from the picture. We do see at the very least we need $p \times q = 10$.



Let’s fill in the rectangle and see what we learn.



We see that not only do we need $pq = 10$, we also need $px + qx$ to equal $7x$.

So, we have that p and q are two numbers such that

$$p + q = 7$$

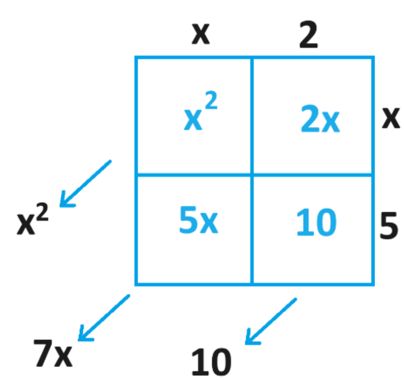
$$pq = 10$$

Now it is a puzzle. Can you think of two numbers that add to 7 and multiply to 10?

The numbers $p = 2$ and $q = 5$ come to mind. Or should it be $p = 5$ and $q = 2$?

We just need something to work. So, let’s try the first option and if we get in a pickle, we can try the second one next.

Here goes:



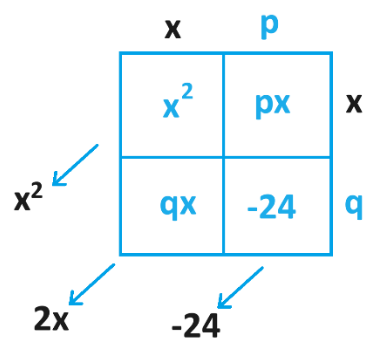
Yes. It worked. We see that $x^2 + 7x + 10 = (x + 2)(x + 5)$.

Practice 100.1 Do we get the same result if we try $p = 5$ and $q = 2$ instead?

Let' try one know for which we don't know the answer first.

Example: Please factor $x^2 + 2x - 24$.

Answer: Here's the picture, assuming the x^2 piece comes from the obvious first guess of $x \times x$.



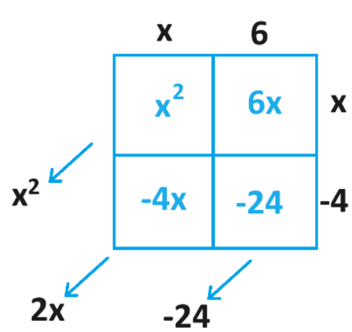
We see we need two numbers p and q such that

$$p + q = 2$$
$$pq = -24$$

Thinking of such numbers is tricky this time.
We do know that one of the numbers must be positive and the other negative.

After a little while, I came up with $p = 6$ and $q = -4$.

Let's see if this works.



We're good!

$$x^2 + 2x - 24 = (x + 6)(x - 4)$$

We can see that factoring requires guess work and some luck: We end up needing to think of two numbers with a given sum and a given product. Is this always easy? Is it always possible?

For example:

Question:
Can you think of two numbers that sum to 10 and have a product of 24?

Can you think of two numbers that sum to 10 and have a product of 25?

Can you think of two numbers that sum to 10 and have a product of 26?

The first of these questions has an answer, the second has a slightly sneaky answer, and the third one has no answer!

Practice 100.2 Use graphing software to graph each of the two graphs $p + q = 10$ and $pq = 26$. Do the graphs intersect?

So, answering schoolbook factoring problems really does rely on presuming that the book author is being kind to you and given you problems that happen to work out if you follow all reasonable assumptions.

Warning: The real-world is not so kind! Most expressions of the form $x^2 + bx + c$ do not factor!

Practice 100.3 Please factor the following expressions that have been designed to factor.

a) $x^2 + 7x + 12$

b) $x^2 - 8x + 12$

c) $x^2 - x - 12$

d) $x^2 - 11x - 12$

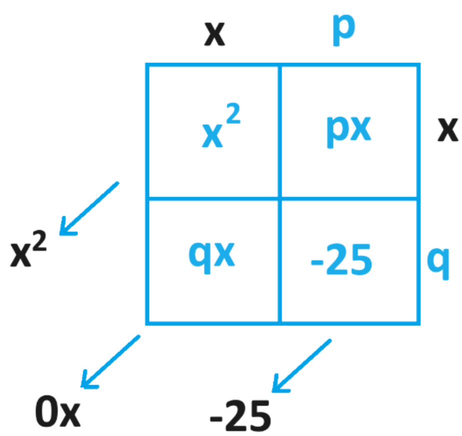
e) $w^2 + 5w - 24$

The Difference of Two Squares – Again!

We can use this technique to factor a difference of two squares.

Example: Factor $x^2 - 25$

Answer: Think of this as $x^2 + 0x - 25$.



We seek two numbers p and q such that

$$p + q = 0$$

$$pq = -25$$

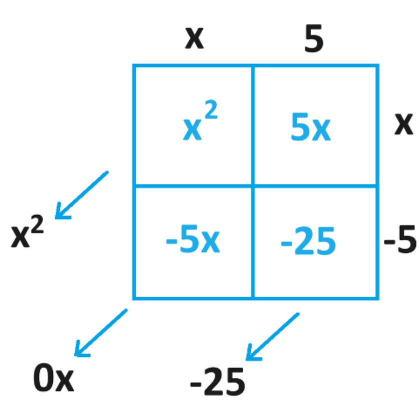
So, we see that we need $p = -q$, so I am thinking two numbers that are "the same" but opposite in sign.

Let's try $p = 5$ and $q = -5$.

It's

It's good!

$$x^2 - 25 = (x + 5)(x - 5)$$

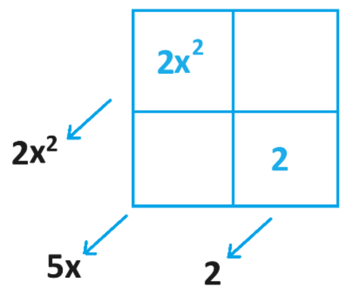


Some Trickier Factoring

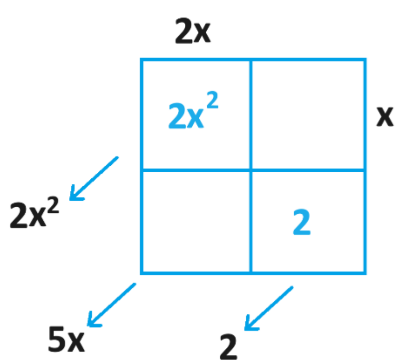
Let's put some numbers in front of the x^2.

Example: Kindly factor $2x^2 + 5x + 2$

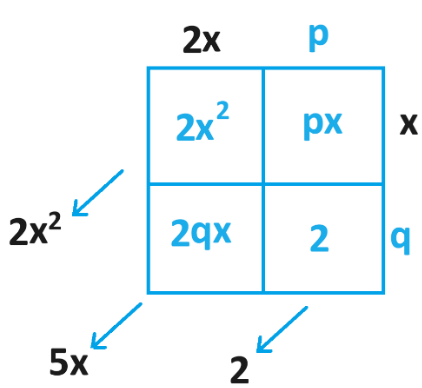
Answer: Here's the setup this time.



Let's presume the author is being kind to us and the $2x^2$ term comes from something straightforward: $2x \times x$.



Let's put in some general numbers p and q.



Now we have to be careful! Did you catch the "$2qx$" in the bottom left corner?

We need two numbers such that

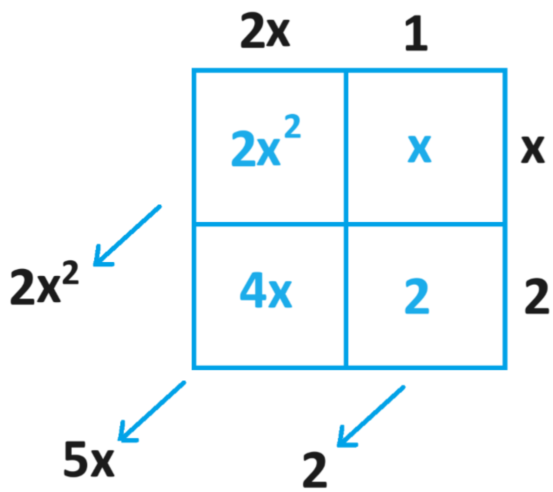
$$p + 2q = 5$$

$$pq = 2$$

Two numbers that multiply to 2, and so that double one plus the other is 5.

I am guessing $p = 1$ and $q = 2$.

Does this work?



Yes!

$$2x^2 + 5x + 2 = (2x + 1)(x + 2)$$

Practice 100.4 Kindly factor each of these expressions that have been designed to factor.

a) $3x^2 - 8x + 4$
b) $5x^2 + 9x - 2$
c) $4x^2 - 4x - 3$
d) $4x^2 - x - 3$
e) $-x^2 + 9$
f) $19x^2 + 20x + 1$
g) $24x^2 + 17x + 3$ (This one is annoying. Which two factors of 24 are best to use?)

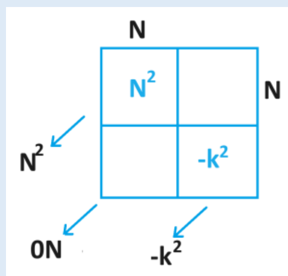
MUSINGS

Musing 100.5 It seems that, by and large, we want mathematical expressions to factor with all numbers involved integers. But we need not insist on this.

Making use of square roots and fractions, please factor each of these expressions.

a) $5x^2 - 3$
b) $x^2 + \frac{3}{4}x + \frac{1}{8}$
c) $3x^2 + 3\sqrt{3}x + 2$

Musing 100.6 Complete the picture to show, again, that $N^2 - k^2$ factors as $(N-k)(N+k)$.



Musing 100.7 EXPLORATION
Roll three dice and get three outcomes a, b, and c. Try to factor the expression $ax^2 + bx + c$. Most times it won't factor using whole numbers, or factor at all.

Can you get a sense the percentage of these expressions that do factor nicely?

Musing 100.8 There is a "difference of two cubes" formula.

$$N^3 - k^3 = (N-k)(something)$$

Can you figure out the "something"?