THINKING WORKSHEETS

James Tanton

THE ARITHMETIC OF FRACTIONS

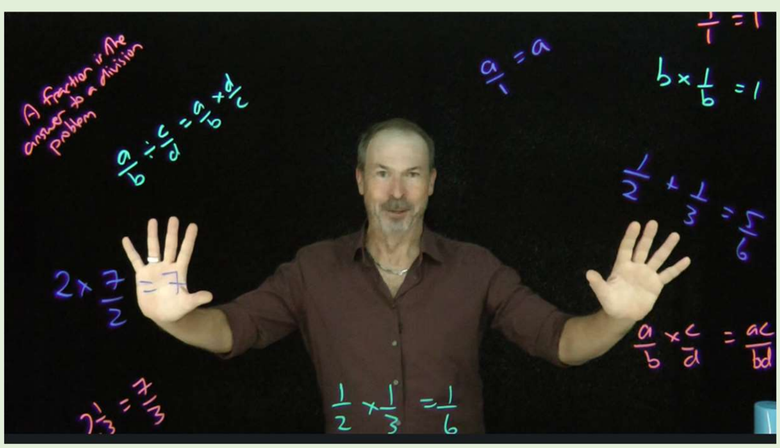
Are you game for the math, and nothing but the math?

High-School/College Content

CONTENTS

The following dozen pages present text and 32 questions (with solutions provided) about fraction arithmetic. This packet is designed to be self-contained, but students might enjoy reading and working though this material in pairs or triples. Or perhaps educators could use this package as a whole class reading and discussion experience.

This video also covers the entire content presented:

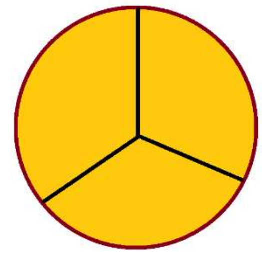


https://youtu.be/qMvYQfsWhH0

Introduction

Young students are initially taught that fractions arise from sharing problems—typically illustrated through equally dividing round pizzas.

For example, when dividing a plain pizza fairly among three people, each person receives a slice referred to as one third, denoted as 1/3. (Keeping the pizza plain avoids debates about what "fair" means; a slice with more anchovies, for instance, might be deemed more valuable by someone like me.)



Students are instructed to interpret 2/3 literally as we read it aloud: "two thirds." It represents two copies of a third (two slices):

$$\frac{2}{3} = 2 \times \frac{1}{3}$$

And three copies of a third return us to one whole pizza:

$$3 \times \frac{1}{3} = 1$$

A considerable amount of work (a daunting amount!) is devoted in the early grades to helping students make sense of fractions in real-world contexts to then, hopefully, grasp their underlying mathematics.

But this approach is challenged by a fundamental paradox.

> Although much of mathematics is motivated by real-world contexts, in the end, the mathematics is bigger and bolder than any one real-world scenario.

Mathematics excels at describing and pinpointing the nuances of real-world examples; however, no singular real-world model can fully encapsulate, and thus explain, mathematics. (Yet school curricula give the impression that real-world models can.) It's time to focus just on the math.

Are you ready for mathematical truth?

This will be a journey into the purity of logic, devoid of real-world examples, focusing on what mathematics "wants" the arithmetic of fractions to be. (And, surprisingly, it aligns perfectly with what we were taught in school and all our real-world intuition!)

We're in for a mathematical adventure.

Let's go!

THE MATHEMATICAL START

Sharing pizza suggests that the real world wants numbers of the form $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ with the property that $2 \times \frac{1}{2} = 1$ and $3 \times \frac{1}{3} = 1$ and $4 \times \frac{1}{4} = 1$ and so on.

That's where mathematicians start. We posit that we can extend our number system, with all its usual rules of arithmetic, to now include a new type of number.

For each non-zero number $\boldsymbol{b}$ there is a number $\boldsymbol{\frac{1}{b}}$ with the property that $\boldsymbol{b \times \frac{1}{b} = 1}$
These numbers obey the standard rules of arithmetic.

Right off the bat we confront a subtle point. Why isn't zero allowed in our considerations? Why can't we bring "$\frac{1}{0}$" into our number system?

> **Answer:** The quantity $\frac{1}{0}$ is defined to be a number that fits this equation:
>
> $$0 \times \frac{1}{0} = 1$$
>
> But the "standard rules of arithmetic" have that any quantity times zero is zero:
>
> $$0 \times \frac{1}{0} = 0$$
>
> The existence of $\frac{1}{0}$ leads to a contradiction with the principles of arithmetic!

For this reason, we'll work only with new numbers of the form $\frac{1}{b}$ with b not zero.

Pizza-sharing also shows that writing $\frac{2}{3}$ is helpful when thinking of two copies of a third, $2 \times \frac{1}{3}$. Mathematicians agree and follow suit.

Understand $\boldsymbol{\frac{a}{b}}$ to be a shorthand for $\boldsymbol{a \times \frac{1}{b}}$

And that's it! Everything we learned in school about the mathematics of fractions follows logically from this one starting belief (and this one convention of notation).

It's stunning!

PROPERTY 1: $\boldsymbol{b \times \frac{a}{b} = a}$

Question 1: Show that $3 \times \frac{2}{3}$ equals 2.

Hint: Use that the fact that $\frac{2}{3}$ means $2 \times \frac{1}{3}$ and that $\frac{1}{3}$ is the number such that $3 \times \frac{1}{3} = 1$.

Question 2: Show mathematically that $6 \times \frac{5}{6}$ equals 5.

Question 3: Show mathematically that $\frac{3}{4} \times 12$ equals 9.

Comment: I was taught to read this as "three-quarters **of** 12" in school.

PROPERTY 2: $\boldsymbol{\frac{a}{b}}$ **is the answer to** $\boldsymbol{a \div b}$

I was told in school that "a fraction is the answer to a division problem." Mathematics agrees! Let's see why.

First, let's be clear on the (purely mathematical) meaning of division: it's "reverse multiplication."

For example, filling in the box for

$$12 \div 3 \; = \; \blacksquare$$

is equivalent to filling in the box for

$$\blacksquare \times 3 = 12$$

Filling in the box for

$$270 \div 15 \; = \; \blacksquare$$

is equivalent to filling in the box for

$$\blacksquare \times 15 = 270$$

In short: **The answer to $a \div b$ is the number that multiplies by b to give the value a.**

Question 4: What number fills in this box?

$$\blacksquare \times 6 = 42$$

What division problem have you just solved?

Question 5: What number fills in this box?

$$\blacksquare \times 20 = 20$$

What division problem have you just solved?

Question 6: What number fills in this box?

$$\blacksquare \times 5 = 0$$

What division problem have you just solved?

Question 7: What number fills in this box?

$$\blacksquare \times 3 = 2$$

What division problem have you just solved?

Question 8: What number fills in this box?

$$\blacksquare \times 5 = 6$$

What division problem have you just solved?

Question 9: Explain why $\frac{a}{b}$ is the answer to $a \div b$.

Question 10: In question 3 you showed that $\frac{3}{4} \times 12 = 9$.
How does this translate to a statement about division?

PROPERTY 3: MULTIPLYING FRACTIONS

Question 11: Show that $6 \times \frac{1}{2} \times \frac{1}{3}$ equals 1.

But hang on! The number that fills in this blank

$$6 \times \blacksquare = 1$$

is $\frac{1}{6}$, not $\frac{1}{2} \times \frac{1}{3}$. Unless …. these are the same number!

We've just established that

$$\frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$$

(If you picked up on something a wee bit sneaky here, read the solution to Question 11 for more commentary on this.)

Question 12: Show that $20 \times \frac{1}{4} \times \frac{1}{5} = 1$ and deduce that $\frac{1}{4} \times \frac{1}{5} = \frac{1}{20}$.

Question 13: Show that $ab \times \frac{1}{a} \times \frac{1}{b} = 1$ and deduce that

$$\boldsymbol{\frac{1}{a} \times \frac{1}{b} = \frac{1}{ab}}$$

We've just shown how to multiply any two basic fractions!

But let's keep going.

Question 14: Show that $\frac{2}{5} \times \frac{3}{7}$ equals $6 \times \frac{1}{35}$, which is $\frac{6}{35}$

Question 15: Show that $\frac{10}{17} \times \frac{3}{4} = \frac{30}{56}$.

Question 16: Show in general that $\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$.

PROPERTY 4: **EQUIVALENT FRACTIONS**

Question 17: Show that $\frac{4}{6}$ is the same as $\frac{2}{3}$.

Hint: Use $\frac{4}{6} = 4 \times \frac{1}{6}$ and that we know $\frac{1}{6} = \frac{1}{2} \times \frac{1}{3}$.

Question 18: Show that $\frac{18}{30}$ is the same as $\frac{3}{5}$.

Question 19: Show, in general, that $\frac{a \times k}{b \times k}$ is the same as $\frac{a}{b}$.

We've established a schoolbook practice:

If you see a common factor in the top and bottom of a fraction, feel free to "cancel" it!

Alternatively, feel free to multiply the top and bottom of a fraction by a common number: you won't affect the value of the fraction.

$$\frac{18}{30} = \frac{3 \times \cancel{6}}{5 \times \cancel{6}} = \frac{3}{5} \qquad \frac{2}{7} = \frac{2 \times 13}{7 \times 13} = \frac{26}{91}$$

PROPERTY 5: DIVIDING FRACTIONS

Recall Property 2: The answer to a division problem $a \div b$ is the fraction $\frac{a}{b}$.

So, when faced with the division problem

$$\frac{1}{2} \div \frac{1}{3}$$

we can state that the answer is the (unusual) fraction

$$\frac{\frac{1}{2}}{\frac{1}{3}}$$

But fractions within fractions are awkward. But we've just learned that we can multiply the top and bottom of a fraction by a common number and not change the value of the fraction.

Let's multiply the top and bottom by 2 and by 3. That should clean things up!

$$\frac{\frac{1}{2}}{\frac{1}{3}} = \frac{\frac{1}{2} \times 2 \times 3}{\frac{1}{3} \times 2 \times 3}$$

Do you see that this equals $\frac{3}{2}$?

We've just shown that

$$\frac{1}{2} \div \frac{1}{3} = \frac{3}{2}$$

Question 20: Use this approach to show that $\frac{6}{11} \div \frac{5}{7}$ equals $\frac{42}{55}$.

Question 21: Show that $\frac{7}{10} \div \frac{7}{9}$ equals $\frac{9}{10}$.

Question 22: Show, in general, that $\frac{a}{b} \div \frac{c}{d}$ equals $\frac{a \times d}{b \times c}$.

(Were you taught "keep change flip" in elementary school? That is, were you taught to rewrite $\frac{a}{b} \div \frac{c}{d}$ as $\frac{a}{b} \times \frac{d}{c}$ and to compute this product instead?)

PROPERTY 6: $\boldsymbol{a = \frac{a}{1}}$

Recall the basic property of fractions: $\frac{1}{b}$ is the number so that $b \times \frac{1}{b} = 1$.

Consequently, $\frac{1}{1}$ is the number so that $1 \times \frac{1}{1} = 1$.

Question 23: What other number fills in the blank to $1 \times \blacksquare = 1$?
Deduce that $\frac{1}{1}$ is 1.

Question 24: Show that $\frac{7}{1}$ is 7.

In general, we have:

$$\frac{a}{1} = a \times \frac{1}{1} = a \times 1 = a$$

PROPERTY 7: ADDING AND SUBTRACTING FRACTIONS

It might seem curious that we have left the addition and subtraction of fractions to this late stage of thinking.

The challenge is that fractions are intimately connected to division (reverse multiplication), not addition and subtraction, and so one has to think more deeply on how to bring addition into their story.

One approach is to use property 6: "Rewrite the quantity over 1 and proceed from there."

Example: Compute $\frac{1}{2} + \frac{1}{3}$.

Perhaps try this on your own before turning the page. Can you see how to obtain the answer $\frac{5}{6}$ using only the properties we've established so far?

Question 25: We can rewrite $\frac{1}{2}+\frac{1}{3}$ as

$$\frac{\frac{1}{2}+\frac{1}{3}}{1}$$

But having fractions within fractions is awkward!

a) Multiply the top and bottom of $\frac{\frac{1}{2}+\frac{1}{3}}{1}$ each by 2. This won't change the value of the fraction. Show that doing so gives you

$$\frac{1+\frac{2}{3}}{2}$$

b) Now multiply the top and bottom each by 3. Show that doing so gives you

$$\frac{5}{6}$$

We've just established that $\frac{1}{2}+\frac{1}{3}=\frac{5}{6}$

Question 26: Compute $\frac{2}{3}+\frac{3}{10}$ via this approach.

Question 27: Compute $\frac{2}{3}-\frac{3}{10}$ via this approach.

Schoolbooks usually encourage students to rewrite fractions to have a common denominator (Property 4) and then just "add numerators."

Question 28: Show that $\frac{a}{n}+\frac{b}{n}$ equals $\frac{a+b}{n}$.

This "putting over 1" technique is very handy!

Question 29: Show again that $\frac{1}{2}\times\frac{1}{3}$ equals $\frac{1}{6}$ but this time start with $\frac{\frac{1}{2}\times\frac{1}{3}}{1}$.

PROPERTY 8: MIXED NUMBERS

A mixed number is a whole number and a fraction added together.

For example, $2\frac{1}{3}$ is read as "two **and** a third" and is indeed really $2+\frac{1}{3}$.

Also,

$$5\frac{3}{4}=5+\frac{3}{4}$$

$$200\frac{1}{200}=200+\frac{1}{200}$$

and so on.
(Society has decided to skip writing the plus sign between the whole number and the fraction.)

Question 30: Show that $2\frac{1}{3}$ is same as the fraction $\frac{7}{3}$.

Hint: Start by rewriting $2\frac{1}{3}$ as $\frac{2\frac{1}{3}}{1}$ which is really $\frac{2+\frac{1}{3}}{1}$.

Question 31: Write $200\frac{1}{200}$ as single fraction.

We can work in the reverse direction and convert fractions to mixed numbers.

For example, let's rewrite $\frac{17}{5}$ as a whole number plus a fraction.

Strategy: *Make use of* $5\times\frac{1}{5}=1$ *as often as possible to have whole numbers appear.*

Here goes:

$$\frac{17}{5}=17\times\frac{1}{5}=(5+5+5+2)\times\frac{1}{5}=1+1+1+2\times\frac{1}{5}=3\frac{2}{5}$$

Question 32: Write $\frac{80}{7}$ as a mixed number.

CLOSING

That was a mighty swift overview of the arithmetic of fractions. Well done for making it through!

This review demonstrates that everything you were taught to do mechanically with fractions in the past is correct and grounded in solid mathematical principles.

There's no obligation to perform the arithmetic of fractions as we did here. Always feel free to use any of the techniques you've learned in the past, confident that they make mathematical sense and are mathematically correct!

Final Question: Have I overlooked a piece of fraction arithmetic you were taught? If so, prove that it is mathematically correct using the techniques in this packet!

Notes and Solutions

1. We have

$$3 \times \frac{2}{3} = 3 \times 2 \times \frac{1}{3} = 2 \times 1 = 2$$

because $3 \times \frac{1}{3} = 1$.

2. We have

$$6 \times \frac{5}{6} = 6 \times 5 \times \frac{1}{6} = 5 \times 1 = 5$$

because $6 \times \frac{1}{6} = 1$.

3. We have

$$12 \times \frac{3}{4} = 12 \times 3 \times \frac{1}{4} = 3 \times 4 \times 3 \times \frac{1}{4} = 3 \times 3 \times 1 = 9$$

because $4 \times \frac{1}{4} = 1$.

4. 7 fills in the box and we have $42 \div 6 = 7$. (Seven is the number that multiplies by six to give 42.)

5. 1 fills in the box and we have $20 \div 20 = 1$.

6. 0 fills in the box and we have $0 \div 5 = 0$.

7. $\frac{2}{3}$ fills in the box and so we must have $2 \div 3 = \frac{2}{3}$.

8. $\frac{6}{5}$ fills in the box and so we must have $6 \div 5 = \frac{6}{5}$.

9. Consider $\blacksquare \times b = a$. We know $\frac{a}{b}$ fills in the box. So this must be the answer to $a \div b$.

10. $9 \div 12 = \frac{3}{4}$

11. We have

$$6 \times \frac{1}{2} \times \frac{1}{3} = 2 \times 3 \times \frac{1}{2} \times \frac{1}{3} = 1 \times 1 = 1$$

Comment: *We have just shown that two different(?) numbers,* $\frac{1}{6}$ *and* $\frac{1}{2} \times \frac{1}{3}$, *fill in this box:*

$$6 \times \blacksquare = 1$$

How can we be sure that that means the two numbers must be the same?

To see that they are, consider this extra-long product.

$$\frac{1}{6} \times 6 \times \frac{1}{2} \times \frac{1}{3}$$

We can see that it equals $1 \times \frac{1}{2} \times \frac{1}{3}$, *which is just* $\frac{1}{2} \times \frac{1}{3}$.

We can also see that is equals $\frac{1}{6} \times 1$, *which is just* $\frac{1}{6}$.

One product can't have two different answers. It must be that $\frac{1}{2} \times \frac{1}{3}$ *and* $\frac{1}{6}$ *are the same number.*

Challenge: Consider the equation

$$a \times \blacksquare = 1$$

If x and y are two numbers that can go into the blank, prove that x and y are actually the same number.

Hint: Compute $x \times a \times y$ two different ways.

12. We have

$$20 \times \frac{1}{4} \times \frac{1}{5} = 4 \times 5 \times \frac{1}{4} \times \frac{1}{5} = 1 \times 1 = 1$$

But $20 \times \frac{1}{20} = 1$. It must be that $\frac{1}{20}$ and $\frac{1}{4} \times \frac{1}{5}$ are the same.

13. We have

$$ab \times \frac{1}{a} \times \frac{1}{b} = a \times b \times \frac{1}{a} \times \frac{1}{b} = 1 \times 1 = 1$$

But $ab \times \frac{1}{ab} = 1$. It must be that $\frac{1}{ab}$ and $\frac{1}{a} \times \frac{1}{b}$ are the same.

14. We have

$$\frac{2}{5} \times \frac{3}{7} = 2 \times \frac{1}{5} \times 3 \times \frac{1}{7} = 6 \times \frac{1}{5} \times \frac{1}{7}$$

But question 13 shows that $\frac{1}{5} \times \frac{1}{7} = \frac{1}{35}$ and so

$$\frac{2}{5} \times \frac{3}{7} = 6 \times \frac{1}{35} = \frac{6}{35}$$

15.

$$\frac{10}{17}\times\frac{3}{4}=10\times\frac{1}{17}\times 3\times\frac{1}{4}=30\times\frac{1}{17}\times\frac{1}{4}=30\times\frac{1}{56}=\frac{30}{56}$$

16.

$$\frac{a}{b}\times\frac{c}{d}=a\times\frac{1}{b}\times c\times\frac{1}{d}=ac\times\frac{1}{b}\times\frac{1}{d}=ac\times\frac{1}{bd}=\frac{ac}{bd}$$

17.

$$\frac{4}{6}=4\times\frac{1}{6}=4\times\frac{1}{2}\times\frac{1}{3}=2\times 2\times\frac{1}{2}\times\frac{1}{3}=2\times 1\times\frac{1}{3}=2\times\frac{1}{3}=\frac{2}{3}$$

18.

$$\frac{18}{30}=18\times\frac{1}{30}=18\times\frac{1}{6}\times\frac{1}{5}=3\times 6\times\frac{1}{6}\times\frac{1}{5}=3\times 1\times\frac{1}{5}=3\times\frac{1}{5}=\frac{3}{5}$$

19.

$$\frac{ak}{bk}=ak\times\frac{1}{bk}=a\times k\times\frac{1}{b}\times\frac{1}{k}=a\times\frac{1}{b}\times 1=\frac{a}{b}$$

20.

$$\frac{6}{11}\div\frac{5}{7}=\frac{\frac{6}{11}}{\frac{5}{7}}=\frac{6\times\frac{1}{11}}{5\times\frac{1}{7}}$$

Multiply top and bottom each by 11 and each by 7. This gives

$$\frac{6}{11}\div\frac{5}{7}=\frac{\frac{6}{11}}{\frac{5}{7}}=\frac{6\times\frac{1}{11}\times 11\times 7}{5\times\frac{1}{7}\times 11\times 7}=\frac{6\times 1\times 7}{5\times 1\times 11}=\frac{42}{55}$$

21.

$$\frac{7}{10}\div\frac{7}{9}=\frac{\frac{7}{10}}{\frac{7}{9}}=\frac{7\times\frac{1}{10}\times 10\times 9}{7\times\frac{1}{9}\times 10\times 9}=\frac{7\times 9}{7\times 10}$$

From question 19 we know we can “cancel” the common factor of 7. So,

$$\frac{7}{10}\div\frac{7}{9}=\frac{9}{10}$$

22.

$$\frac{a}{b} \div \frac{c}{d} = \frac{\frac{a}{b}}{\frac{c}{d}} = \frac{a \times \frac{1}{b} \times b \times d}{c \times \frac{1}{d} \times b \times d} = \frac{a \times 1 \times d}{c \times 1 \times b} = \frac{a \times d}{b \times c}$$

The answer is the same as $\frac{a}{b} \times \frac{d}{c}$, according to question 16. (We kept the first fraction the same, changed the ÷ sign to a ×, and changed the second fraction to its upside-down version.)

23. Both $\frac{1}{1}$ and 1 fill in the box. It must be that these are the same number.

24.

$$\frac{7}{1} = 7 \times \frac{1}{1} = 7 \times 1 = 7$$

25.

a)

$$\frac{(\frac{1}{2} + \frac{1}{3}) \times 2}{1 \times 2} = \frac{\frac{1}{2} \times 2 + \frac{1}{3} \times 2}{2} = \frac{1 + \frac{2}{3}}{2}$$

b)

$$\frac{1 + \frac{2}{3}}{2} = \frac{(1 + \frac{2}{3}) \times 3}{2 \times 3} = \frac{1 \times 3 + \frac{2}{3} \times 3}{6} = \frac{3 + 2 \times \frac{1}{3} \times 3}{6} = \frac{3 + 2 \times 1}{6} = \frac{3 + 2}{6} = \frac{5}{6}$$

26.

$$\frac{2}{3} + \frac{3}{10} = \frac{\frac{2}{3} + \frac{3}{10}}{1} = \frac{\left(2 \times \frac{1}{3} + 3 \times \frac{1}{10}\right) \times 3}{1 \times 3}$$

$$= \frac{2 + 3 \times \frac{1}{10} \times 3}{3} = \frac{(2 + 9 \times \frac{1}{10}) \times 10}{3 \times 10} = \frac{20 + 9}{30} = \frac{29}{30}$$

27.

$$\frac{2}{3} - \frac{3}{10} = \frac{\frac{2}{3} - \frac{3}{10}}{1} = \frac{\left(2 \times \frac{1}{3} - 3 \times \frac{1}{10}\right) \times 3}{1 \times 3}$$

$$= \frac{2 - 3 \times \frac{1}{10} \times 3}{3} = \frac{(2 - 3 \times \frac{1}{10} \times 3) \times 10}{3 \times 10} = \frac{20 - 9}{30} = \frac{11}{30}$$

28.

$$\frac{a}{n}+\frac{b}{n}=\frac{\frac{a}{n}+\frac{b}{n}}{1}=\frac{(a\times\frac{1}{n}+b\times\frac{1}{n})\times n}{1\times n}=\frac{a+b}{n}$$

29.

$$\frac{1}{2}\times\frac{1}{3}=\frac{\frac{1}{2}\times\frac{1}{3}}{1}=\frac{\frac{1}{2}\times\frac{1}{3}\times 2\times 3}{1\times 2\times 3}=\frac{1}{2\times 3}=\frac{1}{6}$$

30.

$$2\frac{1}{3}=\frac{2+\frac{1}{3}}{1}=\frac{\left(2+\frac{1}{3}\right)\times 3}{1\times 3}=\frac{6+1}{3}=\frac{7}{3}$$

31.

$$200\frac{1}{200}=\frac{200+\frac{1}{200}}{1}=\frac{\left(200+\frac{1}{200}\right)\times 200}{1\times 200}=\frac{40000+1}{200}=\frac{40001}{200}$$

32.

$$\frac{80}{7}=80\times\frac{1}{7}=(7+7+7+7+7+7+7+7+7+7+7+3)\times\frac{1}{7}=11\frac{3}{7}$$

Comment: Writing

$$\frac{80}{7}=80\times\frac{1}{7}=(11\times 7+3)\times\frac{1}{7}=11\frac{3}{7}$$

is a bit simpler!