Chapter 3

The Integers

Chapter 3

20. New Numbers: The Opposites of the Counting Numbers

In chapter 1, we made sense of addition and multiplication, but skipped over subtraction. Why? Because I don't believe subtraction exists! To me, subtraction is just addition again—but it is the addition of a new type of number.

Let me explain what I mean by telling a story that is blatantly not true.

> When I was a young child, I spent my entire days playing in a sandbox at the back of my yard. (Not true.)
>
> And being a very quiet and contemplative child (not true), I would start each day leveling out the sand in my box to make a perfectly flat horizontal surface. This appealed to my tranquil sensibilities and calmed my mind. I even gave this special level state a name. I called it **zero**, **0**.

0

> And I spent many an hour admiring my zero state. (Still not true.)
>
> But then, one day, I had a flash of insight. I realized I could reach behind where I was sitting, grab a handful of sand, and make a pile. And I called the one pile, **1**.

1

> And then I realized I could do grab even more sand and make two piles, which I called **2**, and three piles, **3**, and so forth.

2 **3** **4** **etc.**

> I had even more hours of mathematical fun creating and admiring more and more piles of sand.
>
> I had discovered the **counting numbers**.

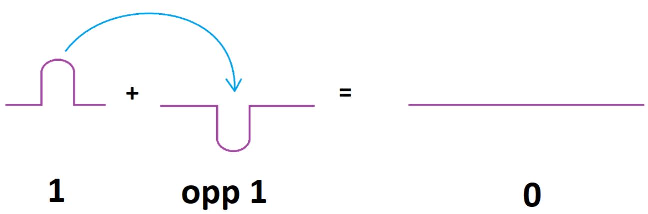
And, by lining up piles, I also discovered **addition** with the counting numbers.

For example, I saw that two piles plus three piles equals five using piles.

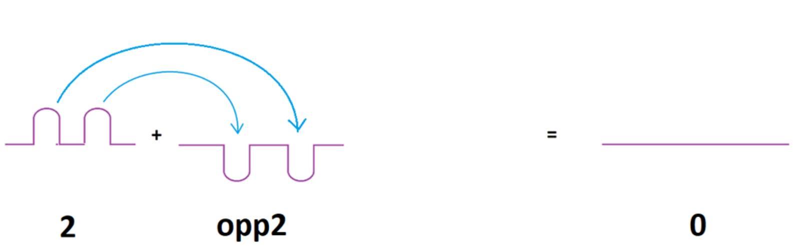
Question: Can you imagine a picture of $7 + 9 + 13 + 26$?

But then, one day I had the most astounding flash of insight of all! Instead of using a handful of sand to make a pile, I realized I *take away* a handful of sand and make the opposite of a pile, namely, a hole!

I wrote **opp 1** for a hole because I realized that one hole truly is the opposite of 1 pile: a pile and a hole together cancel each other out to return to the zero state. Whoa!

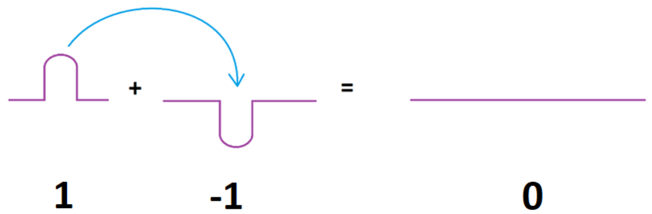


I also wrote **opp 2** for the two holes, **opp 3** for three holes, and so on.



Question: Can you see in your mind's eye that $7 + \mathbf{opp}\ 7$ is zero? That is, can you see how the sand of seven piles perfectly fills seven holes?
(I guess, in this untrue story we are assuming that all piles and all holes are the same size.)

Later in school I was taught draw a little dash for “opposite” and write -1 for the opposite of a one. I was also taught to call this **negative one**. This felt strange and unnatural to me, but so be it.



I was also taught to write -2 for two holes and to call that **negative two**. And so on.

But even though I would say “negative 5” with my words, my brain was still secretly thinking the opposite of five, **opp 5**, and I was imagining five holes.

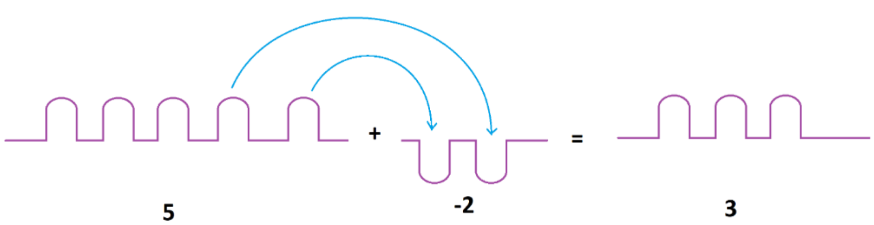
And strange matters went even further.

School also wanted me to do this thing called **subtraction**. But I never bought into it.

For example, my schoolmates would read

$$5 - 2$$

as **five take away two** and they would talk about removing two objects from a set of five objects. I, on the other hand, thought of this as $5 + \mathbf{opp}\ 2$, five piles and the addition of two holes. I could see that makes for three piles.



Of course, knew that my two holes, in effect, “took away” two piles just as my colleagues were doing, but I knew my supposedly unusual way of thinking had an advantage!

Question: Can you see, by drawing a picture of piles and holes, that **10 plus the opposite of 4** (which is, $10 + -4$) is the same as **10 take away 4** $(10 - 4)$?

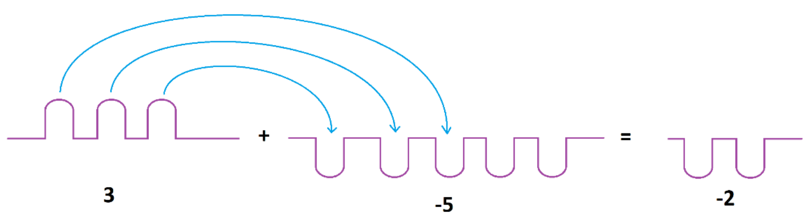
I asked my colleagues:

What's three take away five, $3 - 5$*?*

They all said:

That's impossible!
You can't take away five things if you only have three to begin with.
There is no answer to $3 - 5$.

But I saw that $3 - 5$ does have an answer. It's three piles (3) and five holes (-5), which makes two holes (-2).



Practice 20.1: Many people say that $2 - 6$ has no answer.

But think of this as $2 + -6$.

How many piles are there? How many holes are there? And when we combine them, what are we left with?

I was having fun doing arithmetic that my teacher told me I won't be learning until a few more years' time. (This part is true!)

This whimsical story makes an important point.
It shows that we can view any subtraction problem as an addition problem.

Subtraction is just the **addition** of the opposite.

For example, we have

$10 - 3$ is really $10 + \ -3$ (ten piles and three holes)

$26 - 6$ is really $26 + -6$ (twenty-six piles and six holes)

$100 - 40$ is really $100 + -40$ (one hundred piles and forty holes)

People tend not to write the opposite numbers first in an expression, but something like the following is actually fine and makes good sense.

$$-3 + 7 = 4$$

Three holes and seven piles combine to leave four piles.

In fact, it is very handy to rewrite traditional statements using the subtraction symbol as addition statements.

For example, writing $6 - 9 + 2 - 1$ as

$$6 + \ -9 + 2 + \ -1$$

allows us to see eight piles and ten holes which combine to make two holes: $6 - 9 + 2 - 1 = -2$.

Comment: I personally think it is odd and confusing that we use the same symbol "$-$" both for a verb (*subtract* or *take away*, as in "$8 - 5$") and for an adjective (*negative*, as in "-5").

I do write statements such as $6 + \ -9 \ + 2 + \ -1$, but I understand they look confusing. Many educators won't let students write such things probably because it is hard to decipher in and of itself, and messy handwriting won't help. Writing

$$6 + \mathbf{opp}\ 9 + 2 + \mathbf{opp}\ 1$$

is clear and fun, but it is cumbersome.

Question: What do you think of writing $\overline{6} + \underline{9} + \overline{2} + \underline{1}$ for six piles and nine holes and two piles and one hole? (No one writes this, but I am just wondering what you think.)

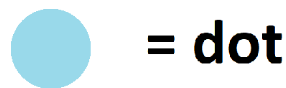
Practice 20.2:
a) A friend wrote $6 - 7 - 1$. How do you think I would rewrite this expression?
b) If I wrote $5 + -1 + 3 + -2 + -1$, what do you think my friends would write?
c) Can you imagine piles-and-holes pictures these two sums? What final value do they each give?

By the way, people call the little dash in front of a number a **negative sign**.

In many parts of the world, it is called a **minus sign**. But folk in the U.S. object to this language. They say that "minus" is a verb and so shouldn't be used as an adjective (but, as I mentioned, folk in the U.S. are still happy to use the one symbol "$-$" for both, nonetheless!)

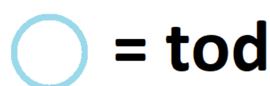
DOTS and ANTIDOTS

Piles and holes are well and good. But we began this book with a dot.



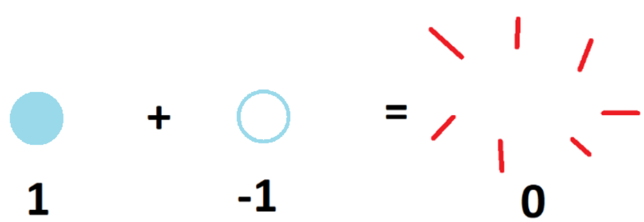
A dot perhaps looks like a pile viewed from above. So, what should we draw for a hole looked at from above? That is, what should we draw for the opposite of a dot?

> **True Story Moment:** When I ask this question to students in countries all over the world, kids always tell me the same one answer.
>
> *Draw an open circle for the opposite of a dot and call it a* ***tod****. That's the word dot, backwards.*



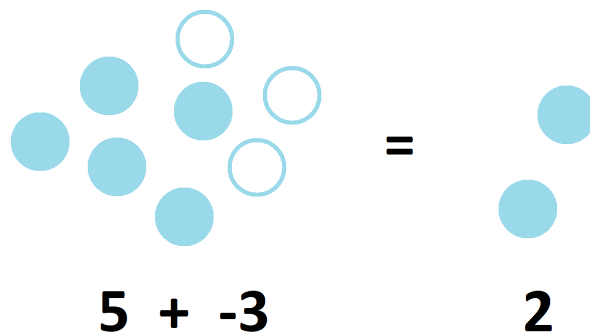
> Adults usually suggest drawing an open circle too but, less imaginatively, suggest calling it an **antidot**.
>
> I personally like the term *tod* very much, but *antidot* have the advantage of evoking a sense oppositeness, which is a help. I'll be adult and go with that name here.

Like a pile and a hole which annihilate one another when brought together, a dot and an antidot annihilate too – POOF! – when brought together and leave nothing behind.

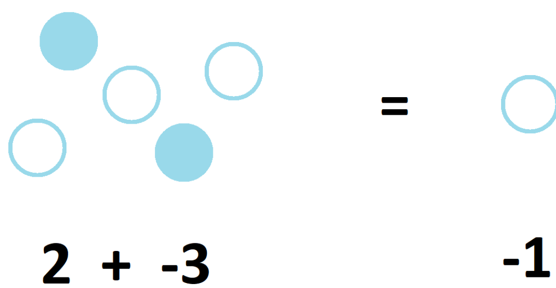


We can conduct arithmetic with dots and antidots, just like we did with piles and holes.

For example, in $5 - 3$ (“five plus the opposite of three”) there are three annihilations – POOF! POOF! POOF! – leaving behind two dots.



In $2 + \; -3$ there are two annihilations—POOF! POOF!—leaving behind one antidot.



Practice 20.3 Draw, or just imagine drawing, a dots-and-antidots picture of $10 - 20$ and find the value of this quantity.

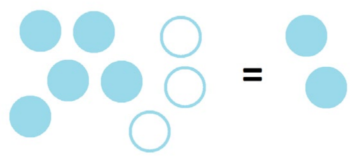
"Take Away" Again

Some people really do prefer the "take away" mindset over the "add the opposite" mindset. Here's a way to bridge both approaches—if you are interested.

Example: $5 - 3$.

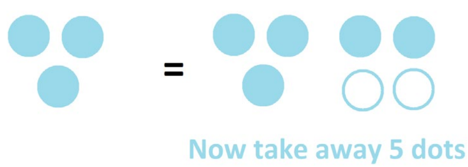
Take Away Approach:
Draw five dots and then take three away.

Adding the Opposite Approach:
Read the problem as $5 + -3$. Draw five dots and then three antidots and see the annihilations. (This, of course, has the effect of "taking three away.")



Example: $3 - 5$.

Take Away Approach:
Draw three dots. We don't have five dots to take away, so draw in an extra two dots and counteract that move by drawing two antidots to balance them out. Now take away five dots. That will leave -2.



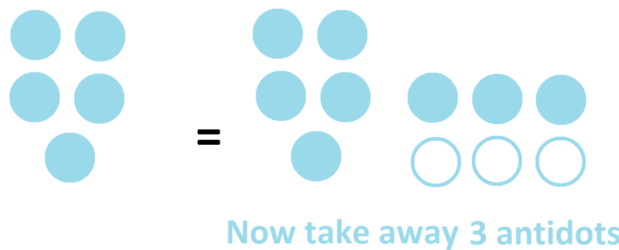
Adding the Opposite Approach:
Read the problem as $3 + -5$ and draw three dots and five antidots. Two antidots result.

Now let's get quirky.

Example: $5 - (-3)$.

Try making sense of this before turning the page.

Take Away Approach:
Think of this as "five dots take away three antidots." We don't have any antidots in a picture of five dots, so draw them in, three of them, and counteract that move by drawing three dots as well.



Now we can take away three antidots. That leaves behind 8 dots.

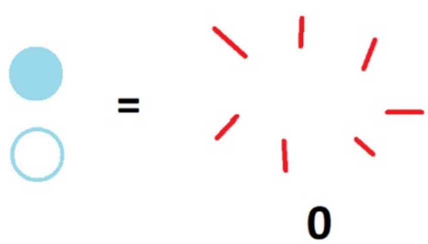
Adding the Opposite Approach:
Read the problem as $5 + -(-3)$, which is "five dots and the opposite of three antidots."

What's the opposite of three antidots? Well, that must be three actual dots!

So, we're being asked to add together five dots and three dots. The result is 8 dots.

Whoa!

This idea of drawing in pairs dots and antidots, with each pair technically being "nothing" and so not affecting the value of a problem, is sneaky and helpful! It creates items you can then take away if you are a "take away" kind of person.



MUSINGS

Musing 20.4 What do you think should be the value of -0?

Musing 20.5 What do you think should be the value of $---5$?
(Maybe think of this as $-\left(-\left(-5\right)\right)$.)

Musing 20.6 Multiple Choice:

A picture of $1-1+1-1+1-1+1-1+1$ would contain …

a) five dots and four antidots, which would combine to leave one dot (1).
b) four dots and five antidots, which would combine to leave one antidots (-1).
c) five dots and four antidots, which would combine to leave one antidots (-1).
d) four dots and five antidots, which would combine to leave one dot (1).
e) This is absurd! Who writes such a sum in the first place?

MECHANICS PRACTICE

Musing 20.7 Rewrite each expression involving subtraction as one involving only addition.
For example, $5-9-1$ can be rewritten as $5+\ -9\ \ +\ -1$.

a) $67-33-88+102-46$.
b) $-9-9$.
c) $-9-(-9)$.

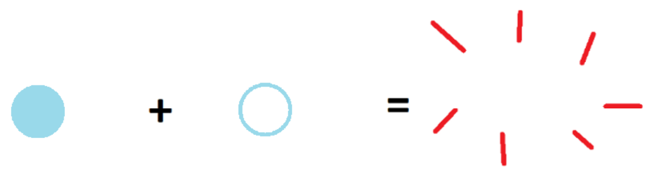
Musing 20.8 Find the value of each of these expressions.
(Practice envisioning dots and antidots, or piles and holes.)

a) $3-7$
b) $2000-1000+65$
c) $15-15$
d) $15-(-15)$
e) $-15-15$
f) $-15-(-15)$

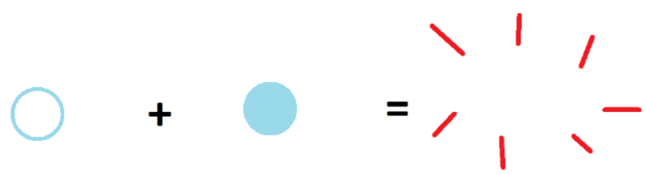
21. Distributing the Negative Sign

A dot (1) and an antidot (-1) annihilate one another.

An antidot is the opposite of a dot because it annihilates it.



And what is the opposite of an antidot? That is, what annihilates an antidot? A dot!



Since society uses a little dash for "opposite" this second statement is written as follows.

$$-(-1)=1$$

We can read this as saying: the opposite of the opposite of a dot is … a dot!

Okay then, what's this quantity?

$$-\,------1$$

(I hope it is okay for me to omit the parentheses. I don't think any confusion results.)

We have here the opposite of the opposite of the opposite of the opposite of the opposite of the opposite of the opposite of a dot. That's seven "opposites." What have we got in the end?

Well, toggling dot-antidot-dot-antidot- … seven times, starting with a dot, we will land us with an antidot.

$$-\,------1=-1$$

We can keep playing this toggle game, starting with any number of dots.

5 = five dots
-5 = five antidots
- - 5 = five dots
- - - 5 = five antidots
- - - - 5 = five dots
- - - - - 5 = five antidots

Let me now ask some trickier questions.

Example: What is the opposite of "three **dots** and two **dots**" altogether?

Answer: That would be three **antidots** and two **antidots**.

Actually, the tricky part is to translate what we just asked, and answered, into mathematics. Parentheses help us make clear what we're making opposite.

The opposite of a group of three dots and two dots is $-(3+2)$. And we computed this as three antidots, -3, and two antidots, -2. So, we have

$$-(3+2) = -3 + -2.$$

Of course, we can go the step further and say this is -5. But let's not worry about carrying the actual arithmetic for now.

Example: What is the opposite of "three dots and two antidots" altogether?
That is, what is $-(3+-2)$?

Answer: That would be three antidots and two dots.
We have $-(3+-2) = -3+2$.

Example: What is the opposite of "a dots and b antidots and 2 dots"?
That is, what is $-(\mathrm{a}+-\mathrm{b}+2)$?

Answer: That would be a antidots and b dots and 2 antidots.
Thus, the expression $-(a+-b+2)$ can be rewritten as $-a+b+-2$

Practice 21.1 What is

$$-(x + -y)$$

expressing in words? How can this expression thus be rewritten?

Next question:

Example: What is the opposite of $10 - T + 7 - 3 + b$?

Answer: Think of the quantity we're given as $10 + -T + 7 + -3 + b$. Then the opposite of this is

$$-(10 + -T + 7 + -3 + b) = -10 + T + -7 + 3 + -b.$$

Since the author expressed the question using subtractions, we can rewrite our answer in their preferred style.

$$-(10 - T + 7 - 3 + b) = -10 + T - 7 + 3 - b$$

People call the game we're playing **distributing the negative sign**. But that sounds too scary for what we are actually doing: we're just identifying the opposite of everything given to us. Nothing more!

This next example is typical of textbook exercises.

Example: Please make $2 - (20 - x)$ look friendlier.

Answer: This is $2 + -(20 + -x)$, which I read as

2 dots and the opposite of "20 dots and x antidots."

Unravelling that, we get

2 dots and 20 antidots and x dots.

This is $2 + -20 + x$.

We can make this friendlier still by doing arithmetic. We have 18 antidots and x dots, so this answer is $-18 + x$.

Since the world likes subtraction, let's write this as $x + \ -18$ and use the subtraction notation. This, finally, gives the answer the world likes best: $x - 18$.

Example: Kindly make $(x - y) - (x + y)$ look friendlier.

Answer: This as $(x + -y) + \ -(x + y)$ which we read as:

x dots and y antidots and the opposite of "x dots and y dots."

That gives

x dots and y antidots and x antidots and y antidots.

Within this collection are x dots and x antidots, which would all annihilate each other. So, that just leaves y antidots and y antidots. That's, $y + y$ antidots. But remember, repeated addition is multiplication, so we have here $2y$ antidots.

Answer: $-\left(2y\right)$

Oh no! Wait! Multiplications come with invisible parentheses. We can make the parentheses invisible.

Final answer: $-2y$.

Whoa!

This example brings up an important point:

Multiplications still come with invisible parentheses, even if there is a negative sign involved.

For example, the expression

-2×3

has hidden parentheses. It is $-(2 \times 3)$, which equals $-(6)$, and that is the opposite of six dots, which is six antidots, -6.

If n is the name of some number (maybe, it's 17 or maybe it's 3? I am not going to say), then $2n$ represents two times that number ($2 \times n$), and

$-2n$

is the opposite of two times that number. If we make the hidden parentheses visible, it's $-(2 \times n)$.

We'll learn soon that this technical fussiness doesn't actually matter: all the different ways you might correctly or incorrectly interpret -2×3 will likely lead you to the same final answer when evaluating it!

MUSINGS

Musing 21.2 What is $- - - - - - - - - - 3$?

Musing 21.3 What is $-x$ if

a) x is three dots?
b) x is three antidots?
c) x is 16 ?
d) x is -16?

MECHANICS PRACTICE

Practice 21.4
There are two ways to evaluate $3 - (3 - 7)$.
Either be evaluating the quantity inside the parentheses first

$$3 - (3 - 7) = 3 - (-4) = \ 3 + 4 = 7$$

or by distributing the negative sign.

$$3 - (3 - 7) = 3 + \ -3 \ + \ \ 7 = 0 + 7 = 7$$

Evaluate each of the following expressions two different ways (and get the same answer!)

a) $(15 - 2) - (13 - 2)$

b) $100 - (100 - 2)$

c) $-9003 - (2 + 1)$

Practice 21.5 In the following, x, y, and R each just represent some unspecified number.

a) Show that $x - (x - 2)$ is really just 2.

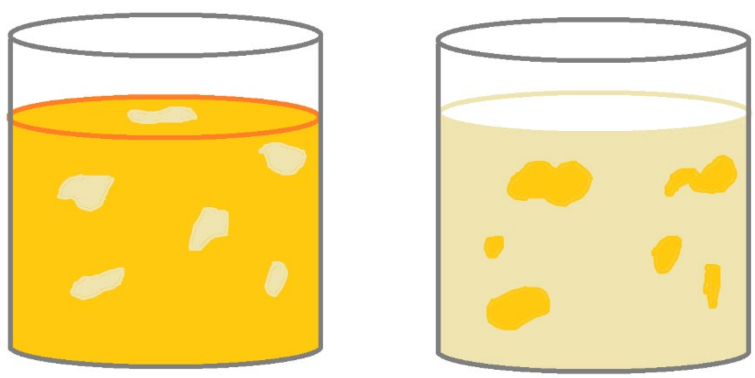
b) Show that $20 - (15 - y)$ is really $5 + y$.

c) Show that $\Big(1 - \big(1 - (1 - (1 - R))\big)\Big)$ is really just R.

22. Interlude: Milk and Soda

Here’s a classic puzzle.

> Penelope has a glass of soda and a glass of milk.
>
> She takes a tablespoon of soda from the soda glass and haphazardly stirs it into the milk. She then takes a tablespoon of the milk/soda mixture and transfers it to the soda. Both drinks are now “contaminated.”
>
> Here’s the question: *Which drink has more foreign substance?*
>
> Is there more foreign milk in the soda than foreign soda in the milk? Or is it the other way round? Or is it impossible to say as it depends on how much or how little mixing took place?



What are your thoughts on this matter?

We can explore this puzzler by modeling it with some playing cards.

Read on!

ACTIVITY

CARD PILE MYSTERY

a) Take 10 red cards and 10 black cards from a deck of cards. Shuffle your 20 cards and arbitrarily split them into two equal piles. Count the number of red cards in the left pile and the number of black cards in the right pile. What do you notice? Repeat this activity two more times.

b) Shuffle your 20 cards and this time split them into a pile of 6 cards and a pile of 14 cards. Count the number of red cards in the small pile and count the number of black cards in the large pile. Take the (positive) difference of those two counts. Did you get 4? Repeat this exercise two more times.

c) Shuffle the 20 cards again and this time split them into a pile of 9 cards and a pile of 11 cards. Count the number of red cards in the small pile, count the number of black cards in the large pile and take the (positive) difference of this count. What did you get? Repeat two more times. What do you notice?

d) Complete the following table.

Small Pile	Large Pile	Difference count of red in small versus count of black in large
10	10	0
9	11	
8	12	
7	13	
6	14	4
5	15	
4	16	
3	17	
2	18	
1	19	
0	20	

Any patterns?

e) Suppose in a game with 5 cards in the small pile and 15 cards in the large pile, I counted three red cards in the small pile. Complete the following table giving the number of cards of each type in the remaining three piles.

	Small Pile 5	Large Pile 15
# Reds	3	
# Blacks		

What is the difference of counts of red cards in the small pile and black cards in the large pile?

f) Let's go back to the first scenario with two piles of equal size, 10 cards per pile. Let's say the number of red cards in one pile is N. Complete the table. What do you notice about the number black cards in the other pile?

	Pile 10	Pile 10
# Reds	N	
# Blacks		

Explanation:

Let me go straight to parts a) and f).

There are ten red cards in total. If N of them are in one pile, the remaining red cards, $10 - N$ of them, must be in the other pile.

	Pile 10	Pile 10
# Reds	N	10 - N
# Blacks		

But that pile only has ten cards. If there are $10 - N$ red cards in that pile, the remaining $10 - (10 - N)$ of them must be black.

	Pile 10	Pile 10
# Reds	N	10 - N
# Blacks		10 - (10 - N)

So, we have $10 - (10 - N)$ black cards in that second pile. And what is this quantity?

$$\begin{aligned} 10 - (10 - N) &= 10 + -(10 + -N) \\ &= 10 + -10 + N \\ &= 0 + N \\ &= N \end{aligned}$$

Lo and behold! There are N black cards in that second pile, the same as the number of red cards in the first pile.

The number of red cards in one pile is always equal to the number of black cards in the second!

Okay. The algebra here provides proof that the number of red cards in one pile matches the number of black cards in the second pile—but the work doesn't feel intuitive and satisfying. Really, why should these two counts be the same?

An Intuitive Explanation:

Count the number of red cards in the first pile.

This tells you how many black cards you need to make the whole pile black.

And where are those missing black cards? They must be in the second pile!

The number of black cards in the second pile must match the number of red cards in the first pile.

Practice 22.1: Use algebra, or intuition, to explain what you observed in the remaining sections of the activity.

We can answer the milk and soda puzzle now in a similar intuitive manner. It must be the case that each glass must have the same volume of contaminant!

> Imagine taking out from the milk glass all the molecules that belong to the soda. This leaves a missing volume from that milk glass. Where must the missing milk be? It must be contaminating the soda—the same volume of it!

Practice 22.2: Must the two glasses have equal volumes of soda and milk at the start of this puzzle? By transferring a tablespoon of liquid one way and then back, are we always guaranteed to have the same volume of contaminant in each cup?

23. The Rules of Arithmetic and Negative Numbers

In chapter 1 we made sense of the counting numbers $1, 2, 3, 4, \ldots$ including 0. These numbers are also called the **natural numbers**.

In this chapter we developed an intuitive model for the opposites of these numbers $-1, -2, -3, -4, \ldots$. (And Musing 20.4, if you thought about it, might have suggested to you that -0, the opposite of zero, is no different from 0. We'll see in a moment that mathematics itself also wants this to be so.)

To be clear, we've been led to the following belief:

For each (counting) number a, there is one other number "$-a$" such that $a + -a = 0$.

If you want to be fancy with language, but there is no need for it, people call $-a$ the **additive inverse** of a.

-5 is the additive inverse of 5 because it is the number you add to 5 to get to zero:

$$5 + -5 = 0$$

-7 is the additive inverse of 7 because it is the number you add to 7 to get to zero:

$$7 + -7 = 0$$

Example: Show that -0, the additive inverse of zero (the "opposite of 0"), is actually 0 itself!

Answer: Well, -0 is the number must we add to 0 to get the answer zero.

$$0 + ? = 0$$

What number works? Well, 0 works!

Ahh, -0 is the number 0.

We've just established that $-0 = 0$.

Example: Show that $-(-5)$, the "opposite of -5," is actually 5.

Answer: Now $-(-5)$ is the number must we add to -5 to get the answer zero.

$$-5+?=0$$

Well, 5 works! So, $-(-5)$ is 5.

We've just established that $--5=5$. This matches our intuition about piles and holes, or dots and antidots. Phew!

Practice 23.1: Explain for yourself, mathematically, why $-(-17)$ must be 17.

Example: Show that $-(2+3)$ is actually $-2+\ -3$.

This shows that all the work we were doing in the last section "distributing the negative sign" is mathematically valid!

Answer: Now $-(2+3)$ is the number we need to add to $2+3$ to get the answer zero. What could that number be?

$$2+3+?=0$$

Well, $-2+\ -3$ does the trick.

$$2+3+\ -2+\ -3=0$$

So, $-(2+3)$ is indeed $-2+\ -3$.

Practice 23.2: Show mathematically that $-(5-7)$ is $5+\ -7$.

People call this entire collection of numbers—the counting numbers and their opposites—the set of **integers** and they use the symbol $\mathbb{Z}$ to denote this set. (This symbol comes from the German word *Zahlen* for "numbers.")

So, we know have the set of natural numbers $\mathbb{N}$ and the set of integers $\mathbb{Z}$.

$$\mathbb{N} = \{1, 2, 3, 4, \dots\} \text{ or } \{0, 1, 2, 3, 4, \dots\}$$

$$\mathbb{Z} = \{\dots, -4, -3, -2, -1, 0, 1, 2, 3, 4, \dots\}$$

People also call the counting numbers $1, 2, 3, 4, \dots$ the **positive integers** and the opposite numbers -1, $-2, -3, -4, \dots$ the **negative integers**. The number zero, with the property that -0 is the same as 0, is deemed neither positive nor negative.

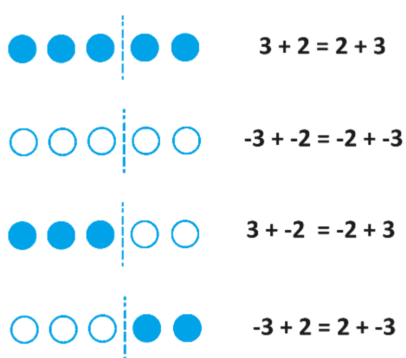
> **Comment:** There is confusion in school textbooks over the term "set of *whole numbers*." Some authors use this term to mean the set of numbers $0, 1, 2, 3, 4, \dots$. Others use it as another name for the set of integers $\{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$. Mathematicians tend not to use this term. (Or, if they do, in a context that causes no confusion.)

In section 14 we listed the rules of arithmetic that seem appropriate for how we think about addition and multiplication with the counting numbers.

These rules were motivated by drawing pictures of dots.

We can also try drawing pictures of antidots to see to what if the same rules might hold for our new opposite numbers too.

For example, we can still imagine a notion of "addition" as coming from lining up dots and/or antidots. And by reading such pictures from left to right and then from right to left, we will come to believe that we still can switch the order in which we add two numbers, even if one or both of those numbers happen to be negative integers.



It feels like at least one of our familiar rules of arithmetic is still valid in the world of integers.

For any two numbers a and b we have that $a + b = b + a$.

Other rules seem valid too.

For example, adding nothing to a picture of dots or to a picture of antidots changes nothing about the quantity that picture represents.

For any number a, we have that $a + 0 = a$ and $0 + a = a$.

We can even play the same not-exciting solitaire game from section 10 with negative numbers.

-2 **5**

0

6 **-7**

-3 **6**

Erase two numbers and replace them with their sum.
Repeat until a single number remains.

(See Musing 23.3 at the end of this section.)

Reasoning just as before, we deduce:

For any string of numbers added together

$$a + b + c + d + e + \ldots + y + z$$

it does not matter in which order one chooses to perform the additions. The same answer will always result.

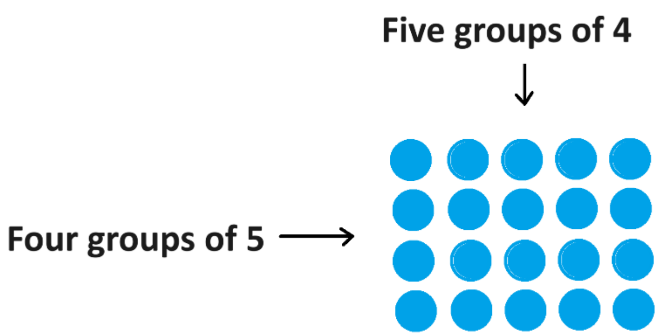
All feels fun and good when thinking about addition with positive and negative integers via dots and antidots. But thinking about multiplication with negative integers, on the other hand, forces us to take a new stance on matters.

Let me explain.

Within the system of counting numbers, multiplication is defined as *repeated addition*.

For instance, 4×5 is "four groups of 5," namely, $5 + 5 + 5 + 5$.

Drawing a rectangular array of dots to match this notion of repeated addition allowed us to deduce some properties of multiplication, that 5×4 is sure to have the same numerical value as 4×5, for instance.



But the notion of "repeated addition" is sometimes meaningless when it comes to working with negative numbers and drawing guiding pictures with dots and antidots is impossible.

Let engage in some mind-bendiness!

Consider this question:

What does $(-4) \times 5$ mean?
That is, what are "negative four groups of 5"?

The issue:

What's a "negative group"?

We know what a negative dot is (it's an antidot). But we've never talked about a "negative group" of something.

Schoolbooks typically "cheat" right at this moment and simply assert

$(-4) \times 5$, *negative four groups of five, is just the opposite of … four groups of* 5.

That is, they assert that $(-4) \times 5$ is the same as $-(4 \times 5)$.

Or they might assert that

$(-4) \times 5$, *negative four groups of five, is the same four groups of negative five.*

That is, they assert that $(-4) \times 5$ is the same as $4 \times (-5)$.

The extra confusing thing is that it turns out mathematics will come to tell us that $(-4) \times 5$ and $-(4 \times 5)$ and $4 \times (-5)$ are all sure to have the same value. So, the schoolbooks are not suggesting incorrect mathematics and so they can get away with this.

But they are sidestepping the question of what $(-4) \times 5$ itself actually means.

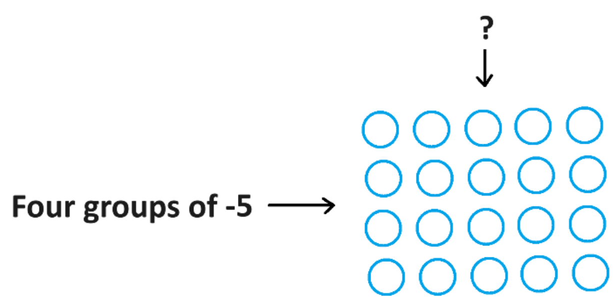
Question: Most people do not understand the issue being pointed out here because they have been told over and over again: "This is how it is in math. It just is. It just is. It just is." The chance to step back and be confused and to question claims made about mathematics has been denied to most students.

So, how are you doing processing the previous page?

Do you see that $-(4 \times 5)$—the opposite of "four groups of five"—means something different from $4 \times (-5)$—four groups of negative five—even though you can imagine them each to give a picture of twenty antidots?

Do you see that there is no *a priori* reason to think that $(-4) \times 5$ should be the same as either one of these?

Practice 23.3: Let's lean into the quantity $4 \times (-5)$ a little bit. It makes sense in our "repeated addition" thinking to see it as "four groups of -5" and represent it as a picture four groups of five antidots in a rectangular array.



What quantity do you see if you look at this picture via columns?

We, and schoolbooks, can make intuitive sense of 4×5 and of $4 \times (-5)$ with dots and antidots. They are four groups of five dots and four groups of five antidots, respectively.

Making concrete sense of $(-4) \times 5$, however, is problematic and leads to some schoolbook shenanigans.

(And we haven't even touched on something like $(-4) \times (-5)$ yet! *Why negative times negative positive* is an age-old question students—and adults—have been crying out for decades and decades.)

Here's the truth about multiplication with negative numbers—and I know you can handle the truth.

> Mathematics is bigger and bolder than the real world. It is therefore bigger and bolder than all schoolbook attempts to make every part of it concrete and real. Mathematics certainly incorporates real-world models and is immensely powerful in helping describe them. But mathematics sits at a higher plane to them.
>
> The concept of multiplication—for numbers beyond just the counting numbers—is one of those higher-plane mathematical concepts.
>
> We choose to believe that there is a general operation on numbers, called **multiplication**, which behaves the same way as we defined multiplication for just the counting numbers.
>
> We recognize that we can sometimes give "real-world" meaning to the notion of multiplication, when it is appropriate to do so, and are not at all phased when we in a context where we cannot.

In other words, we have chosen to believe that there is a notion of "multiplication" that can be applied to all numbers. We make no attempt to assert what it means and what it is in concrete terms. Rather, we define it by how it behaves.

We understand what multiplication is when applied to just the counting numbers and how it behaves for those numbers. We now extend that notion of multiplication to all numbers by insisting its observed behavior continues.

The mathematical operation multiplication is defined by how it behaves, not by what it is.

This is the mathematician's take on matters, and it is a powerful take. It's also very hard for people who have undergone years and years of school training to shift to it.

To be upfront:

Mathematicians have no trouble saying that they don't actually know what $(-4) \times 5$ means in a concrete sense. (They even think it is folly to always insist on "real world" interpretations.)

But they do know how to do mathematics with this quantity and get meaningful and practical results from it, nonetheless.

So, here's the honest way to move from the world of counting numbers to the world of integers (the counting numbers and their opposites).

We work with the same eight rules we know so well from playing with the counting numbers but now use these rules to define the behavior of our two basic operations: addition and multiplication. And to get us beyond just the counting numbers, we add a ninth rule that shows how the new "opposite numbers" should work.

Everything is spelled out on the next two pages.

There is nothing to memorize!

Your job is to just have a quick read-through what comes next and observe how these nine rules are simply bringing together the intuition we've already developed about arithmetic and how it works.

The Integers

There is a set of numbers, called the **integers**, $\mathbb{Z}$, which contains the counting numbers.

For these numbers there are notions of "addition" and "multiplication." Eacb is a way to combine two numbers to produce a new number. These two operations …

a) match the expected addition and multiplication we know for the counting numbers when they are applied to just the counting numbers

b) continue to follow the behavior of addition and multiplication we know for the counting numbers when applied to numbers that are not themselves counting numbers.

This behavior is outlined in the following nine rules.

Addition

Rule 1: We can change the order in which we add any two numbers and not change the final result. That is, for any two numbers a and b we have that $a + b = b + a$.

Rule 2: Adding zero to a number does not change the value of the number. That is, for any counting number a we have that $a + 0 = a$ and $0 + a = a$.

Rule 3: In any string of numbers added together

$$a + b + c + d + e + \ldots + y + z$$

it does not matter in which order one chooses to perform the additions. The same answer will always result.

Multiplication

Rule 4: We can change the order in which we multiply any two numbers and not change the final result. That is, for any two numbers a and b we have $ab = ba$.

Rule 5: Multiplying a number by 1 does not change the value of the number. That is, for each number a we have that $1 \times \mathrm{a} = a$ and $\mathrm{a} \times 1 = a$.

Rule 6: Multiplying a number by 0 gives a result of zero.
That is, for each number a we have that $0 \times a = 0$ and $a \times 0 = 0$.

Rule 7: In any string of numbers multiplied together

$$a \cdot b \cdot c \cdot d \cdot \cdots \cdot y \cdot z$$

it does not matter in which order one chooses to perform the products. The same answer will always result.

Addition and Multiplication Together

Rule 8: "We can chop up rectangles from multiplication and add up the pieces."

"Opposite Numbers"

Rule 9: For each number a there is one other number, denoted $-a$, such that $a + -a = 0$.

Again, we add Rule 9 to make sure we are getting the "opposite numbers" to create all of the integers.

MUSINGS

Musing 23.4 On a scale of 1 to 5, how perturbed are you by the idea that mathematicians are so brutally honest to admit that they do not know what multiplication **is** for numbers beyond the counting numbers, but are comfortable to simply define multiplication as an operation on numbers that behaves a certain way?

> **1** = "What a cop out! Mathematicians are wimps!"
> **5** = "Brutal honesty! At last, someone is admitting the truth about the nature of mathematics. This stance is refreshing, and I can handle it."

Musing 23.5
My mathematical colleagues, who like logical austerity, will object to my phrasing of Rule 9:

> For each number a there is **one** other number, denoted $-a$, such that $a + -a = 0$.

They say that pointing out that there is only **one** other number that deserves to be called $-a$ is unnecessary; logic dictates that there can't be more than one! So, simply say "there is a number …".

For example, here's an argument that shows that there can only be one number that behaves like -5.

Suppose there is another number, let's call is b, that behaves like -5.
That is, when you add it to 5 you get zero: $5 + b = 0$.

Consider the sum $b + 5 + -5$.

We can add this in any order we like. (Rule 3.)
Adding together b and 5 first gives us

$$b + 5 + -5 = 0 + -5 = -5$$

On the other hand, adding together 5 and -5 first gives

$$b + 5 + -5 = b + 0 = b$$

These answers must match as they are both just $b + 5 + -5$, after all.
So, b must -5.

On a scale of 1 to 5, to what extent did you follow the logic here of my colleagues?

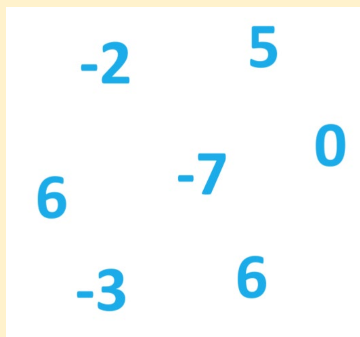
> **1** = "What on Earth are we talking about here? This is all gobbly gook."
> **5** = "Wowza! I get what they are saying."

MECHANICS PRACTICE

Musing 23.6

Let's play a not-exciting game of solitaire again, a la the game presented in section 6. But this time, let's play it with some negative numbers thrown in for fun!

Here are some numbers drawn on a page.



Recall that a "move" in this game consists of erasing two numbers and replacing them with their sum.

For example, if you cross out 5 and -7, you then write -2 on the page in their stead.

You keep repeating such moves until a single number remains on the page.

a) In this particular game, what will that final number be no matter the choice of moves you make along the way? Can you explain why this will be the case? (Perhaps draw pictures of groups of dots and of antidots.)

b) What number should you add to the starting board above so that the final number that remains on the pages is sure to be -10?

24. Making Sense of Rule 8: "Chopping Up Rectangles"

I quietly slipped over Rule 8 in the previous section without any comment or fuss, as though it is all fine and not worthy of any extra comment.

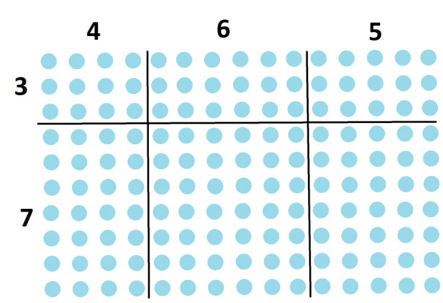
Rule 8: "We can chop up rectangles from multiplication and add up the pieces."

But that was wrong of me.

We really need to talk about what this rule means now that we have negative numbers in our minds.

Playing with just counting numbers in chapter 1 allowed us to represent any multiplication problem we encountered with a picture, a rectangle of dots. And we could chop up these rectangles in any way we liked without a care in the world. The number of dots in each of the small rectangular pieces we created were sure to sum to the total number of dots in the picture to begin with.

For example, here's a picture representing 10×15 broken up into six pieces: 3×4 and 7×4 and 3×6, and so on.



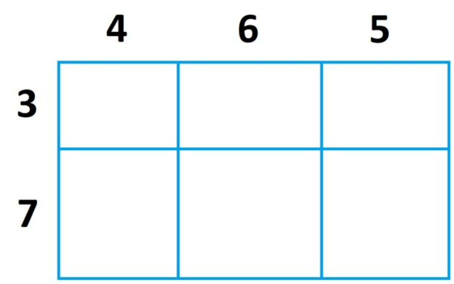
But, as we saw in the previous section, is not at all clear what dot or antidot pictures we could draw for some multiplication statements involving negative numbers. (We couldn't even give a meaningful interpretation of $(-4) \times 5$, for instance, in the last section, let alone draw a picture to represent it!)

So, what then is Rule 8 now saying in the context of negative numbers?

In order to move forward, we need to let go of dot and antidot thinking. Instead, we need to shift to thinking about the behavior of numbers as represented by the statement "we can chop up rectangles" (since, after all, we are now defining multiplication by how it behaves, not by "what it is").

And we kinda did that already in Chapter 1 by drawing just rectangles, not rectangles of dots, and chopping up the pictures of those rectangles.

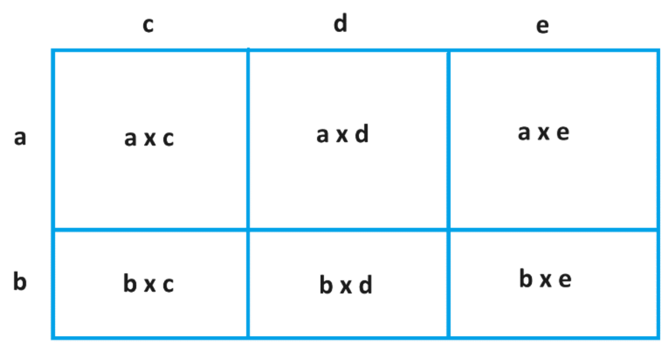
For example, this picture illustrates 10×15 and says that it matches the sum of six smaller products: 3×4 and 7×4 and 3×6 and so on. It's a picture that shows how arithmetic is working rather than showing you literal counts of dots.



Practice 24.1: What is the value of $(3 + 7) \times (4 + 6 + 5)$?
What is the value of $3 \times 4 + 7 \times 4 + 3 \times 6 + 7 \times 6 + 3 \times 5 + 7 \times 5$?
(They should be the same!)

This is how we need to represent Rule 8. Not by attempting to draw literal pictures of dots and antidots in rectangular arrays, but by drawing general rectangles to illustrate the behavior of the arithmetic.

For example, this picture is showing that we can think of a product $(a+b) \times (c+d+e)$ as the same as the sum of six smaller products (just as was the case for $(3+7) \times (4+6+5)$).



But now we are allowing some or all of our numbers to be negative.

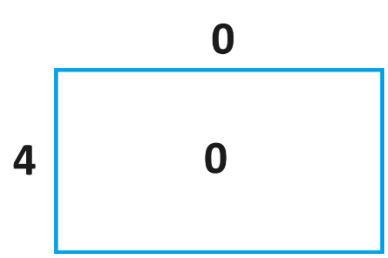
And this will look confusing to some people, “How can a rectangle have a negative side length or a negative area?” they might cry out.

Again, we can’t read the pictures literally: just as they are not representing pictures of actual dots and antidots, and they are also not representing pictures from geometry class of actual rectangles with real-world measurements for side lengths and areas.

The rectangle pictures illustrate how arithmetic is working—nothing more.

(But this, I know, can be confusing because, if all the numbers shown in a picture happen to be positive, then the picture does perfectly match a rectangle that could have been drawn in geometry class!)

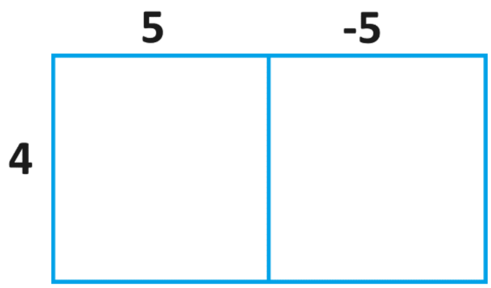
For example, here’s a picture representing 4×0 with the answer 0.



One doesn’t draw side lengths of zero and areas of zero in a geometry class. But this picture is still illustrating for us that $4 \times 0 = 0$ and so is a helpful picture.

It is particularly helpful when we change how we represent that top side length of zero. Let's think of zero as $5 + \ -5$ and chop up the rectangle accordingly.

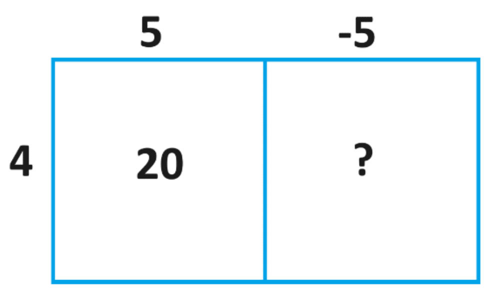
The total "area" of this rectangle is still zero.



But we now see two pieces: a left piece corresponding to 4×5 and a right piece to $4 \times (-5)$.

We know the value of the left piece: it's just multiplication with the counting numbers and we are assuming that hasn't changed. So $4 \times 5 = 20$.

It remains to work out the value of the right piece



Whatever its value, we know that 20 and it have to add to zero.

$$20 + ? = 0$$

Ahh! That right piece has to be -20.

So, rule 8 has just forced us to conclude that $4 \times (-5)$ has value -20.

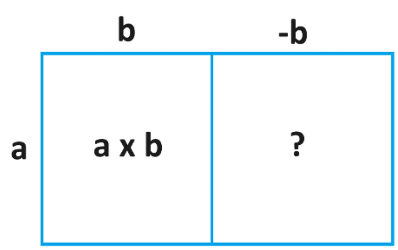
And this is lovely. It matches our intuitive sense that $4 \times (-5)$ might be interpreted as "four copies of -5," which is $-5 + \ -5 + \ -5 + \ -5$. And, yes, that's -20.

Math and intuition are aligned … again!

Practice 24.2 Draw a rectangle that represents the statement $8 \times 0 = 0$, but make the top side length $7 + \ -7$ instead of 0.

Use your picture to then explain why the value of $8 \times (-7)$ must be -56.

If you want to be fancy and use letter names for numbers and not specific numbers (like 4 and 5 or 7 and 8), here's a picture of $a \times 0$, which has value 0, but with the top side length presented as $b + \ -b$, instead of 0.



The value of the question mark must be $-(a \times b)$, the number we add to $a \times b$ to get zero. But it is also $a \times (-b)$ from looking at the picture.

We've discovered:

For any two numbers a and b, we have $a \times (-b) = -(a \times b)$.

For example, $4 \times (-5) = -(4 \times 5)$, and working out what is inside the parentheses, this is -20.

Also, $7 \times (-8) = -(7 \times 8)$, and working out what is inside the parentheses, this is -56.

People say "we can just pull out a negative sign from a product."

Here's a next example.

In the last section, we couldn't give meaning to the quantity $(-4) \times 5$ (although we have all been trained from our school days to say that is has value -20).

We are now ready to see what value math assigns to $(-4) \times 5$.

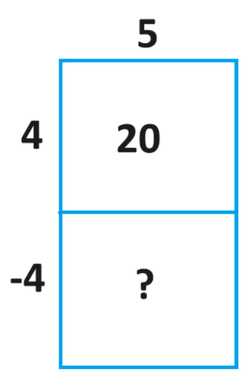
In fact, we can get to it two ways!

Approach 1:
By Rule 4, we can change the order in which we multiply two numbers.
So, $(-4) \times 5$ has the same value as $5 \times (-4)$.

By "pulling out the negative sign" as we just learned, this is $-(5 \times 4)$, which is -20. (Work out what is inside the parentheses.)

Approach 2: We know that 0×5 is zero.

Draw a picture to represent this, but with 0 presented as $4 + -4$.



One piece of the rectangle has value $4 \times 5 = 20$.
The second piece is given by $(-4) \times 5$, whose value we seek.

But the two pieces combined have value 0. It must be that $(-4) \times 5 = -20$.

Practice 24.3 Draw a picture of $0 \times b = 0$ to show that for any two numbers a and b, we have $(-a) \times b = -(a \times b)$.

So,

$4 \times (-5)$ has value -20,

and

$(-4) \times 5$ has value -20,

and

$-(4 \times 5)$ has value -20 too. (Work out what is inside the parentheses first.)

And this is why schoolbooks can get away with being unclear in their attempts to make sense of a product like $(-4) \times 5$ and just assert "It's obviously the same as $4 \times (-5)$" or "It's obviously the same as $-(4 \times 5)$." Our mathematics has shown that all three products have the same value, so no false claims are being made (except for the "obviousness" of it).

You can see too why schoolbooks avoid explaining the mathematics of all this. Look at the amount of hard thinking we had to go through to get this point!

In general, for any two numbers, a and b, we have established that

$$a \times (-b)$$

$$(-a) \times b$$

$$-(a \times b)$$

are all sure to have the same value.

We can just "pull out negative signs" from products in any way we like. Doing so won't affect the value of the product.

Practice 24.4 Explain why $(-1) \times 8$ is -8.

For extra fun, try justify this more than one way, perhaps by drawing a picture of 0×8 but presenting 0 as $1 + \ -1$; or perhaps by "pulling out a negative sign" from $(-1) \times 8$ and see where that leads you; or perhaps by adding $8 + (-1) \times 8$ and factoring (section 13) to see if you get zero.

Where We Are At

We feel we have a concrete understanding of addition and multiplication ("repeated addition") for the counting numbers. This was the work of Chapter 1.

And now we've extended addition and multiplication to a wider class of numbers, namely, to the counting numbers and their opposites—the integers.

We've had to let go of always trying to interpret what addition and multiplication concretely mean ("What is $(-4) \times 5$, really?") and have moved instead to defining addition and multiplication by how they behave rather than what they are.

The nine rules of arithmetic we listed are to define the rules of behavior of arithmetic.
And these nine rules really do capture all the arithmetic we were taught in school.

We saw in this section and the last that we can:

"**distribute negative signs**" ($-(4-5) = -4 + \ -5$, for instance)

"**pull out negative signs**" ($(-4) \times 5 = -(4 \times 5)$ and that's -20, for instance)

multiply by negative one to make a number negative ($-(1) \times 8 = -8$, for instance)

and more.

On the next page, in one place, is a much quicker and cleaner summary of everything we need to know about the arithmetic of the integers, that is, the counting numbers, zero, and their negatives.

There is nothing to memorize or do here!

This list is just to say that everything we've been taught to do in arithmetic is sound.

Numbers come with two operations—addition and multiplication—which behave as we expect when applied to just the counting numbers and, more generally, behave as follows:

Rule 1: For any two numbers a and b we have $a + b = b + a$.

Rule 2: For any number a we have $a + 0 = a$ and $0 + a = a$.

Rule 3: In a string of additions, it does not matter in which order one conducts individual additions.

Rule 4: For any two numbers a and b we have $a \times b = b \times a$

Rule 5: For any number a we have $a \times 1 = a$ and $1 \times a = a$.

Rule 6: In a string of multiplications, it does not matter in which order one conducts individual multiplications.

Rule 7: For any number a we have $a \times 0 = 0$ and $0 \times a = 0$.

Rule 8: "We can chop up rectangles from multiplication and add up the pieces."

Rule 9: For each number a , there is one other number "$-a$" such that $a + -a = 0$.

Some Logical Consequences: For any two numbers a and b

i) $-0 = 0$
("The opposite of zero is zero")

ii) $- - a = a$
("The opposite of the opposite is back to the original")

iii) $-(a + b) = -a + -b$
(We can "distribute a negative sign")

iv) $(-a) \times b$ and $a \times (-b)$ and $-(a \times b)$ all have the same value
(We can "pull out a negative sign")

v) $(-1) \times a = -a$
("Multiplying by -1 gives you the opposite number")

25. Why Negative times Negative is Positive

We're all set to address this age-old question of why multiplying together two negative numbers should give a positive answer.

It is the only "case" remaining for us to consider.

Positive times Positive

Example: What is 2×3?

This is just the product of two counting numbers, and multiplication is still just "repeated addition" in this context. Thus, $2 \times 3 = 3 + 3 = 6$, which is positive six.

The product of two positive integers is positive.

Positive times Negative

Example: What is $2 \times (-3)$?

We no longer have the product of two counting numbers, so we (technically) cannot rely on repeated addition.

But we did learn that we can "pull out a negative sign" from a product. Thus

$$2 \times (-3) = -(2 \times 3)$$

Working out what is inside the parentheses, we get

$$2 \times (-3) = -6$$

(And this answer does happen to match "repeated addition" thinking!)

The product of a positive integer and a negative integer is negative.

Negative times Positive

Example: What is $(-2) \times 3$?

This example of "meatier." Now the even intuition of "repeated addition" fails here.

But we know we can again "pull out a negative sign." Thus

$$(-2) \times 3 = -(2 \times 3)$$

Working out what is inside the parentheses, we get

$$(-2) \times 3 = -6$$

The product of a negative integer and a positive integer is negative.

And now to the challenge case ...

Negative times Negative

Example: What is $(-2) \times (-3)$?

Why should the answer to this product be positive six?

Let's establish why this must be so two different ways.

Explanation 1: Pull out a negative sign

Let's pull out a negative sign once.

$$(-2) \times (-3) = -(2 \times (-3))$$

And let's now work out what is inside the parentheses: $2 \times (-3) = -6$, as we saw.

So,

$$(-2) \times (-3) = -(-6)$$

We showed that the opposite of the opposite of a number is the original number: $- - 6$ is 6. What we have now is that

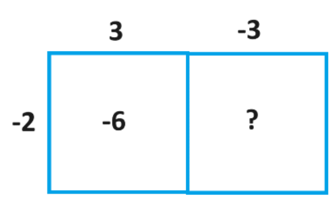
$$(-2) \times (-3) = - - 6 = 6$$

The product of a negative integer and a negative integer is positive.

Explanation 2: Chop up a rectangle

We know that $(-2) \times 0 = 0$ by Rule 7.

Let's draw a picture of this, but with zero presented as $3 + -3$.
This gives a rectangle divided into two pieces.



The left piece is $(-2) \times 3$, which we've already seen has value -6.
The right piece is $(-2) \times (-3)$, which is the product we are wondering about.

But we do know that the two pieces have values that sum to 0.

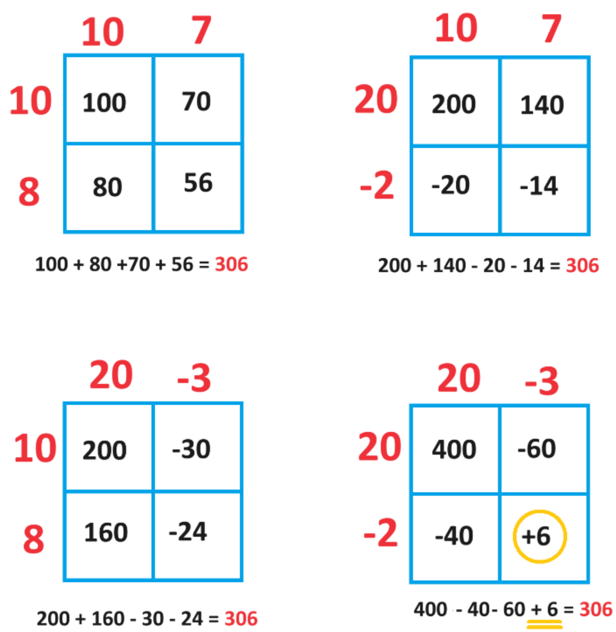
It must be that $(-2) \times (-3) = 6$.

The product of a negative integer and a negative integer is positive.

In summary, we have established what we were taught (told?) in school:

positive times positive is positive
positive times negative is negative
negative times positive is negative
negative times negative is positive

For fun, here are four different ways to work our 17×18, making use of each of these four facts. You can see in the fourth picture that math is telling us that we really do need $(-2) \times (-3)$ to be positive six.



Practice 25.1 Draw four different pictures like these that each compute 14×15. See that mathematics really does want $(-6) \times (-5)$ to be positive thirty.

Practice 25.2 How might you reason why $(-11) \times (-12)$ has to be 132, a positive answer?

Practice 25.3

a) What is the value of $(-1) \times (-1)$?

b) What is the value of $(-1) \times (-1) \times (-1)$? How do you know?

c) What is the value of $(-1) \times (-1) \times (-1) \times (-1) \times (-1) \times (-1) \times (-1) \times (-1) \times (-1)$? How do you know?

Let's practice some ideas from Chapter 1, but with negative numbers involved.

Example: Work out $(4-6)(10-3)$ two different ways.

Answer:

By working out what is inside the parentheses first we get $(-2) \times (7)$, which is -14.

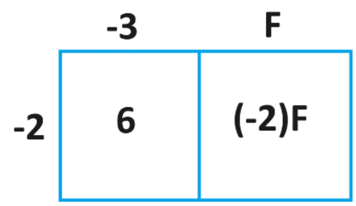
Alternatively, thinking of this as coming from chopping up a rectangle,

	10	**-3**
4	**40**	**-12**
-6	**-60**	**18**

we get $40 + 18 - 60 - 12$, which is -14.

Example: Make $(-2)(-3+F)$ look a bit friendlier.

Answer: Draw, or imagine, a rectangle to see that $(-2)(-3+F)$ is $6+(-2)F$.



Now, $(-2)F$ is really $(-2)\times F$.

Pulling out a negative sign, this is $-(2\times F)$, which most people write as $-2F$. (Do you remember parentheses are usually kept hidden for multiplications?)

So, we have that $(-2)(-3+F)$ is the same as

$$6+\ -2F$$

And we can rewrite this in terms of subtraction

$$6-2F$$

So, $(-2)(-3+F)=6-2F$.

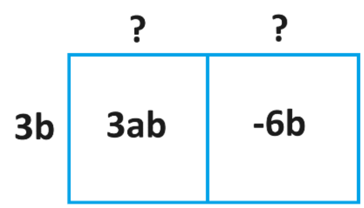
Practice 25.4 Show that $(-5-G)\cdot(-3)$ can be rewritten as $3G+15$.
(Drawing picture really does help!)

Practice 25.5 Show that $2-5(2-3w)$ can be rewritten as $15w-8$.

Example: Factor $3ab - 6b$.

Answer: Think of this as $3ab + -6b$.

There is a common factor of "$3b$" in the terms $3ab$ and $-6b$.



The picture suggests that $3ab - 6b$ is $3b(a + -2)$. That is,

$$3ab - 6b = 3b(a - 2).$$

Practice 25.6 Factor $10xy - 2x^2 - x$.

Practice 25.7 Show that $3(F - 2) - 2(F - 3)$ is the same as just F.

MUSINGS

Musing 25.8 Do you recall how you "negative times negative is positive" was explained to you back in school? (Was it explained?)

Musing 25.9
a) Draw rectangles to show that $(10 + a)^2$ and $10(10 + 2a) + a^2$ are sure to have the same value, no matter what number the letter a represents.

b) Eoin computed 14^2 as $10 \times 18 + 4^2$, which is $180 + 16 = 196$ and is correct, and he computed 15^2 as $10 \times 20 + 5^2$, which is 200+25 $= 225$ and is again correct. Can you explain what he was doing?

c) How might Eoin compute 17^2?

MECHANICS PRACTICE

Musing 25.10
Which of these expressions evaluates to a positive value?
Which evaluate to a negative value?
And which three of them evaluate to zero?

a) $(-2) + (-7)$
b) $4 \cdot (-3)$
c) $(-10) \times 110$
d) $-(-6)$
e) $(-5)(-4)$
f) $(-10)(-3 - 5)$
g) $7 - (10 - 3)$
h) $(4 - 8)(2 - 1)$
i) $(-143) \cdot (542) \cdot 0 \cdot (-1987)$
j) $(5 - 2) \cdot (100 - 50 - 20)$
k) $(f - a)(f - b)(f - c)(f - d)(f - e)(f - f)(f - g)(f - h)$

Musing 25.11 Make each of the following expressions look friendlier.
(Each letter is just a symbol representing some unspecified number.)

a) $3(x-4)$
b) $(-3)(4-x)$
c) $2-y-4(2y-5)$
d) $5(F-4)-4(F-5)$

Musing 25.12 Rewrite each of these expressions by factoring and see if you can match each with the expression given in square brackets.

a) $5c+cd$ [$c(5+d)$]
b) w^2-2w [$w(w-2)$]
c) $px-py$ [$p(x-y)$]
d) $(r^2-7y+43)x-(r^2-7y+43)y$ [$(r^2-7y+43)(x-y)$]
e) $(3z-1)(4-z)-(3z-1)(3-z)$ [$3z-1$]
f) $3(f-2)-(f-2)$ [$2(f-2)$]

26. Interlude: Finger Multiplication

ACTIVITY

FINGER MULTIPLICATION

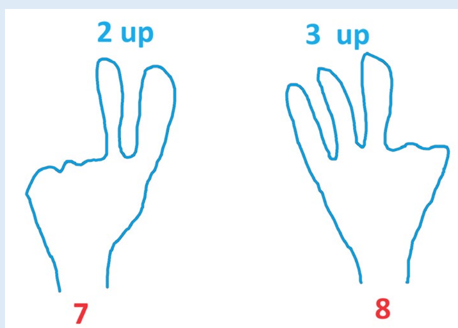
Don't memorize your multiplication table facts. Let your fingers do the work!

If you are comfortable with multiples of one, two, three, four, and five, then there is an easy way to compute the products in the six- through-ten part of the times table.

Start by encoding the two numbers you wish to multiply on your hands as follows.

A closed fist represents "five" and any finger raised on that hand adds one to that value.

For example, a hand with two fingers raised represents $5+2$, that is seven, and a hand with three fingers raised represents $5+3$, that is, eight.



To multiply together the two numbers you've encoded, follow these steps.

1. *Each raised finger is worth ten. Count ten for each finger raised you see.*
2. *Count the number of unraised fingers you see on each hand and multiply together those two small numbers.*
3. *Add the results of steps 1 and 2. The sum in the desired product.*

For example, 7×8 is represented as two raised fingers on the left hand and three on the right hand.

There are five raised fingers in all, yielding the number $\mathbf{50}$ for step 1.
The left hand has **3** lowered fingers and the right hand **2**, and $3 \times 2 = \mathbf{6}$ for step 2.
The product we seek is then $50 + 6 = 56$, which is indeed the value of 7×8!

Magic!

In the same way, one computes 9×7 as follows:

One hand with **four** fingers raised is "nine." $(5 + 4 = 9)$.
One hand with **two** fingers raised is "seven." $(5 + 2 = 7)$.
That's six fingers up in total, making $\mathbf{60.}$

We have 1 finger down on one hand, 3 fingers down on the other, and $1 \times 3 = \mathbf{3}$.
Thus $9 \times 7 = 60 + 3 = 63$, and it does!

And one computes 6×8 as follows:

One hand with **one** finger raised is "six." $(5 + 1 = 6)$.
One hand with **three** fingers raised is "eight." $(5 + 3 = 8)$.
That's four fingers up in total, making $\mathbf{40.}$

We have 4 fingers down on one hand, 2 fingers down on the other, and $4\times 2 = \mathbf{8}$.
Thus $6\times 8 = 40 + 8 = 48$, and it does!

a) Practice this method by computing 9×9 and 6×7.
b) How about going to extremes? Does this method work for 5×5 and 10×10?
c) Can you explain why this strange method of multiplication is working?

Fingers and Toes

We can go to the ten- through twenty-times tables too if we use all of our twenty digits!

Here's how to work out 17×18, as an example.

On the left side of my body I have ten digits (five fingers and five toes). All ten digits down represents 10. To make this 17, raise **seven** of those digits. (Or, at least, imagine doing so.)
On the right, all ten digits down also represents 10. To make 18, say, raise **eight** of those digits.

Each raised digit is now worth 20.
We currently have a total of fifteen raised digits. That makes for $\mathbf{300}$.

Now multiply the count of digits down: 3 on the left and 2 on the right giving $3 \times 2 = \mathbf{6.}$

And, lo and behold, 17×18 does equal $300 + 6 = 306$.

d) Compute 14×18 and 16×19 this way.
e) Again, why is this bizarre method working?

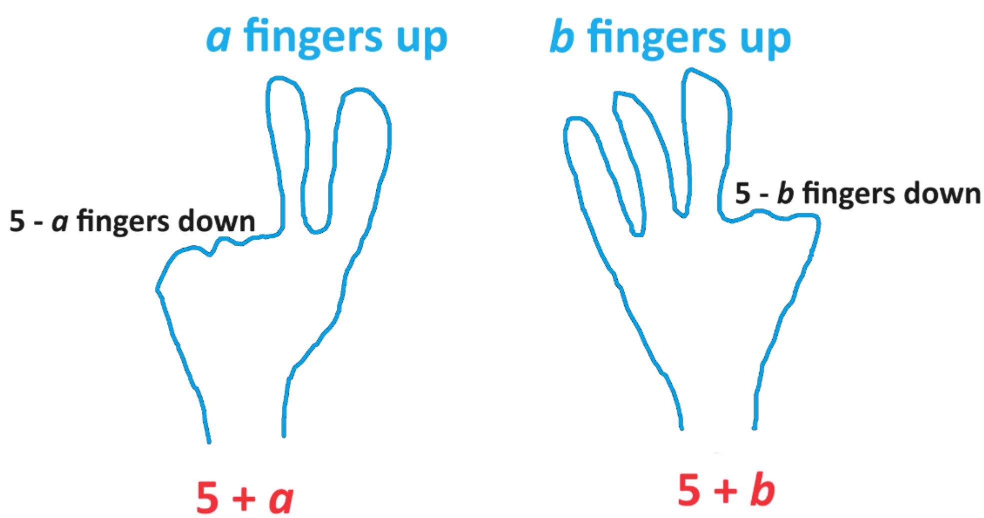
I remember being taught several tricks for recalling math facts when I was a kid, but I don't remember having anyone help me think through why they worked. (Understanding seemed much more interesting to me than memorizing the facts!)

Surprisingly, this finger trick utilizes all the math we've developed so far—including multiplying negative numbers—which seems surprising since we're just doing multiplication of counting numbers a la chapter 1.

Let's first consider just finger multiplication. (No toes.) And let's be a little abstract in our thinking.

Suppose we have a fingers up on the left hand. This would represent the number $5 + a$.
(For instance, if a is the number 2, then $5 + a$ is 7.)

And suppose we have b fingers up on the right hand. This would represent the number $5 + b$.
(For instance, if b is the number 3, then $5 + b$ is 8.)



We are trying to multiply together the numbers $5 + a$ and $5 + b$.
That is, we're computing

$$(5 + a)(5 + b)$$

(For $a = 2$ and $b = 3$, this is the product 7×8.)

But the finger tricks tells us to do this in an unusual way.

Step 1: *Regard each raised finger as "worth ten."*

We currently have $a + b$ fingers raised.
So, step 1 of the algorithm gives us the value: $\mathbf{10(a+b)}$.

Step 2: *Multiply together the counts of fingers down on each side.*

There are 5 fingers in total on our left hands. If a of them are raised up, then $5 - a$ are down.
There are 5 fingers in total on our right hands. If b of them are raised up, then $5 - b$ are down.
We need to multiply these two counts together.
So, step 2 of the algorithm gives us the value $\mathbf{(5-a)(5-b)}$.

Step 3: *The product we seek is these two values added together.*

We're looking for the value of the product $\mathbf{(5+a)(5+b)}$.

The algorithm says that it equals $\mathbf{10(a+b)+(5-a)(5-b)}$.

These don't look the same!

Are they?

Let's play with each of these expressions and see if they are really the same thing in disguise.

Playing with $(5+a)(5+b)$

Let's expand the brackets.

We see that $(5+a)(5+b)$ is the same as $25 + 5a + 5b + ab$.

	5	b
5	25	5b
a	5a	ab

Playing with $10(a + b) + (5 - a)(5 - b)$

We have two pieces here.

Expanding $10(a + b)$ gives us $10a + 10b$.

	a	b
10	10a	10b

Expanding $(5 + -a)(5 + -b)$ gives

$$25 + -5a + -5b + ab$$

	5	-b
5	25	-5b
-a	-5a	ab

Putting these two pieces together we get:

$$10a + 10b + 25 + -5a + -5b + ab$$

But we can add this string of additions in any order we like. And I can't help noticing "$10a + -5a$" and "$10b + -5b$" sitting within the string.

Practice: Factor to show that $10a + -5a$ is the same as $5a$.
Practice: Factor to show that $10b + -5b$ is the same as $5b$.

So, our string of additions is really

$$5a + 5b + 25 + ab$$

Comparing the Two:

$(5 + a)(5 + b)$ is really the same as $25 + 5a + 5b + ab$.

$10(a + b) + (5 - a)(5 - b)$ is really the same as $5a + 5b + 25 + ab$.

Are these the same?

Yes! They are each the same sum of four terms, just in a different order.

The result of the finger trick algorithm is sure to match the product we seek!

Practice 26.1
a) Show that $(10 + a)(10 + b)$ is the same as $20(a + b) + (10 - a)(10 - b)$.
b) Do you see that this observation explains the finger-and-toe version of the trick?

MUSINGS

Musing 26.2 A Martian has two hands, but with six fingers per hand.

a) Describe a finger-multiplication method the Martian can use to compute the values of products in the six- through- twelve times tables.
b) Can you provide an explanation as to why your method works?

Musing 26.3: Optional Plutonians have bodies that are not symmetrical. They have two hands, but with four fingers on one hand and six fingers on the other. Might there be a finger-multiplication method what will work for them?

27. Even and Odd Negative Integers

Let's end this chapter by playing with even and odd numbers again. We met them in Section 16.

Our definitions of *even* and *odd* actually work for negative numbers too.

An integer N, positive or negative (or zero), is said to be **even** if it equals twice another integer. That is, we can write $N = 2a$ for some integer a.

An integer N, positive or negative, is said to be **odd** if it is one more than twice another integer. That is, we can write $N = 2a + 1$ for some integer a.

For example:

-14 is even as it is double -7.

-15 is odd because is double -8, plus 1. (Think though that!)

Practice 27.1
a) If I have 14 antidots, can I split them into two groups of equal size? If so, how many dots are in each group?
b) If I have 15 antidots, what happens if I try to split them into two groups of equal size?
c) Draw a picture of two sets of eight antidots and one dot. Does this picture represent the number -15?

Practice 27.2 Write each of these integers as either double another integer or double another integer plus one.

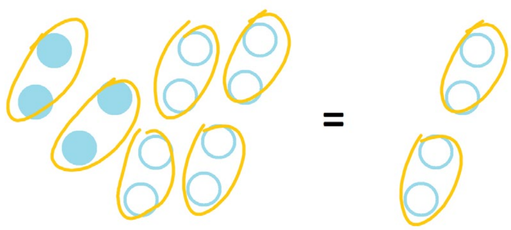
-20 -402 -5 -401 -1

In section 16 we briefly contemplated the validity of the following claims for counting numbers using a "take away" mindset.

EVEN - EVEN = EVEN
EVEN - ODD = ODD
ODD - EVEN = ODD
ODD - ODD = EVEN

We can intuitively justify these claims for integers too by thinking about pairs of dots and pairs of antidots. (Remember, subtraction is really the addition of the opposite.)

For example, this picture of $4 + -8$ is an example of "EVEN – EVEN" being EVEN.



We can certainly give an abstract mathematical proof of "EVEN – EVEN = EVEN," but it is a bit tedious. Can you make sense of this proof?

Claim: If N and M are each even integers, then $N - M$ is sure to be even too.

Proof: Since N and $M$are each even, we can write $N = 2a$ and $M = 2b$ for some integers a and b.

Then, $N - M$ is $2a - 2b$.

This is really $2a + -2b$, which we can think of as $2a + 2(-b)$.

Factoring out the two gives $2(a + -b)$.

So, $N - M$ is $2(a - b)$, which is double and integer. It is even!

Optional Practice 27.3 Prove mathematically that N is an even integer and M is an odd integer, then $N - M$ is sure to be an odd integer.

(Also, feel free to prove than if N is odd and M is even, then $N - M$ is sure to be odd; and also prove that if N and M are both odd, them $N - M$ is sure to be even.)

Here's something fun-ish.

ACTIVITY

THE PLUS/MINUS GAME

Niko and Zoe play the following game with the following seven numbers written in a row.

15	8	12	7	3	4	11

They each take turns inserting a plus sign $+$ or a minus sign $-$between two consecutive numbers until all six spaces are filled. They then compute the sum.

If the sum is odd, Niko wins.
If the sum is even, Zoe wins.

Who is sure to win?

Explanation:

Just like we did in Section 16, we can argue that any sum of integers containing an even number of odd integers is sure to be even, and any sum of integers containing an odd number of odd integers is sure to be odd.

The final sum in this game will contain four odd integers and so is sure to be even, and so Zoe is sure to win!

Practice 27.4 Design a plus/minus game Niko is sure to win.

MUSINGS

Musing 27.5

a) A grasshopper jumps along a line, moving left or right, one inch at a time. After 105 hops to the left and 106 hops to the right, in some random order, could it be that the grasshopper is back at its starting position?

b) A grasshopper again jumps randomly left and right along a line. Its first jump is 1 inch long. Its second jump is 2 inches long. Its third is 3 inches long, and so on. Could it be that after 50 jumps the grasshopper is back at its starting position?

c) Suppose instead that the grasshopper jumps two inches on its first hop, 4 inches on its second, 6 inches on its third, and so on. Could the grasshopper return to its starting position after 50 hops?

Solutions

20.1 Two piles and six holes combine to leave four holes: $2 + -6 = -4$.

20.2 a) and c): $6 + -7 + -1$. This equals -2.
b) and c): $5 - 1 + 3 - 2 - 1$. This equals 4.

20.3 Ten dots and twenty antidots combing to leave ten antidots, -10.

20.4 The opposite of "no piles and no holes" is "no holes and no piles." That's still nothing. So, maybe $-0 = 0$? (Read on.)

20.5 The opposite of the opposite of the opposite of file piles is … five piles? (Read on.)

20.6 a) and e), which emphasis on e)!

20.7

a) $67 + -33 + -88 + 102 + -46$
b) $-9 + -9$
c) $-9 + -(-9)$

20.8

a) -4 b) 1065 c) 0 d) 30 e) -30 f) 0

21.1 We're asking for the opposite of x dots and y antidots, altogether. That would be x antidots and y holes.

$$-(x + -y) = -x + y$$

21.2 3

21.3 a) three antidots b) three dots c) -16 d) 16

21.4 a) $(15 - 2) - (13 - 2) = 13 - 11 = 2$ or $(15 - 2) - (13 - 2) = 15 + -2 + -13 + 2 = 2$
b) $100 - (100 - 2) = 100 - 98 = 2$ or $100 - (100 - 2) = 100 + -100 + 2 = 2$
c) $-9003 - (2 + 1) = -9003 - 3 = -9003 + -3 = -9006$
or $-9003 - (2 + 1) = -9003 + -2 + -1 = -9006$

21.5 a) $x - (x - 2) = x + -x + 2 = 0 + 2 = 2$
b) $20 - (15 - y) = 20 + -15 + y = 5 + y$
c) Let's work with the nested parentheses from the inside out (as one should!)

First $1 - (1 - R) = 1 + -1 + R = R$

So $1-\big(1-(1-R)\big)=1-R$

So $\Big(1-\big(1-(1-(1-R))\big)\Big)=1-(1-R)=1+\ -1+R=R$

22.1 Abstractly: Suppose there are P cards in the small pile. Then there are $20-P$ cards in the large pile.

This table shows what happens if we assume there are R red cards in the small pile.

	Small Pile P	Large Pile 20-P
# Reds	R	10 - R
# Blacks		20 - P - (10 - R)

Now, $20-P-(10-R)=20+\ -P+\ -10+R=10-P\ +R.$

The number of red cards in the small pile (R) differs from the number of black cards in the large pile ($10-P+R$) by $10-P$.

This matches what is seen when conducting the experiments in the question.

22.2 Nope! The exact same reasoning applies even if the initial amounts of each liquid are not the same.

23.1 We're asking for the number we need to add to -17 to get to zero. Well, 17 does the trick! Thus $-(-17))$ is 17.

23.2 $-(5-7)$ is the number we need to add to $5-7$ to get zero. Well, $-5+7$ does the trick! Thus $-(5-7)$ is $-5+7$.

23.3 We're seeing that $4\times(-5)$ and $5\times(-4)$ are the same.

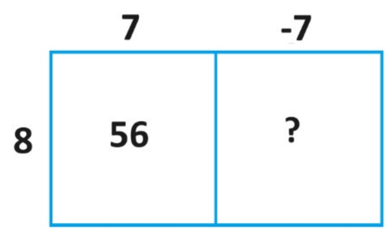
23.4 I am curious as to what you think.

23.5 It is not important to "get" this – but maybe you "got" it?

23.6
a) Imagine combining groups of dots and antidots. The final result will be all these objects combined together, no matter the path you took to combine them. We have, in the end $-2+5+6+\ -7+0+\ -3+6=5$ dots.
b) Include -15 to the start of the game.

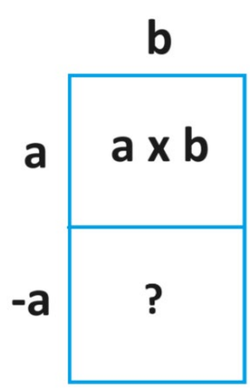
24.1 It is the same.

24.2



We have two pieces that add to zero. We must have $8 \times (-7) = -56$.

24.3 We have two pieces that add to zero. We must have $(-a) \times b = -(a \times b)$.



24.4 i) Draw 0×8 rectangle with zero as $1 +\ -1$ and reason as per the previous two questions.
ii) "Pulling out a negative sign": $(-1) \times 8 = -(1 \times 8) = -(8) = -8$
iii) Factoring: $8 + (-1) \times 8 = 8 \times (1 +\ -1) = 8 \times 0 = 0$.
So, $(-1) \times 8$ adds to 8 to give zero.
It must be -8.

25.1

	10	5
10	100	50
4	40	20
	210	

	10	5
20	200	100
-6	-60	-30
	210	

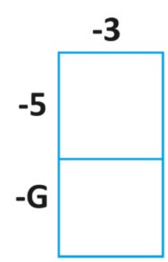
	20	-5
10	200	-50
4	80	-20
	210	

	20	-5
20	400	-100
-6	-120	+30
	210	

25.2 i) Pull out a negative sign, twice, or ii) Look at a $0 \times (-12)$, being zero, with the $11 + -11$ for the first zero, or iii) play with 19×18 four ways, thinking of 19 as $30 + -11$ and 18 as $30 + -12$.

25.3
a) $(-1) \times (-1) = 1$, positive.
b) $(-1) \times (-1) \times (-1) = 1 \times (-1) = -1$
c) $(-1) \times (-1) \times (-1) \times (-1) \times (-1) \times (-1) \times (-1) \times (-1) \times (-1) = 1 \times 1 \times 1 \times 1 \times (-1) = -1$

25.4
$(-5 - G) \cdot (-3) = (-5 + -G) \times (-3)$
We see from that picture that this is $15 + 3G$ or $3G + 15$ if one prefers.

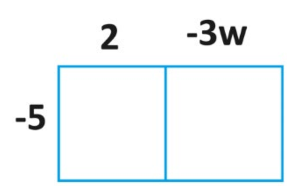


25.5
Now $2 - 5(2 - 3w) = 2 + -5 \times (2 + -3w) = 2 + (-5) \times (2 + -3w)$.

And $(-5) \times (2 + -3w) = -10 + 15w$.

So,

$$2 - 5(2 - 3w) = 2 + -10 + 15w = -8 + 15w = 15w - 8$$



25.6 $x \times (10y - 2x - 1)$

25.7 $3(F + -2) + (-2)(F + -3)$

$$= 3F + -6 + -2F + 6$$

$$= 3F + -2F$$

$$= (3 - 2)F = 1 \times F = F$$

25.8 I am curious how it was explained to you.

25.9

	10	a
10	**100**	**10a**
a	**10a**	**a²**

	10	2a
10	**100**	**20a**

a) $(10 + a)^2 = 100 + 10a + 10a + a^2 = 100 + 20a + a^2$

Also,

$10(10 + 2a) + a^2 = 100 + 20a + a^2$.

The expressions are the same.

b) We have the formula $(10 + a)^2 = 10(10 + 2a) + a^2$ working for any number a.

If we decide that a is the number four, then the formula is reading $14^2 = 10 \times 18 + 4^2$.

If we decide that a is the number five, then the formula is reading $15^2 = 10 \times 20 + 5^2$.

We have a "number trick:"

> *Two square a number in the teens, look at the last digit.*
> *Double that last digit, add* 10*, multiply the result by* 10.
> *Add to this answer the last digit squared.*
> *The result is the answer you seek.*

(It's easier to just get out a calculator or just use the area model directly without this trick!)

25.10

a) $(-2) + (-7) = -9$ **Negative**

b) $4 \cdot (-3) = -12$ **Negative**
c) $(-10) \times 110 = -1100$ **Negative**
d) $-(-6) = 6$ **Positive**
e) $(-5)(-4) = 20$ **Positive**
f) $(-10)(-3 - 5) = 80$ **Positive**
g) $7 - (10 - 3) = 0$ **Zero**

h) $(4-8)(2-1)=-4$ **Negative**
i) $(-143)\cdot(542)\cdot 0\cdot(-1987)=0$ **Zero**
j) $(5-2)\cdot(100-50-20)=90$ **Positive**
k) $(f-a)(f-b)(f-c)(f-d)(f-e)(f-f)(f-g)(f-h)=0$ **Zero** [Look at $f-f=0$ within the product!]

25.11 Of course the term "friendlier" is subjective.

a) $3x-12$
b) $3x-12$
c) $22-9y$
d) F

25.12 Did you get the given answers?

26.1 a) Looking at the rectangles we see

$$(10+a)(10+b)=100+10a+10b+ab$$

and

$$\begin{aligned}20(a+b)+(10+\ -a)(10+\ -b)&=20a+20b+100+-10a+\ -10b+ab\\&=(20+-10)a+(20+\ -10)b+100+ab\\&=10a+10b+100+ab\end{aligned}$$

These are the same!

	10	b
10	100	10b
a	10a	ab

	a	b
20	20a	20b

	10	-b
10	100	-10b
-a	-10a	ab

b) This is finger and toe multiplication.

26.2 a) Same as before, but a closed fist is now worth 6 and each finger up is worth 12.

b) $(6 + a)(6 + b)$ does equal $12(a + b) + (6 - a)(6 - b)$.

26.3 We can show that $(4 + a)(6 + b)$ is the same as $10(a + b) + (6 - a)(4 - b)$, but this is not really the finger multiplication trick: each finger up is worth 10 (for ten fingers in all), but we're not quite multiplying the digits down (which are $4 - a$ and $6 - b$).

27.1 a) Two piles each with seven antidots.
b) One antidot is left over.
c) It does!

27.2
$-20 = 2 \times (-10)$
$-402 = 2 \times (-201)$
$-5 = 2 \times (-3) + 1$
$-401 = 2 \times (-201) + 1$
$-5 = 2 \times (-3) + 1$
$-1 = 2 \times (-1) + 1$

27.3
If N is an even integer, then $N = 2a$ for some integer a.
If M is an odd integer, then $M = 2b + 1$ for some integer b.
Consequently,

$$N - M = N + -M = 2a + -(2b + 1) = 2a + -2b - 1$$

Now let's be a bit sneaky and write the "-1" at the end there as $-2 + 1$

$$N - M = 2a + -2b + -2 + 1 = 2(a - b - 1) + 1$$

So, $N - M$ is one more than twice an integer, and so is odd.

27.4 Any game with and even number of odd integers will do.

27.5
a) Adding together one hundred and five -1s and one hundred and six $+1$s will give the answer $+1$, ni matter in which order they are added together. The grasshopper is sure to end one inch to the right of where it started.

b) To compute the grasshoppers final displacement from start, we're adding together the lengths 1 inch, 2 inches, 3 inches, all the way up to 50 inches, but some of these jumps will be in the negative direction and some in the positive direction. Nonetheless, we'll be adding together twenty five odd integers and twenty five even integers, and the result will be an odd integer.

That number then can't be zero as zero is an even integer.

The grasshopper cannot return to start.

c) There is nothing special about the unit of an "inch" in part b). As long as the grasshopper is jumping 1 unit of length, then 2 units of length, then 3 units of length, and so, the grasshopper won't be able to return to start after 50 jumps.

In part c), let "two inches" be our unit of length. The grasshopper cannot return to start.