# COLLEGE ALGEBRA FOR HUMANS

A Refreshingly Joyous, Human, and Accessible approach to Algebra
for all those who may have experienced it otherwise

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PART 1

# Arithmetic as the Gateway to All


Photo: Erick Mathew, Tanzania

*Algebra is the practice of avoiding the tedium of doing arithmetic problems one instance at a time, to take a step back and see a general structure to what makes arithmetic work the way it does, and so open one's mind to more than the one view of what arithmetic can be.*

# CONTENT FOR PART 1

## Chapter 4: Exploding Dots and the Power of Place Value



## Chapter 5: Fractions

## Chapter 6: Decimals



## Chapter 7: All Bases, All at Once

# 0: It All Starts with a Dot

I remember as a young lad, at around age 12 or so, wondering whether it might be possible to communicate with aliens (assuming they exist). What would it take to do so?

I was quite rational in my thinking and applied a systematic line of reasoning to consider the matter.

Communication, I first noted, requires each being involved to have a sense of “self” and a sense of “other.” If the entity with which I am trying to talk has no sense of anything but itself, all will be pointless. The fact is that I can only hope to communicate with a being who is aware of communication.

Such a being, possessing a sense of self and of other, I then reasoned is likely to have a sense of “nothing” and “something.” Maybe the something could be a physical object, or a waft of smell, or a pulse of sound or light. The “nothing” would be the absence of the something.

*Figure 0:* Nothing



*Figure 1:* Something

I next reasoned that—and this might have been a bit of a leap in thinking—if a being was aware of one thing, it might be aware of more than one thing; specifically two things, and three things, and so on. The being and I might share awareness of the *counting numbers*: 1, 2, 3, 4, … .



*Figure 2:* Two things



*Figure 3:* Three things

I decided that my best bet in communicating with an alien is to assume we each know how to count things and thus to communicate via the counting numbers. So, I decided we should send “blips” of sound or light out into space, in patterns of different counts to somehow say, “Hello! I am here.”

But what counts of these blips should we send? What pattern of counting numbers would be interpreted as deliberate and “intelligent” and undeniably as coming from someone trying to communicate?

I decided we should send blips that match the first few prime numbers—**2** blips, pause, **3** blips, pause, **5** blips, pause, **7** blips, pause, **11** blips, pause, **13** blips, pause, **17** blips, pause, **19** blips, pause, **23** blips, pause, **29** blips, pause, **31** blips pause, **37** blips, pause, **41** blips, pause, **43** blips, pause, **47** blips, pause, **53** blips, pause, **59** blips, pause, **61** blips, pause, **67** blips. (Maybe that's enough?)

Such a sequence of counts would be undeniably intelligent as I knew that these numbers are special and fundamental and not at all random. They take some mathematical sophistication to recognize. Sending a list of prime numbers, I thus thought, is likely to be interpreted as deliberate. (We'll learn about the prime numbers in this book.)

And this idea of mine, I later realized, is not a bad one: several science fiction writers had come up with the same proposal.

But looking back, it was clear as a young lad I was a enamored with mathematics. I could sense power to it. I could sense universality to it. I could sense that mathematics transcended my humanness—and I found that thrilling, and inspiring, and somehow comforting.

My young mind could start to see a marvelous journey to be had by simply contemplating "nothing" and "something," and then multiple copies of that something. (At least a form of mathematics that my human brain could comprehend.)

My first "something" was a dot.

It all starts with a dot.

# Chapter 1

## The Counting Numbers and the Basis of Arithmetic

Chapter 1

# 1. Humankind's First Mathematics

The first mathematical activity humankind ever conducted might have simply been counting. There is historical evidence for this claim.

In 1937, archeologists in Czechoslovakia at Dolní Věstonice in Morovia uncovered the radius bone of a wolf on which 57 notches were carved. The first 25 notches appear in groups of five and are then followed by a long notch, 30 ungrouped short notches, and a final long notch. (Or if you turn the bone the other way, maybe it's 30 ungrouped short notches followed by five groups of 5 short notches with long notches in-between?)

*Figure* | | | | | | | : Tally marks on a bone

This bone dates back some 30,000 years and strongly suggests that pre-historic humans were counting. (Counting what? Deer? Mammoths? Full moons?)

Notched bones have also been found at Border Cave in South Africa too, dating back to possibly 44,000 BCE, suggesting that counting has been happening for a very long time indeed.

And counting certainly seems innate to our human thinking. Small children delight in the act of counting. For instance, they will count stairs going up. They will count the same stairs going down. (And if they get different answers for these two counts they might or might not think something of it.)

Be it for humankind, or for young humans, or for a young lad wondering about communicating with aliens, it does indeed seem that counting is a natural (human) start to mathematics. And we'll officially start this journey too with the positive whole numbers that count things.

**1 2 3 4 5 6 7 8 9 10 11 12 …**

*Figure* 1, 2, 3, 4, …*:* The set of counting numbers

People sometimes call the set of counting numbers the set of **natural numbers**, probably because they are so natural to us. A funny "blackboard-script" capital N is used to denote them as a collection: $\mathbb{N}$.

No human or group of humans or a multitude of humans can write down *all* the counting numbers. The list of them simply does not stop. If you think you've written the biggest counting number there is, you are mistaken: simply add $143$ to it and you'll have a bigger counting number still.

We humans always have to “cheat” when listing the counting numbers. We do that by writing “dot dot dot” after a start to the list to mean “and keep on going forever.”

But on the other end of matters, namely, at the start of the list, there is a tricky question to be asked.

*Is 1 the first counting number? Or is there another number just before it?*

Humans have mused over the meaning of zero, 0 for millennia, wondering if it deserves to be called a counting number not. Does zero count something?

Think about it. If I say that there are zero sparkly purple giraffes in the room I am sitting in right now as I type this very sentence (and it is true, there are no sparkly purple giraffes here with me), do you think it is because I actually *counted* zero sparkly purple giraffes, or did I not count and just *observe* a lack of sparkly purple giraffes? That is, does one **count** zero or does one just **observe** zero?

To do this day, people—mathematicians even—choose not to give a definitive answer to this question and live with both options. Sometimes people say that zero should be included in the list of counting numbers and sometimes they choose to exclude it. There really is no standard convention on this. In fact, the same mathematician might one time write a paper in which she does regard zero as a counting number for the purposes of that work, and, later, write a separate paper, in which she states she won’t. It’s just a matter of what is appropriate for the problem being explored. Some problems naturally allow for zero. Others don’t.

Whether or not you choose to include zero in the list of counting numbers really does not matter. You just have to state to your readers in any work you do whether or not you are including zero in your list of counting numbers.

**0 1 2 3 4 5 6 7 8 9 10 11 12 ...**

*Alternative Figure* 0, 1, 2, 3, 4, …*:* The set of counting numbers perhaps

In this book, we'll follow what their name suggests and use the counting numbers for counting things. Specifically, we'll count dots. And to be clear, we'll consider zero to be a number too.

For instance, "$5$" shall represent five dots



and "$2$" shall represent two dots



and "$12,876,290,980,771,006,629,932,183$" shall represent a dreadfully large count of dots.

We'll end this section with a picture of zero dots.

Here it is:

## MUSINGS

**Musing 1.1** In 1938, Milton Sirotta, nine-year old nephew of American mathematician Edward Kasner, coined the term *googol* for the number given by 1 followed by one hundred zeros.

$$1\ 000000000000000000000000000000000000000000\ 0000000000000000000000000000000000000000000000000000000000$$

At the same time, he coined the name *googolplex* for the number given by 1 followed by a googol of zeros.

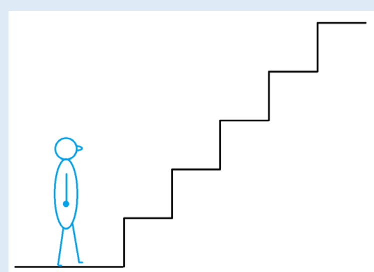
Can you describe a number bigger than a googleplex? Can you describe another number bigger still?

**Musing 1.2** Many words in the English language are obvious associated with numbers. For example, a *monologue* is a speech by a single person and to speak in *unison* is to speak in one voice. (The prefix mono comes from the Greek *mónos* meaning “alone” or “single”, and the prefix uni from the Latin *unis* for “one.) To have a *dilemma* is to be caught between two choices and a *bicycle* is a vehicle with two wheels. (*Bi* comes from the Latin for “two” and *di* the Greek for “two.”)

a) Can you think of two English words using prefixes associated with each of the numerals three through ten?
b) September is the ninth month of the year, yet the prefix *sept* comes from the Latin for “seven.” October is the tenth month of the year, yet the prefix *oct* comes from the Latin for “eight.” November and December are the eleventh and twelfth months of the year, despite *nov* and *dec* being derived from the Latin words for “nine” and “ten.”

   Can you find out why these months of the year are off their counts by two?

**Musing 1.3** Image standing at the base of a large set of stairs.



There is $1$ way to take one step up and one step down. We’ll denote this as $UD$.

There are $2$ ways to take two steps up and two steps down in some order, namely, $UUDD$ and $UDUD$. (Why isn’t $UDDU$ an option?)

a) There are $5$ ways to take three steps up and three steps down in some order. List them all.
b) There are $14$ ways to take four steps up and four steps down in some order. Can you find them all?
c) Care to list all $42$ ways to take five steps up and five steps down in some order? (The answer can be no!)

## 2. Addition

What does "$2+3$" mean in terms of dots?

It seems natural to regard this as "two dots placed together with three dots." And if I draw a picture of such a thing, I see five dots.



*Figure* $2+3$

I followed a "please read left-to-right" bias in this picture, placing all my dots in a row with two dots on the left followed by three on the right. But there is no need to read this picture left to right. If I look right to left, instead, I see $3+2$. (And this is, of course, still is the same five dots.)

This is philosophically deep!

I realize now, for instance, that $4+7$ must give the same answer as $7+4$, without ever having to say, or even think, "eleven." Simply imagine placing four dots and seven dots next to each other in a row and looking at that picture left-to-right and then right-to-left. The count of dots in the picture does not change even if your perspective looking at it does.

I can even "see" why computing $176203982761+87799998699$ and $87799998699+176203982761$ must give the same answer without doing a lick of actual arithmetic. (I don't want to do such arithmetic!)

It seems we've stumbled upon a fundamental truth about the counting numbers.

| |
|---|
| $a+b$ and $b+a$ are sure to have the same value,<br>no matter which counting numbers $a$ and $b$ represent. |

What about zero? Does this "truth" hold for number zero as well?

Let's explore.

Here's a picture of $5+0$ (or is it of $0+5$?)



*Figure* $5+0=0+5$

Five dots followed by no dots is just, well, five dots.

$$5+0=5$$

Also, no dots followed by five dots is five dots.

$$0+5=5$$

It seems that not only does our first fundamental truth seem to hold if one (or both?) of the counting numbers is zero, but we've also stumbled upon a second truth.

$a+0=a$ and $0+a=a$ no matter which counting number $a$ represents.

It's fun to imagine a picture of $0+0$. (Can you?)

And one of $0+0+0$.

And one of

$$0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+$$
$$0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+$$
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$$0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0.$$

**An Additional Thought (Ha!)**

There is another way to interpret $2 + 3$ in terms of dots. Simply look at a set of dots with different characteristics among them: say two are purple and three are blue.

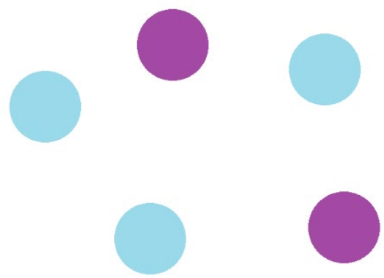
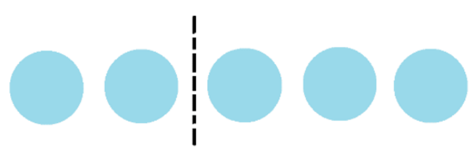


*Figure:* 2 + 3 again

Then the answer to "2 + 3" is the result of recounting the dots choosing to ignore differences.

We did precisely this with our first picture of 2 + 3: we had two left dots and three right dots and then chose to ignore leftness and rightness.



We can say that 7 apples and 9 oranges make for 16 pieces of fruit by choosing to ignore fruit details.

Perhaps this is leading us to a curious, and somewhat philosophical, idea of what addition actually is?

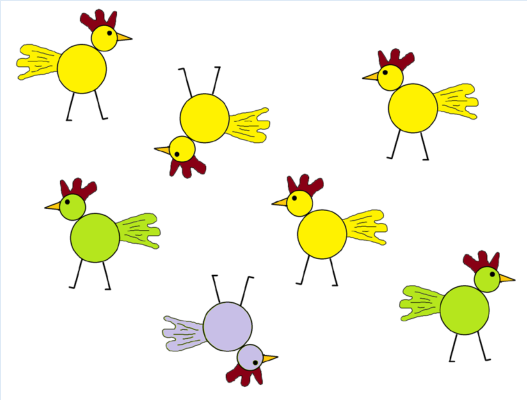
*Addition is the result of recounting a set of objects after choosing a set of differences to ignore.*

I don't think many people think of addition this way.

(I thank my colleague Joe Norman for opening my eyes to this!)

## MUSINGS

**Musing 2.1** Here is a picture of some chickens.



a) Can you interpret this as a picture for computing 2 + 5? In what way?
b) Can you interpret this as a picture for computing 1 + 2 + 4? In what way?
c) Can you interpret this as a picture for computing 4 + 3? In what way?

**Musing 2.2** When we write a number such as $523$ we are aware of the importance of "place." The $5$ here denotes five hundreds (and not five tens nor five thousands), the $2$ two tens, and the $3$ three units. This spares us the need to invent new symbols beyond the ten with which we are familiar: $0$, $1$, $2$, $3$, $4$, $5$, $6$, $7$, $8$, and $9$. This positional notation also helps us calculate sums of numbers.

The Romans, on the other hand, used different symbols for units, tens, hundreds, thousands, as well as for five, fifty, and five hundred.

I = one
V = five
X = ten
L = fifty
C = one hundred
D = five hundred
M = one thousand

These numerals can still be seen on clock faces, monuments, and during the credits of television shows and movies.

In the system of Roman numerals, the number $523$, for instance, was written DXXIII. Positional notation did not come into play (except for the convention that symbols were listed in decreasing order of size).

During the Medieval period in Europe, it became popular to use a subtractive principle to denote numbers, such as 4, 9, and 90.

4 = IV
9 = IX
90 = XC

Medieval scholars set the following rule for this subtractive principle.

> *One can only subtract a single I from a single V or a single X; or subtract a single X from a single L or a single C; or subtract a single C from a single D or a single M.*

So, writing VL for 45 would not be allowed. Nor would writing IIX for 8 or XCC for 190, for instance. (Well, perhaps this third one can be interpreted as "XC + C." But to avoid the seeming transgression of subtracting X from CC, scholars would write CXC for 190.)

With this subtraction principle the position of a symbol was made important, but the meaning of "place" was still different from what we mean it to be today.

a) Three movies were made in the years 1978, 1983, and 1999. How do their dates appear in the movie credits?

b) To appreciate our place-value system for writing numbers, try computing the following sum without mentally converting the numbers you see to our decimal system. Can you do it?

```
     XVI
+     IX
+   XXIV
+     XL
---------
=
```

c) Look at a clock face with Roman numerals. What do you notice about the number four? Is the same true for the number nine?

d) How did Romans represent extremely large numbers, numbers in the hundreds of thousands and the millions? Find out.

# 3. Repeated Additions

Last section we had fun adding together multiple copies of zero. The result is always zero. Let's now add together multiple copies of a non-zero counting number, say $5$.

Here are four copies of $5$ placed together.

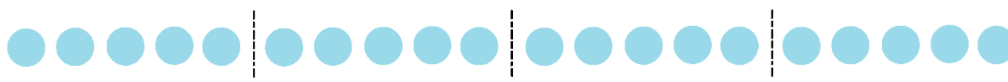


*Figure* $5+5+5+5$

In the world of counting numbers, people use the word **multiplication** for repeated addition.

The standard shorthand for $5+5+5+5$ is $4\times 5$, using the multiplication symbol $\times$ to denote the repeated addition.

The term $4\times 5$ is read as "four groups of five" in the U.S., It is read as "four lots of five" in Australia, or perhaps as "four copies of five." But watch out, folk in Europe interpret "$4\times 5$" as meaning something different: they say it is "the number 4, five times," and so read it as $4+4+4+4+4$.

Luckily $5+5+5+5$ and $4+4+4+4+4$ have the same value, so folk on all continents are thinking "20" in the end.

But was that luck?

We'll follow the U.S. language and thinking in these notes. In which case

$5+5+5+5$ is four groups of five: $4\times 5$,

$4+4+4+4+4$ is five groups of four: $5\times 4$.

And at first glance, these two quantities are philosophically different.

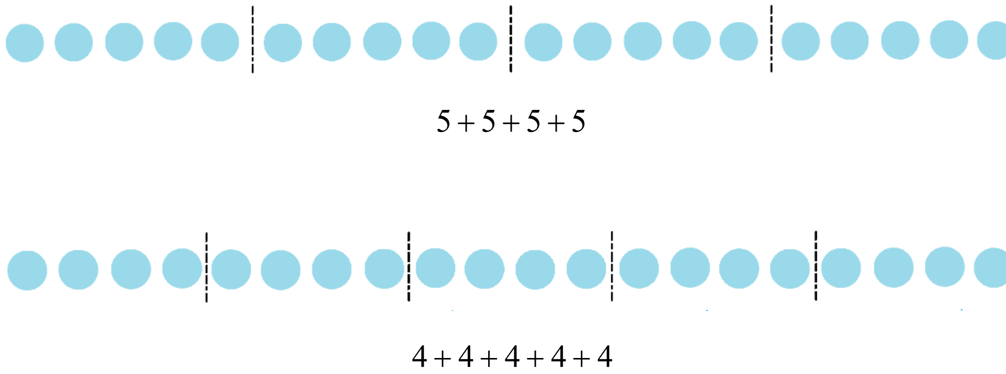



*Figure* $4 \times 5$ and $5 \times 4$

Is it remarkable coincidence that both pictures have twenty dots?

Pause. This really is remarkable!

For example, is there any reason to believe that a picture of $173$ groups of $985$ dots should contain the same count of dots as a picture of $985$ groups of $173$ dots? (Do you have the patience to draw out each of these pictures to check? Please say you don't!)

If you compute the products $173 \times 985$ and $985 \times 173$ via the long multiplication algorithm taught in school, it is not at all obvious that one is going to obtain the same final number in each: the computations look so very different in their middles! It is quite a shock to see the common answer $170{,}045$ appear.

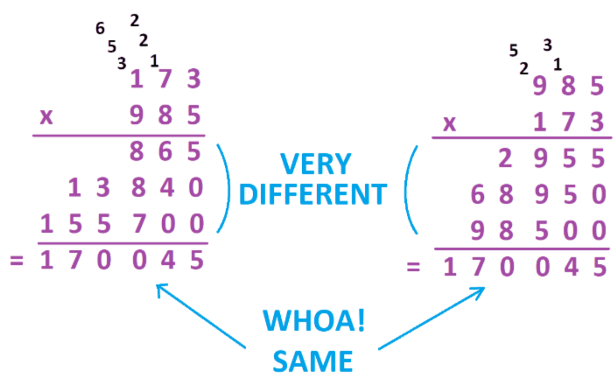



*Figure* $173 \times 985$ versus $985 \times 173$

What magic is this?

**TRY IT!**

Take this in!

Work out $87 \times 43$ and then $43 \times 87$ using the school algorithm. The final answer of 3741 will appear for each, but the middles of the computations are quite different.

Is it obvious to you that the school algorithm is certain to give the same answer if you switch around the two numbers you multiply?

This is the allure and delight of mathematics. As one thinks about and plays with mathematics, little mysteries and surprising "coincidences" start to arise, and one begins to suspect there is something deep and hidden lurking behind the scenes. And then, out-of-the-blue, a flash of brilliant insight suddenly makes everything stunningly clear. All hidden machinations are revealed.

The flash of insight needed to reveal the workings of repeated addition is this: instead of drawing groups of a repeated quantity in a single row, stack those quantities instead to make a rectangle of dots.

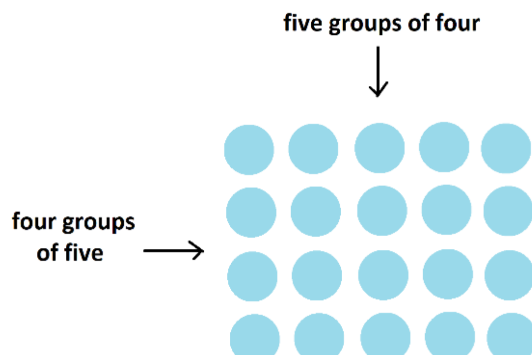



*Figure* $4 \times 5 = 5 \times 4$

Look at the figure from the left, focusing on the rows and you see four copies of five dots: $4 \times 5$. Now look from above to focus on the columns to see five copies of four dots: $5 \times 4$. It's the same collection of dots just viewed from two different perspectives. It simply must be the case, then, that $4 \times 5$ and $5 \times 4$ represent the same count of dots. (No mention of the number $20$ needed!)

In the same way, a $173$-by-$985$ rectangle of dots looked at two different ways would reveal that $173 \times 985$ and $985 \times 173$ simply must give the same count of dots. This means that the school multiplication algorithm simply can't change its answer if we switch the order of the numbers we multiply.

We seem to have stumbled upon another fundamental truth about the counting numbers.

> $a \times b$ and $b \times a$ are sure to have the same value,
> no matter which counting numbers $a$ and $b$ represent.

## The Role of Zero with respect to Multiplication

What's "five groups of no dots"?

Well … that's no dots and no dots and no dots and no dots and another set of no dots! We have $5 \times 0$ is $0+0+0+0+0$ and this is $0$. Five groups of nothing is nothing.

We can argue this way that $17 \times 0$ and $62 \times 0$ and $70986798766519273 \times 0$ should all be zero as well.

What about $0 \times 5$? Is that zero as well? If I have no groups of five dots, does that mean I have no dots at all? (Check out **Musing 3.2**. Terell is worried about this.)

We just said at the top of the page that we can switch the order we multiply numbers and make no change to the answer. If we think that should apply to the number $0$ as well, then we'd have to say that $0 \times 5$ has the same value as $5 \times 0$. We said that $5 \times 0$ equals $0$. This means that $0 \times 5$ should be $0$ as well.

We have

> $a \times 0$ and $0 \times a$ each have the value $0$,
> no matter which counting number $a$ represents

Did invoking the idea that "we can switch the order we multiply numbers" feel okay to you? Or maybe it already felt obvious to you that "0 groups of 5" has to be zero?

Zero is a tricky number. It can cause all sorts of brain-hurty troubles. Consider this question:

**What is $0 \times 0$?**

Here's an argument suggesting that $0 \times 0$ should equal zero.

> We just set the rule that $a \times 0$ equals zero, for all numbers $a$. So, it works for $0$ too.
> We have $0 \times 0 = 0$.

Here's an argument suggesting that $0 \times 0$ can't be zero.

> In the "real world," $0 \times 0$ reads as "zero groups of nothing."
> How much is that?
>
> Well, if I have no nothing, it must be because I have something.
> $0 \times 0$ thus cannot be nothing.
> It better be something!

The number $0$ really has befuddled humankind for millennia—not just on the philosophical matter of whether or not it deserves to be considered a counting number in its own right, but also with regard to understanding on what doing arithmetic with zero actually means.

It really can hurt one's brain!

**Where mathematicians have landed on this.**

Seventh-century Indian mathematician and astronomer Brahmagupta was the first to lay out rules for working with the number zero and led the world to show that the mathematics is logically consistent if we do indeed assume that our four properties of numbers are valid, even if we include the number zero.

If $a$ and $b$ are counting numbers, including possibly being zero, then we have that

$a + b$ and $b + a$ are sure to have the same value,

$a + 0$ and $0 + a$ both have the value $a$,

$a \times b$ and $b \times a$ are sure to have the same value,

$a \times 0$ and $0 \times a$ both have value $0$.

The logical consequence of the fourth property is that $0 \times 0$ must be $0$.

Mathematicians have decided to follow the mathematics. They don't feel that all quantities and actions must always have real-world interpretations. And this is surprising to many people who experience only school mathematics.

> **Mathematics is exceptionally good for describing and making sense of real-world scenarios, but real-world scenarios are not good at "explaining" all mathematics.**

Mathematics is bigger than the real world!

## One more thing. (That's a little joke too as you will see.)

What's one group of five? Clearly five!

$$1 \times 5 = 5$$

What do five groups of one make? Clearly five!

$$5 \times 1 = 5$$

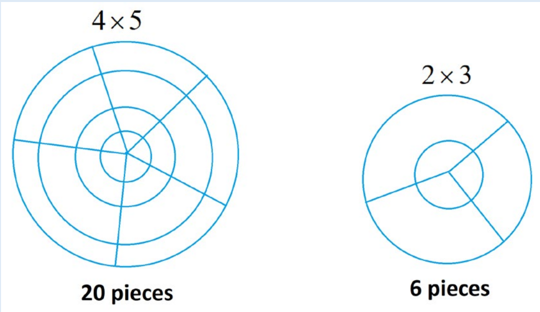
In the same way we can argue that $17 \times 1 = 17$ and $1 \times 299 = 299$ and $30012 \times 1 = 30012$.

It seems we have

| |
|---|
| $1 \times a = a$ and $a \times 1 = a$<br>no matter which non-zero counting number $a$ represents (including zero). |

## MUSINGS

**Musing 3.1** Here is an unusual (and inefficient!) way to compute the product of two positive counting numbers. To compute $4 \times 5$, say, first draw $4$ concentric circles and then draw $5$ radii for those circles. The number of pieces you get is $4 \times 5 = 20$.




In the same way, $2$ concentric circles and $3$ radii give $2 \times 3 = 6$ pieces.

a) Draw a picture for $3 \times 4$ and verify one does indeed see $12$ pieces.

b) Can you see in your mind's eye that a picture for $20 \times 1$ must have $20$ pieces?

c) Can you see in your mind's eye that a picture for $1 \times 20$ must also have $20$ pieces?

d) We drew the picture for $4 \times 5$ and saw $20$ pieces. Draw the picture for $5 \times 4$ and verify that it gives $20$ pieces too.

It is not obvious to me why computing $a \times b$ and $b \times a$ this weird way should give the same counts of pieces for all possible non-zero counting numbers $a$ and $b$. I am curious, is it obvious to you?

For that matter, is it all obvious to you why drawing circles and radii this way and counting pieces should precisely match the ordinary multiplication of the two numbers? This is weird!

Any thoughts?

**Musing 3.2** Terell is a bit worried about saying that $0 \times 5$ should be $0$. He says, "Draw a picture of three dots and ask how many groups of five you see?" He reasons that you could legitimately answer that you see no groups of five in a picture of three. So, maybe, $0 \times 5$ is $3$?

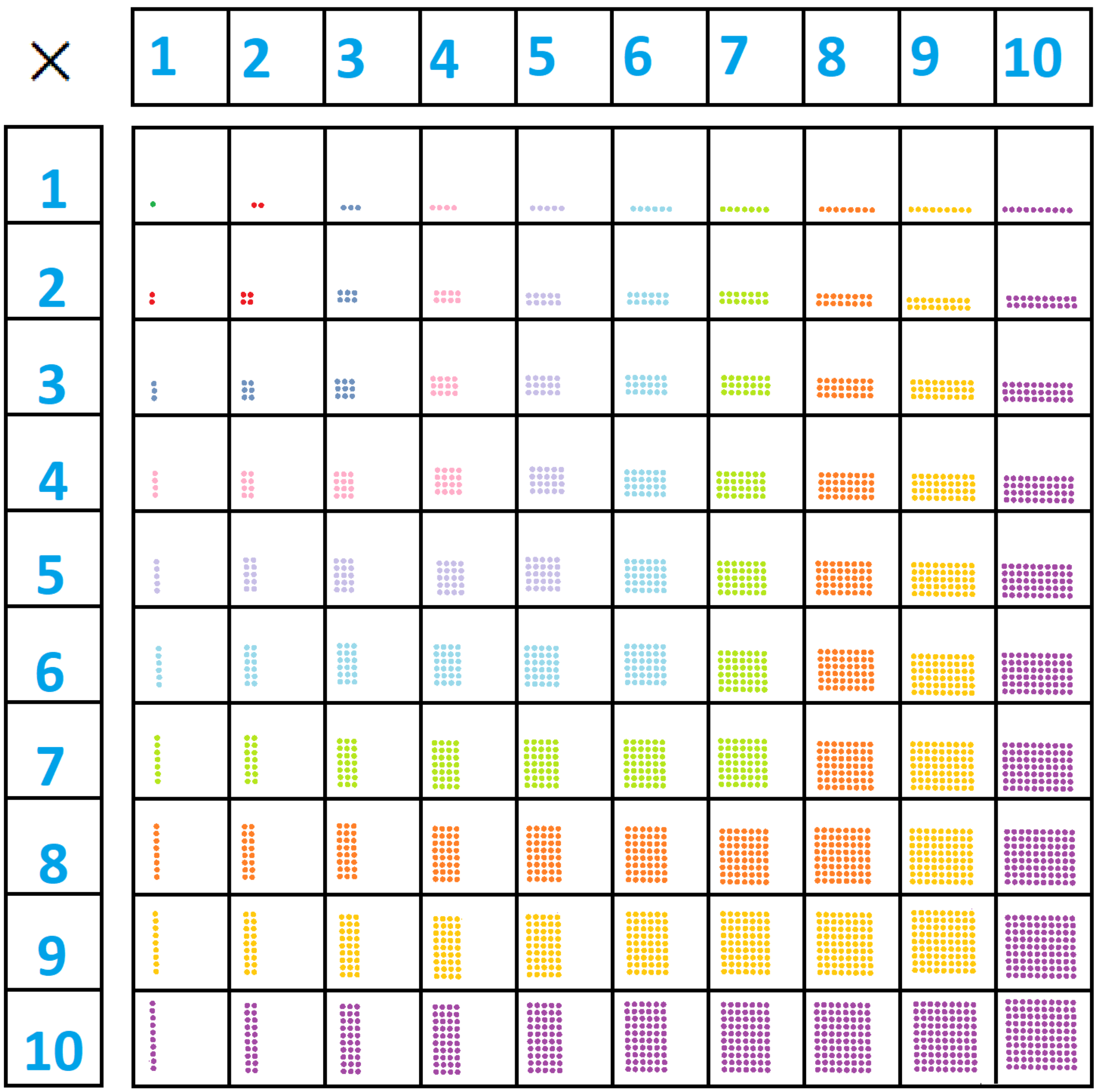


Terell has just made my brain hurt. What do you think? Or is your brain hurting too?

## 4. The Repeated Addition Table

Most people call this a “multiplication table,” except each entry here is a rectangular array of dots representing the multiplication fact appropriate for its cell. For example, in the third row, seventh column of the table we have a 3-by-7 rectangle of 21 dots.

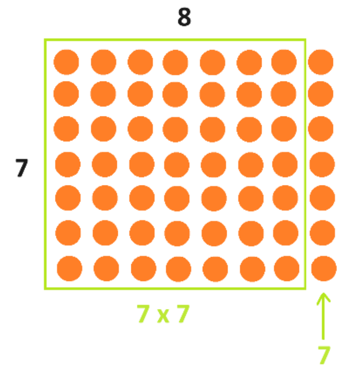
(It has become a societal convention to always mention rows first and columns second.)



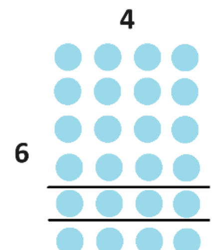
Having this imagery for multiplication answers in your mind can help you figure out new multiplication facts from known ones.

For example, if you happen to remember that $7 \times 7$ is $49$, then you can figure out the answer to $7 \times 8$ somewhat readily: imagine the rectangle picture of $7 \times 8$ and identify a picture of $7 \times 7$ within it.

We then see that $7 \times 8 = 49 + 7 = 56$.

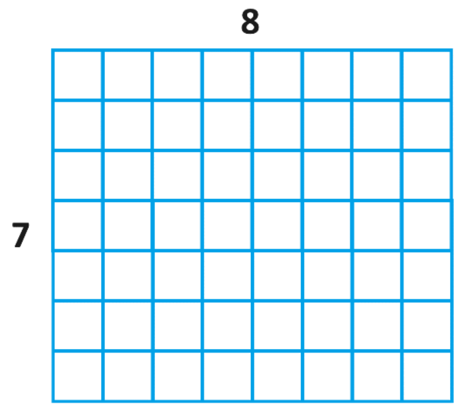



In the same way, $6 \times 4$ is $16 + 4 + 4 = 24$.



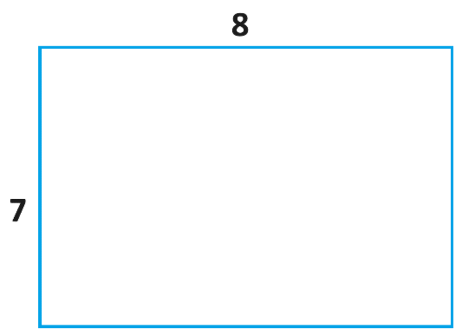
Actually, drawing dots gets tiresome pretty quickly. It's easier to draw "unit squares" (squares of area one).

For example, here's a picture of $7 \times 8$ again, but with seven rows of eight unit squares per row.
We have $7 \times 8 = 56$ squares of area one, and so this rectangle has area $7 \times 8 = 56$ square units.

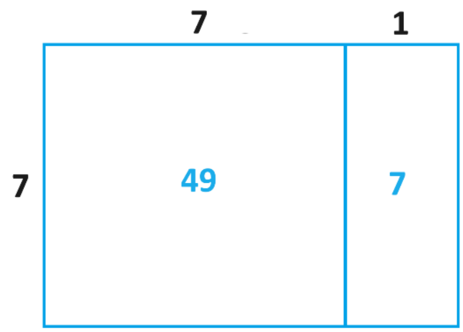



Actually … This picture is tedious to draw too!

We can just draw a rectangle and label one side as length $7$ and the other side as length $8$. The area of the rectangle is $7 \times 8 = 56$ and we can just imagine 56 square units.



Now we can readily see how to chop up the rectangle into two pieces and to deduce that it's area if given by $49$ (which is $7 \times 7$) plus $7$ (which is $7 \times 1$).




$$7 \times 8 = \quad 7 \times 7 \;+\; 1 \times 7 \quad = \quad 49 + 7 \quad = 56$$

In 1998, then Labour Schools Minister for the U.K. Stephen Byers was asked on the fly during an interview, "What is $7 \times 8$?" In a flustered moment he responded "$54$" and become a bit of a laughingstock for the nation.

It would have been lovely if he had the presence of mind to answer along these lines:

> "Ooh! Seven times eight is the hard one. Let me think.
> Well, I know that seven times seven is 49. So, adding another 7 to this gives $7 \times 8$.
>
> The answer is 56."

It would have been just brilliant for a nation to see a demonstration of beautiful mathematical thinking.

> **Practice:** Draw a picture to show that the value of $32 \times 16$ equals the sum of these four multiplication pieces: $30 \times 10 + 2 \times 10 + 30 \times 6 + 2 \times 6$.

## MUSINGS

**Musing 4.1** The numbers from computing $1 \times 1 = \mathbf{1}$, $2\times 2 = \mathbf{4}$, $3\times 3 = \mathbf{9}$, $4\times 4 = \mathbf{16}$, and so on are called the **square numbers**. Can you see why? (Look at the multiplication table.)

**Musing 4.2** The entry for $3 \times 7$ in the multiplication table is a rectangle of dots with three rows and seven columns. Describe the entry for $7 \times 3$.
In general, how would you describe the relationship between the entries for $a \times b$ and $b \times a$?

**Musing 4.3** How many dots are there along the first row of the table? Along the second row? Along the third, fourth, and tenth rows?

How many dots are there altogether in the table?

**Musing 4.4** What's common about all the entries of the same color?
How many dots are there of each color?

## MECHANICS PRACTICE

**Practice 4.5** Match each quantity on the left with its matching quantity on the right. (Try to imagine rectangles here.)

| | |
|---|---|
| 8 x 9 | 9 + 3 + 3 |
| 7 x 6 | 25 +5 |
| 6 x 8 | 64 + 8 |
| 3 x 5 | 36 + 6 |
| 5 x 6 | 36 + 6 + 6 |

## 5. Repeated Addition in the "Real World"

Suppose there are essentially only two different routes for driving from city A to city B, and essentially three different routes for driving from city B to city C.

How many different options do I have for driving from city A to city C?

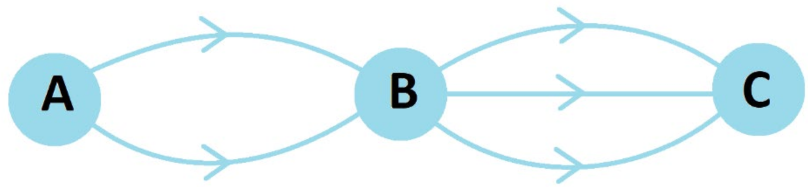
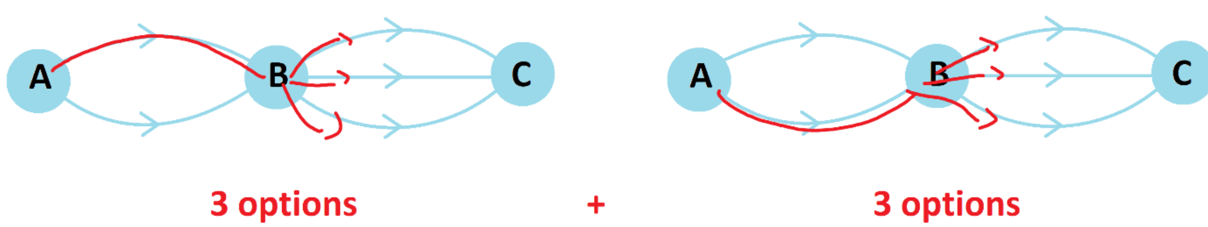



*Figure* 3 cities

**Question:** Some people look at the picture without thinking and say that the answer to this question is 5. Do you see why the number 5 might first come to mind?

If I take the top route from A to B, then I am presented with $3$ options on how to proceed next. And if I take the bottom route from A to B, I am again presented with $3$ options for how to continue. I thus have $3+3=6$ possible ways to travel from A to C.




If, instead, there were 5 choices of route from city A to city B, and still 3 choices of route from city B to city C, then each option I choose for the first leg of my journey offers 3 choices on how to follow that choice. I'd thus have $3+3+3+3+3$, that is, 5 groups of 3 options. This is repeated addition, and so, in this revised scenario, there $5 \times 3$ routes in total from city A to city C.

**Question:** Draw a picture of this of this and convince yourself that there are indeed $3+3+3+3+3=15$ routes from city A to city C.

Now, let me tell you something about my wardrobe.

*I own just $5$ shirts, all different, and just $4$ different pairs of trousers. How many different shirt-trouser outfits could you see me in?*

Again, this is a repeated addition challenge. For each choice of shirt, I have $4$ choices of trousers to go with it. Thus, I have a total of $4+4+4+4+4=5\times 4$ choices for my outfit.

Of course, this count would change if I gave you some further information that restricts my options. (For example, I will never wear my chartreuse trousers with my acid-green shirt.) But without knowledge of such restrictions, counting options is a matter of thinking through repeated addition.

We have

**The Multiplication Principle**

If there are $a$ ways to complete a first task and $b$ ways to complete a second task, and assuming that a choice made for one task in no way influences the choice made for the other, then there are

$$a\times b$$

ways to complete both tasks together.

For example, if there are $8$ different movies I could watch tonight and $3$ different snacks I could eat while watching them, then there are $8\times 3=24$ different movie/snack combinations for me to consider.

If there $4$ answers to a select from for a first question in an exam and $4$ to select from for a second question, then there are $4\times 4=16$ different ways I could answer those two questions. (Hopefully, I choose the answers that are correct for both questions.)

In the travel example above there are $6$ ways to move from city A to city C. If there are also $5$ routes from city C to a new city D, then we can travel from A to B to C to D in $6 \times 5 = 30$ different ways. (This is really the product $2 \times 3 \times 5$.)

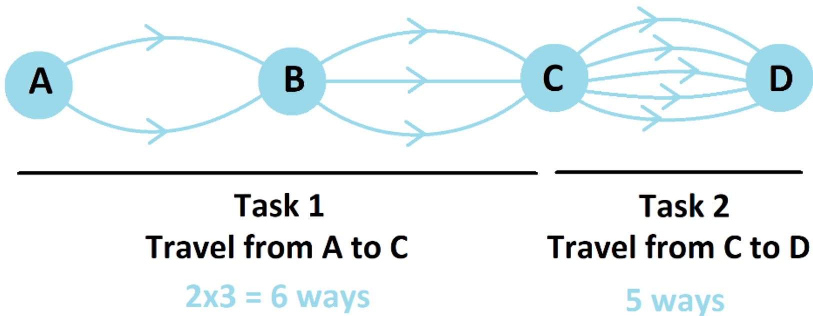



*Figure* $30$ routes from city A to city D

**Question:** On a menu, there are 10 choices for a starter, 12 choices for a main meal, and 5 choices for dessert. How would you explain to a friend why that provides $10 \times 12 \times 5 = 600$ options for a three-course meal?

We're seeing how to use the multiplication principle multiple times to handle counting multiple tasks!

**The Full Multiplication Principle**

If there are $a$ ways to complete a first task and $b$ ways to complete a second task, up to $z$ ways to complete a final task, and assuming that a choice made for any one task in no way influences the choices made for any other task, then there are

$$a \times b \times \cdots \times z$$

ways to complete all the tasks together.

## MUSINGS

**Musing 5.1** I own $5$ different shirts and $4$ different trousers. If there are no restrictions on which shirt I might wear with each pair of trousers, then you could see me in $20$ different shirt/trouser outfits.

a) I also own $3$ different pairs of shoes. How many different shirt/trouser/shoes outfits could I wear?
b) I own $1$ hat, which I might or might not wear. With the hat option, how many different shirt/trouser/shoes/hat-no-hat outfits could you see me in?

(Assume in these questions that there are no restrictions on my choices in putting together an outfit.)

**Musing 5.2**

a) If I were to roll a die and flip a coin, how many different outcomes are there for me to possibly see?

b) If I were to roll a red die and a blue die, how many different outcomes are there for me to possibly see?

Here's an annoying question.

c) If I were to roll two identical white dice, how many different outcomes are there for me to possibly see?

When Amit considered this third question, he said that this is just the same problem as part b and thus has the same answer. "After all, why should the color of each die matter?" he responded.

Beatrice, on the other hand, wasn't so sure. She was worried that because one can no longer tell the dice apart, the outcomes you see might be interpreted differently and the answer to the question thus might change.

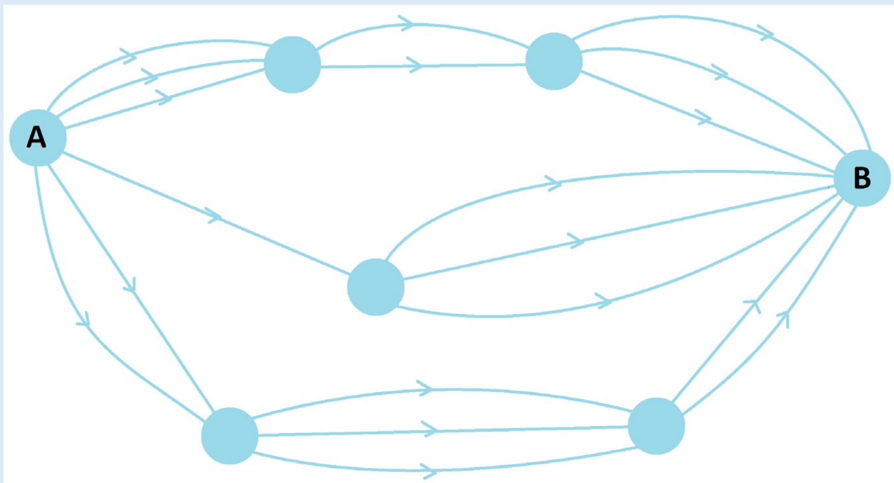
Chi thought about this and asked: "What if one white die is rolled first and the other is rolled second? Then we could tell the rolls apart and maybe the answer is the same as for part b?"

Debjyoti said that his brain hurts and he doesn't know what to think.

Here, finally, is my question to you:

Is your brain now hurting too?

**Musing 5.3** Can you see that there are $33$ ways to make your way from city A to city B in this complicated road map?




## MECHANICS PRACTICE

**Practice 5.4** Make up a "choice" problem whose answer is $6 \times 2 \times 3 \times 2$.

## 6. Ordering Additions

Try this!

**ACTIVITY**

**A NOT-EXCITING GAME OF SOLITAIRE**

Write the numbers 1, 2, 3, 4, 5, and 6 on a page.


A "move" in this game of solitaire consists of erasing two numbers and replacing them with their sum.

For example, if you cross out 3 and 5 you will then write 8 on the page and have the numbers 1, 2, 4, 6, and 8 to work with. If you next cross out 1 and 8, you will replace them with 9 and be left with 2, 4, 6, and 9. And so on.

Each move has you erasing two numbers and writing one number, so the count of numbers on the page steadily decreases. The goal of this game is to end up with the single number 21 on the page.

Do try it. Can you win?

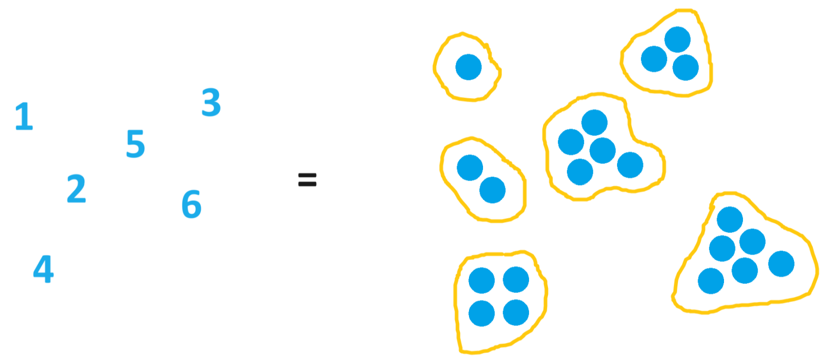
It really is worth playing the game.

I bet you can get 21 when you try it.
I bet you can get 21 again playing a second time but making different choices along the way.

**Next challenge**: Play the game yet again and try to **not** get the answer 21.

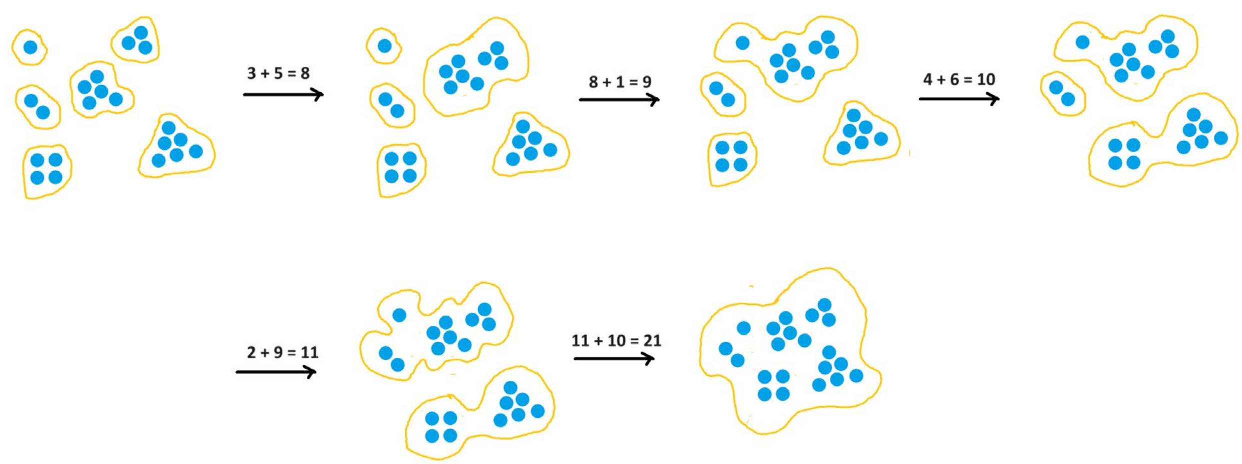
As you probably suspect, this game is rigged: you are sure to get a final single number of 21 each and every time you play no matter what choices you make along the way.

Replacing the numbers with groups of objects makes it clear why this is the case. I'll draw dots.




The act of erasing two numbers and replacing them by their sum simply combines dots in two separate groups to make one group.

Here's play of the game starting by combining 3 and 5, and then combining 1 and 8, and going from there.




Without looking at the details you can see that all we are doing is slowly combining the dots initially in separate groups into one big group. The count of dots never changes as we play this game. Thus, every game ends with the one same final state: all the dots at the start of the game combined into one big group.

As there were 21 dots to begin with, this game is destined to end with the number 21.

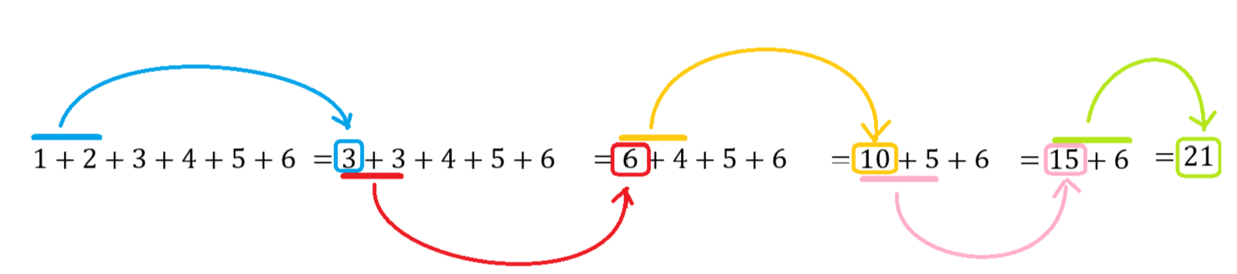
**Question:** This game is solitaire is played starting with the numbers 4, 8, 8, 10, and 20 written on a page. What final single number will remain on the page at the end of the game?

Do you think you could adequately explain why this is so to another person?

In these notes so far, we've only ever added two things at a time. (Well, I did ask you to add together an absurd number of zeros in section 2.) But we know from our school days we can add together any number of numbers we like. Let's consider

$$1 + 2 + 3 + 4 + 5 + 6.$$

School teaches us to compute a string of additions like this by only ever adding two numbers at a time, starting at the left and working to the right.




But we've just seen from the solitaire game that we add pairs of numbers in any order we like, and we are sure to obtain the same final answer.

In $1 + 2 + 3 + 4 + 5 + 6$, we can add the 4 and 6 together first if we like and turn the sum into $1 + 2 + 3 + 5 + 10$. And now we could add the 2 and 5 and make it $1 + 7 + 3 + 10$, and so on. (Adding 3 and 7 next gets us to $1 + 10 + 10$ and the final answer of 21 is now apparent.)

We also learned in section 2 that when we add 4 and 6 we can think $4 + 6$ or can think $6 + 4$, it doesn't matter.

So, in all possible interpretations of "order does not matter," we have the following powerful realization.

| |
|---|
| In any string of counting numbers added together, $a + b + c + d + e + \ldots + y + z$, it does not matter in which order one chooses to perform the additions. The same answer will always result. |

This gives us the means to sometimes be clever when presented with a long sum.

For example, in the sum $31+7+84+3+9+16$ I can see:

31 and 9 together make 40,
7 and 3 make 10, and
84 and 16 make 100.

So, this sum can be computed as $40 + 10 + 100$, which is 150.
This is much better than working left to right!

Breaking a number down into a sum of two numbers can be helpful too. For example, seeing 47 as $3 + 44$ makes $97 + 47$ manageable.

$$97 + 47 = 97 + \underline{3 + 44} = 100 + 44 = 144$$

**Question:** Can you see $16 + 92 + 4 + 39$ as the same as $20 + 100 + 31$?

## MUSINGS

**Musing 6.1** Can you see that each of these sums has value one hundred?

a) $3+50+47$
b) $46+18+4+15+2+15$
c) $17+17+17+17+17+2+2+2+2+2+1+1+1+1+1$
d) $48+3+49$

**Musing 6.2**

a) Can you see that

$$1+2+3+4+5+6+7+8+9+10+11+12+13+14+15+16+17+18+19+20$$

is ten copies of $21$, and so has value $210$?

b) What is the value of $1+3+5+7+9+11+13+15+17+19$?

c) There is a story of a famous eighteenth-century mathematician Carl Friedrich Gauss who, as a schoolboy, dumbfounded his teacher by finding the sum of all the numbers $1$, $2$, $3$, $4$ and so on up to $100$ in a matter of seconds. Can you see that the sum of these numbers equals $50$ copies of $101$? (This makes the sum equal to $5050$.)

(Care to look up the story on the internet? You'll find several different versions of it leaving one to wonder how true the story might be in the first place!)

d) You decide to play the solitaire game described in this section with the numbers $1$, $2$, $3$, $4$ up to $100$ written on a chalk board. You keep erasing two numbers and replacing them with their sum until a single number remains on the board. What number will that be?

## MECHANICS PRACTICE

**Practice 6.3** Compute each of these sums in a way that feels efficient to you.

a) $19+18+16+2+4+1$
b) $46+294$
c) $998+875$
d) $199+199+199+199+199+7$
e) 19 + 18 + 18 + 19 + 17 + 9
f) $37+72+11$

## OPTIONAL ADDENDUM

You may have heard the terms **commutative property** of addition and **associative property** of addition. These are not important to know, but if you are curious, they are both to do with the notion that "order does not matter" when computing addition problems.

The **commutative property** refers the fact that we can change the order of two numbers we are adding. We have that $a + b = b + a$ no matter which counting numbers $a$ and $b$ represent. We saw this in section 2.

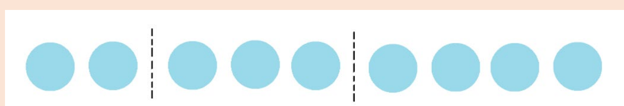
**4 +6 = 6 +4**

**12 +10 = 10 +12**

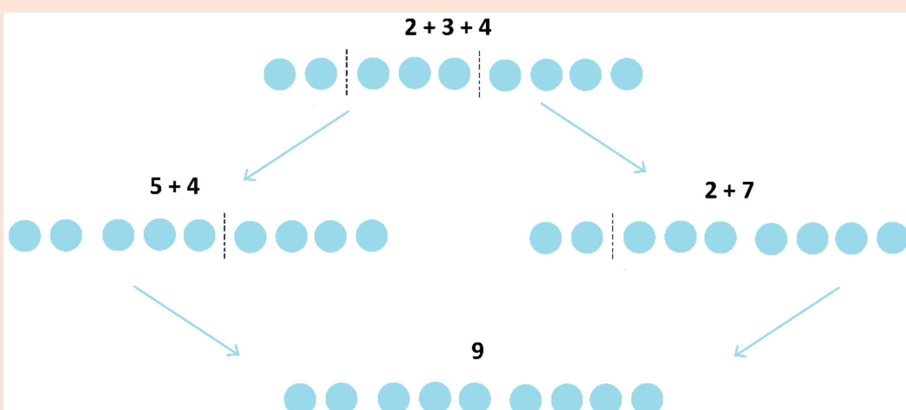
**8 +92 = 92 +8**

**1656 + 207 = 207 +1656**

The **associative property** refers to the fact that we can change the order we choose to conduct the two summations in a sum of three numbers. To illustrate what I mean, here's a picture of $2 + 3 + 4$.



We can compute the left addition first and the right addition second, if we like, or the right addition first and the left addition second. They both lead to the same result in the end (as we know they must from our solitaire game).

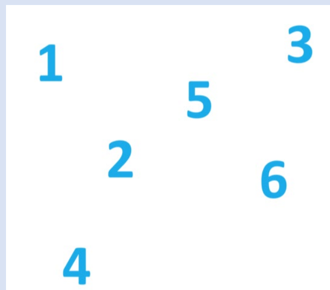


## 7. Ordering Multiplications

Here's another game of solitaire.

**ACTIVITY**

Again, write the numbers 1, 2, 3, 4, 5, and 6 on a page.




This time, instead of erasing two numbers and replacing them with their sum, replace them with their **product**, the two numbers multiplied together.

For example, if you cross out 3 and 5 you will then write 15 on the page and have the numbers 1, 2, 4, 6, and 15 to work with. If you next cross out 1 and 15, you will replace them with 15, their product (got that?) and be left with 2, 4, 6, and 15. And so on.

Each move has you erasing two numbers and writing one number, so the count of numbers on the page steadily decreases.

With which single number is this game sure to end?
(How do you know it is going to be the same number each and every time you play?)

It seems the game ends with the number 720 no matter how you choose to play.

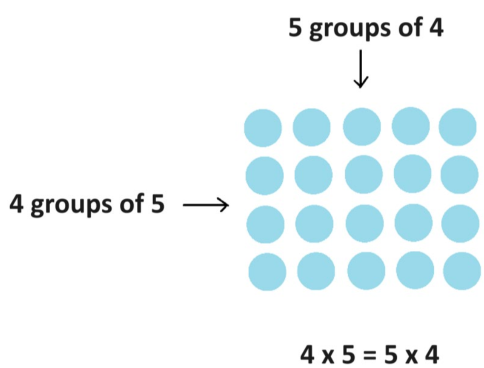
This is the value of product $1 \times 2 \times 3 \times 4 \times 5 \times 6$ computed from left to right, two numbers at a time, just as school teaches you how to evaluate a string of numbers multiplied together.

And again, the solitaire game seems to be leading us to say:

> In any string of counting numbers multiplied together, $a \times b \times c \times d \times e \times \ldots \times y \times z$ it does not matter in which order one chooses to do the individual multiplications. The same answer will always result.

Can we justify this?

We certainly have that the order in which you multiply two counting numbers together does not matter: $a \times b$ and $b \times a$ are sure to have the same value no matter which numbers $a$ and $b$ represent. We saw this in section 3.




What about three counting numbers multiplied together? Can we explain why "order doesn't matter" when computing $2 \times 3 \times 4$ for instance?

Remember, school world has us compute $2 \times 3 \times 4$ from left to right. We look at $2 \times 3$ first to get $6$, and then compute $6 \times 4$ next to get $24$.

$$\overline{2 \times 3} \times 4 = \boxed{6} \times 4 = 24$$

How do we draw a picture of this?

Start, we can draw $2 \times 3$ as two groups of three dots drawn in a rectangle.

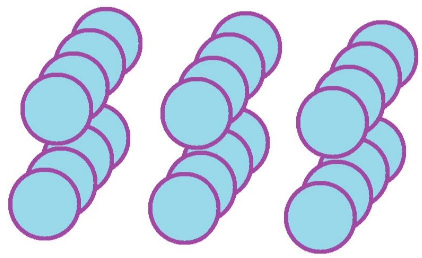


**2 x 3**

And now we need to draw $2 \times 3 \times 4$, which is "$2 \times 3$ groups of $4$," whatever that means.

> **Question:** Think about this before turning the page. It took me a little while to figure out what one could draw here. I am curious if we come up with the same approach.

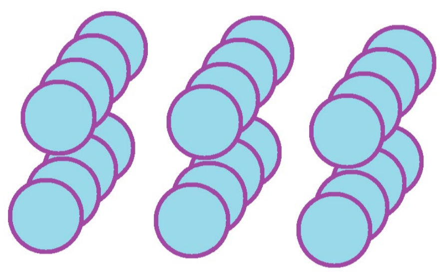
I went to the third dimension!



**2 x 3 groups of 4**

Here's a three-dimensional picture of 24 dots arranged in a box-like arrangement. The front face is picture of $2 \times 3$ and each dot in that picture extends $4$ dots deep into the page.

But when I look at my picture, I also can't help seeing two horizontal layers of dots, and that each layer is a picture of $3 \times 4$. (Can you see that too?) The picture is also one of $2$ copies of $3 \times 4$.



**2 groups of 3 x 4**

This picture explains why we can compute $2 \times 3 \times 4$ by focusing on $2 \times 3$ first to get $6 \times 4$ or by focusing on $3 \times 4$ first to get $2 \times 12$. Both are these $24$ dots viewed two different ways.

And we can go further.

If I focus in columns first and rows first, I can interpret the front face of dots as $3 \times 2$. The whole three-dimensional picture is thus also "$3 \times 2$ groups of $4$," which is $3 \times 2 \times 4$.

Or, I can look at the right face of dots and see "$2 \times 4$." And each of those dots is the end of a row of $3$ dots. So, the picture also looks like "$2 \times 4$ groups of $3$," which is $2 \times 4 \times 3$.

Or, I can see $3$ copies of that vertical $2 \times 4$ face of dots, which is $3 \times 2 \times 4$ but with computing the second multiplication first.

Or, I can see the picture as $4 \times 3$ groups of $2$ dots, which $4 \times 3 \times 2$. Or as $2$ copies of $4 \times 3$, which is $2 \times 4 \times 3$ but with the second multiplication computed first.

**Question:** Come up with yet another way to interpret the three-dimensional picture.

We can keep playing this confusing game of switching perspectives on this single picture and explaining why we are not changing anything. All these values are the same

$$\begin{aligned} 2 \times 3 \times 4 &= 3 \times 2 \times 4 = 3 \times 4 \times 2 \\ &= 2 \times 4 \times 3 = 4 \times 2 \times 3 = 4 \times 3 \times 2 \end{aligned}$$

even with compute the second multiplications first (which goes against the grain of the school way).

We're seeing, for sure, that the order in which you compute $2 \times 3 \times 4$ —be it the order of the numbers or the order of which of the two multiplications you compute first—just doesn't matter. And, of course, there is nothing special about the numbers 2, 3, and 4 here: we can imagine a three-dimensional picture of any size.

This is great!

By looking at a two-dimensional figure we've explained why order doesn't matter for computing a product of two numbers, $a \times b$.

And by looking at a three-dimensional picture we've explained why order doesn't matter for computing a product of three numbers, $a \times b \times c$.

But I am nervous. For a product of four numbers do we need to draw a four-dimensional picture? (I don't know what that would mean!)

Let's hold off on justifying matters for products of four or more numbers for right now. (We'll come back to it later as there's got to be a better way than going to the fourth dimension!)

But if we do trust that order does not matter for multiplication, then we can sometimes use that idea to our advantage.

For example, here's a nice way to compute $35 \times 14$. Think of $35$ as $5 \times 7$ and $14$ as $2 \times 7$.

Then

$$35 \times 14 = 5 \times 7 \times\ 2 \times\ 7$$

And can you see that this is $10 \times 49$ to give $490$?

**Question:** Compute each of these products in a similar manner.

a) $15 \times 12$
b) $28 \times 25$
c) $68846 \times 50$

## MUSINGS

**Musing 7.1** Katya wondered about the two ways to interpret $2\times3\times4$ and thought she could go back to the idea of repeated addition. She thought about matters and then wrote

$3\times4$ is $4+4+4$

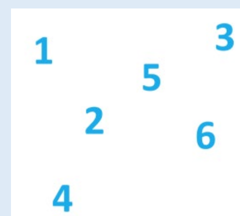
$2$ copies of $3\times4$ is $\underline{4+4+4}+\underline{4+4+4}$ (using an underline to make each copy clear)

and

"$2\times3$ copies of $4$" is $\begin{matrix}4 & 4 & 4\\ 4 & 4 & 4\end{matrix}$.

But then she wasn't sure if what she was writing was helpful, or even meaningful.

What do you think?




**Musing 7.2 OPTIONAL**

Here's a third variation of the solitaire game that combines addition and multiplication. Again, start with the numbers 1 through 6 on a page.

Erase two numbers and replace them with their **sum** and their **product** **added together**.

For example, if you cross out 3 and 5 you will then write 23 on the page. (This is 8, their sum, and 15, their product, added together.) If you then cross out 2 and 4 you will write 14 on the page. (This is 6 and 8 added together.)

a) Do you see the same final number each and every time you play the game?

b) What final number do you if you start with just 1 through 5 instead? Just 1 through 4? 1 through 3? Just 1 and 2? Is there a pattern to these final numbers?

**Hard Very Optional Challenge:** Going back to starting with the numbers 1 through 6, can you explain *why* the same final number appears each time you play the game, no matter the choices you make along the way?

## MECHANICS PRACTICE

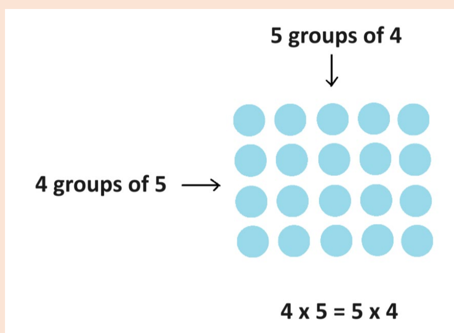
**Musing 7.3** Compute each of these products in a way that feels efficient to you.

a) $25 \times 36$
b) $5 \times 216$
c) $5 \times 846044288$
d) $15 \times 6 \times 15 \times 6$
e) $72 \times 125 \times 35 \times 84 \times 55 \times 0 \times 25 \times 15 \times 8$

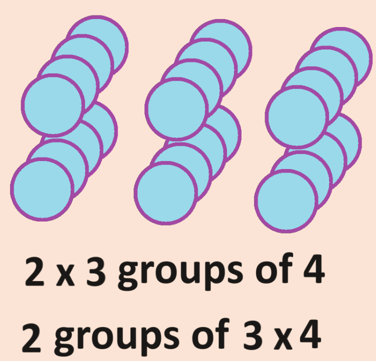
## OPTIONAL ADDENDUM

You may have also heard the terms **commutative property** of multiplication and **associative property** of multiplication. If you are curious, they are the official names of the two properties we discussed and explained in this section.

The **commutative property** refers the fact that we can change the order of two numbers we are multiplying. We have that $a \times b = b \times a$ no matter which counting numbers $a$ and $b$ represent.




The **associative property** refers to the fact that we can change the order we choose to conduct the two multiplications in a product of three numbers.



## 8. The Vinculum

Going back to additions, consider again $2+3+4$.

If I were a fussy sort, I might insist that you think this through by adding $2$ and $3$ together first to get $5$ and then add $4$. Or I might insist that you compute $3+4$ first to think $7$, and then compute $2+7$.

The question is: *How might I communicate to you the order I insist you conduct the two additions?*

Back in 1484, French mathematician Nicolas Chuquet wrote a manuscript in which he used a horizontal bar to denote an intended order to operations. Its use became the rage among a good number of European mathematicians for the centuries that followed.

For example, mathematicians would write

$$2+\overline{3+4}$$

if they intended the reader to think $2+7$ to get to $9$, or

$$\overline{2+3}+4$$

if they want the line of thought to be $5+4=9$.

The horizontal bar was—and still is—called a *vinculum*, from the Latin word for "bond" or "tie," as it suggests which terms of an expressions are "tied together" and to be handled first. Some mathematicians liked to write their vincula on top of an expression (like I do) and others preferred to write them as underlines (like Chuquet preferred).

If I had to choose a favorite mathematical symbol, I would choose the vinculum. I just think it is neat! For example, here is a mighty complicated expression loaded with nested vincula. But despite its complexity, it is clear how I am meant to think my way through it.

$$\overline{2+\overline{4+\overline{2+\overline{\overline{8+\overline{17+\overline{\overline{1+9}+10}}}+7}}}}+1$$

(Can you see—literally see!—that one is to think $10$, then $20$, then $37$, then $45$, then $52$, then $54$, then $58$, then $60$, and then, finally, $61$?)

There are some natural conventions people have settled on for working through vincula.

1. *If a mathematical expression has a vinculum placed on it, compute what is under the vinculum first.*

For example, $5\times\overline{3+7}$ is $50$. (Can you see this?)

2. *If there are nested vincula, work with the innermost vinculum first and proceed from there.*

For example, $10+\overline{\overline{4\times\overline{3+2}}\times 3}$ is $70$. (Check this.)

3. *If there are two or more "equally nested" vincula, work them out in any order you like (left to right, or right to left, or simultaneously).*

For example, $\overline{2+3}\times\overline{4+1}$ has two "equally deep" vincula. This is to be computed as $5\times 5$, giving $25$.

Unraveling $5+\overline{\overline{\overline{4\times 4}\times 2}+\overline{3\times 2}}$ first gives

$5+\overline{\overline{16\times 2}+\overline{3\times 2}}$ (the innermost vinculum)

and then

$5+\overline{32+6}$ (from equally nested vincula),

and then $5+38$, to give $43$.

**Question:** Can you see that that value of $\overline{5+\overline{\overline{\overline{5+6}\times 4}-4\times 2}}$ is ninety?

## Reducing the Number of Vincula

To cut down on the abundance of vincula in an expression, mathematicians have settled on another convention.

> *Assume every multiplication sign comes with its own hidden vinculum above the two numbers being multiplied.*

For instance, $2+3\times 4$ is to be understood as $2+\overline{3\times 4}$, and so equals $14$.

And $7\times 5+3\times 1$ is to be understood as $\overline{7\times 5}+\overline{3\times 1}$, which equals $38$.

A tricker example is

$$2\times\overline{3+4}+5\times 4.$$

This is to be unraveled as

$$2\times 7+5\times 4,$$

which is

$$\overline{2\times 7}+\overline{5\times 4},$$

giving $14+20$, which is $34$.

> You may have been taught an "order of operations" rule in school which says something like:
>
> *Do multiplications before doing additions.*
>
> So, in $2+3\times 4$, one is to compute $3\times 4=12$ first and then compute $2+12=14$.
>
> This is just our rule for vincula with the (hidden) vinculum over the product: *Always do vincula first!*

## Getting a bit ahead on matters …

People don't use the vinculum anymore—except in three places.

In 1631, English mathematician Thomas Harriot suggested it might be a good idea to attach a vinculum to this symbol √ used for square roots. (This symbol is called a *radix*, by the way.) For example, an expression like

$$\surd 9+16$$

is ambiguous. Is this the square root of $9$ (which is $3$) with $16$ later added to give the answer $19$? Or is this the square root of the entire quantity $9+16$, which is $25$, to give the answer $5$?

The vinculum clarifies matters.

$$\sqrt{9}+16=3+16=19$$
$$\sqrt{9+16}=\sqrt{25}=5$$

Most people today do not realize that $\sqrt{\phantom{x}}$ is two separate symbols combined.

In 1647 Italian mathematician Bonaventure Cavalieri used the vinculum in his geometry book. If $A$ and $B$ are the names of two points in space, he suggested using the notation $\overline{AB}$ for the line segment that "ties" them together. This is now standard notation in geometry books.

Today, people in some countries (the US included) use a vinculum to denote a group of digits that repeat in an infinitely long decimal. For example, we have

$$\frac{1}{7}=0.\overline{142857}=0.142857142857142857142857...$$
$$\frac{1}{3}=0.\overline{3}=0.3333333......$$

Folk in other parts of the world might write $0.\dot{1}4285\dot{7}$ or $0.(142857)$ or $0.\overparen{142857}$, for instance, for the repeating decimal representation of $\frac{1}{7}$.)

## MUSINGS

**Musing 8.1** There are $\mathbf{2}$ ways to place a vinculum in the expression $1+2+3$.

$$\overline{1+2}+3 \qquad\qquad 1+\overline{2+3}$$

There are $\mathbf{5}$ ways to place vincula in the expression $1+2+3+4$ so that one is only adding two quantities at a time.

$$\overline{\overline{1+2}+3}+4 \qquad \overline{1+\overline{2+3}}+4$$

$$1+\overline{2+\overline{3+4}} \qquad 1+\overline{\overline{2+3}+4}$$

$$\overline{1+2}+\overline{3+4}$$

Care to list the $\mathbf{14}$ ways to list vincula in the expression $1+2+3+4+5$?

Care to list the $\mathbf{42}$ ways to list vincula in the expression $1+2+3+4+5+6$?

Some people like to say there is $\mathbf{1}$ way to place vincula in the sum $1+2$, namely, to not do anything and just leave it as it is as there is no need for one.

We are developing a list of "vinculum numbers": $1,\ 2,\ 5,\ 14,\ 42$.

Do you care to guess what the next vinculum number after $42$ might be?
(Remember, the answer can always be: "No. I do not care to guess.")

## 9. Parentheses/Brackets

Use of the vinculum remained popular all through the fifteen- and sixteen-hundreds and well into the seventeen-hundreds, which seems to befuddle some mathematical historians. After all, around the year 1440, Johannes Gutenberg invented the printing press allowing for the first time the mass production of books. Printing lines of text and mathematical symbols was straightforward. But inserting horizontal bars between lines of text was awkward and hard to do. Why stay with a notational system that was so difficult to print, especially since other symbols for grouping terms were being proffered at the time?

One alternative was to use **parentheses** (many people in the world call them **brackets**) to group terms. Instead of writing $\overline{2+3}+4$ and $2+\overline{3+4}$, we could write $(2+3)+4$ and $2+(3+4)$. And instead of writing

$$\overline{2+\overline{4+\overline{2+\overline{\overline{8+\overline{17+\overline{\overline{1+9}+10}}}+7}}}}+1,$$

which is awfully hard to print by lining up letter and symbol tiles on the printing press table, one could write instead

$$\left(2+\left(4+\left(2+\left(\left(8+\left(17+\left(\left(1+9\right)+10\right)\right)\right)+7\right)\right)\right)\right)+1.$$

Although this is a bit harder to unravel (but, really, who in their right mind would be writing so many nested parentheses in the first place?), it is certainly straightforward to print with a press.

In the mid-1700s, the prominent and prolific Swiss mathematician Leonhard Euler often used parentheses for grouping. It seems he helped accustom European mathematicians to their use and they remain the preferred grouping symbol to this day.

## Translating our Vincula Conventions to Parentheses/Brackets Conventions

Here are our vincula conventions rewritten in terms of parentheses.

1. *If a mathematical expression has a set of parentheses in it, compute what is in the parentheses first.*

For example, $5\times(3+7)$ is $50$.

2. *If there are nested parentheses, work with the innermost parentheses first and proceed from there.*

For example, $10+((4\times(3+2))\times3)$ is $70$.

3. *If there are two or more "equally nested" parentheses, work them out in any order you like (left to right, or right to left, or simultaneously).*

For example, $(2+3)\times(4+1)$ has two "equally deep" parentheses. This is to be computed as $5\times5$, giving $25$.

Unraveling $5+(((4+4)\times2)+(3\times2))$ first gives

$5+((16\times2)+(3\times2))$ (the innermost parentheses)

and then

$5+(32+6)$ (from equally nested parentheses),

and then $5+38$, to give $43$.

To cut down on the abundance of parentheses in an expression, mathematicians have settled on another convention.

*Assume every multiplication sign comes with its own set of parentheses immediately around the two numbers being multiplied. (They've simply been made invisible.)*

For instance, $2+3\times4$ is to be understood as $2+(3\times4)$, and so equals $14$. And $7\times5+3\times1$ is to be understood as $(7\times5)+(3\times1)$, which equals $38$.

A tricker example is

$$2 \times (3+4) + 5 \times 4 .$$

This is to be unraveled as $2 \times 7 + 5 \times 4$ ,which is $(2 \times 7) + (5 \times 4)$, giving $14 + 20$, which is $34$.

> You may have been taught an "order of operations" rule in school which says something like:
>
> *Do multiplications before doing additions.*
>
> So, in $2 + 3 \times 4$, one is to compute $3 \times 4 = 12$ first and then compute $2 + 12 = 14$.
>
> This is just our rule for parentheses with the (hidden) parentheses around the product:
>
> *Always do what's inside parentheses first.*

**Question:** Do these last two pages feel like *déjà vu?*

## Notations for Multiplication

As I am sure you are aware, the letter $x$ has quite the favored status in algebra class. (We'll get to algebra.) But that letter looks awfully similar to the traditional multiplication symbol ×. To avoid possible confusion, mathematicians tend to use a raised dot · to denote multiplication.

For example, $2 \cdot 3$ means $2 \times 3$, and $42 \cdot 17$ means $42 \times 17$.

And sometimes they will not write a multiplication symbol at all—just placing the two quantities to be multiplied next to each other if no possible confusion could result.

For example, instead of writing $2 \cdot (3+7)$ mathematicians will drop the dot, and write

$$2(3+7).$$

If $a$ represents a number, instead of writing $3 \cdot a$, mathematicians will write

$$3a.$$

**Question**: Mathematicians would never drop the dot in $23 \cdot 17$. Can you see why?

This leads to odd looking expressions every now and then. For example, in computing

$$(3+7)(4+9)$$

one might write $(10)(13)$ and wonder why there are parentheses around the single numbers. Nonetheless, one recognizes this as $10 \cdot 3$ to get the answer $30$.

In computing $5(83+17)$ one might find oneself writing $5(100)$, which is to be recognized as $5 \cdot 100$.

To prepare students for this, some elementary school curricula explicitly state that $a(b)$ and $(a)(b)$ and $(a)b$ are each alternative notations for $a \times b$. I personally suspect that this must seem weird and confusing to young'uns.

## MUSINGS

**Musing 9.1** Try writing these two expressions with vincula instead of parentheses. Show the hidden vincula as well.

a) $\left(\left(4+2\cdot 3\right)+\left(3+7\right)\right)+\left(1+2\right)$

b) $1+\left(1+\left(1+\left(1+1\cdot 1\right)\right)\right)$

Try writing these two expressions in terms of parentheses. Let’s keep the hidden parentheses hidden this time.

c) $\overline{2\cdot 3+4}\cdot\overline{5+6}$

d) $8\cdot 8+8\cdot\overline{\overline{8\cdot\overline{8+8}}\cdot 8}$

## MECHANICS PRACTICE

**Musing 9.2** Evaluate the following expressions in the order indicated via the parentheses and the multiplications.

a) $3+2\cdot 11$
b) $4\cdot 3+3\cdot 5+2$
c) $6\left(2+3\right)$
d) $\left(1+1\right)\cdot 3\cdot 8$
e) $2+\left(6+4\left(1+5\right)\left(3+2\right)\cdot 7+6+3\cdot 3\right)+10\cdot 4$

## 10. A String of Sums; A String of Products

We argued in section 6 that

> *In any string of sums of counting numbers* $a+b+c+d+e+...+y+z$ *it does not matter in which order one chooses to perform the additions. The same answer will always result.*

This means that it does not matter how you might choose to place parentheses in a string of sums such as

$$2+3+4+5+6,$$

the final result will always be the same.

We have that $\left(2+\left(\left(3+4\right)+5\right)\right)+6$ gives the same answer as $2+\left(\left(3+4\right)+\left(5+6\right)\right)$, which gives the same answer as $2+\left(\left(\left(3+4\right)+5\right)+6\right)$, and so on. For this reason:

> ***People never bother to put parentheses in a string of sums.***

We also wondered in section 7 if the following is true.

> *In any string of products of counting numbers* $a\times b\times c\times d\times e\times ...\times y\times z$ *it does not matter in which order one chooses to do the individual products. The same answer will always result.*

Often people just assume this is true as well. In which case, there is no need for grouping terms with parentheses.

> ***People never bother to put parentheses in a string of products.***

For example, in computing $2\times 3\times 4\times 5\times 6$ it does not matter in which order one chooses to conduct the products. There is no need for parentheses.

Back in section 7 we drew pictures to justify why "order does not matter" when computing a product of two or three numbers: A two-dimensional picture explains why, philosophically, $4 \times 5$ and $5 \times 4$ have the same answer, and a three-dimensional picture explains why $(2 \times 3) \times 4$ and $2 \times (3 \times 4)$ must be the same too.

We have that

$(a \times b) \times c = a \times (b \times c)$ for any three of counting numbers $a$, $b$, and $c$.

We were worried back in the section we would have to somehow draw four-dimensional pictures to justify why "order does not matter" when computing a product of four numbers.

Let's attend to this now without going to the fourth dimension!

**WARNING:** The remainder of this section is optional reading and is not for the faint-hearted!

Feel free to just accept that it is possible to justify our intuition that "order doesn't matter" for any string of numbers multiplied together, no matter the how many numbers are in that string.

There are five ways to group a string of four numbers multiplied together.

$$\left(\left(a\times b\right)\times c\right)\times d$$
$$\left(a\times\left(b\times c\right)\right)\times d$$
$$a\times\left(b\times\left(c\times d\right)\right)$$
$$a\times\left(\left(b\times c\right)\times d\right)$$
$$\left(a\times b\right)\times\left(c\times d\right)$$

Our job is to explain why all five presentations of $a \times b \times c \times d$ must have the same value.
Here goes!

The first and second items in the list give the same answer because we have that $\left(a\times b\right)\times c = a\times\left(b\times c\right)$. (This is what we know about a product of three numbers.)

The third and fourth items in the list give the same answer because we have that $\left(b\times c\right)\times d = b\times\left(c\times d\right)$. (Again, what we know about a product of three numbers.)

The second and fourth items in the list give the same answer because we have that $\left(a\times M\right)\times d = a\times\left(M\times d\right)$ where $M$ just happens to be $(b \times c)$. (Sneaky!)

So, this means the first four items on the list are sure to give the same answer.

The third and fifth items give the same answer too, as $a\times(b\times W)=(a\times b)\times W$ where $W$ just happens to be $(c \times d)$.

This means that all five possibilities do indeed give the same answer!

Our belief about products of three terms led us to believe the same for products of four terms.

In the same way, one can show that all the ways of interpreting a product of five terms must give the same answer (by seeing all the possible products as examples of what we just showed is true about products of four terms), as do all the ways to interpret a product of six terms (by seeing what all the possible products as instances of what we will have just showed true about products of five terms), and one of seven terms, and so on.

## MUSINGS

**Musing 10.1** Show that there are $\mathbf{14}$ ways to compute $a\times b\times c\times d\times e$ so that one is conducting only a product of two terms at a time.

(Actually, don't bother with this question. It is really a repeat of Musing 8.1. Do you see why? Also, this section showed that all the ways to compute this product lead to the same answer, so who cares about parentheses anyway in this context?)

## 11. Chopping up Rectangles

Let's revisit an idea from section 4.

Here's a depiction of $4 \times 5$.

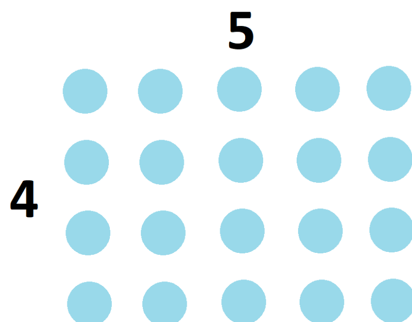



*Figure* $4 \times 5$ again

If we chop up the rectangle, we see we could also interpret the figure as $4 \times 3 + 4 \times 2$, or as $3 \times 3 + 1 \times 3 + 3 \times 2 + 1 \times 2$, or as many other combinations of products.



*Figure* $4 \times 3 + 4 \times 2$

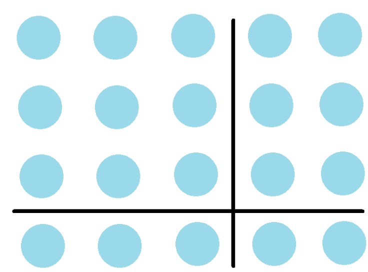


*Figure* $3 \times 3 + 1 \times 3 + 3 \times 2 + 1 \times 2$

**Notice**: We're making use of the hidden parentheses/vinculi that come with multiplication signs.

Each sum of products is, of course, $20$.

$$4 \times 3 + 4 \times 2 = 12 + 8 = 20$$

$$3 \times 3 + 1 \times 3 + 3 \times 2 + 1 \times 2 = 9 + 3 + 6 + 2 = 20$$

Consider this next figure.

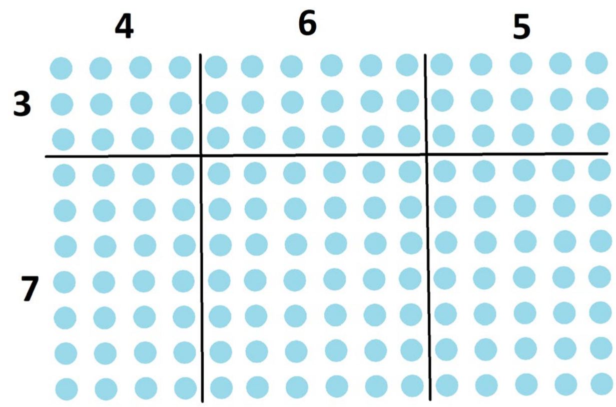



*Figure:* A lot of dots

It can be interpreted as

$$(3+7)\times(4+6+5)$$

(which is $10\times15$, showing that there are $150$ dots in this picture), or as sum of six products

$$3\times4+3\times6+3\times5+7\times4+7\times6+7\times5$$

(which corresponds to $12+18+15+28+42+53$, adding to $150$).

Rather than keep drawing rectangles of dots, let's just draw rectangles, viewing each rectangular region is an array of dots. In this figure we imagine there are $3\times4=12$ dots in the top left region, and $7\times6=42$ dots in the middle bottom region, and so on.

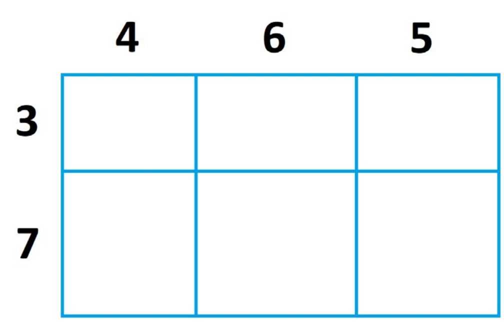



*Figure* $(3+7)\times(4+6+5)$

As mentioned in section 4, people call these rectangle pictures examples of the **area model** of multiplication. If we replace each dot with s unit square tiles, a $4$-by-$5$ array has area $4\times 5=20$ square units.

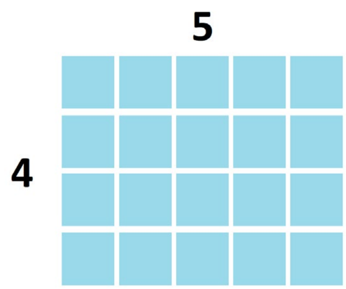



*Figure* $4\times 5$ as a region of area $20$

The previous picture shows a $10$-by-$15$ array, of area 150 square units, divided into six pieces, one of area $3\times 4=12$ square units, and one of area $7\times 6=42$ square units, and so on. Whether we count dots, or count square units and imagine areas, our arithmetic is the same.

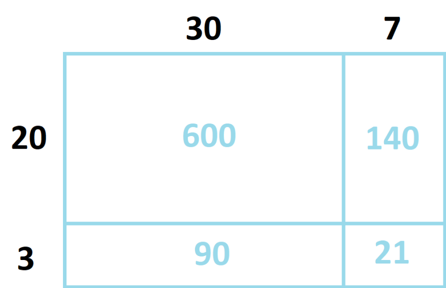
This visual representation of multiplication help us with multi-digit multiplication.

**Example:** Compute $23\times 37$.

<u>Answer 1</u>: Ask Alexa or Siri.

<u>Answer 2</u>: The question is asking "How many dots are there in a 23-by-37 array of dots?" or, equivalently, "What is the area of a 23-by-37 rectangle?"

The numbers are awkward. But let's simplify matters by chopping up the rectangle into regions that involve friendlier numbers. Let's think of $23$ as $20+3$ and $37$ as $30+7$. This divides the rectangle into four regions whose areas are easier to compute.




We see that the area of the rectangle (or, the total number of dots in the array if you are thinking dots) is $600+140+90+21=600+(130+100)+21=851$ square units.

That is, $23\times 37=851$.

(I could almost do this computation in my head by visualizing the rectangle.)

**Example:** Compute $371 \times 42$.

Answer 1: Use your smartphone.

Answer 2: This picture does the trick.


$$371 \times 42 = 12000 + 2800 + 40 + 600 + 140 + 2 = 15582$$

It doesn't matter that our rectangles are not drawn to scale. We just need to make sure the information presented on each diagram is correct.

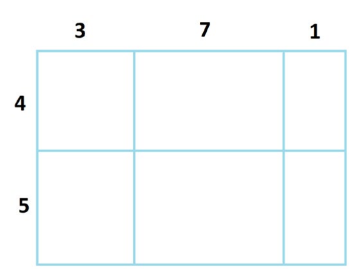
**Question:** Does it matter if the first number mentioned in the product is written along the rows or along the columns of the rectangle? (Were you expecting a slightly different picture for $371 \times 42$?)

**Example:** Compute $(4+5)(3+7+1)$.

Answer 1: This is just $9 \times 11 = 99$.

Answer 2: In terms of chopping up rectangles, we also see the answer 99—with a lot more work along the way! We are summing the areas of six individual pieces to get there.

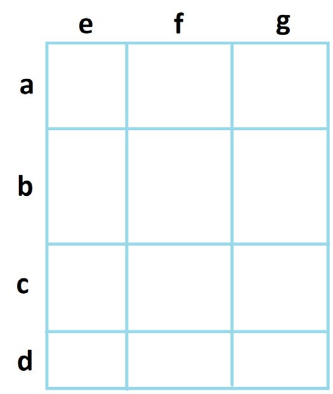
$$\begin{aligned}(4+5)(3+7+1) &= 4\times3+5\times3+4\times7+5\times7+4\times1+5\times1 \\ &= 12+15+28+35+4+5 \\ &= 99\end{aligned}.$$



Let’s make matters a tad abstract

**Example:** If $a$, $b$, $c$, ..., $g$ are numbers, what is $(a+b+c+d)(e+f+g)$?

Answer**:** Geometrically, it is a rectangle chopped into twelve pieces.




Those pieces are:

$$(a+b+c+d)(e+f+g) = ae + af + ag + be + bf + bg + ce + cf + cg + de + df + dg\,.$$

Examine the sum of twelve terms we see in the previous example. Each term in the sum matches a piece of the rectangle and it comes from multiplying one number displayed along the left side of the diagram with one number displayed along the top.

That is, to “expand” $(a+b+c+d)(e+f+g)$, we must select one term from the first set of parentheses, one term from the second, multiply them together, and add the results. We need to make sure to attend to all possible combinations.

$$(a+b+c+d)(e+f+g)$$
$$= ae + af + ag + be + bf + bg + ce + cf + cg + de + df + dg$$

*Figure* The mechanics of expanding brackets

As there are 4 ways to choose an entry in $(a+b+c+d)$ (task 1) and $3$ ways to choose an entry in $(e+f+g)$ (task 2), there are indeed a total of $4 \times 3 = 12$ products to write in the sum.

**Question:** Computing $(7+3)(4+5))$ corresponding dividing a rectangle into four pieces. One piece is $7 \times 4$, another is $3 \times 4$, and so on. Can you see this? (Draw a picture.)

Can you also see that the products $7 \times 3$ and $4 \times 4$ do **not** correspond to valid pieces?

**Question:** Computing $100 \times 100$ as $(50+20+20+7+3)(20+20+20+20+19+1)$ corresponds to dividing a rectangle (a square, actually) into $5 \times 6 = 30$ pieces.

There are four pieces of size $7 \times 20$.
There is just one piece of size $50 \times 19$.

How many pieces of size $20 \times 20$ are there?

Let's have some fun.

What does $(2+3)(4+5)(6+7)$ correspond to geometrically?

This is just the product $5 \times 9 \times 13 = 585$ written in a complicated way, but what is the picture to go with this product? Are we chopping up rectangle?

Think about this before turning the page.

We have that $(2+3)(4+5)(6+7)$ corresponds to chopping a three-dimensional rectangular box into eight pieces.

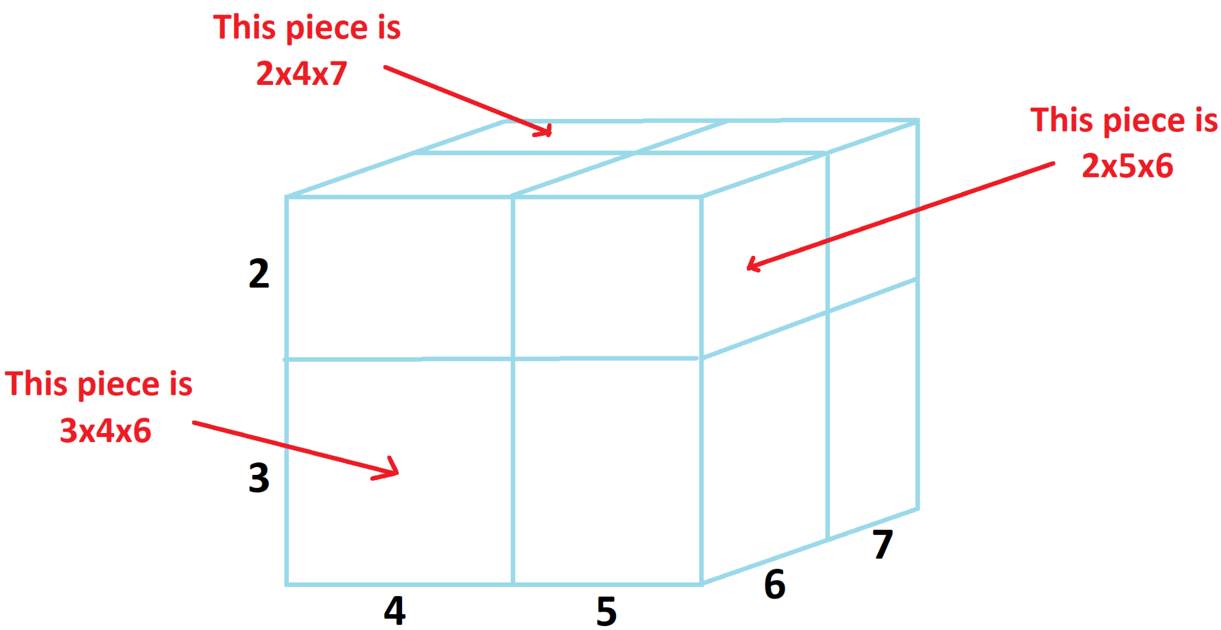



*Figure* $(2+3)(4+5)(6+7)$

Here are the eight pieces.

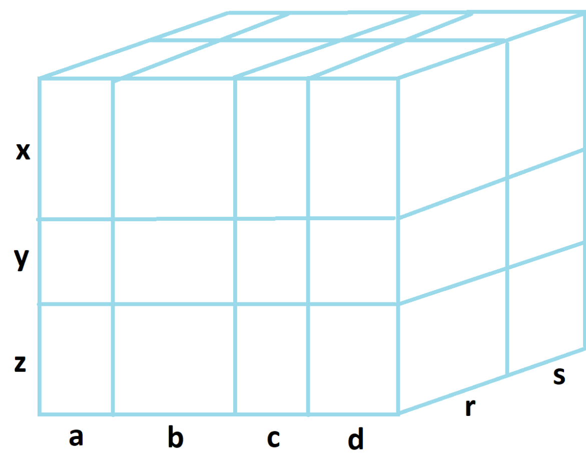
$$\begin{aligned}(2+3)(4+5)(6+7) &= 2\times4\times6+2\times4\times7+2\times5\times6+2\times5\times7\\ &\quad+3\times4\times6+3\times4\times7+3\times5\times6+3\times5\times7\end{aligned}$$

And this sum does indeed add to 585 if you have the patience to check.

$$5\cdot9\cdot13 = 48+56+60+70+72+84+90+105 = 585$$

Again, we are selecting one term from each set of parentheses, multiplying them together, and adding the results, making sure to attend to all possible combinations.

**Example:** Expanding $(x+y+z)(a+b+c+d)(r+s)$ corresponds to chopping a three-dimensional box into $3\times4\times2=24$ pieces. One of the pieces is $xar$ and another is $ycs$, and so on.




**Example:** Imagine expanding $(x+y)(x+a+b)(a+c+p)$.

a) How many pieces would there be?
b) Would $xac$ be one of those pieces? How about $cay$ ? $xcp$ ? $xax$ ? $xyc$ ?

Answers:

a) There would be 18 terms in the final sum.
b) Yes, $xac$ would appear.

$cay$ appears as $yac$.

$xcp$ does not appear.

$xax$ appears as $xxa$

$xyc$ appears as $yxc$.

**Example:** If you were to expand

$$(a+b+c+d+e)(w+x)(a+b+x+t+r)(e+f)$$

how many terms would there be? (Is there a geometric interpretation for this scenario?)

Answer: It seems we're in four-dimensions now! But we do suspect the same arithmetic will be at play: There will be $5\times2\times5\times2=100$ terms and the resulting expansion would look like $awae+bwae+\cdots$.

If we trust that the mechanics of two-dimensional rectangles and three-dimensional rectangular boxes continues to hold in all situations (even if I don't know that the fourth dimension is) it seems we have another natural belief about arithmetic.

> To compute the product of sums of terms, select one term from each set of parentheses, multiply them, and then sum all the results. Make sure to attend to all combinations.
>
> For example,
>
> $$(a+b+c)(x+y+w+z)=ax+bx+cx+ay+by+\cdots$$
>
> $$(x+2)(a+b)=xa+xb+2a+2b$$
>
> $$(x+y)(p+q)=xp+yp+xq+yq$$

Many textbooks focus on just one example of "chopping up a rectangle."

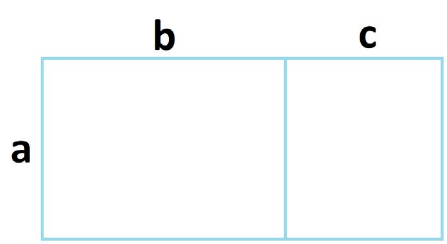



*Figure* $a(b+c)=ab+ac$

If we take the expression $a(b+c)$ and place parentheses around the single term to write

$$(a)(b+c)$$

then the process of "selecting one term from each set of parentheses" has us selecting the term $a$ from the first set of parentheses each and every time and so the sum $ab+ac$ results, just as the diagram suggests.

**Question:** Draw a picture of $(a+b)c=ac+bc$.

### Have you noticed …?

When a number is multiplied by itself, we say that the number is squared. (This came up in Musing 4.1.)

We use a superscript two to denote this. For example,

$5^2 = 5 \times 5 =$ "five squared."

This choice of wording makes sense as the area of a five-by-five square is computed as $5 \times 5$.

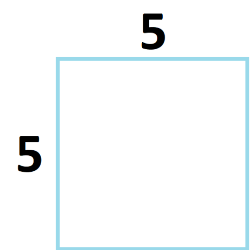



*Figure* $5^2 = 5 \times 5$

We use the superscript three to denote a three-fold multiplication of the same number. We call that the number cubed.

$5^3 = 5 \times 5 \times 5 =$ "five cubed."

This choice of wording makes sense as the volume of a five-by-five-by-five cube is computed as $5 \times 5 \times 5$.

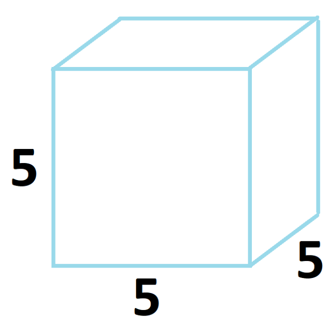



*Figure* $5^3 = 5 \times 5 \times 5$

There is a reason why we humans don't have special words for $5^4 = 5 \times 5 \times 5 \times 5$ and $5^5 = 5 \times 5 \times 5 \times 5 \times 5$, and so on. We just can't envision matters beyond the third dimension!

## MECHANICS PRACTICE

**Musing 11.1** One can compute $(2+3)(7+4)$ two ways:

Short way: $(2+3)(7+4)=5\cdot 11=55$

Long way: $(2+3)(7+4)=2\cdot 7+2\cdot 4+3\cdot 7+3\cdot 4=28+8+21+12=55$

Compute each of the following both the short way and the long way.

a) $(3+4)(5+1)$

b) $(2+3+5)(2+8)(1+9)$

**Musing 11.2**

a) Expand $(a+x+b)(x+y)$.

b) If one were to expand $(x+y+z+w+t+r)(a+b+c+d+e+f+g+h)$ how many terms would there be in the resulting sum?

**Musing 11.3**

I can compute $13\times 26$ by imagining drawing (or actually drawing, that is okay too!) a rectangle that looks at this product as $(10+3)(20+6)$. I can then see that the answer is

$$200+60+60+18=320+18=338.$$

Compute each of the following products the same way.
Use technology to check that your answers are correct.

a) $23\times 14$

b) $106\times 21$

c) $213\times 31$

**Musing 11.4**

If one were to expand $(p+q+2)(p+2)(a+q+p+3)(x+q+3)$, which of the following would be a term you would see in the sum of 72 terms that result?

a) $3p^3$ b) $6p^2$ c) $3q^3$ d) $18$ e) $xq^2p$

# 12. Fun with Long Multiplication

We saw that a natural way to compute a multi-digit multiplication problem such as $23 \times 12$ is to use the area model.

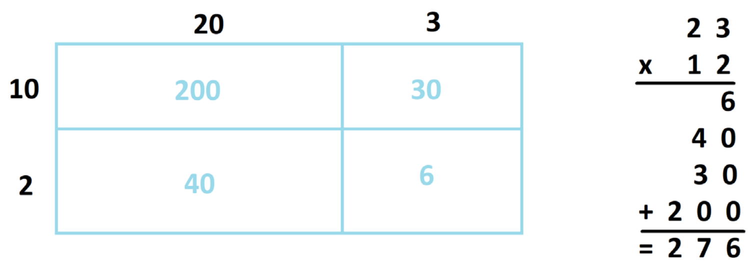



*Figure* $23 \times 12 = (20+3)(10+2)$

Some school students today are taught to compute long multiplication via a method of "partial products," which is just the area model in disguise. (It would be easier if students were shown and then allowed to draw chopped up rectangles.)

> **Question**: Do you see how the computation on the right in the figure above is indeed just the area model? (How are students taught this approach without drawing a rectangle? Hmm.)

During the 1500s and the centuries that followed, paper and ink were precious. The "partial product" algorithm was compactified to save ink. It is the algorithm most students are still taught, even though ink is no longer precious. (In fact, asking Siri for the answer uses no ink whatsoever!)

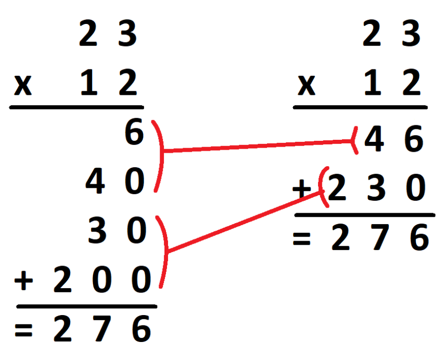



*Figure* $23 \times 12$ the traditional way.

> **Question:** Were you taught to compute $23 \times 12$, say, this compactified way?

**Question:** In section 3 we looked at $173 \times 985$ versus $985 \times 173$.

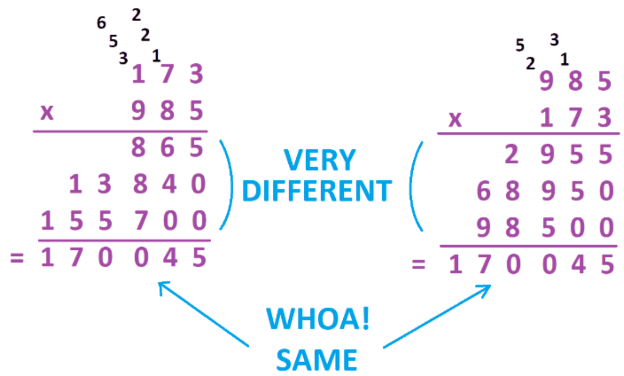



*Figure* $173 \times 985$ versus $985 \times 173$

Can you see now that if we were to compute each of these products via the "partial products" method (that is, via the area model), that there is actually nothing mysterious going on here?

One nice thing about drawing rectangles when computing long multiplications is that the units, tens, hundreds, and so on, all line up nicely along diagonals. (It helps to write numbers on the right rather than the left to see this.)

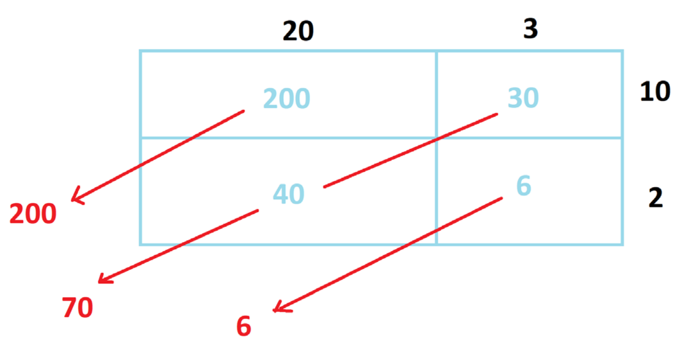



*Figure* $23 \times 12 = (20+3)(10+2) = 200 + 70 + 6$

One can have fun with this as the two musings for this section show.

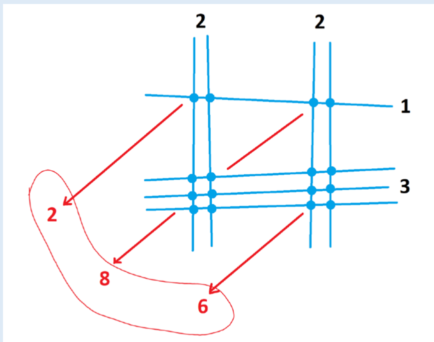
**Warmup:** Draw an area model picture for $218 \times 43$ and add up the diagonals of the picture. Do you see the answer of $9374$ appearing?

## MUSINGS

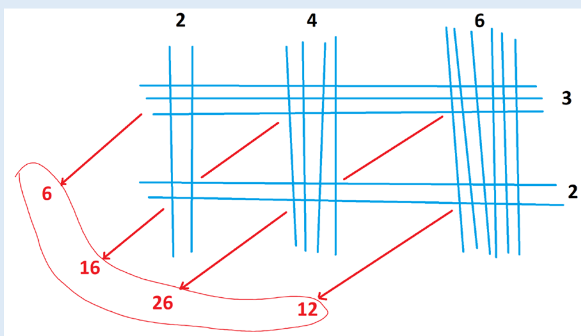
**Musing 12.1** Here's a mighty strange way to conduct long multiplications.

To compute $22 \times 13$, for instance, start by drawing two sets of vertical lines, the left set containing 2 lines and the right set also containing two lines. (These match the digits of the number $22$.) Also draw two sets of horizontal lines, the upper set with just 1 line and the lower set 3 lines (to match the digits of the number $13$).

There are four clusters of intersection points. Count the number of intersection points in each cluster and add the counts diagonally as shown. The answer $286$ appears.




There is one caveat as illustrated by the computation of $246 \times 32$.



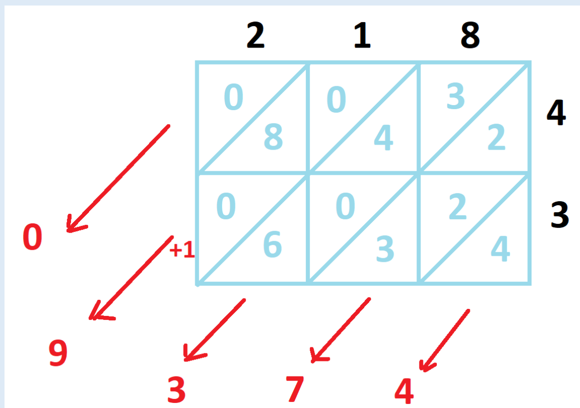

The answer of 6 thousands, 16 hundreds, 26 tens, and 12 ones, that is, $6000 + 1600 + 260 + 12 = 7872$, appears. One might have to "carry" some digits to read off the answer.

a) Compute $131 \times 222$ via this method.
b) Compute $54 \times 1332$ via this method.
c) How best should one compute $102 \times 3001$ (which equals $306102$) via this method?
d) Why does this line method work?

**Musing 12.2** During the 1500s in England, students were taught to compute long multiplication following the *galley method*, also called the *lattice method*. (Today, it is also referred to as the *Elizabethan method*.)

To compute $218 \times 43$, say, draw a two-by-three grid of squares. Write the digits of the first number above the columns of the grid and the digits of the second number to the right of the rows as shown.

Divide each cell of the grid with a diagonal line and write the product of the column digit and the row digit of each cell as a two-digit answer in that cell, but with its two digits split across the diagonal of that cell. (If the product is just a one-digit answer, write a $0$ for the first digit of the "two-digit answer.")



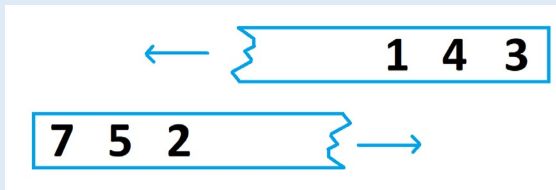

Add the entries in each diagonal, "carrying" any digits over to the next diagonal, if necessary, and read off the final answer. (Okay, this uses a lot of ink!)

In our example, we get $0 \mid 8 \mid 13 \mid 7 \mid 4$, but a carry of a "1" makes this $09374$. We see that $218 \times 43 = 9374$.

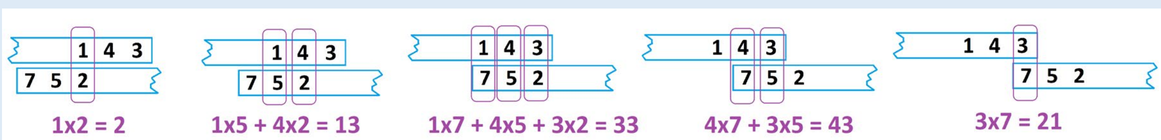
a) Compute $5763 \times 345$ via the galley method to get the answer $1988235$.
b) Explain why the galley method is really the area model in disguise. (What is the specific function of the diagonal lines?)

**Musing 12.3 (DEFINITELY OPTIONAL!)**
Here's a mighty unusual way to compute long multiplication. Let's illustrate it by computing $341 \times 752$ to obtain the answer $256\,432$.

Start by writing each number on a strip of paper, but write the first number backwards (and the second number forwards).

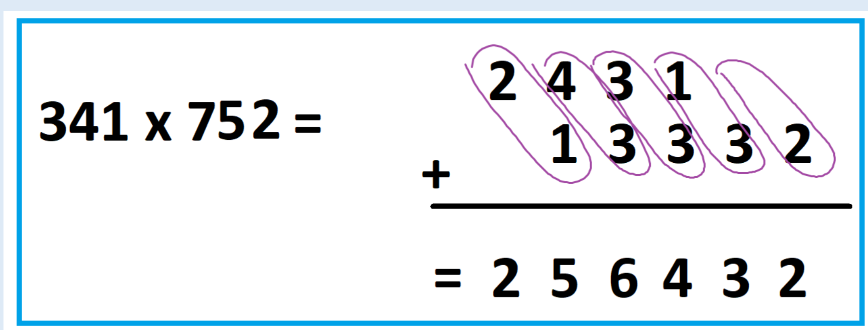



Starting with the first strip to the right and the second strip to the left, slide the first strip leftwards and the second strip rightwards until two digits align vertically. Record their product.




Keep sliding the strips recording the sums of the products the digits that align as you go along.

Write the answers you obtained in diagonally, as shown, and sum the columns. The answer $256432$ appears!

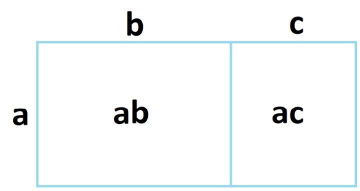



a) Compute $231 \times 121$ via the traditional school method and then again with this paper-strip method. Do you see similarity between the two methods?
b) Calculate $341 \times 752$ via the traditional school method. Do you still see similarity that approach and the paper-strip approach?
c) Can you explain, in general, why the paper-strip approach works?

## 13. Some Factoring

We’ve seen how to chop up rectangles to play with multiplication problems in clever ways.

School books tend to focus on just one type of rectangle-chopping, namely, rewriting $a(b+c)$ as $ab+ac$.




And they also focus on applying this arithmetic fact backwards, namely, to recognize a quantity of the form $ab+ac$ as $a(b+c)$.

People call this backward act **factoring**. (Folk of British descent call it **factorising**.)

For example, for $6+14$, we can “factor out a 2” by recognizing this quantity as $2\times 3+2\times 7$ , and so rewrite it as $2(3+7)$. (Why one would want to do this is not at all clear!)

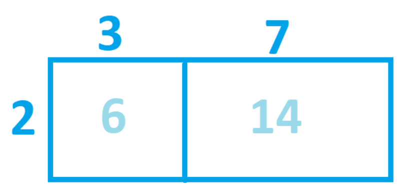



*Figure* $6+14=2(3+7)$

In the same way we can see

$$3a+6b=3(a+2b)$$

by “pulling out a common factor of 3,” and that

$$10p+5pq=5p(2+q)$$

by “pulling out a common factor of $5p$.”

## Division

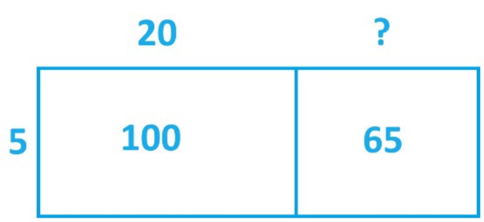
Some elementary school curricula have students compute division problems essentially this way.

Students are often told that division is “multiplication backwards,” so it seems plausible that playing with the area model backwards will help. And it does!

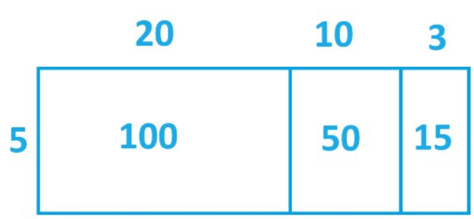
Consider problem $165 \div 5$. Here we are being told that two numbers multiple together to give the answer $165$ and that one on the numbers is $5$. Our challenge is to find the second number.

Here's a picture of the situation.


Students are advised to build up the total area of the rectangle with multiples of five they know. For example, we can get to 100 units of area by using $5 \times 20 = 100$. That leaves us with 65 units to contend with.



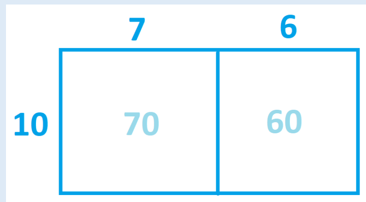

We can now use $5 \times 10 = 50$ to leave $15$ units of area, which, we recognize as $3 \times 5$. (Great!)




We see $165 \div 5 = 33$.

## MUSINGS

**Musing 13.1** If desired, one could factor $70+60$ by writing it as $10(7+6)$.




Factor each of the following expressions in a similar way. (Drawing rectangles helps.)

a) $50+60+90$
b) $33+99$
c) $5x+5y$
d) $awq + pbw$
e) $z^2+3z+xz$

**Musing 13.2** The previous question 'factored out a 10' from $70 + 60$ to write it as $10(7 + 6)$.

a) What expression would you write if you "factored out a 1" instead?
b) What expression would you obtain if you factored out a $\frac{1}{2}$?

## MECHANICS PRACTICE

**Musing 13.3** Compute each of these division problems via the area method. Feel free to check your answers with a calculator.

a) $1491 \div 7$
b) $555 \div 15$
c) $516 \div 4$
d) $299 \div 13$
e) $2001 \div 23$

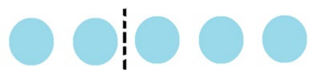
**Musing 13.4** Does the area model for division show remainders? Can you see that $875 \div 6$ is $145$ with a remainder of $5$? Try it!

# 14. Summarizing the Rules of Arithmetic

Our mathematical journey has begun with a study of the counting numbers and their arithmetic properties.

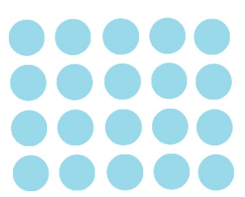
We discovered an operation on the counting numbers called **addition** that creates from any two counting numbers $a$ and $b$ a new number which we write as $a + b$.

We (did this by drawing two counts of dots in a row, left and right, and then recounting the entire row.



We also created an operation called **multiplication** that produces from any two counting numbers $a$ and $b$ a new number which we write as $a \times b$.

We drew rectangular arrays of dot and thought of such an array as an organized picture of repeated groups. (We thus thought of multiplication as repeated addition.)



By viewing our pictures in different ways, we discovered various "rules" of arithmetic that seem natural and right for the counting numbers.

Here's a list of all the rules together in one spot.

## Addition

**Rule 1:** We can change the order in which we add any two counting numbers and not change the final result.
That is, for any two counting numbers a and b we have that $a + b = b + a$.

**Rule 2:** Adding zero to a counting number does not change the value of the counting number.
That is, for any counting number $a$ we have that $a + 0 = a$ and $0 + a = a$.

**Rule 3:** In any string of counting numbers added together

$$a + b + c + d + e + ... + y + z$$

it does not matter in which order one chooses to perform the additions. The same answer will always result.

(Rule 3 is encompasses Rule 1.)

## Multiplication

**Rule 4:** We can change the order in which we multiply any two counting numbers and not change the final result.
That is, for any two counting numbers $a$ and $b$ we have $ab = ba$.

**Rule 5:** Multiplying a counting number by one does not change the value of the counting number.
That is, for each counting number $a$ we have that $1 \times \text{a} = a$ and $\text{a} \times 1 = a$.

**Rule 6:** Multiplying a counting number by zero gives a result of zero.
That is, for each counting number $a$ we have that $0 \times a = 0$ and $a \times 0 = 0$.

(We had to fuss a little bit to make full sense of this Rule 6.)

**Rule 7:** In any string of counting numbers multiplied together

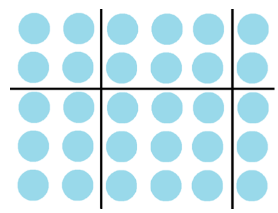
$$a \cdot b \cdot c \cdot d \cdot \cdots \cdot y \cdot z$$

it does not matter in which order one chooses to perform the products. The same answer will always result.

(Rule 7 encompasses Rule 4, and we had to go to quite some fuss to properly explain it.)

## Addition and Multiplication Together

**Rule 8: "**We can chop up rectangles from multiplication and add up the pieces."



These eight rules give the entire the scoop on how basic arithmetic works!

**Question:** Can you remember the gist of how we got to each rule? (For example, rule 1 comes from reading a row of dots from left to right versus from right to left.)

Of course, feel free to look back at the earlier sections if you want o remind yourself.

## FORMAL JARGON

Some school curricula insist that students know and use some formal language to describe each of these eight principles. If you are interested, here it is. (If you are not, skip this page!)

Rule 1 says that **addition is commutative**.
(What is the etymology of the strange word "commutative"? Google it!)

Rule 2 says that $0$ is acting as an **additive identity**.

The aspect of Rule 3 that refers changing the order you conduct additions is described as "**addition is associative**." Mathematics books usually focus just on "$(a+b)+c=a+(b+c)$."

Rule 4 says that **multiplication is commutative**.

Rule 5 says that $1$ is acting as a **multiplicative identity**.

Rule 6 doesn't have an official name in the mathematics community. Some school textbook authors call it **the zero property**.

The aspect of Rule 7 that refers changing the order you conduct multiplications is described as "**multiplication is associative.**" Mathematics books usually focus just on "$(ab)c=a(bc)$."

Rule 8 is called the **distributive property**. Mathematics books usually focus just on "$a(b+c)=ab+ac$."

## Solutions

Coming!