Chapter 4

Exploding Dots and the Power of Place Value

Contents

28. A Mind-Reading Trick

Let's start off this chapter with a classic piece of math magic.

ACTIVITY

A MIND-READING TRICK

In front of an audience present the following five groups of numbers. Write them on a chalk board or on a Power-Point slide or write out the groups of numbers on big cards.

GROUP A	GROUP B	GROUP C	GROUP D	GROUP E
16 20 24 28	8 12 24 28	4 12 20 28	2 10 18 26	1 9 17 25
17 21 25 29	9 13 25 29	5 13 21 29	3 11 19 27	3 11 19 27
18 22 26 30	10 14 26 30	6 14 22 30	6 14 22 30	5 13 21 29
19 23 27 31	11 15 27 31	7 15 23 31	7 15 23 31	7 15 23 31

The numbers 1 through 31 appear throughout these five groups, numbers that match the days of a month. Ask your audience members to each silently think of the day of the month they were born (a number) and look for their birthday among each of the five groups.

Now perform the mindreading trick by having the following question-and answer session with individual audience members.

"Suzzy. Is the number you are thinking of in group A?"	"Yes."
"Is the number you are thinking of in group B?"	"Yes."
"Is it in group C?"	"No."
"Is it in group D?"	"No."
"Group E?"	"Yes!"
"Ahh … you were born the 25th day of the month!"	"Wow! How did you know?"

GROUP A	GROUP B	GROUP C	GROUP D	GROUP E
16 20 24 28	8 12 24 28	4 12 20 28	2 10 18 26	1 9 17 25
17 21 25 29	9 13 25 29	5 13 21 29	3 11 19 27	3 11 19 27
18 22 26 30	10 14 26 30	6 14 22 30	6 14 22 30	5 13 21 29
19 23 27 31	11 15 27 31	7 15 23 31	7 15 23 31	7 15 23 31

Have the same question-and-response conversation with a few more audience members, noting each time which groups elicit a "yes" answer. Any given birthday number is then simply the sum of the top-left corner numbers in each group with a YES answer. For example, Suzzy answered YES YES NO NO YES. Groups A, B, and E have top left numbers 16, 8, and 1, respectively, and indeed $16 + 8 + 1 = 25$ was sure to be Suzzy's birthday.

As practice, check that if Sameer is thinking the number 13, he will answer NO YES YES NO YES and indeed $8 + 4 + 1 = 13$.

Do this as many times as your audience desires. Invite them to figure out what you are doing, and then, why what you are doing works!

Note: Rather than ask each audience member five questions, it is easier to just ask each person "In which group or groups does your birthday appear?"

Question: Examine each of the cards. What do you notice about the numbers in each group? What do you wonder about them?

Here are some things I personally notice and question.

- Group E contains all the odd numbers.
- Group A contains all the numbers 16 and above.
- The top left corner numbers of these groups, 1, 2, 4, 8, and 16, are doubling.

The doubling numbers 1, 2, 4, 8, 16 are clearly the key to this trick.

To explain their magic, let me tell you another personal story that isn't true.

MUSINGS

Musing 28.1

a) Derarcha says that her birthday appears in groups A, D, and E. On what day of the month was she born?

b) Erik says that his birthday appears in every group. On what day of the month was he born?

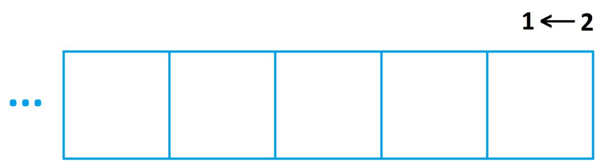
Musing 28.2 Before reading on the rest of this chapter, do you want to try to figure out why the mind-reading trick works all on your own? (This is a YES/NO question!)

29. A Story that is Not True

When I was a child, I invented a machine (not true!). It isn't a physical machine, but a machine I can work with using pencil and paper, or better yet, a board and erasable markers.

The machine is nothing more than a row of boxes that extend as far to the left as I please. For example, I could have 5 boxes heading off to the left, or 7 boxes, or 777 boxes.

And in this untrue story, I gave this machine a name. I called it a "two-one machine" both written and said in a funny backwards way. (I knew no different as a child.)



And what do you do with this machine? You put in dots, of course! I like dots.

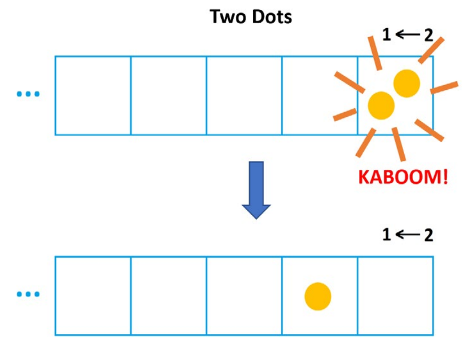
The thing note is that **dots always go into the rightmost box.**

Put in one dot, and, well, nothing happens: it stays there as one dot. Ho hum.

One Dot

But put in a second dot – always in the rightmost box – and then something exciting happens.

Whenever there are two dots in a box they explode and disappear – KAPOW! – to be replaced by one dot, one box to the left.



Question: Do you see now why I called this a "1 ← 2 machine" written in this funny way?

We see that two dots placed into the machine yields one dot followed by zero dots.

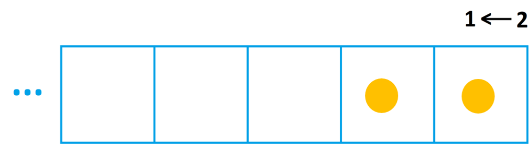
Actually, it's zero dots and zero dots and zero dots and one dot and zero dots for my picture with five boxes (this would be a much longer sentence if I happened to draw 707 boxes). But let's ignore all the zeros for empty boxes to the left and write the result of putting two dots in the machine as "**1 0**"—one dot followed by zero dots.

The 1 ← 2 machine code for the number *two* is **1 0**.

Warning: The code "1 0" looks like the ordinary number ten. But when we are thinking of codes, let's read 1 0, for instance, as "one zero."

Putting in a third dot – always the rightmost box – gives the picture one dot followed by one dot.

Three dots



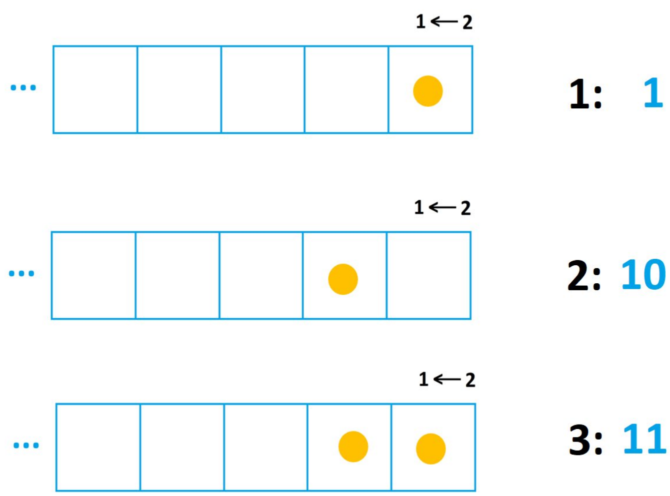
3: 11

We're getting codes for numbers.

Just one dot placed in the machine stayed as one dot.
The 1 ← 2 machine code for the number *one* is **1**.

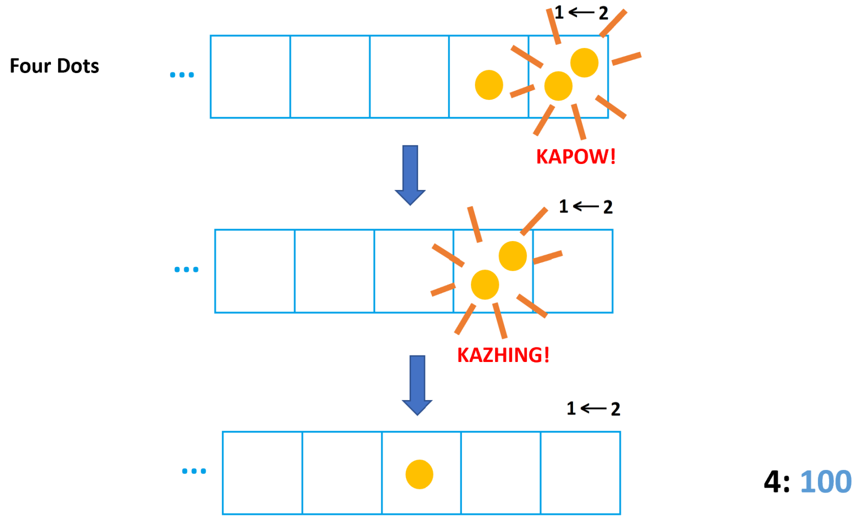
Two dots placed into the machine, one after the other, yielded one dot in a box followed by zero dots.
The 1 ← 2 machine code for the number *two* is **1 0**.

Adding a third dot in the machine gave the code **1 1** for *three*.



Adding a fourth dot into the machine after three dots have exploded is particularly exciting: we are in for multiple explosions!

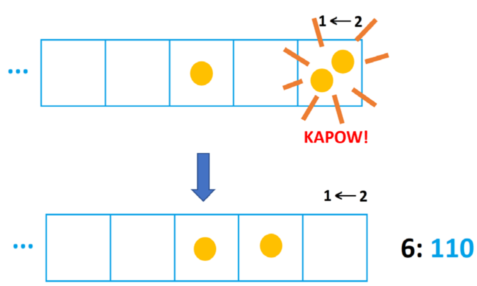
The 1 ← 2 machine code for *four* is **1 0 0**. (Remember: Read this is “one zero zero.”)



We can keep going, adding dots to the rightmost box one at a time.

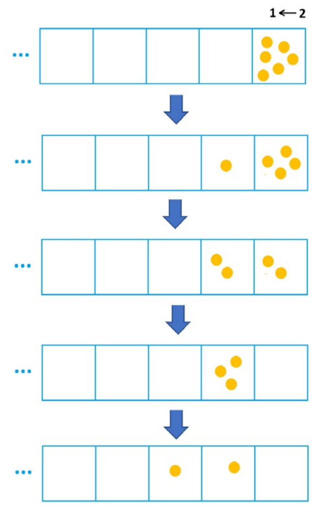
The 1 ← 2 machine code for *five* is **1 0 1**.

And the code for *six*? Adding one more dot to the code for five gives **1 1 0** for *six*.



We can also get this code for *six* by clearing the machine and putting all six dots in at once. Pairs of dots will explode in turn to each produce one dot, one box to their left.

Here is one possible series of explosions. (Sound effects omitted!)



Practice 29.1: Do you get the same final code of **1 1 0** if you perform explosions in a different order? (Try it!)

Practice 29.2: What is the 1 ← 2 machine code for the number *thirteen*?
(It turns out to be **1101**. Can you get that answer?)

There are hours of fun to be had playing with codes in a 1 ← 2 machine.

Practice 29.3 Here are the 1 ← 2 machine codes for the first ten numbers.
Care to work out the code for all the numbers up to twenty?
(This is technically a YES/NO question.)

1: 1	**4: 100**	**7: 111**	**10: 1010**
2: 10	**5: 101**	**8: 1000**	
3: 11	**6: 110**	**9: 1001**	

Practice 29.4 Experiment and try to find the number which has 1 ← 2 machine code **10111**.

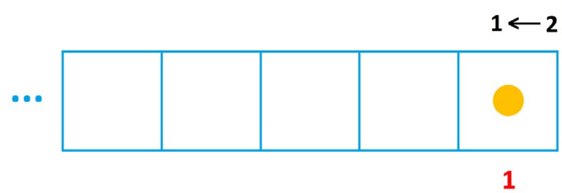
Explaining the 1 ← 2 machine

Let's figure out what's really going on with this machine and the codes it produces.

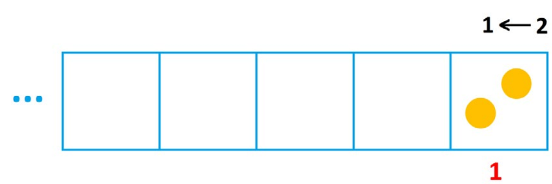
We set up matters in our 1 ← 2 machine so that …

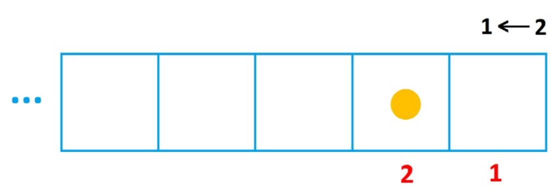
> *Whenever there are two dots in any one box, they "explode," that is, disappear, to be replaced by one dot, one place to their left.*

Also, the machine is set up so that dots in the rightmost box are always worth one.

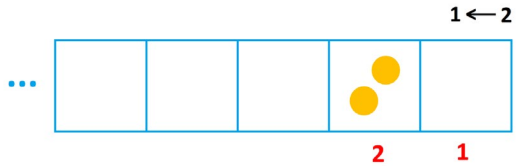


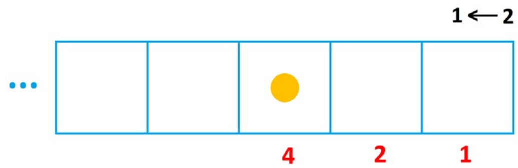
With an explosion, two dots in the rightmost box are equivalent to one dot in the next box to the left. And since each dot in the rightmost box is worth 1, each dot one place over must be worth two 1s, that is, 2.



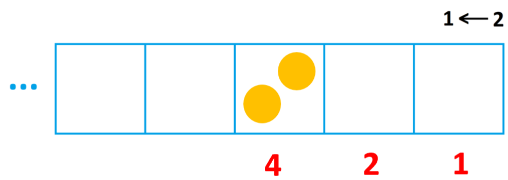


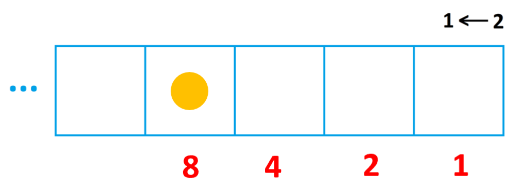
And if you happen to have two dots in this second box, they world explode to make one dot, one place to the left. That new dot is with two 2s, so it is worth 4.





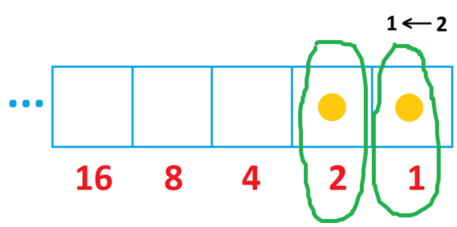
And two 4s makes 8, so a dot in the next place to the left is worth 8.



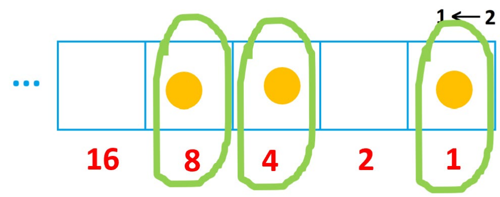


And two 8s make 16, and two 16s make 32, and two 32s make 64, and so on.

Earlier we saw that the $1 \leftarrow 2$ machine code for the number *three* is **1 1**. And one dot and one dot in the last two boxes does indeed make give a picture of dots with total value three: $2 + 1 = 3$. Neat!



If you tried Practice 29.2, you would have seen that the 1 ← 2 machine code for *thirteen* is **1 1 0 1**. Now we can see too—literally see!—that this code is correct: one 8 and one 4 and no 2s and one 1 does indeed make thirteen!



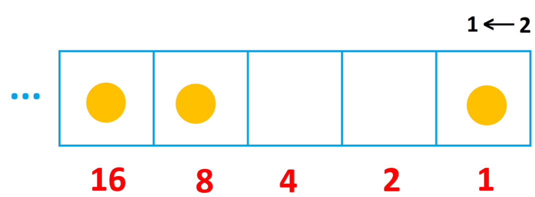
Practice 29.5 Answering question 29.4 this new way, which number which has 1 ← 2 machine code **1 0 1 1 1**?

Practice 29.6 The number seventeen equals 16 + 1. What is the 1 ← 2 machine code for *seventeen*?

Practice 29.7 What is the is the 1 ← 2 machine code for *thirty*?

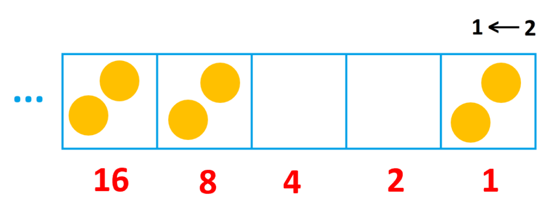
The 1 ← 2 machine shows how to write any number as a sum of the doubling numbers 1, 2, 4, 8, 16, and so on: just put in that many dots into the rightmost box, let them explode, and the code of **0**s and **1**s tells you which doubling numbers to use to make that number.

Practice 29.8 The 1 ← 2 machine code for the number *twenty-five* is **11001**.



Double twenty-five is fifty.

What code do you get if you double the number of dots in each box in the picture and the conduct the explosions. Do you get the 1 ← 2 machine code for the number *fifty*?



Practice 29.9
The largest number with a five-digit 1 ← 2 machine code is *thirty-one* with code **11111**. What is the smallest number with a five-digit code?

Some people might answer this question by noting that we could write the code for *one* as **00001** if we decided to write the **leading zeros** to the left. (For that matter, we could write the code for *zero* as **00000**.) The answer to this question could be one (or it could be zero).

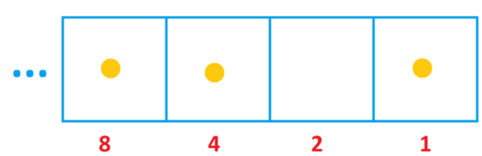
But let's do indeed follow the convention of *not writing leading zeros* in the codes for numbers. In which case, the number with the smallest five-digit code has code **10000**. That corresponds to the number *sixteen*.

Some Language

People call the 1 ← 2 machine codes for numbers the **binary** representations of numbers (with the prefix *bi*- meaning "two"). They are also called **base two** representations. One only ever uses the two symbols **0** and **1** when writing numbers in binary.

Computers are built on electrical switches that are either on, or off. So, it is very natural in computer science to encode all arithmetic in a code that uses only two symbols: say **1** for "on" and **0** for "off." Thus, base two, binary, is used at the heart of computer science.

BASE TWO



thirteen = 1101

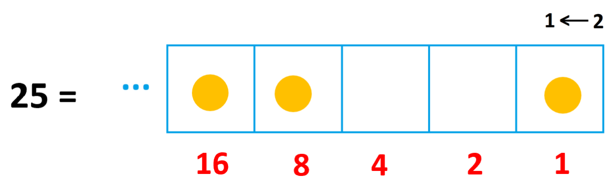
Explaining the Mind-Reading Trick

Let's end this section by explaining the mind-reading trick.

Recall that Suzzy was thinking of the number 25 and saw it groups A, B, and E.

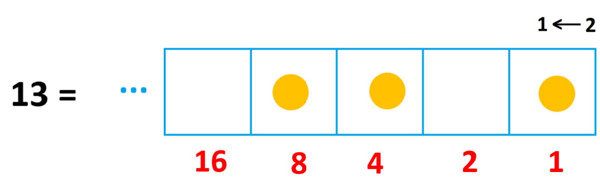
GROUP A	GROUP B	GROUP C	GROUP D	GROUP E
16 20 24 28	8 12 24 28	4 12 20 28	2 10 18 26	1 9 17 25
17 21 25 29	9 13 25 29	5 13 21 29	3 11 19 27	3 11 19 27
18 22 26 30	10 14 26 30	6 14 22 30	6 14 22 30	5 13 21 29
19 23 27 31	11 15 27 31	7 15 23 31	7 15 23 31	7 15 23 31

The $1 \leftarrow 2$ machine code for twenty-five shows us that $25 = 16 + 8 + 1$.



And in creating the cards, I made sure the number 25 appeared in the card with 16 at its top left, in the card with 8 at its top left, and in the card 1 at its top left. That is, I made sure that the number 25 appears in groups A, B, and E so that if Suzzy were to say "A, B, and E" I'd know to compute $16 + 8 + 1$ and deduce her number.

Here's the binary code for *thirteen.*



And look! I made sure 13 appears in cards B, C, and E so that I could quickly compute $8 + 4 + 1$ if those three groups are mentioned!

Practice 29.10: The binary code for *ten* is **1010**. In which cards should ten appear? Does it? (And does it appear only in those groups?)

Practice 29.11: Does it make sense that the number 31 appears in each and every card?

It is a happy coincidence that largest number we can make with the doubling numbers 1, 2, 4, 8, and 16, namely 31, matches the number of days in a (long) month of the year. Having people think of their birthdays makes this trick feel a little more mysterious and special.

Practice 29.12: Why does card A contain all the numbers 16 and higher?

Practice 29.13: Why does card E contain all the odd numbers?

Practice 29.14: The number with the biggest six-digit binary code **111111** is

$$32 + 16 + 8 + 4 + 2 + 1 = 63.$$

Create a six-card mind-reading trick where an audience member is to pick a number between 1 and 63 and, after telling you in which cards they see their chosen number, you read their mind.

What numbers do you place on each card?

MUSINGS

Musing 29.15 Could a number ever have code **100211** in a $1 \leftarrow 2$ machine, assuming we always choose to explode dots if we can?

Musing 29.16 The list of doubling numbers continues indefinitely.

1, 2 4, 8, 16, 32, 64, 128, 256, 512, 1024, ….

Since $100 = 64 + 32 + 4$, we see that the binary code for one hundred is **1100100**.

a) What is the binary code for two hundred?
b) What is the binary code for one thousand?

Musing 29.17 What is the largest number with a ten-digit binary code?

Musing 29.18 Here's a fun question.

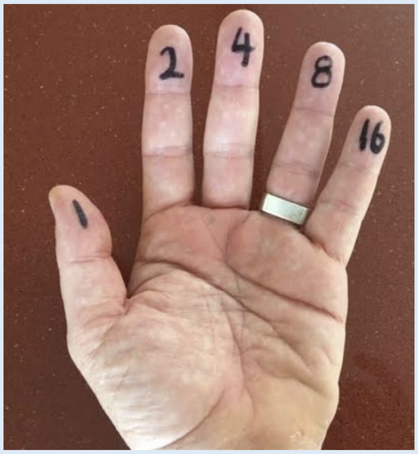
As we noted, the codes from a $1 \leftarrow 2$ machine are called the *binary* codes with the prefix "bi" meaning two. Can you guess what each of these English words have to do with the number two?

bicycle **binoculars** **bisect** **biped**
bivalve (an oyster and a clam are examples of bivalves)

ACTIVITY

Musing 29.19

Imagine the numbers 1, 2, 4, 8, and 16 painted on your fingers as shown.



You make any number from 1 to 31 by raising some fingers and lowering others.

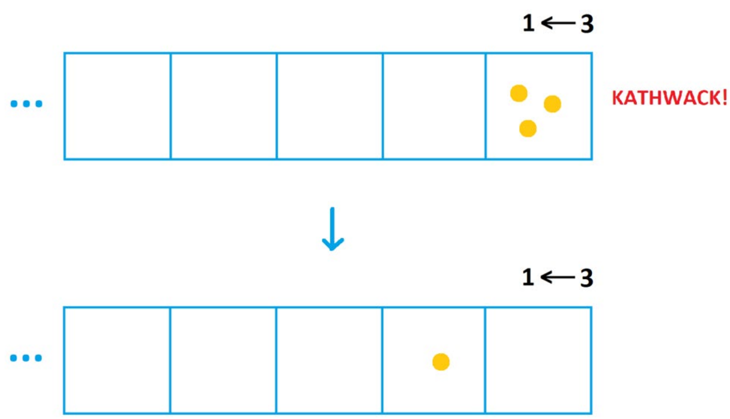
It's quite fun to run through the numbers 1, 2, 3, 4,5, ..., 30, 31 in turn on your fingers. Try it!

How high can you count in binary if you use two hands?

30. More Machines

Let me continue the story of the machines.

After playing with the 1 ← 2 machine for a while, I suddenly had a flash of insight. Instead of playing with a 1 ← 2 machine, I realized I could also play with a 1 ← 3 machine. (Again, this is written and read backwards: a “three-one “machine.) Now, whenever there are three dots in a box, they explode away to be replaced with one dot, one box to the left.



Question: Work out the codes for the numbers *one*, *two*, *three*, *four*, *five*, and *six* in a 1 ← 3 machine. Double-check that my list here is correct.

1:	**1**	**4:**	**11**
2:	**2**	**5:**	**12**
3:	**10**	**6:**	**20**

Practice 30.1 What’s the code for *thirteen* in a 1 ← 3 machine?
(Perhaps try putting thirteen dots all at once into the rightmost box.)

Practice 30.2 Which number has1 ← 3 machine code **222**?

And hours of fun are to be had playing with numbers in a 1 ← 3 machine.

But then …
Another flash of insight!

Instead of playing with a 1 ← 3 machine, I could create a 1 ← 4 machine!

And then …
Another flash of insight!

Instead of playing with a 1 ← 4 machine, I could create a 1 ← 5 machine!

And then …
Another flash of insight!

Instead of playing with a 1 ← 5 machine, I could create a 1 ← 6 machine!

Practice 30.3: In a 1 ← 4 machine, how many dots in a box explode to make one dot, one place to the left?

Practice 30.4: What is the code for *ten* in a 1 ← 5 machine?

Practice 30.5: Multiple Choice
Would the code **30721** be stable in a 1 ← 6 machine?

a) No. In the group of seven dots in the middle, six of them would explode to make one dot, one place to the left.
b) Answer a) is correct.

And then …

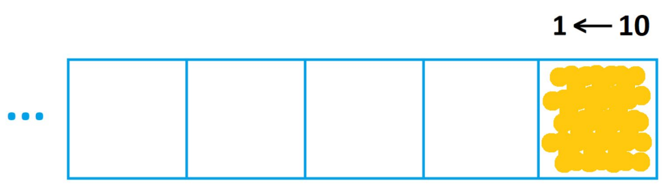
I decided to go wild!

The 1 ← 10 Machine

Okay. Let’s go all the way up to a 1 ← 10 machine. Crazy!

And let’s put in 273 dots. Crazier!

What is the 1 ← 10 machine code for the number 273?



273:

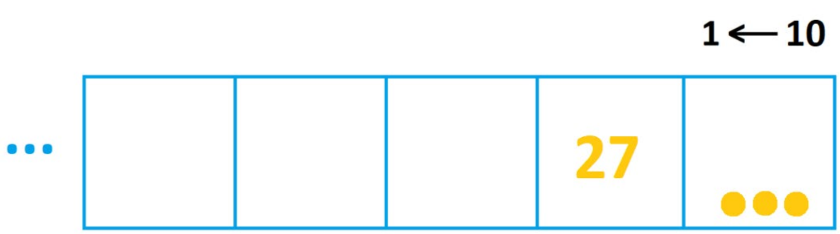
I thought my way through this by asking a series of questions.

Are there any groups of ten that will explode? Certainly!

How many explosions will there be initially in that rightmost box? Twenty-seven.

Will there be any dots left behind? Yes. Three.

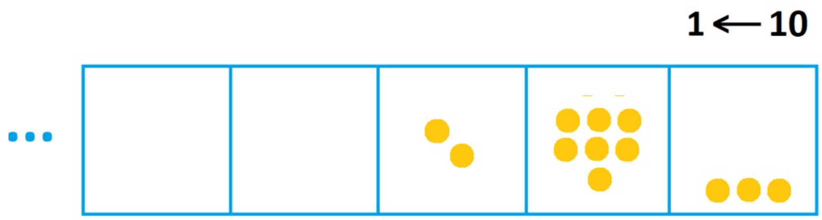
Okay. So, there are twenty-seven explosions, each making one dot one place to the left, leaving three dots behind. (I hope it is okay that just write the number “27” rather than draw twenty-seven dots in a box.)



Next questions …

Will there be any more explosions? Yes. Two more.

Any dots left behind? Seven left behind.



273: 273

And look at what we have!

The 1 ← 10 machine code for two hundred seventy-three is … **273**.

Whoa!

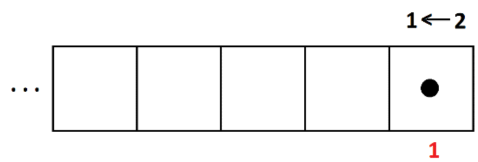
Something curious is going on!

Practice 30.6 Draw twenty-four dots in the rightmost box of a 1 ← 10 machine. Do all the explosions and see the code "24" appear for the number twenty-four.

Practice 30.7
a) Imagine three-hundred-eighty-two dots in the rightmost box of a 1 ← 10 machine. Can you think your way through all the explosions to see that code **382** will appear?
b) What do you think will be the 1 ← 10 machine code for the number one-thousand, eight-hundred, and forty-nine?

Explaining the Machines

We saw what was going on with the 1 ← 2 machines in the last section. This machine is set so that dots in the rightmost box are always worth one.



And as we keep adding dots to the machine, we follow the rule:

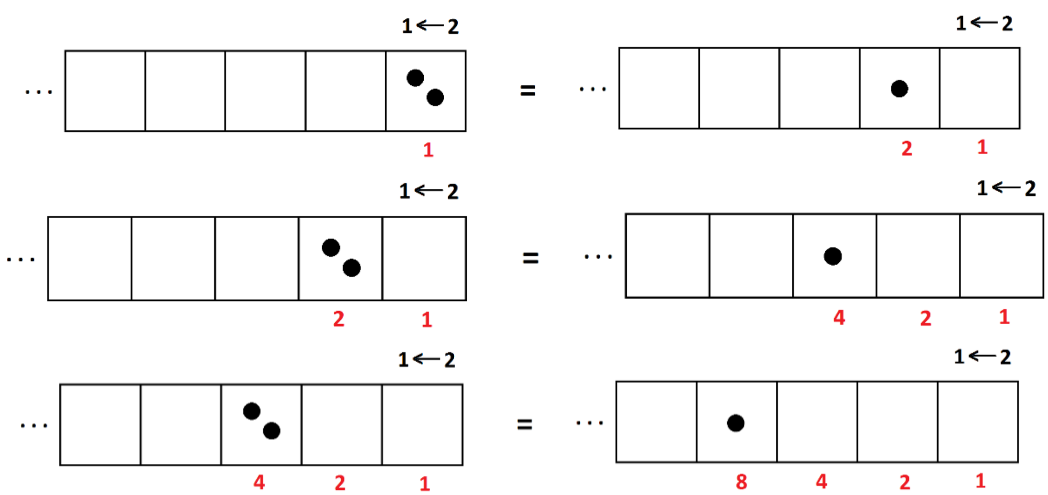
> *Whenever there are two dots in any one box, they "explode," that is, disappear, and are replaced by one dot, one place to their left.*

This means, with an explosion, two dots in the rightmost box are equivalent to one dot in the next box to the left. And since each dot in the rightmost box is worth 1, each dot one place over must be worth two 1s, that is, 2.

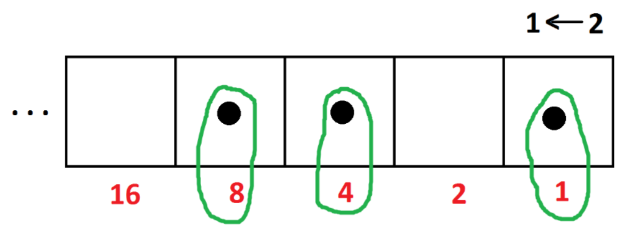
And two dots in this second box are equivalent to one dot, one place to the left. Such a dot must be worth two 2s, that is, worth 4.

And two 4s makes 8 for the value of a dot the next box over.

And two 8s make 16, and two 16s make 32, and two 32s make 64, and so on.

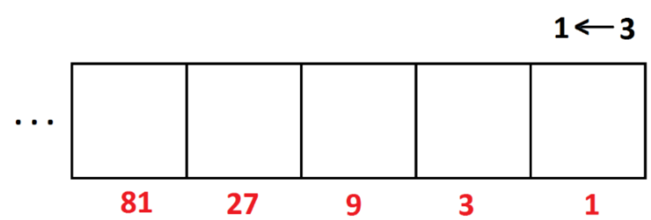


And recall, the 1 ← 2 machine code for thirteen is **1101** and we saw this is correct by looking at the values of the dots in this machine.



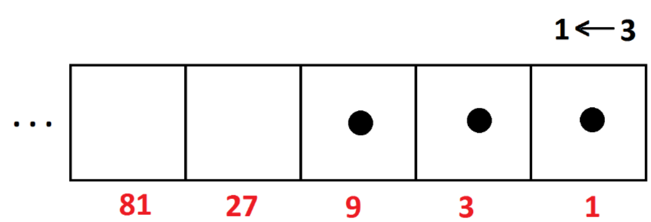
The same idea must be at play for the 1 ← 3 machine.

Here dots in the rightmost box again are each worth one, but now three dots in any one box are equivalent to one dot, one place to the left. We get the dot values in this machine by noting that three 1s is 3, and three 3s is 9, and three 9s is 27, and so on.



Practice 30.8 If one more box was added to the machine, what would be the value of a dot in that sixth box?

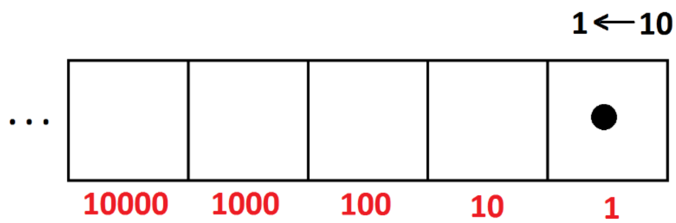
Earlier I asked for the 1 ← 3 machine code of number thirteen. It's **111**, and we see that this is correct because one 9 and one 3 and one 1 do indeed make thirteen.



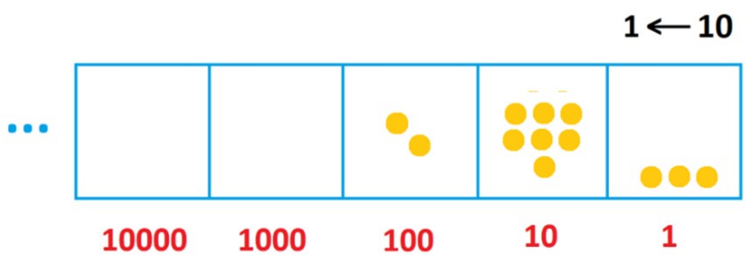
The 1 ← 3 machine codes for numbers are called **ternary** or **base three** representations of numbers. Only the three symbols **0**, **1**, and **2** are ever needed to represent numbers in this system.

Practice 30.9 What is the base-3 code for the number *fifteen*?

In a 1 ← 10 machine, dots in the rightmost place are worth 1, and we have that ten ones make 10, ten tens make 100, ten one-hundreds make 1000, and so on. A 1 ← 10 machine has 1, 10, 100, 1000, and so on, as dot values.



The code for the number 273 in a 1 ← 10 machine is **273** and we see that this is absolutely correct because two 100s, seven 10s, and three 1s do make 273.



273: 273

In fact, we even speak the language of a 1 ← 10 machine. In words, 273 is

273 = two hundred seventy three

We literally say, in English at least, two hundreds and seven tens (that “ty” is short for *ten*) and three.

We call the 1 ← 10 codes for numbers the **base ten** or **decimal** representation of numbers.

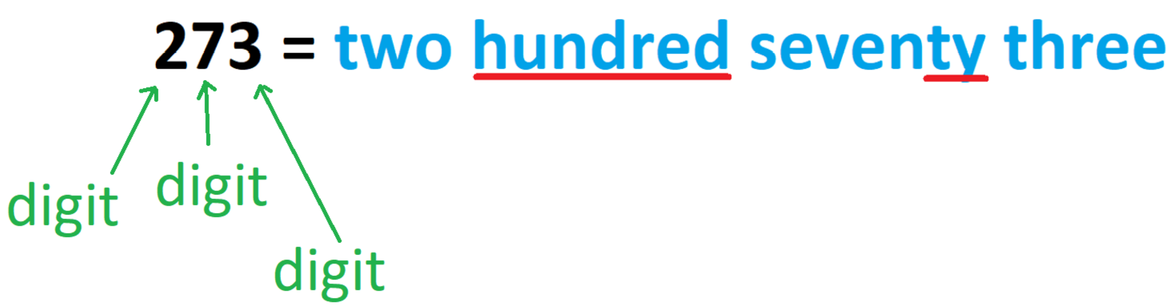
The prefix *dec*- means “ten” and we have, for example, that a *decagon* is a figure with ten sides and a *decade* is a period of ten years. Also, *December*, at one time, was the tenth month of the year. (Care to research the history of how we came up with twelve months for the year and why these months have the names they do?)

Practice 30.10 What’s a *decapod*?

Number Bases in Society

We have discovered **number bases:** base two, base three, base ten, and so on.

And we have noticed that society has decided to speak the language of the base ten machine.



> **Question:** Why do you think we humans settled on 1 ← 10 machine to play with? Why do we like the number ten so much for matters of arithmetic and counting?

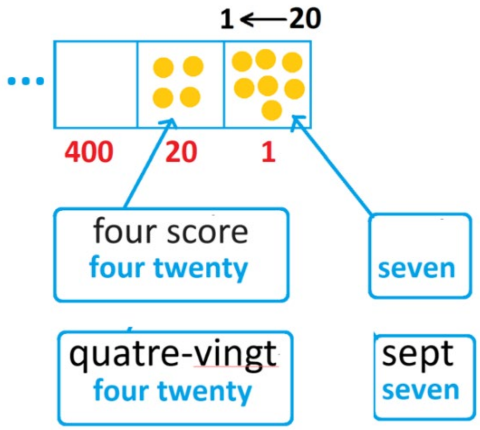
One answer could be because of our human anatomy: we are born with ten digits on our hands (thumbs and fingers—and toes too—are called *digits*) and we are thus naturally prone to think in terms of groups of ten. The connection is particularly strong when you notice we use the word **digit** for the individual symbols when we write in long numbers!

> **Practice 30.11:** Some cultures on this planet used base twenty. Why might they have chosen that number, do you think?

In fact, there are vestiges of base twenty thinking in the western European culture of today.

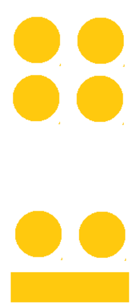
Abraham Lincoln's famous Gettysburg address begins: "Four score and seven years ago." The word *score* is an old word for "twenty" and so Lincoln was saying: "four-twenties and seven years ago." That's 87 years.

And, in French, the number 87 is said just this way too: *quatre-vingt-sept* translates, word for word, as "four twenties seven."



Ibo, a Nigerian language of the south-east region, says 87 as *ogu anon na asaa*, which, word-for-word, translates to "twenty into four and seven." It's the same idea of thinking of four groups of twenty and seven more.

The Maya of the Mayan Civilization of Mesoamerica used a base twenty system, but they wrote their numbers vertically. They used dots and bars in their number system. Can you see how the picture below is meant to be read as "four twenties and seven"?



Question: Do you know another language? How is 87 said in your language: in a base-ten way, like English, in a base-twenty way, like French, or some other way?

It seems natural for humans to develop base ten and base twenty number systems given our anatomy. But some cultures on Earth actually developed a base-12 number system instead!

This could be because there is also a very natural way to count to twelve on one hand. We have four long digits naturally broken into three segments each and a natural pointer to point at them—a thumb!

In some parts of the world, often in India and south-east Asia, it is still very common for people to count this way.

And there is still "twelveness" in our everyday life.

> How many items in a dozen? Answer: 12
> How many inches in a foot? Answer: 12
> How many hours are in a day—literally? Answer: 12!

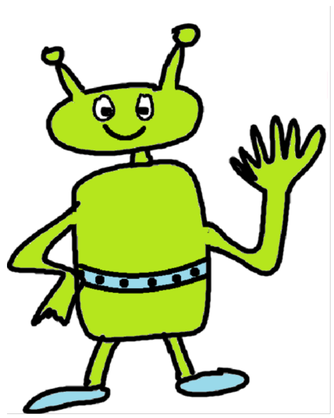
The first clocks humans constructed were sundials, which, of course, work only during daylight hours. The ancient Egyptians divided the day into ten main hours and two extra twilight hours—early morning and early evening. With the invention of water clocks and mechanical clocks people could start measuring time during the night as well. Since the day was divided into 12 hours, they divided the night into 12 hours as well.

Question: The number twelve is very handy in matters of weights and measures. For instance, one might not want to purchase a full unit of some quantity, but perhaps only a half, or a third, or a quarter of that quantity. (These fractions are common in everyday operations.)

a) Eggs are sold in quantities of twelve. How many eggs is half a dozen of them? A third of a dozen? A quarter of a dozen?
b) If eggs were sold in quantities of ten (to follow base ten thinking), how many eggs is half of that quantity? A quarter of that quantity? A third of that quantity?

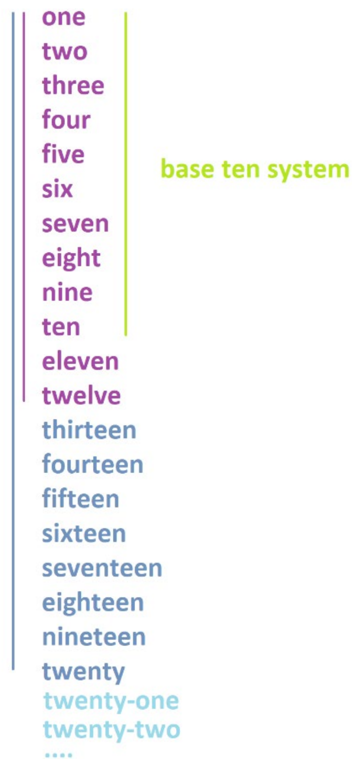
Question: Multiple Choice
I happen to know that Martians have six fingers on each of two hands. What base do you think they might use in their society?

A. If they focus on one hand, perhaps base six?
B. If they focus on two hands, perhaps base twelve?
C. If they focus on fingers and toes (assuming they have two feet and six toes on each foot), perhaps base twenty-four?
D. Everything stated in A, B, and C is reasonable.



The English Language is a Bit of Everything

Even though we write our numbers with ten digits—base ten—we have special words for the first twelve numbers. We apparently still think twelve-ness is important.



After twelve, we fall into a systematic naming pattern: thirteen, fourteen, fifteen, and so on, up to nineteen, using the special suffix "teen."

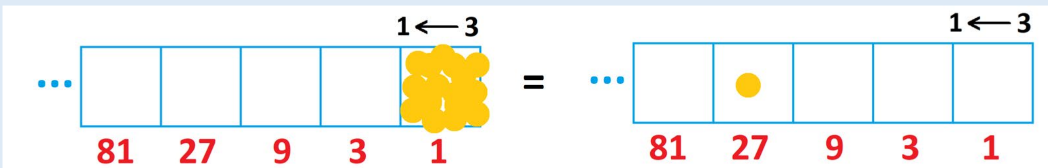
But then at twenty, we change to a different systematic pattern, which we then stay with—twenty-one, twenty-two, …, thirty-one, …, sixty-one, …, and so on. So, we think the first twenty numbers are somewhat special too, but that the numbers thereafter don't need special reference.

So, in speaking English, we follow a base-10 number system, but use special words for the first 12 numbers and a special pattern for numbers up to 20, and then change to a different pattern for saying numbers twenty and larger.

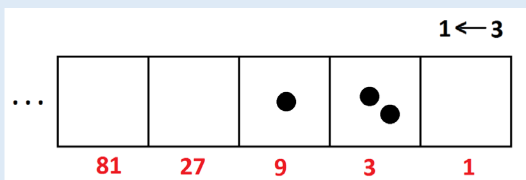
English is trying to do it all. That makes English hard and strange! (How does anyone learn English?)

MUSINGS

Musing 30.12 Place 27 dots into a 1 ← 3 all at once. Verify that after all the explosions one dot four boxes to the left remains. (This shows that a single dot in that box really is "worth" 27.)



Musing 30.13 The 1 ← 3 machine code for *fifteen* is **120**. What number has base-three code **1200**? What number has base-three code just **12**?



Musing 30.14 What is the code for the number 67,620,938 in a 1 ← 10 machine?

Musing 30.15 Write the codes for the number *thirty-one* in each of the following machines.

a) A 1 ← 10 machine.
b) A 1 ← 2 machine.
c) A 1 ← 3 machine.
d) A 1 ← 4 machine.
e) A 1 ← 5 machine.
f) A 1 ← 6 machine.
g) A 1 ← 31 machine.

Musing 30.16

In section 28 we presented a five-card mindreading number trick.
Would you like to consider another trick like that?

Present the following six groups of numbers to a friend and ask them to silently think of a number between 1 and 26.

GROUP A	GROUP B	GROUP C
1 4 7	2 5 8	3 4 5
10 13 16	11 14 17	12 13 14
19 22 25	20 23 26	21 22 23
GROUP D	**GROUP E**	**GROUP F**
6 7 8	9 10 11	18 19 20
15 16 17	12 13 14	21 22 23
24 25 26	15 16 17	24 25 26

As before, have the following conversation with your friend.

> "Shankar, tell me in which groups your secret number lies."
> "It's in groups A, C, and F."
> "Ahh. Your secret number is 22."

You might suspect that we are looking at the top left values in each group mentioned and adding them to find the secret number. And this is the case. Shankar's secret number 22 appears in the groups with corner numbers 1, 3, and 18 and indeed $18 + 3 + 1 = 22$.

GROUP A	GROUP B	GROUP C
1 4 7	2 5 8	3 4 5
10 13 16	11 14 17	12 13 14
19 22 25	20 23 26	21 22 23
GROUP D	**GROUP E**	**GROUP F**
6 7 8	9 10 11	18 19 20
15 16 17	12 13 14	21 22 23
24 25 26	15 16 17	24 25 26

As another example, 14 appears in groups B, C, and E with respective top left numbers 9, 3, and 2, and $9 + 3 + 2 = 14$.

The number 5 appears in groups B and C with top left numbers 3 and 2, respectively, and $3 + 2 = 5$.

The top left numbers of the cards are 1 and 2 (which is two copies of 1), 3 and 6 (which is two copies of 3), 9 and 18 (which is two copies of 9.

What *Exploding Dots* machine might be at play here, do you think?

Write the codes for the numbers 1 through 26 in the machine you have in mind.
Can you now explain why each number appears on the card that it does and why the trick works?

31. English is Weird

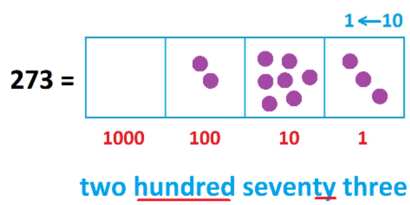
We noted in the last section that English is quirky. But it is actually down and outright weird!

For starters …

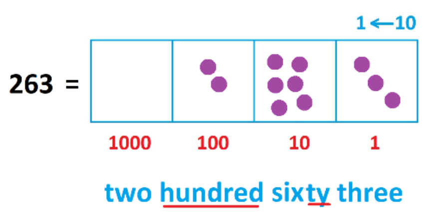
> **Question:** Have you ever noticed that the spelling of "weird" is weird?
> What happened to "i before e, except after c"?

The strangeness of English applies to how we write and say numbers too.

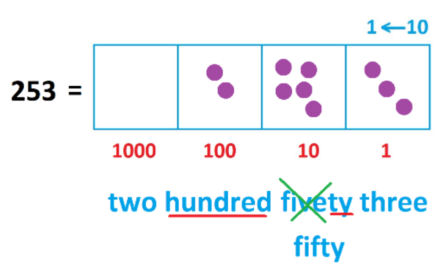
Here's the number 273 in a 1 ← 10 machine. It is literally two hundreds, seven tens (*ty* is short for "ten" in English), and three. Nothing too strange there. (Well, "ty" is a bit strange.)



And here is 263. Nothing strange here either.

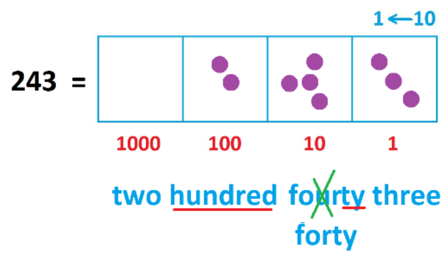


But listen to the number 253. We should say "two hundred five-ty three" but we don't. English has us say "fifty" instead of five-ty.

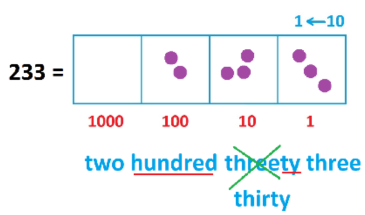


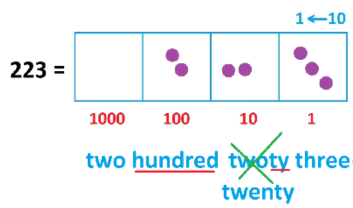
Now saying 243 out loud sounds right, but when writing it out we should write "four-ty," but English insists we write "forty" instead.

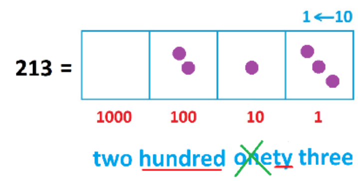
Weird!



And there are 233 and 223 and 213, each particularly weird.

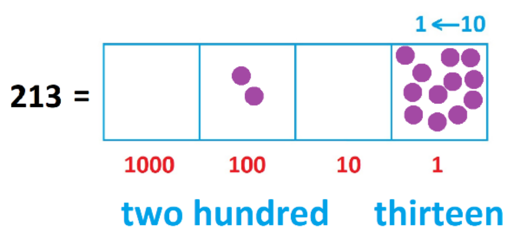




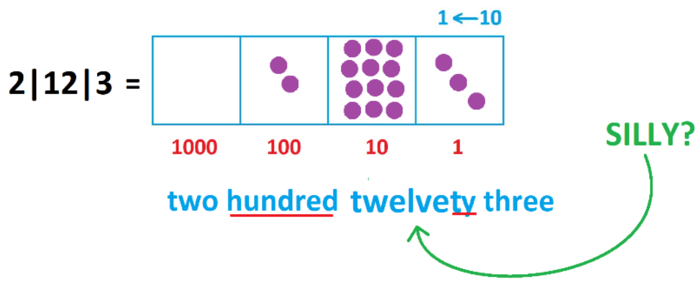


Question: Something additionally strange happens with the number 213. What do we say instead of "two hundred onety three"? Does what we say make sense?

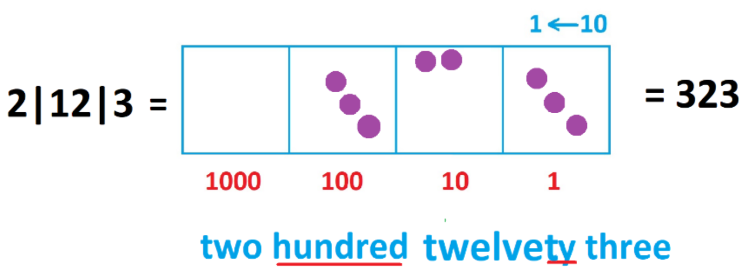
Question: When Katya thought about this question, she said that "two hundred and thirteen" is really this picture: two hundreds and an extra thirteen dots all in the ones place! What do you think? Are we allowed to have 13 dots in a single box?



Most people in society think it is silly to have a large number of dots in any single box. You would never say "two hundred and twelvety three," for example.



But, of course, we know ten dots in any box explode away to produce one dot, one place to their left. So, this number is really 323 in disguise.



Question 31.1: Draw a dots-and-boxes picture of "two-hundred eleventy three." What number is this?

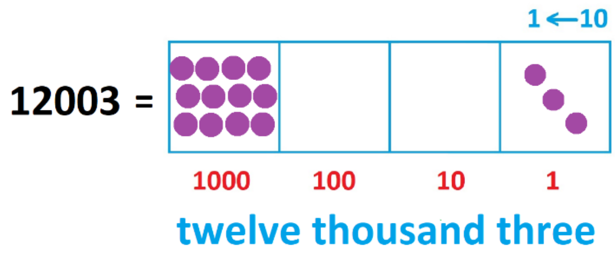
Question 31.2: Over a thousand years ago people spoke a version of English we today call "Old English." It included words equivalent to twelvety and eleventy.

What number is twelvety? What number is eleventy?
(Would saying "thirteen-ty" be just too silly?)

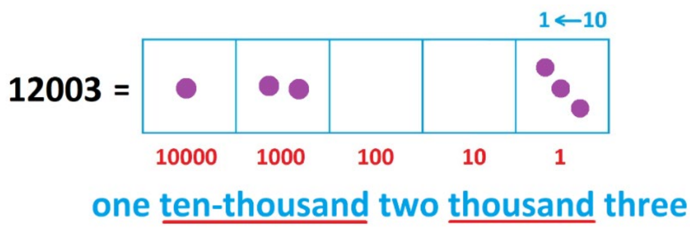
But society is inconsistent.

Question: How do you say the number 12,003? Draw a dots and boxes picture of the number.

The number 12,003 is pronounced “twelve thousand three” as though we have twelve thousands and three ones. Here we **do** allow more than nine dots in a box, it seems.



But we don’t write what we say. Here’s a picture of what we write:



If English and society were consistent, we would say “one ten-thousand two thousand three.” But we don’t!

What we’re learning here is that society and the English language has all sorts of strange demands on how we write and say numbers. The math is always solid and clear. It is just society that makes different, and sometimes strange, choices about speaking the math.

So, if society has permission to be strange, I say … let’s go for it! Let’s be a little strange too and use the strangeness to our advantage for doing math.

Let’s go through the arithmetic we’ve learned in school and see how ignoring society actually makes the mathematics so much simpler! We can always fix up what we do to make society happy at any time.

This is going to be fun!

Question: How is the number 12,003 pronounced in other languages?

MUSINGS

Musing 31.3 Here are four boxes of the $1 \leftarrow 10$ machine. Let's put twelve dots in each box. (We have 12 thousands, 12 hundreds, 12 tens, and 12 ones.)

12	12	12	12

a) How might you say this number out loud?

b) Did you notice as you said this out loud that saying "twelve thousand" sounds acceptable to our ear? And saying "twelve hundred" is okay too? Saying "twelvety" is odd, but saying "and twelve" sounds okay?

That is, did you notice that 75% of what you said in part a) sounds acceptable to our ear?

c) What number is 12|12|12|12 really?

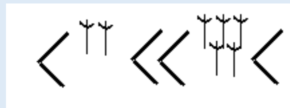
d) How do you say your answer to part c) out loud? Do you start with "One ten-thousand, three thousand, ..."?

Musing 31.4 What number is "eleven-thousand eleven-hundred eleventy eleven"?

Musing 31.5 The Babylonians of ca. 1700 B.C.E. were adept astronomers and based their representations of numbers on multiples of 60. (Is this because 60 is a number that unites base 10, base 12, and base 20 thinking? Is it because the number of days in a year is approximately six 60s? Mathematics historians are not sure.)

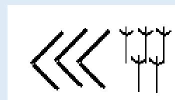
They used the symbols  and , respectively, for one and ten, and imagined they had numbers in boxes like a $1 \leftarrow 60$ machine. The values of the boxes were 1, 60, $60 \times 60 = 3600$, $60 \times 360 = 21600$m and so on.. For example,

represented $23 \times 60 + 11 = 1391$, and



represented $12 \times 360 + 25 \times 60 + 10 = 44710$.

a) Translate each of the following numbers into our base ten system.



b) Write the numbers 323 and 2016 using the Babylonian system.

c) There were problems with the Babylonian's method. Try to write the numbers 61 and 1200 and 3600 in their system. What do you notice?

Remnants of the Babylonian system still exist today. We measure time in units of 60 (sixty seconds in a minute, sixty minutes in an hour) and divide circles into 360 degrees (°). Furthermore, each degree is divided into sixty minutes (‘) and each minute into sixty seconds (“).

EXTRA
As we saw in this question, the Babylonians had a system for representing numbers that used only two symbols and relied on the placement of symbols to represent counts of units, counts of 60, counts of 360, and so on. They developed a **place value numeral system**.

But their lack of a symbol for zero to indicate a lack of any one of these particular quantities causes confusion.

Our current base ten place-value system does use zero to indicate a lack of ones, or tens, or hundreds, and so on, in writing numbers. This system was invented and used by Indian scholars around 100 C.E. and was widely adopted throughout the middle east by the year 900 CE. Some 300 years later it was noticed and used by scholars in Europe.

d) Chinese scholars of ancient times also used a place-value numeral system based on the number ten. Did they too make use of a symbol for zero within their place-value system?

32. Addition

Let’s now focus on the base-ten place-value system, the one our society today likes best. That is, let’s look at doing arithmetic numbers written in $1 \leftarrow 10$ machine codes.

And let’s have some quirky fun with it all!

Here’s an addition problem:

> Compute $251 + 124$.

Such a problem is usually set up this way.

$$\begin{array}{r} 251 \\ +\,124 \\ \hline \end{array}$$

This addition problem is easy to compute: $2 + 1$ is 3, and $5 + 2$ is 7, and $1 + 4$ is 5.
The answer 375 appears.

$$\begin{array}{r} 251 \\ +\,124 \\ \hline 375 \end{array}$$

But did you notice something curious just then?

I worked from left to right—just as I was taught to read. But this is the opposite to what most people are taught to do in a math class: always work from right to left.

> **Question:** Does it matter? Compute the problem from right to left instead. Do you get the same answer 375?
>
> Why are we taught to work right to left in mathematics classes?

Perhaps the issue at hand is hidden because the problem we just did is "too nice." We should do a more awkward addition problem, one like $358 + 287$.

$$\begin{array}{r} 358 \\ +\,287 \\ \hline \end{array}$$

Okay. Let's do it!
But I'll be naughty and go left to right again.

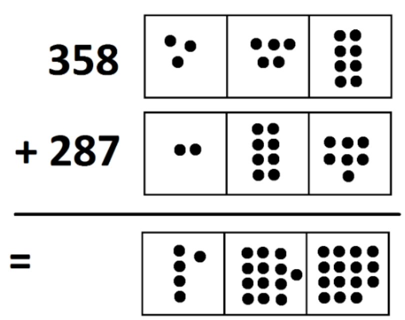
$3 + 2$ is 5 and $5 + 8$ is 13 and $8 + 7$ is 15

The answer five-hundred thirteen-ty fifteen appears. (Remember, "ty" is short for *ten*.)

$$\begin{array}{r} 358 \\ +\,287 \\ \hline 5\,|\,13\,|\,15 \end{array}$$

And this answer is absolutely, mathematically correct! You can see it is correct in a $1 \leftarrow 10$ machine.

Draw pictures of 358 and 287.



5 | 13 | 15

Adding together 3 hundreds and 2 hundreds really does give 5 hundreds.
Adding together 5 tens and 8 tens really does give 13 tens.
Adding together 8 ones and 7 ones really does give 15 ones.

“Five-hundred thirteen-ty fifteen” is absolutely correct as an answer – and I even said it correctly. We really do have 5 hundreds, 13 tens, and 15 ones. There is nothing mathematically wrong with this answer. It just sounds weird.

But, as we realized in the last section, English is inconsistent about what it thinks is weird and what it thinks is not. I say, let’s use weirdness to our mathematical advantage and be weird when it makes matters natural and easy.

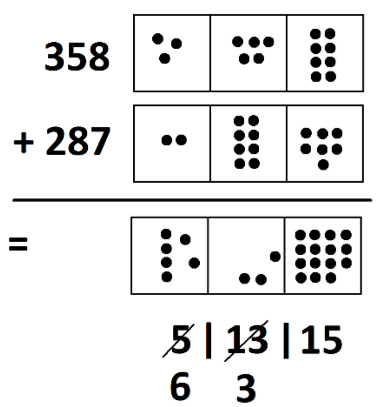
> Can we fix up this strange-sounding answer for society’s sake—not mathematics’ sake, but for society’s sake?

The answer is yes! We can do some explosions. (This is a $1 \leftarrow 10$ machine, after all.)

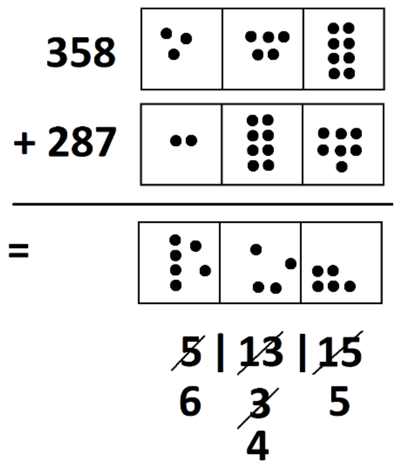
Which do you want to explode first: the 13 or the 15?

It doesn’t matter! Let’s explode from the 13.

Ten dots in the middle box explode to be replaced by one dot, one place to the left.



The answer “six hundred three-ty fifteen” now appears. This is still a lovely, mathematically correct answer. But society at large might not agree. Let’s do another explosion: ten dots in the rightmost box.



Now we see the answer "six hundred four-ty five," which is one that society understands. (Although, in English, "four-ty" is usually spelled *forty*.)

Practice 32.1 Write down the answers to the following addition problems working left to right and not worrying about what society thinks! Then, do some explosions to translate each answer into something society understands. You can check your final answers if you like by doing the calculations the traditional way or by using a calculator.

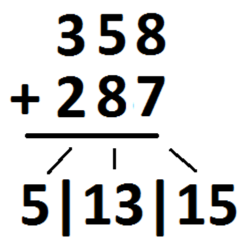
$$\begin{array}{r} 148 \\ +\ 323 \\ \hline = \end{array} \qquad \begin{array}{r} 567 \\ +\ 271 \\ \hline = \end{array} \qquad \begin{array}{r} 377 \\ +\ 188 \\ \hline = \end{array} \qquad \begin{array}{r} 582 \\ +\ 714 \\ \hline = \end{array}$$

$$\begin{array}{r} 310462872 \\ +\ 389107123 \\ \hline = \end{array} \qquad \begin{array}{r} 87263716381 \\ +\ 18778274824 \\ \hline = \end{array}$$

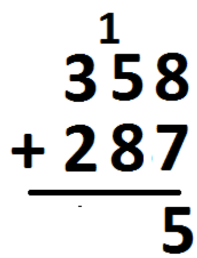
The Traditional Algorithm

How does this *Exploding Dots* approach to addition compare to the standard algorithm most people know?

Let's go back to the example $358 + 287$. Most people are surprised (maybe even perturbed) by the straightforward left-to-right answer $5 \mid 13 \mid 15$.



This is because the traditional algorithm has us work from right to left, looking at $8 + 7$ first.
But, in the algorithm we don't write down the answer 15. Instead, we explode ten dots right away and write on paper a 5 in the answer line together with a small 1 tacked on to the middle column. People call this **carrying the one** and it – correctly – corresponds to adding an extra dot in the tens position.



Now we attend to the middle boxes. Adding gives 14 dots in the tens box ($5 + 8$ gives thirteen dots, plus the extra dot from the previous explosion).

And we perform another explosion.

$$\begin{array}{r} {}^{1}\,{}^{1} \\ 358 \\ +\,287 \\ \hline 45 \end{array}$$

On paper, one writes “4” in the tens position of the answer line, with another little “1” placed in the next column over. This matches the idea of the dots-and-boxes picture precisely.
And now we finish the problem by adding the dots in the hundreds position.

$$\begin{array}{r} {}^{1}\,{}^{1} \\ 358 \\ +\,287 \\ \hline 645 \end{array}$$

The traditional algorithm works right to left and does explosions (**carries**) as one goes along. On paper, it is swift and compact, and this might be why it has been the favored way of doing long addition for centuries.

The *Exploding Dots* approach works left to right, just as we are taught to read in English, and leaves all the explosions to the end. It is easy to understand and kind of fun.

Both approaches, of course, are good and correct. It is just a matter of taste and personal style which one you choose to do. (And feel free to come up with your own new, and correct, approach too!)

MUSINGS

Musing 32.2 When doing a long addition problem the traditional way, we work from right to left and conduct explosions as we go along. Most people use the word "carrying" instead of "exploding." For example, the problem shown has us "carry a 1" two separate times.

$$\begin{array}{cccc} & {}^{1}4 & {}^{1}8 & 7 \\ + & 1 & 2 & 4 \\ \hline = & 6 & 1 & 1 \end{array}$$

a) When adding two numbers the traditional way, why won't we ever carry a 2?

b) Give an example of adding a three-digit number and a two-digit number that results in carrying a 1 exactly one time.

c) Give an example of adding a three-digit number and a two-digit number that results in carrying a 1 exactly two times.

d) Could there be an example of adding a three-digit number and a two-digit number that results in carrying a 1 three times?

Musing 32.3 Multiple Choice
Which of the following are mathematically correct answers to the following addition problem?

$$\begin{array}{cccc} & 5 & 3 & 9 \\ + & 2 & 8 & 9 \\ \hline = & & & \end{array}$$

a) 7|11|18
b) 8|1|18
c) 8|2|8
d) These are all the same number, really, and all are mathematically correct. It's just that society doesn't like options a) and b).

33. Multiplication

Without regard to what society thinks what would be a good—and correct—three-second answer to this multiplication problem?

2784 x 3

Can you see that 6 | 21 | 24 | 12 is the natural answer to this? After all, if we have 2 thousands and 7 hundreds and 8 tens and 4 ones and we triple everything, we'd have 6 thousands and 21 hundreds and 24 tens and 12 ones as a result. Easy! The answer is "six thousand, twenty-one hundred, twenty-four-ty, twelve."

2	7	8	4	x 3	=	6	21	24	12

Now, how can we fix up this answer for society? With some explosions of course!

Let's do two explosions from the 24 first, say. (It doesn't matter which explosions we do when). It gives

6 | 23 | 4 | 12

Now maybe explode from the 12.

6 | 23 | 5 | 2

Now from the 23.

8 | 3 | 5 | 2

We have that $2784 \times 3 = 8352$.

Practice 33.1 Compute each of the following in the same manner.

a) 2784×2
b) 2784×4
c) 2784×5
d) 2784×10

"To Multiply by Ten, Add a Zero." Huh?

Here's a true story this time.

When I was in school, I was told a rule for multiplying by ten: *just add a zero.*

This rule made no sense to me as stated. To compute 213×10, for instance, you don't add zero.

$$\begin{array}{r} 213 \\ +\quad 0 \\ \hline = \end{array} \qquad \text{HUH?}$$

Of course, I realized that people meant, "tack a zero to the end of the number."

$$213 \text{ x } 10 = 2130$$

Why does multiplying a number by ten seem to have the effect of appending a zero to the digits of the number? *Exploding Dots* thinking explains why.

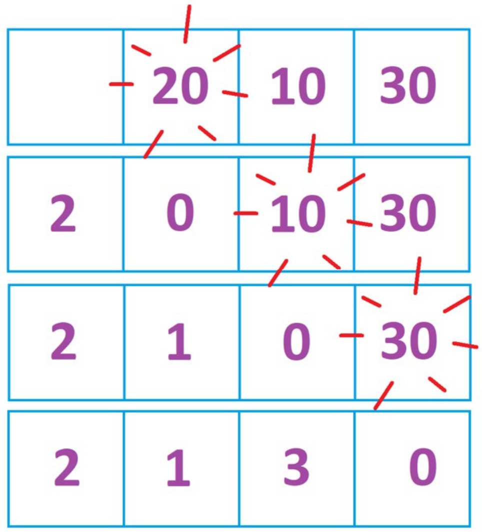
Here's the number 213 in a $1 \leftarrow 10$machine.

	2	1	3

And here is 213×10.

	20	10	30

Now let's perform the explosions, one at a time. We'll need the extra box to the left.



We have that 2 groups of ten explode in the hundreds place to give 2 dots one place to the left, and 1 group of ten explodes in the tens place to give 1 dot one place to the left, and 3 groups of ten explode in the ones place to give 3 dots one place to the left. The net effect of what we see is all digits of the original number shifting one place to the left to *reveal* zero dots in the ones place.

Indeed, the end result looks like we just tacked on a zero to the right end of 213. But really it was a whole lot of explosions that pushed each digit one place to the left to leave an empty box at the very end.

Question: How would you explain to a young student why 2222×10 is 22220?

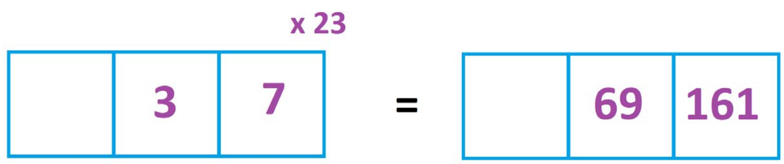
Practice 33.2
a) What must be the answer to 476×10?
b) What must be the answer to 476×100? Why?

Practice 33.3
a) What number was multiplied by 10 to give the answer 9190?
b) What must be the answer to $55740 \div 10$?
c) What must be the answer to $3310000 \div 100$? Why?

Long Multiplication

People wonder if there is a way to conduct long multiplication with dots and boxes. For example, can we compute 37×23 with *Exploding Dots*?

There is, but it is not very nice. This particular example requires you to know your multiples of 23.



Question: After explosions, what number is 69|161? Do you see it is 851?

Question 33.4 Fill in the question marks.

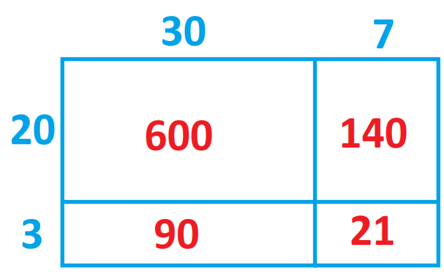
$37 \times 2 = 6 \mid 14 = 74$ $\qquad$ $37 \times 11 = 33 \mid 77 = ?$ $\qquad$ $375 \times 11 = 33 \mid ? \mid ? = ?$

The most natural way to conduct long multiplication, as we saw in Chapter 1, is to use the area model.

Here's a picture of 37×23.

We get $37 \times 23 = 600 + 140 + 90 + 21 = 851$.

Let’s be very clear about what is going on in this picture of 37×23.



We found a piece of the rectangle of area 600.

This came from computing 30×20. This is essentially the same as computing $3 \times 2 = 6$, but our answer is in the hundreds because we are really multiplying $30 = 3 \times 10$ and $20 = 2 \times 10$ and are seeing 3×2 multiplied by ten twice.

$$30 \times 20 = \mathbf{3} \times \mathbf{2} \times 10 \times 10 = 6 \times 10 \times 10$$

We found a piece of the rectangle of area 140.

This came from computing 7×20. This is essentially the same as computing $7 \times 2 = 14$, but our answer is in the tens because we are really multiplying 7 and $20 = 2 \times 10$ and are seeing 7×2 multiplied by ten once.

$$7 \times 20 = \mathbf{7} \times \mathbf{2} \times 10 = 14 \times 10$$

We found a piece of the rectangle of area 90.

This came from computing 30×3. This is essentially the same as computing $3 \times 3 = 9$, but our answer is in the tens because we are really multiplying $30 = 3 \times 10$, and 3. We see 3×3multiplied by ten once.

$$30 \times 3 = \mathbf{3} \times \mathbf{3} \times 10 = 9 \times 10$$

And we found a piece of area 21, which comes from computing 7×3.

$$\mathbf{7} \times \mathbf{3} = 21$$

In short, we’re just multiplying the digits of the two numbers in turn and are keeping track of whether the answers—6, 14, 9, 21—are in the hundreds, tens, or units.

This is what the school long-multiplication algorithm does—without the visuals of the area model to show what is happening. (Section 12.)

But look at this. What do you think of this area model/dots-and-boxes/long-multiplication mash up?

		3	7
x		2	3
		9	21
+	6	14	
=	6	23	21

We see the numbers 6, 14, 9, 21 in the right spots for being hundreds, tens, or units.

We can just add up what we see and get the answer 6 | 23 | 21.

Some explosions show that this is 851 in disguise.

Whoa!

> **Question:** There's a lot to "see" here. Is this making sense at all?

Practice 33.5 My calculator says that 3125×832 is 2600000. (That's the sensible way to do long multiplication in the 21st century!)

Can you get that answer the area/dots-and-boxes/long-multiplication mash-up way?

```
   3 1 2 5
x    8 3 2
__________
=
```

Just so you can compare efforts, this is what I wrote when I tried this question.

```
                3    1    2    5
        x            8    3    2
=               6 |  2 |  4 | 10
           9 |  3 |  6 | 15 |  0
     24 |  8 | 16 | 40 |  0 |  0
     24 | 17 | 25 | 48 | 19 | 10

= 2 | 6 | 0 |  0 |  0 |  0 | 0
```

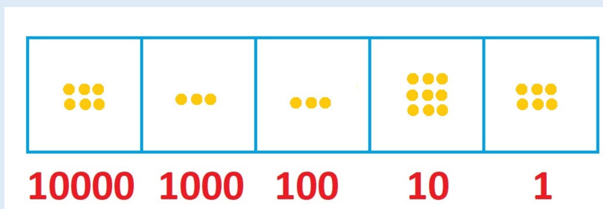
MUSINGS

Musing 33.6 Compute each of the following.

a) 26417×4
b) 26417×5
c) 26417×9.
d) 26417×10

Musing 33.7 Compute 236×34 first with a calculator (as any sensible human would do), and then with the hybrid area/dots-and-boxes approach.

Musing 33.8 Which number was multiplied by 3 to give this picture?



Musing 33.9

a) Evgenya thought of 37×11 as $37 \times (10 + 1)$, which is $37 \times 10 + 37 \times 1$, to get the value $370 + 37 = 407$.

   How might she compute 125×11?

b) What's 313131×11? What's $3131313131313131 \times 11$?

c) What's $11 \times 11$and $11 \times 11 \times 11$ and $11 \times 11 \times 11 \times 11$?

Musing 33.10 Here's a hack for multiplying a two-digit number by 11.

To compute 14×11, say, split apart the digits 1 and 4 and write their sum $1 + 4 = 5$ between them.

$$14 \times 11 = 154$$

To compute 26×11, say, split apart the digits 2 and 6 and write their sum $2 + 6 = 8$ between them.

$$26 \times 11 = 286$$

Some more examples: $54 \times 11 = 594$, $80 \times 11 = 880$, *and* $08 \times 11 = 088$. (Ooh!)

One might have to explode some dots doing this trick.

$$67 \times 11 = 6|13|7 = 737$$

$$99 \times 11 = 9|18|9 = 1089$$

a) Can you explain why this hack works?

b) What's $693 \div 11$?

c) What's $924 \div 11$?

34. Subtraction

Recall from Chapter 3 that I do not believe that subtraction exists.

Subtraction is the addition of the opposite.

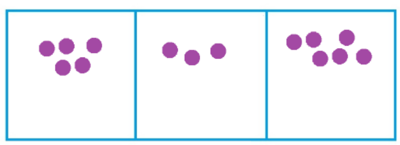
Let's play with some multidigit subtraction. Consider this subtraction problem $536 - 123$.

I think of this as being asked to compute "536 plus the opposite of 123," but schoolbooks don't and will present the problem this way and expect students to fill in the bottom line.

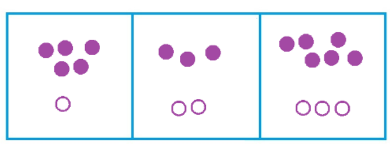
$$\begin{array}{r} 536 \\ -\ 123 \\ \hline = \quad \end{array}$$

But let's think about this the *Exploding Dots* way.

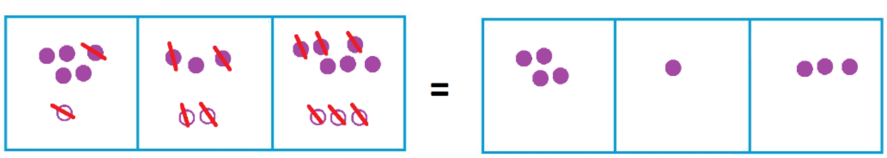
The first number, 536, looks like this in a $1 \leftarrow 10$ machine: five dots, three dots, six dots.



We must add to this the opposite of 123. That is, we're adding one anti-hundred, two anti-tens, and three anti-ones.



And now there are a lot of annihilations: POOF! and POOF POOF! and POOF POOF POOF!



The answer 413 appears.

And notice, we get this answer as though we just work left to right and saying

5 take away 1 is 4,

3 take away 2 is 1,

and

6 take away 3 is 3.

$$
\begin{array}{r}
536 \\
-\ 123 \\
\hline
=413
\end{array}
$$

Yes. Left to right again!

All right. That example was too nice. How about 512 – 347?

$$
\begin{array}{r}
512 \\
-\ 347 \\
\hline
=
\end{array}
$$

Going from left to right, we have

5 take away 3 is 2,
1 take away 4 is -3,

and

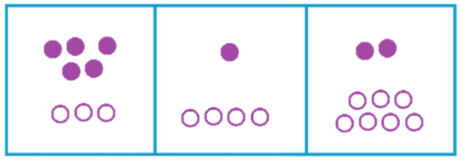
2 take away 7 is -5.

$$
\begin{array}{rccc}
 & 5 & 1 & 2 \\
- & 3 & 4 & 7 \\
\hline
= & 2 & |\,-3\,| & -5
\end{array}
$$

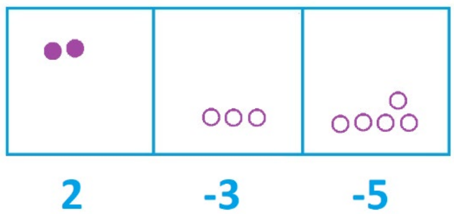
The answer is two-hundred negative-three-ty negative-five.

This answer is absolutely, mathematically correct!

Here's five hundreds, one ten and two ones together with three anti-hundreds, four anti-tens, and seven anti-ones.



And after lots of annihilations we are left with two actual hundreds, three anti-tens, and five anti-ones.



The answer really is "two-hundred negative-three-ty negative-five"!

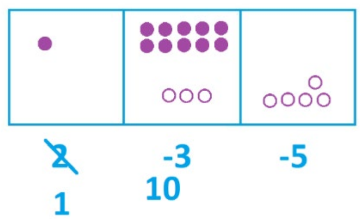
Question: But, of course, giving this answer to the subtraction problem will seem mighty weird to society. Can we fix up this mathematically correct answer to one acceptable by society?

Think about this for a moment. What could we possibly do to fix this answer?

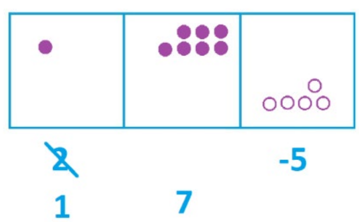
After a moment it might occur to you to unexplode dots. Any dot in a box to the left must have come from ten dots in the box just to its right, so we can just unexplode it to make ten dots.

Important Question: What sound effect should we make for unexploding?

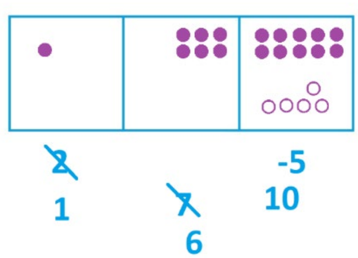
Okay. Let's unexplode one of the two dots we have in the leftmost box. Doing so gives this picture.



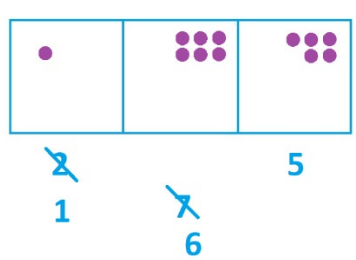
After annihilations, we see we now have the answer one-hundred seventy negative-five. Beautiful!



Let's unexplode again.



And with some more annihilations we see an answer society can understand: one-hundred sixty-five.



Practice 34.1 Show that the number $4\ |-2\ |1| -7$, "four-thousand, negative-two-hundred, onety, negative seven" is really the number 3803.

Practice 34.2 When Raj saw

$$\begin{array}{r} 5\ \ 1\ \ 2 \\ -\ 3\ \ 4\ \ 7 \\ \hline = 2|-3|-5 \end{array}$$

he wrote on his paper the following lines.

$$\begin{array}{r} 200 \\ -30 \\ -5 \end{array}$$

He then said that the answer has to be 165.

a) Can you explain what he is seeing and thinking?

b) What would Raj likely write on the page for $7109 - 3384$?

The Traditional Algorithm

How does the dots-and-boxes approach to long subtraction compare to the standard algorithm we were taught in school?

Consider again $512 - 347$.

$$\begin{array}{cccc} & 5 & 1 & 2 \\ - & 3 & 4 & 7 \\ \hline = & & & \end{array}$$

The standard algorithm has you start at the rightmost column to first consider *2 take away 7*.
This is deemed "not possible."

So, what do you do? You "borrow" a one from the left.

That is, you take a dot from the tens column and unexplode it to make ten ones. In this case, that leaves zero dots in the tens column.

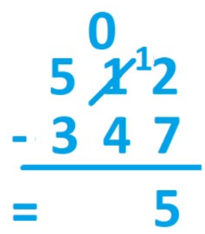
Now, we should write ten ones to go with the two in the ones column like this, but we don't.

$$\begin{array}{cccc} & 5 & \overset{0}{\cancel{1}} & \overset{10}{2} \\ - & 3 & 4 & 7 \\ \hline = & & & \end{array}$$

We do something a bit sneaky and write "12" rather than $10 + 2$, by putting a little 1 next to the digit 2 to make it look like twelve.

$$\begin{array}{cccc} & 5 & \overset{0}{\cancel{1}} & {}^{1}2 \\ - & 3 & 4 & 7 \\ \hline = & & & \end{array}$$

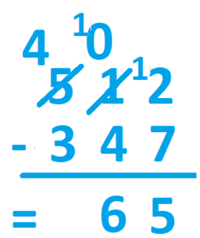
Then we say, “twelve take seven is five,” and write that answer.

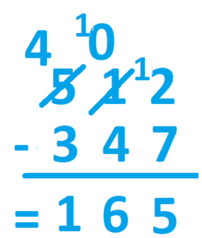


The rightmost column is complete. Shift now to the middle column.

We see “zero take away four,” which can’t be done. So, perform another unexplosion, that is, another “borrow,” to see $10 - 4$ in that column. We write the answer 6.

We then move to the last remaining column where we have $4 - 3$, which is 1.





This is complicated and looks mysterious at face value.

But if you draw picture of dots and antidots side-by-side with the steps of the standard algorithm, you can see what is going on.

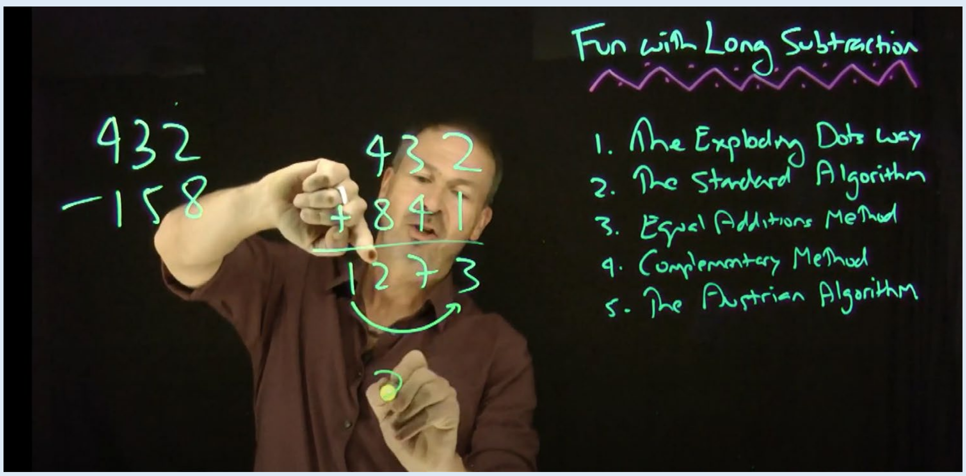
Of course, all correct approaches to mathematics are correct. It is just a matter of style as to which approach you like best for long subtraction. The traditional algorithm has you work from right to left and do all the unexplosions as you go along. The dots-and-boxes approach has you “just do it!” and conduct all the unexplosions at the end.

Both methods are fine and correct.

ACTIVITY

OTHER METHODS OF LONG SUBTRACTION

We might think our traditional algorithm for multi-digit subtraction is "standard," but at other times and in other cultures alternative approaches were considered standard. Can you make sense of the "Equal Additions Method," the "Complementary Method," and the "Austrian Method" I describe in this video? Perhaps pause the video before I go on to give the explanations away.



https://youtu.be/KJb5nfGvXp0

MUSINGS

Musing 34.3 Compute each of the following arithmetic problems three ways:

i) Just use a calculator and see what the answer is. (This is the smart approach.)
ii) Use a method you were taught in school for conducting long subtraction.
iii) Try the *Exploding Dots* way, working from left to right and fixing up the answer for society's liking afterwards.

You should get the same answer via each of these approaches.

$$\begin{array}{r} 6328 \\ -\ 4469 \\ \hline = \end{array} \qquad\qquad \begin{array}{r} 78390231 \\ -\ 32495846 \\ \hline = \end{array}$$

Thinking question along the way: As you fix up your answers for society, does it seem easier to unexplode from left to right, or from right to left?

Additional question: Do you think you could become just as speedy the dots-and-boxes way as you currently are with the traditional approach? (Not that one should become speedy at this. We have calculators, after all, if our goal is to just get answers.)

Musing 34.4 Create a subtraction problem whose answer is $1 \mid -2 \mid -3 \mid -4$. (What number is this answer?)

Musing 34.5 How might you handle this subtraction problem? How would you interpret its answer?

$$\begin{array}{r} 148 \\ -\ 677 \\ \hline = \end{array}$$

Musing 34.6 Something Mind-Bendy

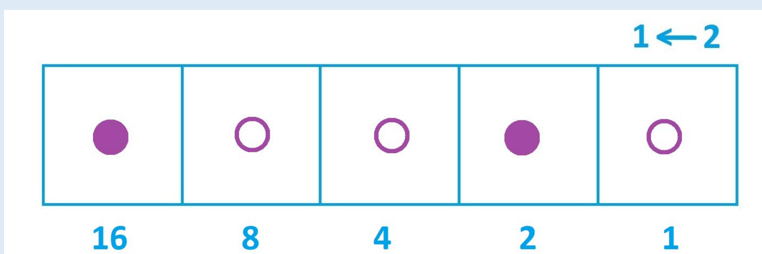
The number *thirteen* in base two is 1101. The number *six* is 110. Compute this long subtraction problem in base two. Do you get an answer that is the base two code for *seven*?
(After all, $11 - 6 = 7$ in ordinary arithmetic.)

$$\begin{array}{r} 1\,1\,0\,1 \\ -\quad 1\,1\,0 \\ \hline = \quad\quad \end{array}$$

Musing 29. 6 WILD EXPLORATION

Here's a puzzle to explore if you like.

Let's go back to a $1 \leftarrow 2$ machine and just work with its five rightmost boxes.



And let's fill each box with either one dot or one antidot.

The picture shows the number $16 + (-8) + (-4) + 2 + (-1)$, which is 5.

a) Show the that the number 17 can be represented in the same way, with each box containing either one dot or one antidot (and with no box empty).

b) Show that the number -17 can also be represented this way.

c) Find all the numbers that can be formed by placing single dots and antidots in the five rightmost boxes of this machine. How do you know your list is complete?

35. Division

Archeologists often find artifacts from an ancient past and wonder how they were made. They don't see the process that produced the object, just the end of result of the process.

In much the same way, some people like to think of division as the reverse of multiplication. (We saw this in section 17.)

For instance, from

$$3 \times 7 = 21$$

we could focus on the answer 21 and ask about the process: *What times 7 makes 21?*

This would now be considered a division problem and we'd write $21 \div 7 = 3$ after recognizing multiplying 7 by 3 is what makes 21.

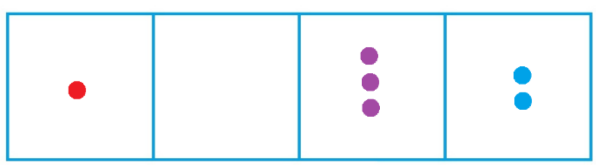
Let's revisit multiplication for a moment to then see if we can follow it backwards to get to division. We'll start with a straightforward multiplication problem, say,

$$1302 \times 3$$

(with answer 3906).

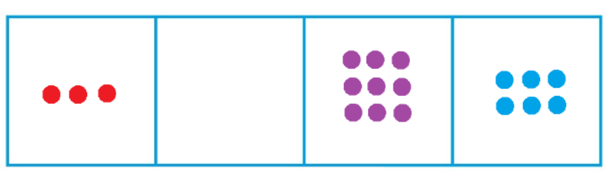
Here's what 1302 looks like in a $1 \leftarrow 10$ machine. (I've colored the dots for fun.)

1302 =



To triple this quantity, we just need to replace each dot in the picture with three dots. We see the answer 3906.

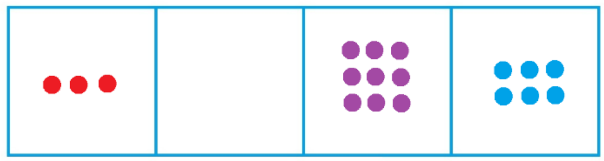
1302 x 3 =



Now let's go backwards and start with the answer to ask:

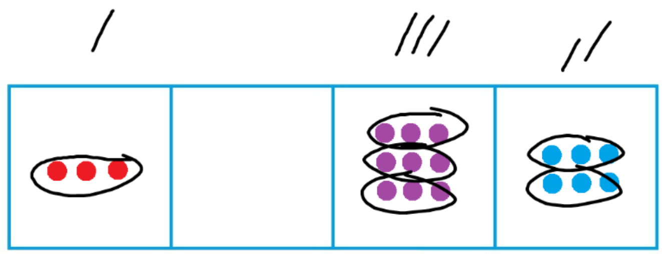
Here's a picture of 3906*. What times three gives this picture?*

3906 =



To answer it, you would look for triples of dots that must have come from single dots. And you'd see plenty of those.

3906 =



We see the picture as the result of tripling one dot at the thousands level, tripling three dots at the hundreds level, and tripling two dots at the ones level. That is, we see 3906 as the number 1302 tripled.

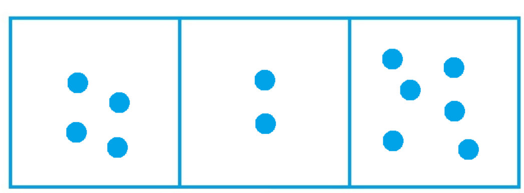
We have just deduced, from the picture, that $3906 \div 3 = 1302$!

So, to divide a number by three, all we need to do is to look for groups of three in the picture of the number. Each group of three corresponds to a dot that must have been tripled. We can just read off the answer to the division problem then by looking at the groups we find!

And we can do the same for any single-digit division problem.

Question: Here's a picture of 426. Can you see in the picture that $426 \div 2$ must equal 213? (What was doubled to give this picture?)

426 =



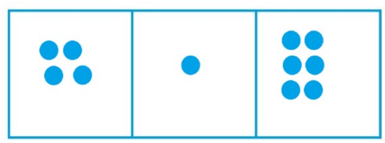
Practice 35.1: Draw a dots-and-boxes picture of 848.
Use your picture to show that $848 \div 4$ must be 212.

How would you explain what is happening to a curious friend?

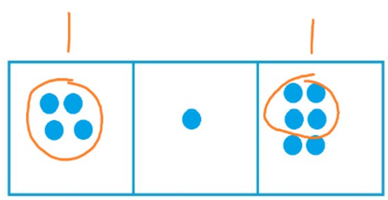
There could be a hiccup in our approach.

Let's try computing $416 \div 4$. (The answer is going to be 104.)

This means we are looking for groups of four (dots that got quadrupled) in a picture of 416.

416 = 

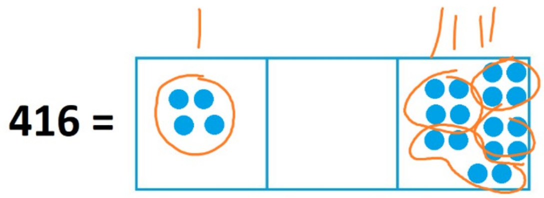
We see one group of four at the hundreds level, and one at the ones level. But no more. Hmm.

416 = 

But an unexplosion might help!

416 = 

This allows to see more groups four.



All dots are now accounted for.

We have one group of four at the hundreds level and four at the ones level. We see that $416 \div 4 = 104$. (That is, 104 was quadrupled to give 416.)

Practice 35.2 Try finding $402 \div 3$ with just a dots-and-boxes picture. How can you get to the answer 134?

Practice 35.3 Use a dots and boxes picture to show that $102 \div 2$ equals 51.

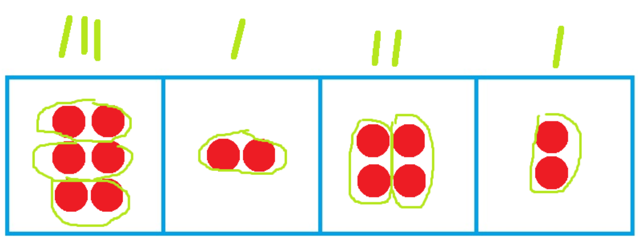
Practice 35.4

a) How does the *Exploding Dots* approach show that $10 \div 5$ equals 2? Draw the picture.

b) Can you work out $1000 \div 8$ with a dots-and-boxes picture? You might be drawing a lot of dots!

Practice 35.5 We saw that $416 \div 4 = 104$. What's $417 \div 4$? What does a dots-and-boxes picture show?

Practice 35.6 Genelle did a division problem the dots-and-boxes way and drew this picture. But she forgot what the problem was.



What division was she solving and what answer did she get for it?

Practice 35.7 Is computing $452 \div 1$ the dots-and-boxes way weird?

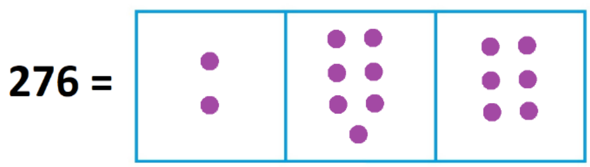
Long Division

Division by single-digit numbers is all well and good. What about division by multi-digit numbers? People usually call that **long division**.

Let's consider the problem

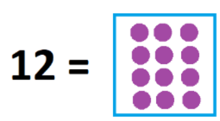
$$276 \div 12.$$

Here is a picture of 276 in a $1 \leftarrow 10$ machine.

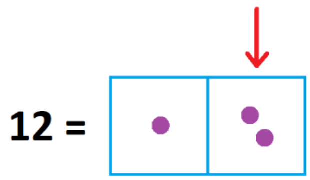


And we are looking for groups of twelve in this picture of 276. That is, we are looking for what got multiplied by twelve to create this picture of 276.

Here's what twelve looks like.



Actually, this is not quite right as there would be an explosion in our $1 \leftarrow 10$ machine. Twelve will look like one dot next to two dots.

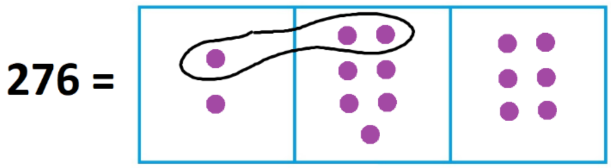


This is going to be confusing, as we don't actually see twelve dots in this picture. We'll have to remember that there really are twelve dots in the rightmost box and that an explosion caused some "spillage."

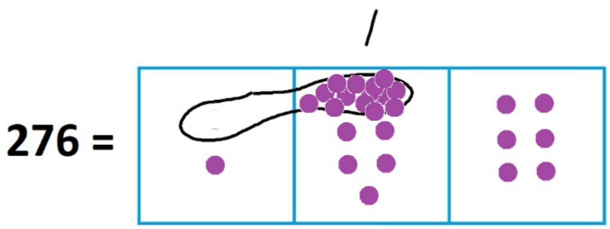
Okay. We are looking for groups of 12 in our picture of 276.

Do we see any "one-dot-next-to-two-dots" in the diagram?

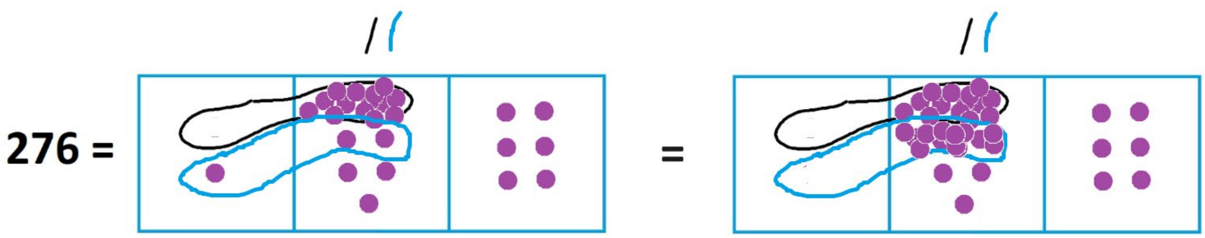
Yes. Here's one.



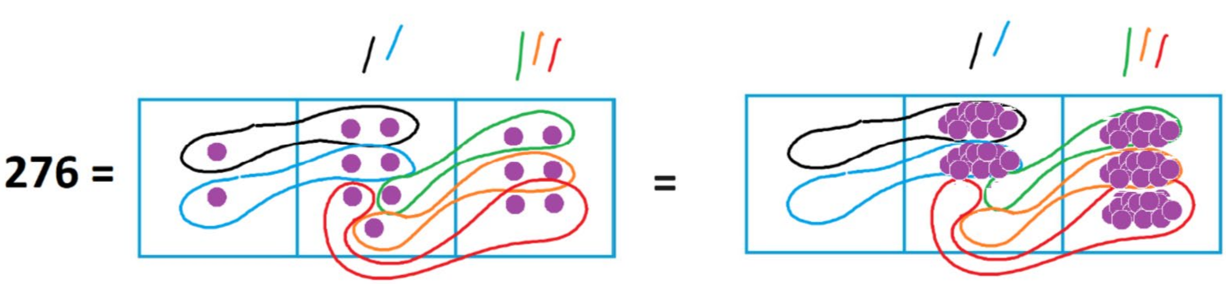
If I do the unexplosion within this loop, we really do see twelve dots at the tens level. One dot at the tens level was multiplied by 12.



There's a second group of twelve at the tens level. A second dot was multiplied by 12 at the tens level.

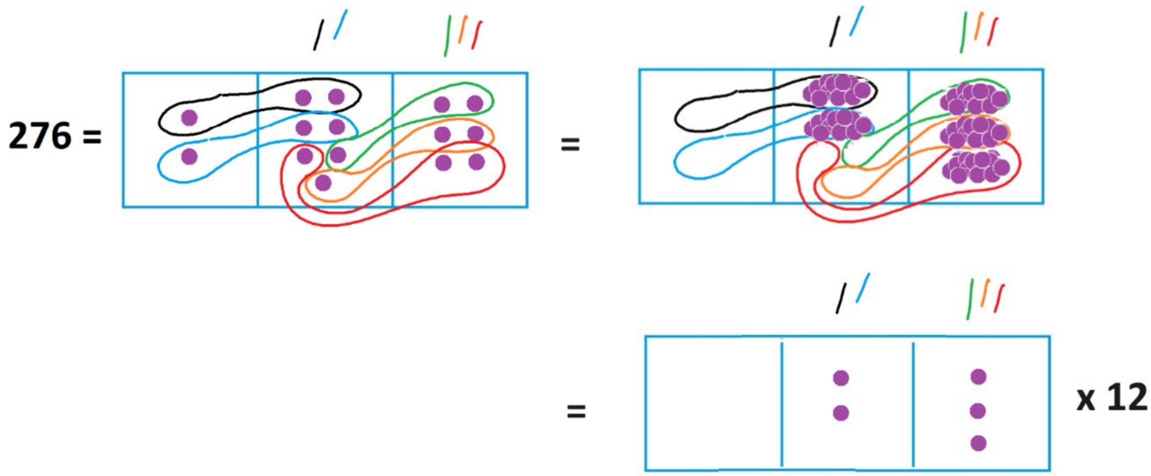


And we can see three more "one-dot-next-to-two-dots" loops, but now at the ones level.



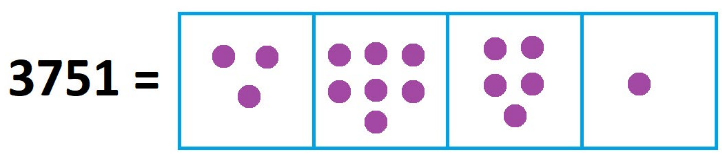
Our picture shows that 276 is really the number 23, two dots in the tens place and three dots in the ones place, with each dot replaced by a group of 12. We see that $276 = 23 \times 12$ and so

$$276 \div 12 = 23.$$

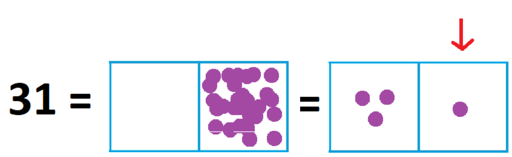


As another example, let's compute $3751 \div 31$this dots-and-boxes way.

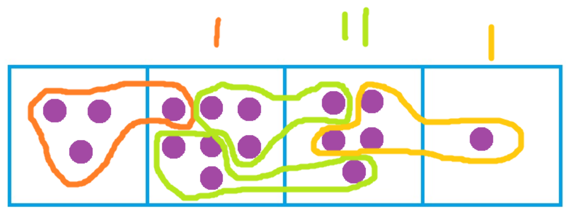
Here's a picture of 3751.



And here's what a group of 31looks like. (All 31 dots are in the rightmost box.)



Here are all the 31s I can find in the picture of 3751.

3751 = 

Question: Can you imagine the unexplosions within each loop?
Can you see the picture as one dot at the hundreds level, two dots at the tens level, and one at the ones level that each got multiplied by 31.

Can you see that it must be that 121 was multiplied by 31 to give the answer 3751?

We have:

$$3751 \div 31 = 121$$

Practice 35.8: See if you can compute

$$2783 \div 23$$

using dots and boxes. Do you see the answer 121?

How would computing $2784 \div 23$ be different?

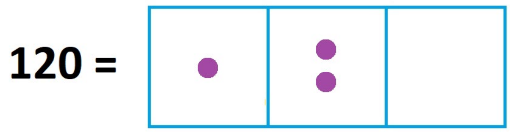
Practice 35.9:

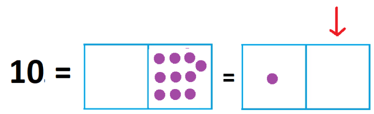
a) Compute $4473 \div 21$ with dots and boxes to get the answer 213.
b) Now compute $4473 \div 213$. Do you see the answer 21?

Practice 35.10: SOMETHING TRICKY

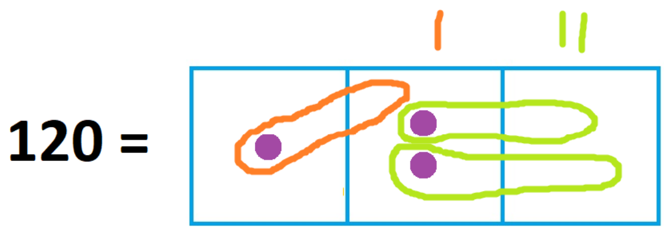
a) Ricky is wondering about 120 ÷ 10. He knows the answer is going to be 12, but he is wondering how the dots-and-boxes approach is going to show this.

He starts by drawing this.





He says he is looking for "one dot next to no dots" and then hunts for them. He thinks he's finding them.



Is he finding them? Is he seeing the answer 12 to 120 ÷ 10?
Is everything he is doing good, and fabulous, and correct?

What do you think?

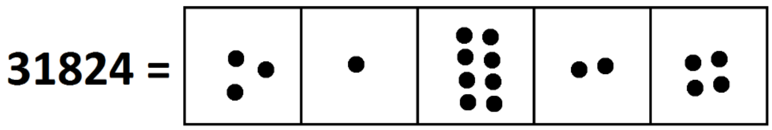
b) Can you see how a dots-and-boxes picture shows that 700 ÷ 70 equals 10?

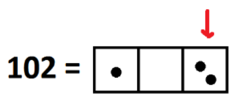
Try computing

$$31824 \div 102$$

before I do it on the next page. Notice the zero in that "102."

Here the pictures to set us up for computing $31824 \div 102$.

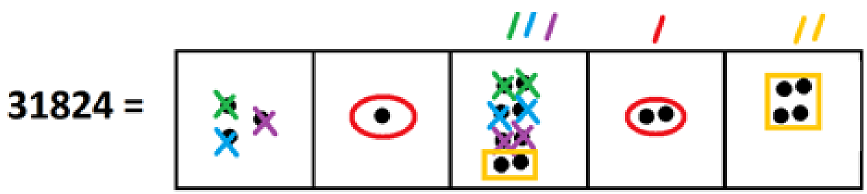




(In the picture of 102, all one-hundred-and-two dots are actually sitting in the rightmost position. But after many explosions, lots of dots “spilled” over to the left.)

We are looking for groups of “one dot–no dots–two dots” in our picture of 31824.

And we can spot a number of these groups.



Question: Can you make sense of what I am doing here? I found drawing loops to be messy, so I drew Xs and circles and boxes instead. Is that okay?

And do you also see how I circled a double group in one hit at the very end?

We see that $31824 \div 102$ equals 312.

Practice 35.11 If I am looking for a pattern of “three dots-no dots-one dot-two dots” when doing a long division problem the dots-and-boxes way, what number am I dividing by?

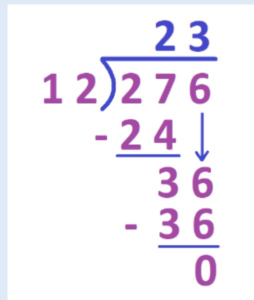
TRADTIONAL LONG DIVISION

I remember as a young lad being taught an algorithm for conducting long division.

It looked like this.

Example: Compute $276 \div 12$.

Answer: We see $276 \div 12 = 23$.

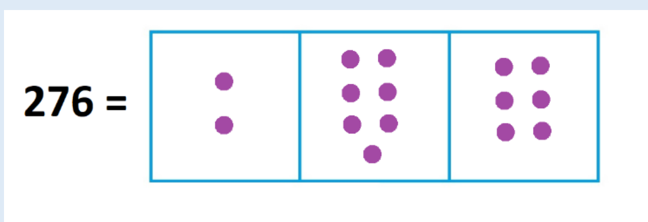


And I remember as a young lad being thoroughly perplexed by this algorithm!

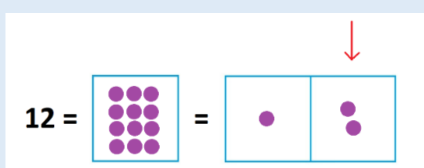
Did you learn an algorithm like this one too?
Can see for yourself what this algorithm is doing?
Can you explain why it works?

I couldn't then! But you and I can now.

The dots-and-boxes approach to computing $276 \div 12$ starts with a picture of 276 in a $1 \leftarrow 10$ machine



and a picture of 12 (which appears as one dot next to two dots, but it is really twelve dots in the rightmost box).



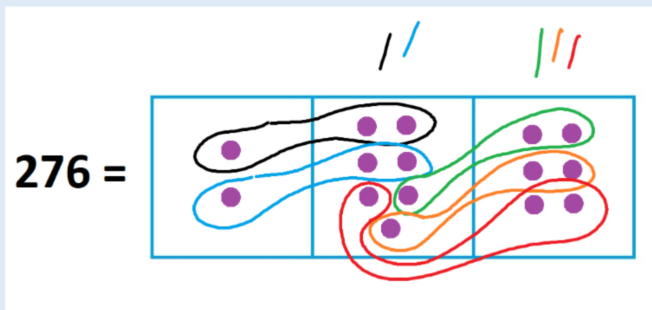
We looked for groups of 12 in the picture of 276.

We first saw two groups among the left two boxes. That is, we first found 2 groups of twelve at the tens level. This accounts for two dots in the left most box, and four of the dots in the middle box.

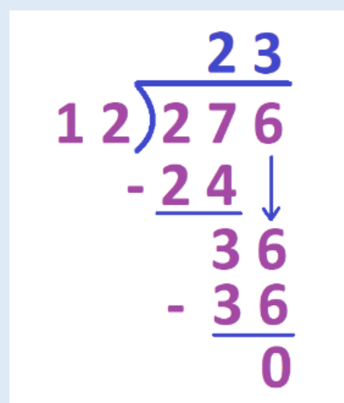
That left three dots still to consider in the middle box and the six in the rightmost box.

Then we found 3 more groups of twelve at the ones level.

This now accounts for all the dots in the picture. Zero dots are left over.



Now think through the school algorithm. Can you see now that it is actually the same process—just without the pictures?



The algorithm first identifies 2 groups of twelve at the tens level just by looking at the first and second boxes.

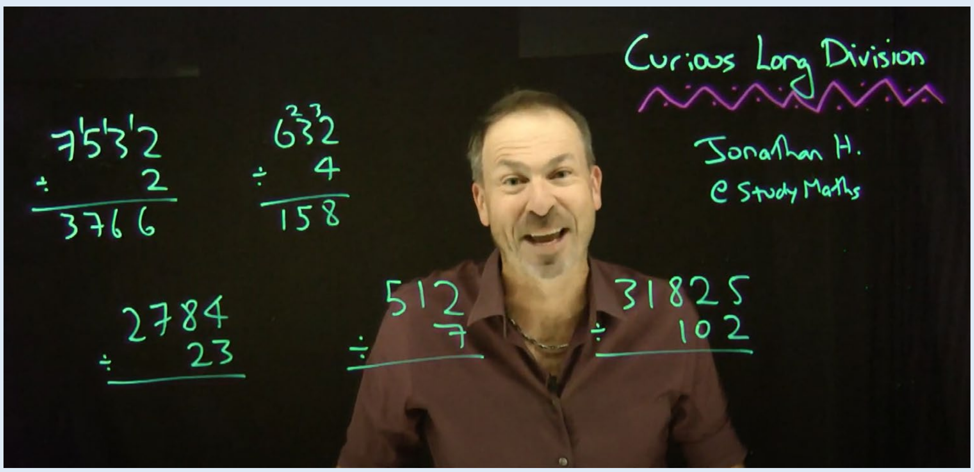
Then the subtraction simply observes that there are 3 dots in the middle box unaccounted for after doing this.

The algorithm then draws our attention to the second and third boxes, where we can next see 3 groups of twelve at the ones level with no dots are left unaccounted for.

Practice 35.12 Compute 2756 ÷ 13 the traditional way and the dots-and-boxes way at the same time, side by side. Can you see that both approaches are doing the same thing?

SPEED DIVISION

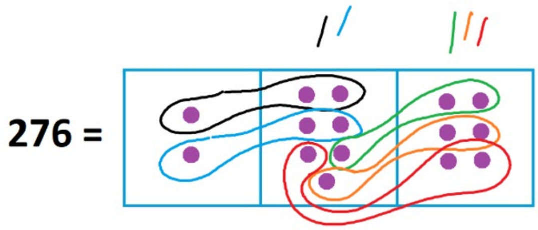
Here's another video. It shows another fun way to conduct long division! (Fun, if you are the sort of person who enjoys doing computations just with pencil and paper.)



https://youtu.be/sWrjGjM2Vjc

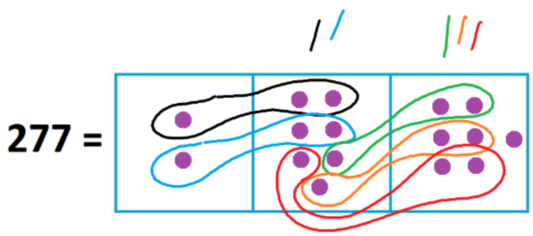
Remainders

We saw that 276 ÷ 12 equals 23



Suppose we tried to compute 277 ÷ 12 instead.
What picture would we get? How should we interpret the picture?

Well, we'd see the same picture as before except for the appearance of one extra dot, which we fail to include in a group of twelve.



This shows that 277 ÷ 12 equals 23 with a remainder of 1.

You might write this as

$$277 \div 12 = 23R1$$

or with some equivalent notation for remainders. (People use different notations for remainders in different countries.)

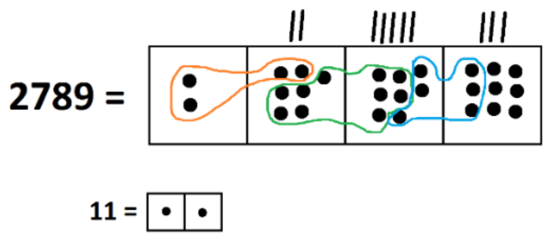
Or you might be a bit more mathematically precise and say that 277 ÷ 12 equals 23 with one more dot still to be divided by twelve.

$$277 \div 12 = 23 + \frac{1}{12}.$$

(And some people might like to think of that single dot as one-twelfth of a group of twelve. All interpretations are good!)

Practice 35.13 Compute 2789 ÷ 11 the dots-and-boxes way.

Do you get a picture like this?



How do you interpret this picture?

Practice 35.14 Use dots and boxes to show that 4366 ÷ 14 equals 311 with a remainder of 12.

Practice 35.15 Use dots and boxes to show that 5481 ÷ 131 equals 41 with a remainder of 110.

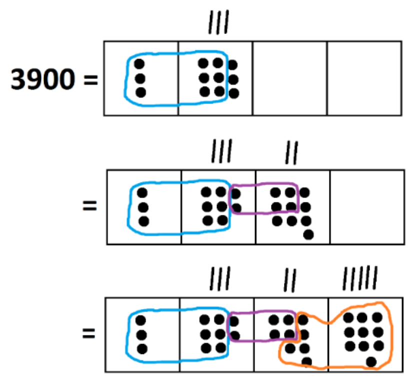
Question: Here are some tricky practice problems you might or might not want to try.

1. Compute 3900 ÷ 12.
2. Compute 46632 ÷ 201.
3. Show that 31533 ÷ 101 equals 312 with a remainder of 21.
4. Compute 2789 ÷ 11.
5. Compute 4366 ÷ 14.
6. Compute 5481 ÷ 131.
7. Compute 61230 ÷ 5.

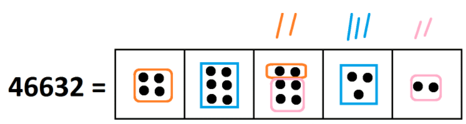
I give my answers to all of these on the next two pages.

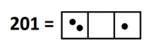
Answers:

1. 3900 ÷ 12 = 325. We need some unexplosions along the way. (And can you see how I am getting efficient with my loop drawing?)

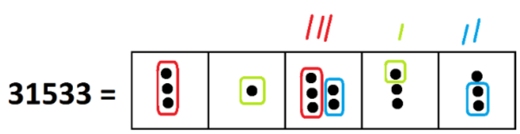


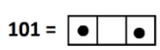
2. 46632 ÷ 201 = 232.



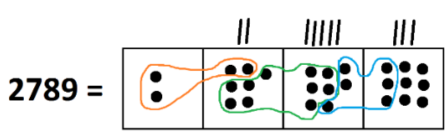


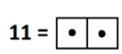
3. 31533 ÷ 101 = 312 with a remainder of 21. That is, $31533 \div 101 = 312 + \frac{21}{101}$



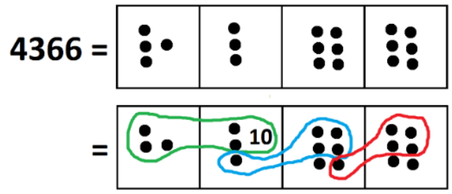


4. We have 2789 ÷ 11 = 253 with a remainder of 6. That is, $2789 \div 11 = 253 + \frac{6}{11}$.

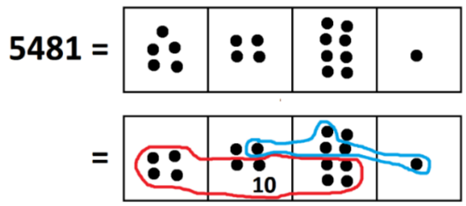




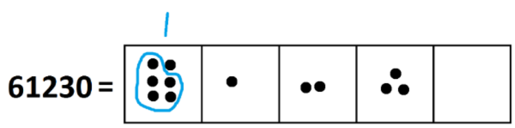
5. $4366 \div 14 = 311 + \frac{12}{14}$.



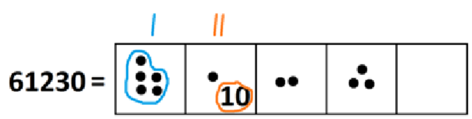
6. $5481 \div 131 = 41 + \frac{110}{131}$.

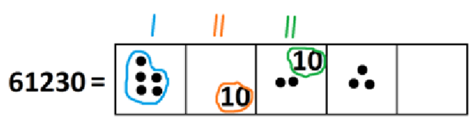


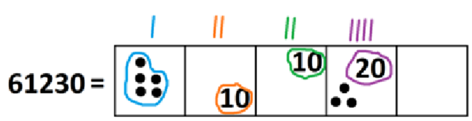
7. We certainly see one group of five right away.

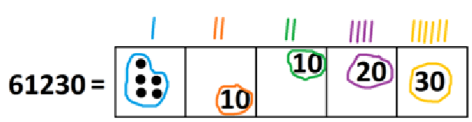


Let's perform some unexplosions. (And let's write numbers rather than draw lots of dots. Drawing dots gets tedious!)





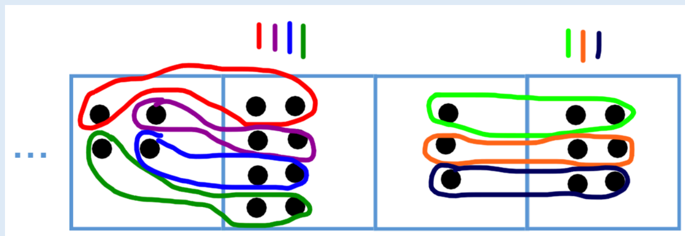




We see $61230 \div 5 = 12246$.

MUSINGS

Musing 35.16 Ebube was working on a division problem, but lost all the details of his work. He just had the picture below.



What division problem was Ebube working on? What answer did he get for it?
(Assume Ebube is working with a $1 \leftarrow 10$ machine.)

Musing 35.17 Here are some division problems you might or might not want to try.
Just pick a few to do! There are too many of these!

a) Compute $4840 \div 4$.

b) Compute $721 \div 7$.

c) Compute $126 \div 6$.

d) Compute $126 \div 3$.

e) Compute $126 \div 2$.

f) Compute $126 \div 1$.

g) Compute $3641 \div 11$.

h) Compute $3642 \div 11$.

i) Compute $3649 \div 11$.

j) Compute $3900 \div 12$.

k) Compute $100 \div 9$.

l) Compute $1000000000 \div 9$.

Musing 35.18 It turns out that $222 \div 3$equals 74. What then is the answer to each of the following?

a) $223 \div 3$?
b) $225 \div 3$?
c) $3222 \div 3$?
d) $444 \div 3$?
e) $2220 \div 3$?

Musing 35.19 **Challenge** Ebube just informed me that his picture back in question 30.1 is actually of a $1 \leftarrow 9$ machine! Okay then. What division problem was he working on, what answer did he get, and how does all that translate to base-ten numbers?

Musing 35.20 **DIVISIBILITY RULE FOR NINE**

Some people know a rule for quickly determining whether or not a number is divisible by nine.

A number is divisible by 9 only if the sum of its digits is divisible by 9.

For example, 387261 is divisible by 9—apparently—since $3 + 8 + 7 + 6 + 2 + 1 = 27$ is. (And if we weren't sure about the number 27, we could test that it is divisible by 9 by noting that $2 + 7 = 9$ certainly is.)

And to check: $387261 \div 9 = 43029$ with no remainder.

Actually, this rule can be made a little stronger.

A number leaves the same remainder upon division by 9 as does the sum of its digits.

For example, 40061 is two more than a multiple of nine (it equals $4451 \times 9 + 2$) and its sum of digits, $4 + 0 + 0 + 6 + 1 = 11$ is also two more than a multiple of nine.

Also, 77 is five more than a multiple of 9, just as $7 + 7 = 14$, the sum of it digits, is.

Also, 2808 is a multiple of nine, that is, leaves a remainder of 0, and $2 + 8 + 0 + 8 = 18$ leaves a remainder of 0 too upon division by nine.

Let's see if we can explain these rules. (Follow along if you like.)

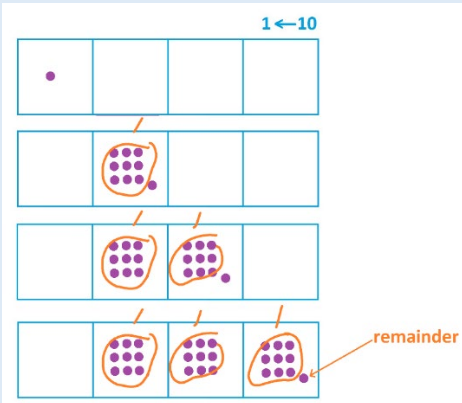
Let's look at division by 9 in a $1 \leftarrow 10$ machine.

a) Draw a dots and boxes picture to compute $210 \div 9$. Show that your work leaves a remainder of $2 + 1 = 3$ dots in the rightmost box.

b) Here is a claim.

Each dot in a $1 \leftarrow 10$machine leaves a remainder of 1 upon division by 9.

Does this picture convince you the claim is true?



Consider these two observations.

The dots-and-boxes picture of 210 uses $2 + 1 + 0 = 3$ dots.
Each dot leaves a remainder of one dot in the right most box.
Thus $210 \div 9$ must have a remainder of 3.

The dots-and-boxes picture of 2213 uses $2 + 2 + 1 + 3 = 8$ dots.
Each dot leaves a remainder of one dot in the right most box.
Thus $2213 \div 9$ must have a remainder of 8.

c) Do these observations make sense to you?
d) What are the matching observations for $2005 \div 9$ and $11111 \div 9$?
e) What is the matching observation for $473 \div 9$?

f) Is it true that each number, upon division of by nine, has the same remainder as the sum of its digits divided by nine?

g) Is it consequently also true that if the sum of the digits of a number is divisible by nine, then the number itself is divisible by nine?

36. Advanced Algebra is not that Advanced Really.

Everything we do in current society is based on the number 10.

We humans are drawn to the number ten for counting, most likely because we were born with ten digits on our hands.

But there is absolutely nothing special about the number ten for doing mathematics. We could do arithmetic in base 6 or base 12 (Martian), or base 4 or base 8 (Venution), or base 2 (computer), or in any other number machine we care to work in. The mathematics is exactly the same!

Algebra is about doing mathematics in any system whatsoever and not being locked into our humanness. (Did you read the introduction to this book? I really do believe that mathematics is universal.) Moreover, not only can we do addition and subtraction and multiplication and division in any base in any base we like, we can do arithmetic before we've even decided which base we want to work in!

The way to do that is to just use a symbol to represent a base number.

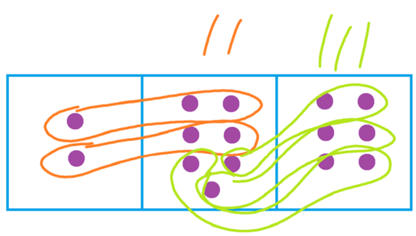
But there is one unfortunate thing. People seem to use the same symbol over and over again when they think "any old number." They use the letter x. And now that letter has been so overused for so many decades that people are now actually scared of that one letter of the alphabet in math class. So unfortunate. (I personally like to use the symbol n for "number" or maybe b for "base.")

Take a deep breath and look at this picture. It has the symbol x.

Can you see that it is showing that division in base 10, as we've been doing, and division in algebra look exactly the same?

Arithmetic

$$276 \div 12 = 23$$



Algebra

$$\left(2x^2 + 7x + 6\right) \div (x+2) = 2x + 3$$

IT'S THE SAME!

And remember, the symbol x here just means "any base you like." We can go back to being human and choose x to be 10 if we like, and this takes us back to the familiar calculation $276 \div 12 = 23$. But it is also fun to play with other possibilities as to what x could be. We could be human or Martian or Venutian or a computer, or something else entirely, with ease.

Algebra is about just having fun not being locked into our humanness.

Solutions

28.1 a) 19 b) 31

28.2 Do feel free to read on to check if your thinking is on mark.

29.1 You do.

29.2 Try it. Put thirteen dots in the rightmost box. You do get the code $1\ 1\ 0\ 1$.

29.3 Try it.

29.4 It's twenty three.

29.5 Can you see $16 + 4 + 2 + 1 = 23$?

29.6 $\mathbf{1\ 0\ 0\ 0\ 1}$

29.7 $\mathbf{1\ 1\ 1\ 1\ 0}$

29.8 $\mathbf{1\ 1\ 0\ 0\ 1\ 0}$

29.9 Read on!

29.10 Ten appears in B and D.

29.11 It does. The number 31 equals $16 + 8 + 4 + 2 + 1$.

29.12 The highest number you can "make" without the doubling number 16 is $8 + 4 + 2 + 1 = 15$. Each of the numbers from 16 to 31 must "use" the number 16 and so these numbers appear in group A.

29.13 The numbers 2, 4, 8, and 16 are even. So, if you want to "make" an odd number, you must use the number 1. The odd numbers (and only the odd numbers) appear in group E.

29.14 This is a great project.

Your leftmost card will contain all the numbers from 32 to 63 (these numbers "use" the doubling number 32). And the rightmost card will contain all the odd numbers as they "use" the number 1.

For the remaining cards, you'll have to look at the binary codes of all the numbers to see which use a 2, which use a 4, which use an 8, and which use a 16.

29.15 In the code $\mathbf{100211}$ there are two dots in a box we can explode. We won't see this code as our final answer.

29.16 a) Two hundred is 11001000.
b) One thousand is 1111101000

29.17 1023 with code 1111111111.

29.18 A **bicycle** is a vehicle with two wheels; **Binoculars** are viewing devices with two lenses; To **bisect** something is to cut it into two equal parts; A **biped** is a creature that walks on two legs; A **bivalve** is a mollusk with shells composed of two parts.

29.19 You can count as high as $1 + 2 + 4 + 8 + 16 + 32 + 64 + 128 + 256 + 512 = 1023$ on two hands.

30.1 Thirteen as $1 \leftarrow 3$ machine code $\mathbf{111}$.

30.2 Twenty six.

30.3 Four

30.4 It's code is $\mathbf{20}$.

30.5 Both a) and b) are correct!

30.6 Do do this!

30.7 Really try to think your way through this.

30.8 $81 \times 3 = 243$

30.9 Fifteen has code $\mathbf{120}$.

30.10 A creature with ten appendages resembling legs.

30.11 Were they thinking fingers … and toes?

30.12 Really do it.

30.13
120 is the code for fifteen; **1200** is the code for forty-five; **12** is the code for five.

30.14 67,620,938

30.15 a) **31** b) **11111** c) **1011** d) **133** e) **111** f) **51** g) **10**

30.16 We are playing with a $1 \leftarrow 3$ machine here. Can you figure out the details?

31.1 2|11|3 is really the number 313.

31.2
"Twelvety" is twelve tens. That's 120.
"Eleventy" is eleven tens. That's 110.
"Thirteenty," if you were to say it, is thirteen tens. That's 130.

31.3

a) Twelve-thousand twelve-hundred, twelvety, twelve.

b) Indeed. The only part of this that sounds weird in English is "twelvety."

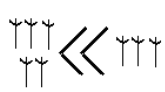
c) It's 13,332.

This is read as "thirteen-thousand, three-hundred, thirty, two."
We don't say "one ten-thousand, three-thousand, three-hundred, threety, two" as we should!

31.4 12,221

31.5
a) 35 and $11 \times 60 + 50 = 710$

b)

$323 = 5 \times 60 + 23 \times 1 =$ 

$2016 = 33 \times 60 + 36 \times 1 =$ 

c) 61 =  (Or is this two?)

1200 =  (Or is this twenty?)

3600 =  (Or is this one or sixty?)

The Babylonians lacked some sort of "space filler" to help the reader know which groups of symbols represent units, which represent 60s, which 360s, and so on.

It was probably assumed that readers could deduce from the context of the text how to correctly interpret the numerical symbols.)

For example, writing

I have  *siblings*

likely means I have two siblings, not sixty-one of them nor 361 of them!

d) Do explore the history of place value on the internet.

32.1

$$148 + 323 = 4 \,|\, 6 \,|\, 11 = 471$$

$$567 + 271 = 7 \,|\, 13 \,|\, 8 = 838$$

$$377 + 188 = 4 \,|\, 15 \,|\, 15 = 5 \,|\, 5 \,|\, 15 = 565$$

$$582 + 714 = 12 \,|\, 9 \,|\, 6 = 1 \,|\, 2 \,|\, 9 \,|\, 6 = 1296$$

$$310462872 + 389107123 = 6 \,|\, 9 \,|\, 9 \,|\, 5 \,|\, 6 \,|\, 9 \,|\, 9 \,|\, 9 \,|\, 5 = 699569995$$

$$87263716381 + 18778274824 = 9 \,|\, 15 \,|\, 9 \,|\, 13 \,|\, 11 \,|\, 9 \,|\, 8 \,|\, 10 \,|\, 11 \,|\, 10 \,|\, 5 = \ldots \; = 106041991205$$

32.2

a) The largest sum one computes in a column of two numbers is $9 + 9$, plus maybe a 1 from a previous explosion, giving $9 + 9 + 1 = 19$. This requires carrying only a 1.

b) $199 \; + \; 9$, for example, works.

c) $10009 + 2$, for example works.

d) $9999999999 + 1$ works.

32.3 d)

33.1
a) $2784 \times 2 = 4|14|16|8 = 5568$
b) $2784 \times 4 = 8|28|32|16 = 11136$
c) $2784 \times 5 = 10|35|40|20 = 13920$
d) $2784 \times 10 = 20|70|80|40 = 27840$

33.2
a) 4760
b) 47600. Multiplying by one hundred is the same as multiplying by ten (which gives 4760) and then by ten again (which gives 47600).

33.3 a) 919
b) 5574 was multiplied by 10 to give this answer. $313 \times 10 = 3130$.
c) 33100 was multiplied by 10 and 10 again to give the answer 3310000.

33.4

$37 \times 2 = 6 \mid 14 = 74$ $\quad$ $37 \times 11 = 33 \mid 77 = 407$ $\quad$ $375 \times 11 = 33 \mid 77 \mid 55 = 4125$

33.5 Do try it.

33.6
a) $2784 \times 2 = 4|14|16|8 = 5568$
b) $2784 \times 4 = 8|28|32|16 = 11136$
c) $2784 \times 5 = 10|35|40|20 = 13920$
d) $2784 \times 10 = 20|70|80|40 = 27840$

33.7 Try it!

33.8 21132

33.9 a) $125 \times 11 = 125 \times (10 + 1) = 1250 + 125 = 1{,}375$

b) $313131 \times 10 + 313131 = 3131310 + 313131 = 344{,}441$
$31313131313131 \times 10 + 31313131313131 = 344444444444441$

c) $110 + 10 = 121$; $1210 + 121 = 1331$; $13310 + 1331 = 14641$.

33.10

a) We're seeing this in the previous question. If "ab" is a two-digit number, then $ab \times 11$ equals

$$ab \times (10 + 1) = ab0 + ab$$

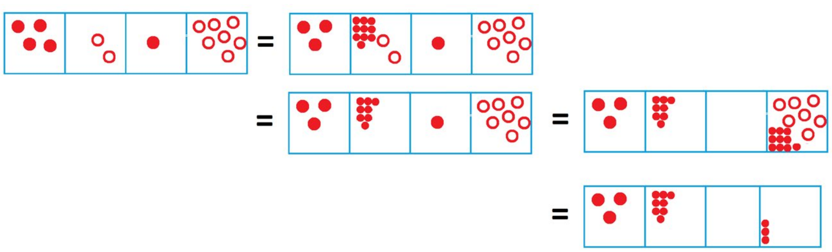
Doing the addition, we see the result is a three-digit number with the sum of the digit in the middle.

$$\begin{array}{cc|c|c} & a & b & 0 \\ + & & a & b \\ \hline = & a & a+b & b \end{array}$$

b) $693 \div 11 = 63$

c) $924 \div 11 = 8|12|4 \div 11 = 84$

34.1



34.2

a) I am guessing that Raj realizes that the 2 in $2|-3|-5$ is really 200, the -3 is really -30, and the -5 is, well, -5. So, the answer is $200 - 30 - 5$, which is 165.

b) $7109 - 3384 = 4|-2|-8|5 = 4000 + -200 + -80 + 5 = 3725$

34.3

$$
\begin{aligned}
6328 - 4469 &= 2\,|\,{-1}\,|\,{-4}\,|\,{-1}\\
&= 1\,|\,9\,|\,{-4}\,|\,{-1}\\
&= 1\,|\,8\,|\,6\,|\,{-1}\\
&= 1\,|\,8\,|\,5\,|\,9 \quad = 1859
\end{aligned}
$$

$$
\begin{aligned}
&78390231 - 32495846\\
&\quad = 4\,|\,6\,|\,{-1}\,|\,0\,|\,{-5}\,|\,{-6}\,|\,{-1}\,|\,{-5}\\
&\quad = 4\,|\,5\,|\,9\,|\,0\,|\,{-5}\,|\,{-6}\,|\,{-1}\,|\,{-5}\\
&\quad = 4\,|\,5\,|\,8\,|\,10\,|\,{-5}\,|\,{-6}\,|\,{-1}\,|\,{-5}\\
&\quad = 4\,|\,5\,|\,8\,|\,9\,|\,5\,|\,{-6}\,|\,{-1}\,|\,{-5}\\
&\quad = 4\,|\,5\,|\,8\,|\,9\,|\,4\,|\,4\,|\,{-1}\,|\,{-5}\\
&\quad = 4\,|\,5\,|\,8\,|\,9\,|\,4\,|\,3\,|\,9\,|\,{-5}\\
&\quad = 4\,|\,5\,|\,8\,|\,9\,|\,4\,|\,3\,|\,8\,|\,5\\
&\quad = 45894385
\end{aligned}
$$

I personally find it much easier to do the unexplosions from left to right.

34.4 $1000 - 234$ works. We get $1000 + -200 + -30 + -4 = 766$.

34.5 The answer is $-5\,|\,{-3}\,|\,1$, which is -500 and -30 and 1, which makes -529. (Is this what a calculator gives?)

Or, maybe you might say that $-5\,|\,{-3}\,|\,1$ is the opposite of $5\,|\,3\,|\,{-1}$ (which is 529). So, the answer is -529.

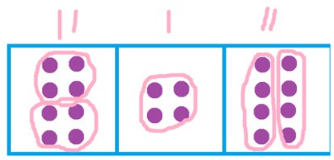
34.6 We get the answer $1|0|-1|1$. We need to unexplode, but we unexplode to two dots.

$$1|0|-1|1 = 0|2|{-1}|1 = 0|1|1|1$$

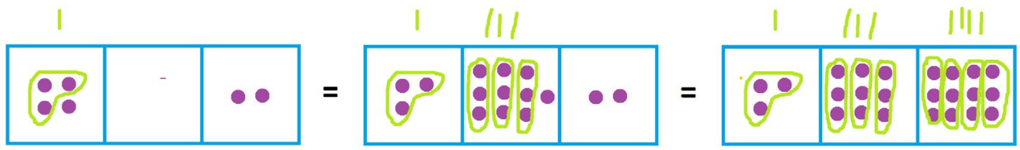
A 111 is indeed seven in base two!

34.7 a) Seventeen is $1|1|-1|{-1}|-1$.
b) Negative seventeen is $-1|-1|1|1|1$.
c) All odd integers between -31 and 31 can be created this way.
(Every expression has to be odd as it must include the a dot or antidot in the 1s box. And one can check every odd number can be created.)

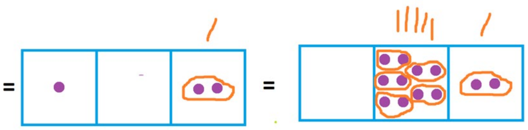
35.1

848 = 

35.2

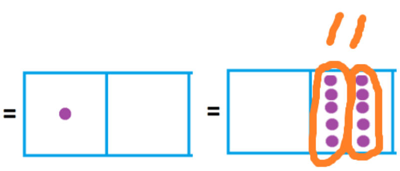
402 = 

35.3

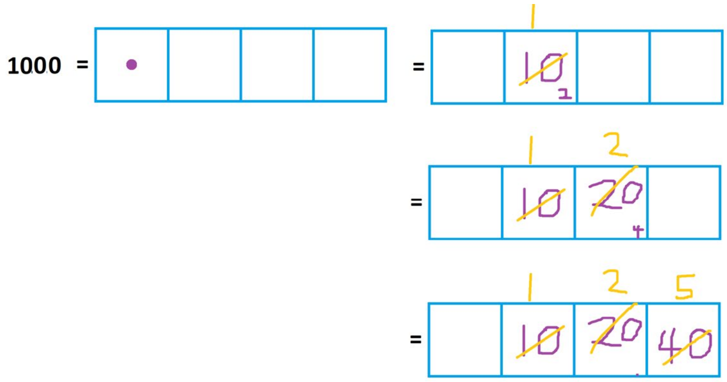
102 = 

35.4

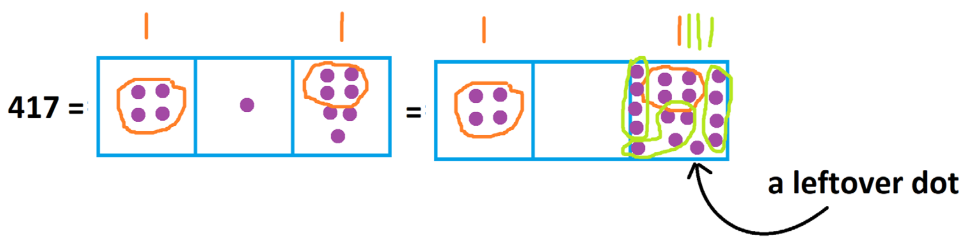
a)

10 = 

b) Can you see what I am doing here as I hunt for groups of eight?



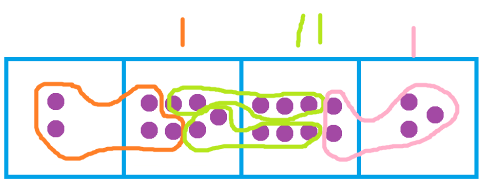
35.5 We see that $417 \div 4 = 104$ with a remainder of 1.



35.6 $6242 \div 2 = 3121$

35.7 You end up circling all the single dots.

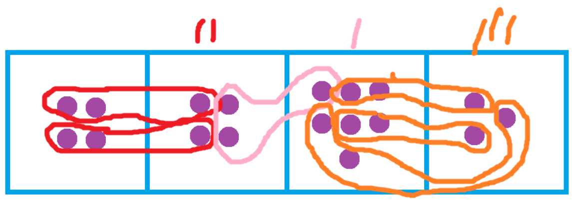
35.8



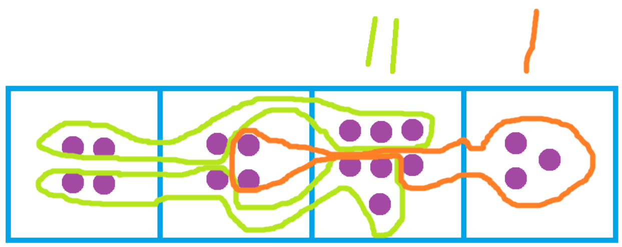
If we did $2784 \div 23$ we'd see an extra dot in the picture and conclude that the answer is 121 with a remainder of 1.

35.9

a)



b)



35.10 Everything Ricky has done is absolutely beautiful and correct.

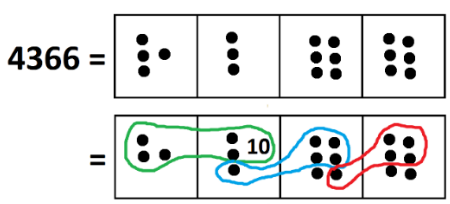
Do test your understanding and draw a picture for $700 \div 70$. (There is just one loop in the picture.)

35.11 302

35.12 Really do try this.

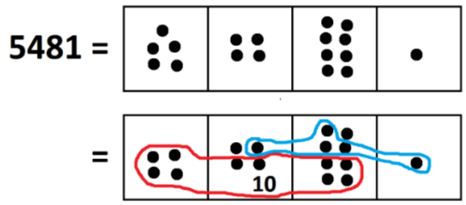
35.13 We see 2789 ÷ 11 = 253 with a remainder of 6. That is, $2789 \div 11 = 253 + \frac{6}{11}$.

35.14 $4366 \div 14 = 311 + \frac{12}{14}$.



35.15

$5481 \div 131 = 41 + \frac{110}{131}$.



35.16 $4836 \div 12 = 403$

35.17

a) 4840 ÷ 4. This equals 1210.

b) 721 ÷ 7. This equals 103. (One needs to unexplode two dots.)

c) 126 ÷ 6. This equals 21. (Unexplode one dot.)

d) 126 ÷ 3. This equals 42. (Unexplode one dot.)

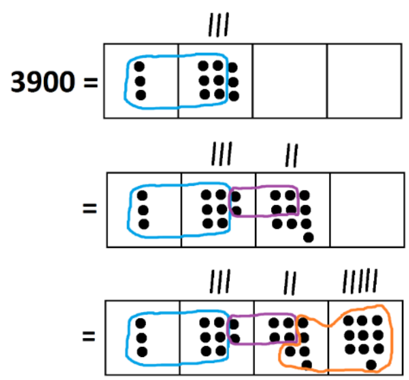
e) 126 ÷ 2. This equals 63. (Unexplode one dot.)

f) 126 ÷ 1. This equals 126. (There is one group of one at the hundreds level, two at the tens level, and six at the ones level!)

g) 3641 ÷ 11. This equals 331.

h) 3632 ÷ 11. This equals 331 with a remainder of 1, or $331\frac{1}{11}$.

i) 3639 ÷ 11. This equals 331 with a remainder of 8, or $331\frac{8}{11}$ .j) 3900 ÷ 12. This equals 325. (I got efficient with my loops.)



k) 100 ÷ 9. This equals 11 with a remainder of 1, or $11\frac{1}{11}$.

l) 1000000000 ÷ 9. This equals 111111111 with a remainder of 1, or $111111111\frac{1}{9}$.

35.18 a) $74\frac{1}{3}$ b) $74\frac{3}{3} = 75$ c) $1000 + 74 = 1074$ d) Double 74, which is 148. e) 740

35.19 He still got "$4836 \div 12 = 403$" which we have to translate now from base nine.

$$4836 = 4 \times 9 \times 9 \times 9 + 8 \times 9 \times 9 + 3 \times 9 + 6 = 3597$$

$$12 = 1 \times 9 + 2 = 11$$

$$403 = 4 \times 9 \times 9 + 3 = 327$$

So, the calculation in base ten is

$$3597 \div 11 = 327$$

which is correct.

35.20 I hope you were able to follow along.