Chapter 5

Fractions:
Not Getting Them is Not Your Fault

Contents

37. It's Not Your Fault

Many people fear fractions. And this fear seems somewhat universal.

I have a theory as to why.

People first experience learning about fractions as youngsters at home, typically from sharing goods and desserts. Understanding portions of items—such as *halves*, *thirds*, *quarters*—is intuitive and natural.

The typical school curriculum then picks up on this intuitive start and over the course of several years next attempts to build a robust and coherent mathematical story of fractions from this start.

But the school curriculum is caught between two contradictory demands: the demand to be age appropriate and to develop "real world" meaning to the mathematics of fractions each step of the way, and to be honest about the mathematics itself, which, for fractions, very quickly steps beyond real world loyalties.

As I said in Section 23,

> *Mathematics is bigger and bolder than the real world. It is therefore bigger and bolder than all schoolbook attempts to make every part of it concrete and real. Mathematics certainly incorporates real-world models and is immensely powerful in helping describe them. But mathematics sits at a higher plane to them.*

The teaching of fractions is particularly challenged by this fact.

As a result, many elementary- and middle-school curriculums on the topic are muddled and confounding. One doesn't usually see that confusion right away as each step of the curriculum brings in a real-world idea that makes sense for that one step. It is only when you look back and try to make sense of the story as a whole do you say: *Hang on! So, what is a fraction really? Which real-world model applies when, and why not to everything?*

To see what I mean, what comes next is an overview of how the story of fractions is often presented from grades K to 7.

Read this section, but don't take it too seriously. The only message I hope you glean from by its end is this:

It is not your fault!

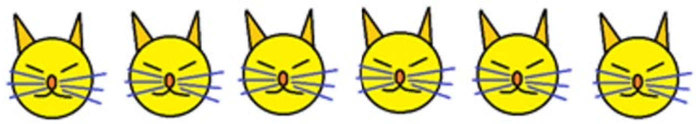
If you are befuddled, confused, and scared by fractions, it is not your fault for not "getting" them. (Actually, not getting them is a sign of your intelligence: you've picked up that something is awry.)

Here's the school story of fractions as you may well have experienced it.

VERY EARLY GRADES: Fractions are Parts of Things

Here's a task for you, right off the bat.

*Please circle **one third** of these six kittens.*

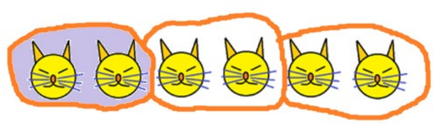


*Now please circle **half** of these stars.*

These tasks represent how young students often first experience fractions.

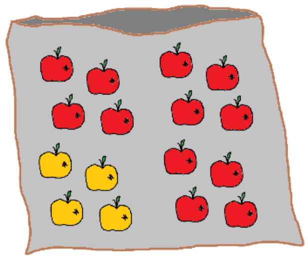
They are taught the notions of *half*, *third*, *quarter*, *fifth*, and so on, and practice dividing sets of objects into equal-sized parts. And they learn that equal-sized parts have these special names.

And you just did this (mentally, at least). You saw that the kittens naturally divide into three sets of equal size, and you selected one of those sets. That's a third of the kittens.



The stars naturally divide into two sets of equal size, and you selected one of those sets: half the stars.

Which word describes the proportion of yellow apples in this bag?



Answer: One quarter—also called a fourth.

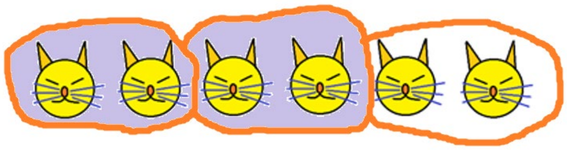
It can be hard for young students to recognize equal-sized groups, and there is genuine confusion to be had here.

For example, in this bag of apples, one can see four equal groups of size 4, two equal groups of size 8, eight equal groups of size 2, and sixteen equal groups of size 1. We can even say there is one group of 16.

There are many sets of equal-sized groups!

Students are expected to intuit which size of equal sized groups make sense for a question and respond appropriately.

Sometimes young students might be asked to go a step further. Rather than select one third of the kittens, they might be asked to select **two thirds** of the kittens.



The English language helps here.

“Two houses” is one house and another house. “Two kumquats” is one kumquat and another kumquat. And “two thirds” is one third and another third.

Similarly, “three quarters” is literally three quarters, and “seven eighths” is literally seven eighths.

Still at this early stage, students learn to write the symbols $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, … and so on for the words one half, one third, one quarter, and so forth.

They also learn to write $\frac{2}{3}$, for "two thirds" and $\frac{3}{8}$, for "three eights," for example.

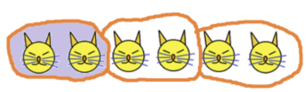
Weirdly, many curricula then seem to soon lose sight of what the English language is suggesting: $\frac{3}{8}$, for instance, really is three eighths: an eighth and an eighth and an eighth, that is, it's three copies of a basic fraction called one eighth.

Where Does this First Story Leave Us?

It's not clear what fractions are at this point.

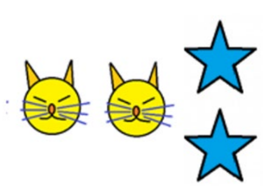
We certainly use numbers to describe them—$\frac{1}{2}$ and $\frac{1}{3}$ and the like–and we count things to find them. But they are not really numbers in-and-of themselves.

A fraction is more like a *call to action* or a thing to do: "circle a third of the kittens" or "select half the stars."



And right now, the idea of doing arithmetic with fractions makes no sense.

> **Question:** *What is a third plus a half?*
> Which of these might be a reasonable answer a young student might give to this question?



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HUH?

All three seem like reasonable answers to me.

An older student will be expected to give the answer $\frac{5}{6}$ for a half plus a third, but this answer makes no sense whatsoever in this part of the story. (Where is that 5 coming from? Where is that 6 coming from? The fraction $\frac{5}{6}$ is a bizarre and nonsensical answer to this problem.)

The upshot here is that people are first typically taught intuitive and practical understanding of the notions of halves, thirds, sevenths, tenths, and so on.

But there is no reason to think that fractions are numbers in any way, and there is certainly no reason to think we should be able to do arithmetic with them.

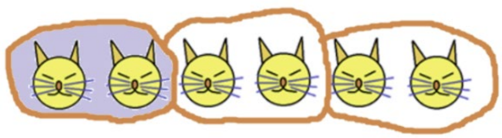
But the typical curriculum wants to change that.

And it does so by next making an edict.

LESS EARLY GRADES: Fractions Should Come from the Same "Whole."

We just saw that a third could be two kittens and that a half could be two stars.

But as it is not possible to compare kittens and stars, the idea of combining fractions or doing arithmetic with fractions has no meaning—at least not in this context.



So, the typical curriculum next takes a stand and makes an edict.

> **In a conversation about fractions, the fractions must come from the same whole.**

Let's think about what this means.

First, what type of word is "whole"?
It's used both as an adjective and a noun in everyday life.

> "I spent the whole day doing math."
> "On the whole, life is good."

But it is a bit weird to use it as a stand-alone noun: "They come from the same whole."

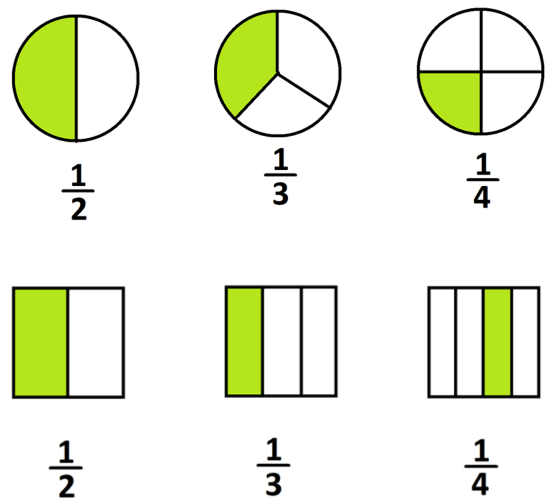
The language of the edict feels a bit strange.

It is easier to just give examples of "wholes" and slide in the use of that word with those examples.
And the stereotypical example of whole is a pie, actually, to be clear, a whole pie. (Nouns and adjectives again!)

People tend to think of pizza pie, or an apple pie, or just a generic round pie.

Drawing circles for pies and dividing up circles is actually hard! It is so much easier to draw square or rectangular pies.

But no matter what type of pie you draw and divide into equal-sized parts, the basic fractions $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ are then seen as slices of pie, each just one of an equal-sized part.

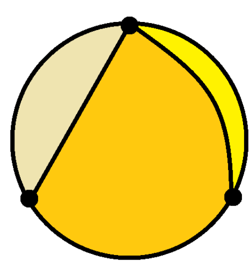


But there is confusion to be had here.

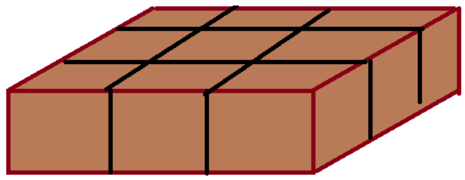
We all know in the real world that not all people deem all slices as "equal."

A slice of pizza with more anchovies on it might be more desirable to some. (To me, for instance!)

People who only like the cheese-stuffed crust and don't care one whit about the topping might agree that this pizza shown is divided into equal thirds.



If we are talking about square cake, then corner pieces have more icing on the side and might be deemed more valuable.



Young students like to think like lawyers and find loopholes and exceptions.
And they are right to do so. That is mathematical thinking!

So, in our discussion of fractions with pie as “the whole,” we have to make some points absolutely clear.

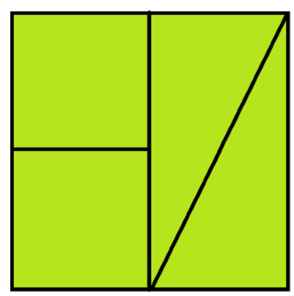
1. We’re talking about the top surface area of the pie.
2. We are assuming that the pie is perfectly uniform in its topping.
3. The perimeter of the pie (its crust) is irrelevant.

We are also assuming that no one cares about the shape of the pieces, just as long as each piece has the same area.

This is a lot for a young student to take in. (And this is especially hard if a student hasn’t yet properly learned what area is from a geometry class.)

Question: Here’s a square pie divided into four pieces.

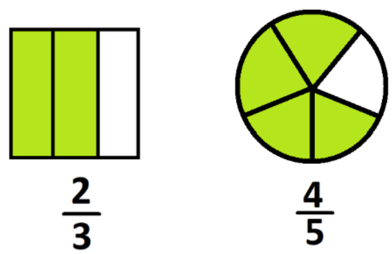
Does it look like it has been divided into four **equal** pieces according to the parameters just outlined?



Next, students are reminded that $\frac{2}{3}$ means "two copies of $\frac{1}{3}$" and $\frac{4}{5}$ means "four copies of $\frac{1}{5}$," for instance. It is understood that all the copies (slices) are coming from the same pie and all the slices are "equal."

In general,

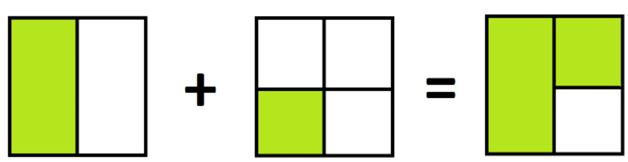
The fraction $\frac{a}{b}$ is interpreted as "a copies of $\frac{1}{b}$," where $\frac{1}{b}$ represents one slice that comes from dividing a pie into b equal slices.



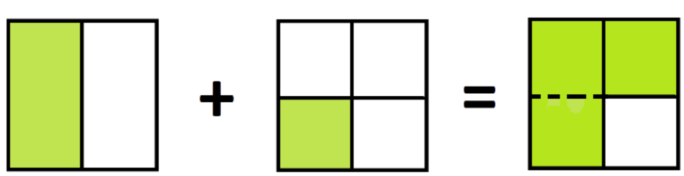
So, fractions are represented by portions of pie. But it is still not yet clear whether or not fractions are numbers in and of themselves.

But if two fractions come from the same pie, it does start to feel possible to do some legitimate arithmetic with them.

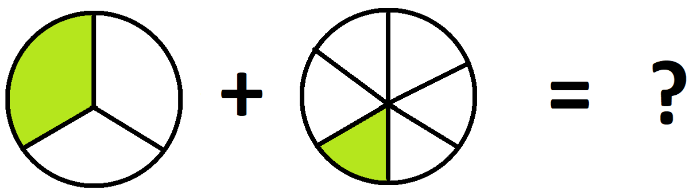
For example, $\frac{1}{2}+\frac{1}{4}$ can be interpreted as putting half of a pie and a quarter of that same pie together on one plate.



In this example, students might be encouraged to extend the horizontal line in the middle of the right picture to see that the answer is the same as $\frac{3}{4}$, three copies of a quarter of the pie.



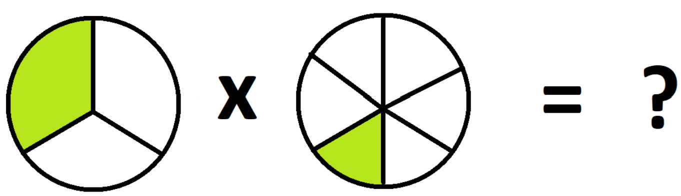
Question: What's $\frac{1}{3} + \frac{1}{6}$ with this type of thinking?



We're now doing arithmetic with fractions. They are starting to feel like numbers.

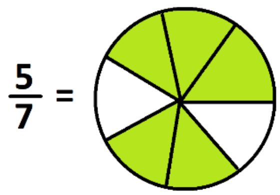
But are they numbers?

Question: If $\frac{1}{3}$ and $\frac{1}{6}$ truly are numbers, then we should be able to multiply them as well. Can we? Does "a third of a pie times a sixth of a pie" make sense to you? (It sounds like nonsense to me.)



Here's a little more on jargon.

The fraction $\frac{5}{7}$ means "five copies of a seventh."
It counts out five copies of a certain "type of thing," namely, a *seventh*.



The late Latin word for "a count of things" is *numerare* and the word for "a type of a thing" is *denominare* with *de-* meaning "completely" and *nominare* meaning "to name".

These led to words we use in everyday life. For example, to *enumerate* means to count out something and a *denomination* is the name of a certain class or type.

These Latin words also led to the modern names of the two numbers we use to write fractions.

For the fraction $\frac{5}{7}$, for instance, the top number "5" is called its **numerator**. It tells us the count of things (slices) we have. The bottom number "7" is called its **denominator**. It tells us the type of thing we are counting, namely, sevenths.

While we are at it …

The word **fraction** itself comes from the late Latin word *fractionem* which means "the act of breaking into pieces." One might break one's arm and have a bone *fracture*, or one might break the law and commit an *infraction*, or one might break a pie into pieces and create examples of *fractions*.

Where Does This Second Story Leave Us?

We have the edict that in any conversation about fractions, the fractions should come from the same "whole" (pie). This allows us to legitimately compare fractions and we can start to do what feels like some arithmetic with fractions.

But it is only a feeling. Fractions aren't numbers here – they are portions of pie – and the "addition" of fractions is just the physical act of bringing portions of pie together.

Still, students can be asked some mind-bendy questions.

Question 1: Betta ate $\frac{1}{6}$ of a pie and Cuthbert ate $\frac{1}{7}$ of a pie. Who ate the most pie? Select all options that could be true.

a) Betta did if we know the pies were the same size.
b) Cuthbert did if we know the pies were the same size.
c) Cuthbert could have eaten more if his pie was sufficiently larger than Betta's.
d) Perhaps neither. They could have eaten the exact same amount of pie if their pies were differently sized.

Question 2: A pie is divided into five pieces. Only one piece can legitimately be dubbed as $\frac{1}{5}$ of the pie. Which of the following statements must be true?

a) There is at least one piece larger than $\frac{1}{5}$ of the pie.
b) There is at least one piece smaller than $\frac{1}{5}$ of the pie.
c) Neither of these statements need be true.

To fix up the worry of whether or not fractions are actually numbers, the curriculum takes a new turn.

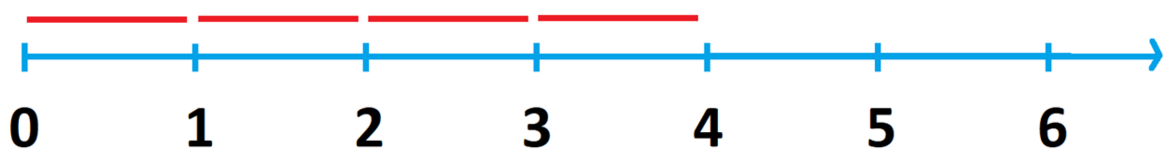
Answers to the Two Questions:

1. a) c) d)
2. a) b)

SLIGHTLY LATER GRADES: Fractions are Points on a Line

School math likes to stack numbers along a **number line**.

It's a line that is built from laying sticks of length 1 end-to-end with the left end of the line labeled 0. The point labeled "4" on the line, for instance, tells us that four sticks laid together from 0 on the left reach that point on the line.



Up to this point, fractions have been portions of a whole, with that whole typically being a pie.

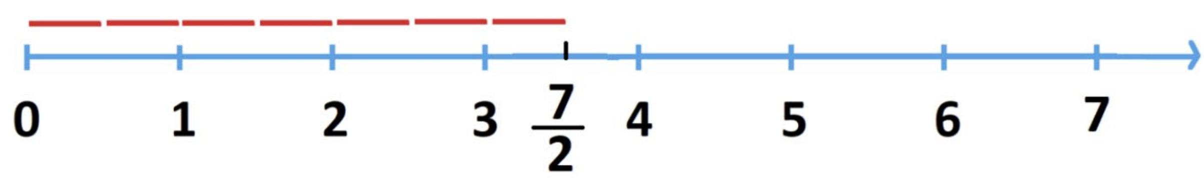
But as soon as schools introduce the number line, a switch in fraction thinking occurs.

Make your stick of length 1 your new pie!

You are now slicing up a stick. (You can think of the stick as a very thin, flat, linear pie. Or maybe a strand of uncooked spaghetti?)

The fraction $\frac{1}{2}$ now represents half a stick and we mark on the number line the point where that portion of the stick reaches. We label that point $\frac{1}{2}$ as well.

The fraction $\frac{7}{2}$ means "seven copies of $\frac{1}{2}$" and we mark on the line the point where seven half sticks reach.



And this number-line thinking suggests that $\frac{7}{2}$ is the same as $3 + \frac{1}{2}$ (though we haven't officially talked about what addition means in this context of a number line).

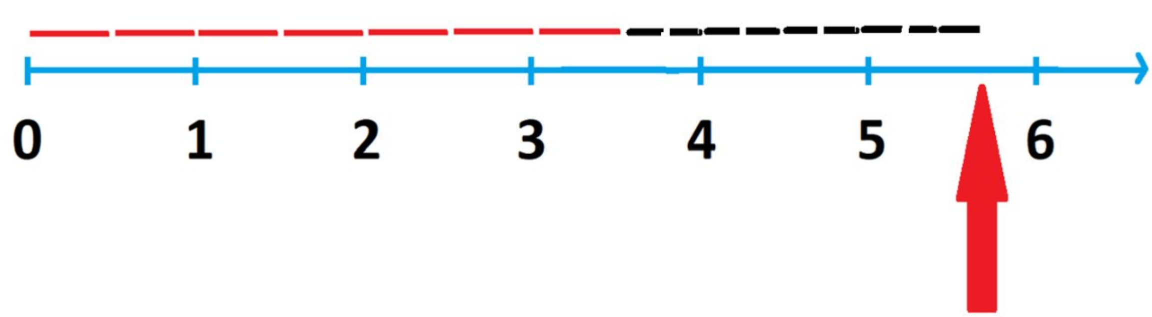
> **Aside:** Using a number line for visualizing addition (and subtraction) came quite late in the history of mathematics. English mathematician John Wallis (1616-1703) seems to have been the first person to write about a number line in this context.

Having students work with fractions on the <u>number</u> line could be taken as a psychological ploy.

> Well, if fractions are on the number line, they must be numbers!

Hmm.

Following Wallis' ideas, we can do arithmetic with fractions on the number line. For example, by placing down a length of $\frac{7}{2}$ and a length of $\frac{9}{4}$ side by side, it looks like we're landing at the mark $5 + \frac{3}{4}$ on the number line.



We marked fractions on the number line. We're doing some arithmetic with fractions.

So, are fractions numbers?

Where Does this Third Story Leave Us?

I am not sure!

I feel like I am being coerced to say that fractions are numbers.
But even here fractions are still portions of pie– just a long thing spaghetti-like pie.

Sure, we can label off the lengths along a number line. But the labels themselves are not the objects we're talking about. Or are they now?

This is confusing.

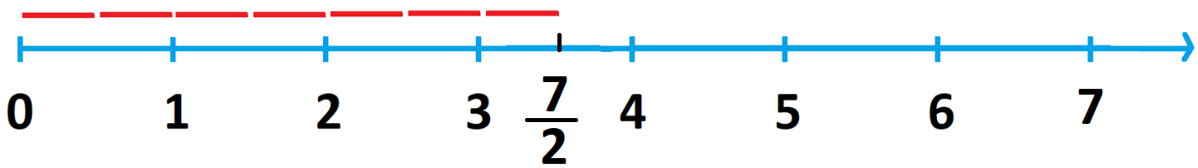
Plus ... If fractions are numbers, I still don't know what it means multiply two fractions.
How do I multiply two labels to make a third label?

My brain hurts!

But never mind, the school curriculum pushes on.

LATER GRADES: Fractions are Numbers

We're at the point of our curriculum story where fractions, that were once portions of pie, are now being seen almost solely as labels on the number line.



And the coercion is strong: Because fractions are on the number line, they must be numbers.

Are they?

The typical school curriculum next just turns this question into an assertion.

Fractions are numbers. They are answers to division problems.

For example, $\frac{7}{2}$ is no longer "seven copies of $\frac{1}{2}$," *per* se. It is now the answer to the division problem $7 \div 2$. And $\frac{1}{3}$ is the answer to the division problem $1 \div 3$.

A fraction $\frac{a}{b}$ is the number that answers the division problem $a \div b$.

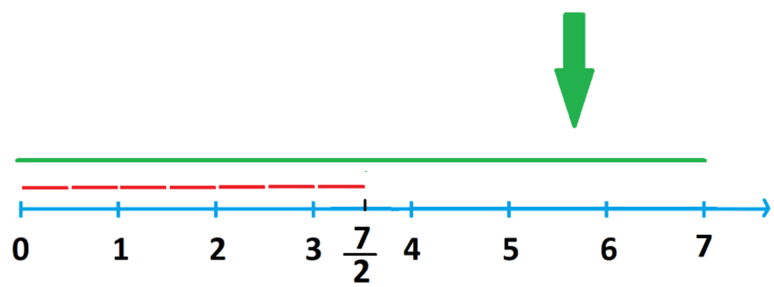
This is typically justified by thinking of division as "division by sharing" (from Section 17).

For example, a textbook might be naughty and have students ignore the number line and go back to portions-of-pie thinking (despite weening students off this thinking).

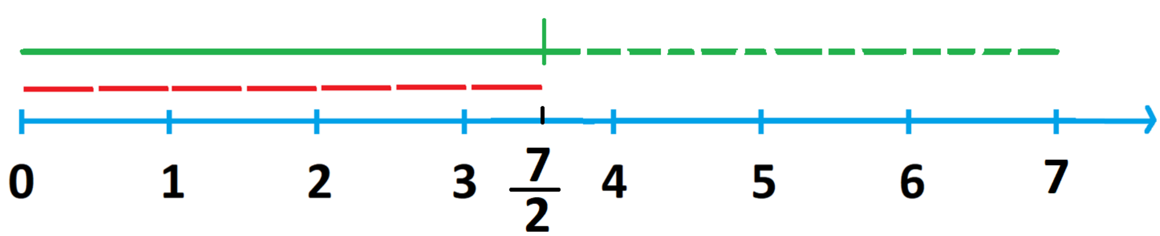
> *Interpret $1 \div 3$, say, as the result of sharing one pie equally among three students. The result is one third of pie per student. We called that $\frac{1}{3}$. So, $1 \div 3 = \frac{1}{3}$.*

Or textbooks attempt to conduct division by dividing (sharing) lengths on the number line.

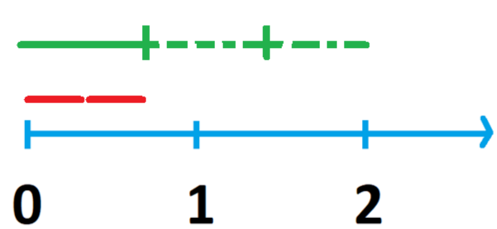
For example, let's try to find the value of $7 \div 2$ by diving a length of 7 sitting on the number line into two equal parts. Can you imagine sliding the green arrow to the appropriate spot?



What is surprising—or should come across as surprising—is that the arrow lands on the spot that matches "seven copies of $\frac{1}{2}$," which is how we previously interpreted $\frac{7}{2}$.



Similarly, if we divide a line segment of length 2 into three equal parts, the length of left piece seems to match the position of the fraction $\frac{2}{3}$, which we previously understood to be two copies of one third.



But this leaves me wondering, in general:

Will the result of constructing $a \div b$ always land at the same spot as "a copies of $\frac{1}{b}$"?

Thinking through this question feels weird and hard! (Plus, I am not sure I "get" the question!)

Okay. I'll just swallow that fractions are numbers and that they do match the answers to division problems.

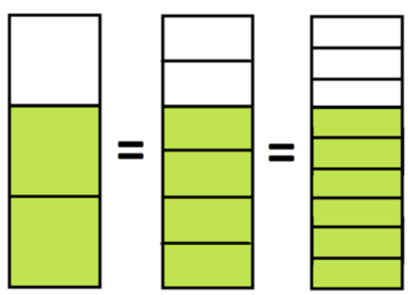
Let's just accept that somehow $\frac{a}{b}$ and $a \div b$ and "a copies of $\frac{1}{b}$" all match up to be the same thing.

But, at the same time, don't!

When introducing a new fraction concept, textbooks often revert back to "portions of pie" when it feels convenient and simply drop the "fractions are answers to division problems" definition.

For example, you might remember learning about *equivalent fractions*.

To explain why $\frac{2}{3}$, for instance, is the same as $\frac{4}{6}$ and as $\frac{6}{9}$ textbooks draw pies. Pies might be vertical rectangles now.



Is there a connection to division problems here? It is not obvious if there is one.

So, despite its claims and edicts, the typical school curriculum oscillates between old and new fraction concepts, pulling up whatever seems magically convenient at the time.

But then things get even worse!

Mysterious “OF” and Magical “KEEP CHANGE FLIP”

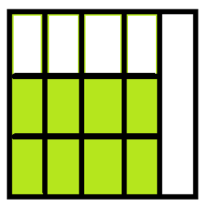
Back in your early school days did you hear the mantra: *“of” means multiply*?

It is natural to use the word “of” when doing work with fractions, at least when we go back to thinking about pie.

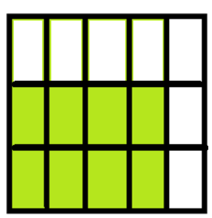
For example, we can imagine figuring out something like “two thirds of four fifths of a pie.”

To compute it, start with a picture of $\frac{4}{5}$ of a pie.

Then select $\frac{2}{3}$ of that portion of the pie.



If we draw in extra lines, it seems we have a pie that has been divided into 15 pieces and we’ve selected 8 of them. It is natural to say that *two-thirds* ***of*** *four-fifths is eight-fifteenths*.



Question: Still thinking in terms of portions of pie, does it seem natural to you to say that $\frac{1}{2}$ **of** $\frac{1}{3}$ of a pie is $\frac{1}{6}$ of the pie?

If we are indeed allowed to go back to thinking of fractions as portions of pie, then this use of the word "of" makes perfectly good sense.

But what does "of" have to do with multiplication?

Well, the seems to be two curriculum answers to this question.

Answer 1: *We've been using the word "of" with multiplication all along. Why stop now?*

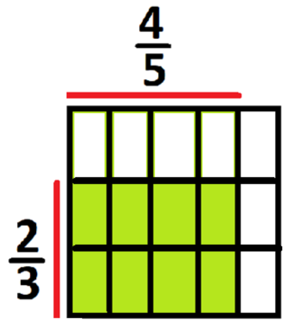
For example, back with counting numbers we've been reading something as simple as 2×3 as "two copies **of** three." So, if "3" is represented as three dots

then 2×3 is six dots.

So, it makes sense that $\frac{1}{2} \times 3$ should be interpreted as "half a copy **of** three dots," and that would match $\frac{3}{2}$, three copies of half a dot.

Answer 2: *Look at our pictures of rectangular pie when we calculate fractions of fractions. They look like Rule 8 for chopping up rectangles, which is the area model for multiplication!*

For example, here's our picture of "$\frac{2}{3}$ of $\frac{4}{5}$" again.



It is mighty lucky that we happened to draw a rectangular pie to work this out. (Would we see the same thing if we had been drawing circular pies?)

Authors also point out that, for this example at least, $8 = 2 \times 4$, $15 = 3 \times 5$, and our answer is $\frac{8}{15}$. Curious!

> **Aside:** Allow me to share a personal concern I have about this picture. If we are thinking "pie," then what exactly is our pie here?
>
> We have one thin spaghetti pie for $\frac{4}{5}$, and another thin spaghetti pie for $\frac{2}{3}$, and our final answer is about a rectangle of green pie coming from a two-dimensional rectangular pie.
>
> I thought the edict was: *In a conversation about fractions, the fractions must come from the same whole.*

Also, what happened to our "fractions are answers to division problems" thinking?
What happened to our number line?

The conversations about "of" and "multiplication" ignore what students were just taught.

Some curriculums don't try to justify a multiplication rule for fractions and just assert a rule:

> To multiply two fractions, multiply together the numerators and the multiply the denominators and use those products to create a new fraction: $\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$.

And then comes the division of fractions!

Were you told a rule like this?

To divide two fractions, just "keep change flip."

For example, to compute $\frac{2}{3} \div \frac{4}{5}$,

keep the first fraction as $\frac{2}{3}$ as it is,
change the division symbol to multiplication sign $\times$, and
flip the second fraction upside down and make it $\frac{5}{4}$.

That is, to work out $\frac{2}{3} \div \frac{4}{5}$ just compute $\frac{2}{3} \times \frac{5}{4}$ instead.

Got that? Does this make perfectly good logical sense to you?
It doesn't to me! Where is this coming from?

Where Does this Final Story Leave Us?

At this point one's education on fractions is essentially complete.

But what are we actually left holding on to? I am not sure!

We have some intuitive models of fractions—portions of pie, answers to sharing problems, points on the number line—and were told that these models are not "complete," that fractions are actually numbers and that one can perform the full range of arithmetic with them.

We have a vague sense that, maybe, the arithmetic of fractions is motivated by real-world models, but it all feels haphazard and hazy.

But, in any case, many students by this stage of the story have stopped questioning the haziness and have memorized and mastered the mechanics of fraction arithmetic.

And then we send them off to high school where educators presume fractions are well understood and proceed to speed on to more abstract ideas that make use of fractional quantities.

This reality of K-12 mathematics education deeply perturbs me both as a professional mathematician and as a somewhat functional human being. One needs to look back and take stock of these stories and find a framework to make sense of the jumble of ideas as one whole. (Ha!)

That we don't is a serious disservice to thinking humans. Not "getting" fractions is a natural and valid response to these curriculum stories if they are just left hanging as they are.

So, let's turn matters around together now, in this chapter and in the next.

We'll be gentle and start with one more real-world model of fractions—but use that model as a transition. It will lead us to the true, robust and logically complete mathematical story of fractions.

There is rhyme and reason to fractions.

We'll explore both the rhyme and the reason.

MUSINGS

Musing 37.1 It is curious that we need to write fraction, which is a single number, with two numbers—a numerator and a denominator. For example, the number $\frac{5}{7}$ is described with a five and a seven. We also place the two numbers vertically with a horizontal bar between them.

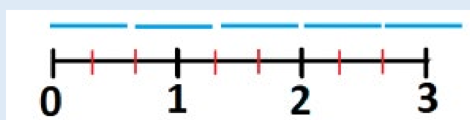
a) Look up the general history of fraction notation. Who first started writing the two numbers to describe a fraction vertically?

b) What is the official name of the horizontal line used in fraction notation? (Most people call it a "fraction bar," but what is its Latin name? Why that name?)

Musing 37.2 Some curriculums have students play with the division of fractions using a number line. They have students draw a picture and ask: "How many groups of what I am looking for do I see?"

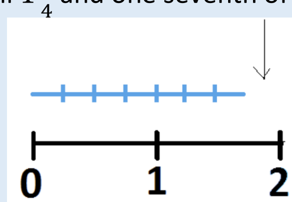
Here are two examples.

Example: *Draw a picture to evaluate* $3 \div \frac{2}{3}$.
Answer:



We see four-and-a-half copies of $\frac{2}{3}$ in 3. We have $3 \div \frac{2}{3} = 4\frac{1}{2}$.

Example: *Draw a picture to evaluate* $2 \div 1\frac{3}{4}$.
Answer: We see that if we had one more seventh of the blue segment shown, we'd have a complete "2." Thus, we see one full $1\frac{3}{4}$ and one seventh of $1\frac{3}{4}$ in 2.



This means $2 \div 1\frac{3}{4} = 1\frac{1}{7}$.

Of course, this is another seemingly out-of-the-blue idea to toss into the general jumble of ideas for working with fractions.

a) Do you like what's going on in the two examples?
(The answer can be: "No I don't! I don't get it. I don't like it. And I am turning the page.)

b) Would you like to complete each of these with number-line pictures?
(The answer can be no.)

i) $2\frac{1}{2} \div \frac{3}{4}$ *ii)* $\frac{1}{2} \div 3$

38. One Somewhat Robust Model of Fractions

Let's really lean into the schoolbook statement:

A fraction is a number. It is the answer to a division problem.

As we saw in section 17, there are three different (but equivalent) ways to think of division. So, let's be specific.

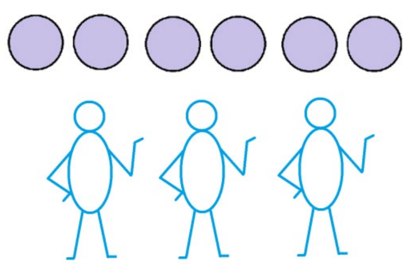
A fraction is a number. It is the answer to an equal-sharing problem.

And what shall we share?

Well, to connect our thinking with standard school mathematics that is obsessed with pies, let's also share pies. We'll share pies equally among students.

Here goes.

Question: Suppose I have 6 pies to share equally among 3 students.
How many pies per student does that give?

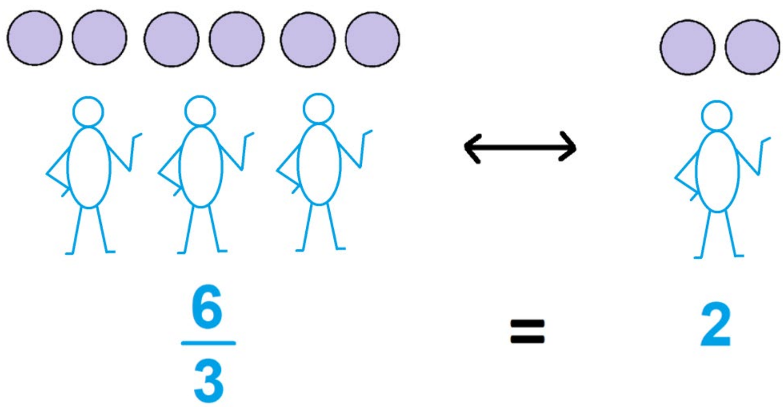


Of course, the answer is 2. Each student gets two pies.

And most people summarize the results of this sharing task by writing $6 \div 3 = 2$.

But let's be direct.

We are saying that a fraction is the answer to a division problem, so let's use fraction notation and write $\frac{6}{3} = 2$ for this sharing task.



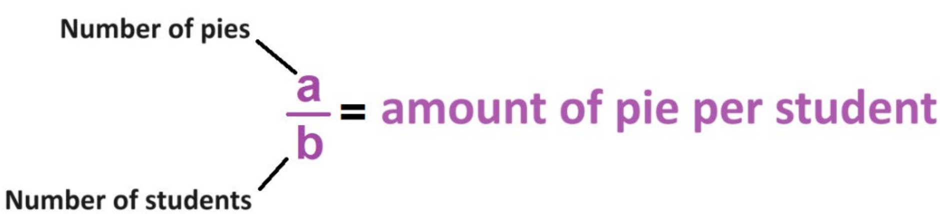
In the same way:

Sharing 20 pies equally among 5 students yields $\frac{20}{5} = 4$ pies per student

and

Sharing 100 pies among 2 students yields $\frac{100}{2} = 50$ pies per student

Here's how we now read fraction notation.

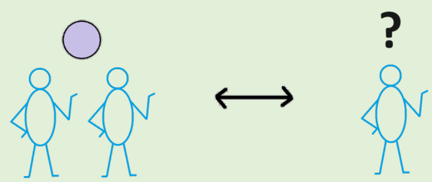


INTUITION CHECK

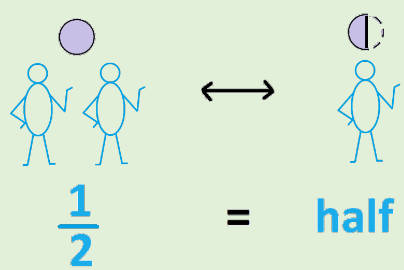
Most people do not think of $\frac{6}{3}$ and $\frac{20}{5}$ and $\frac{100}{5}$ as examples of fractions. The school curriculum starts its story with a numerator smaller than the denominator.

So, what does this pies-per-student model have to say about something we might normally identify as a fraction? For example, what is $\frac{1}{2}$ in this model?

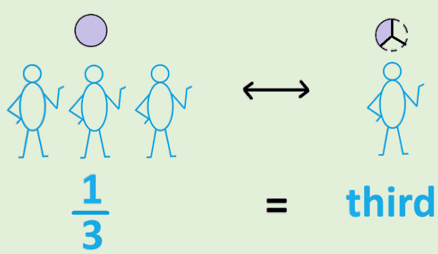
Well, $\frac{1}{2}$ represents the amount of pie each student gets if 1 pie is shared equally among 2 students.

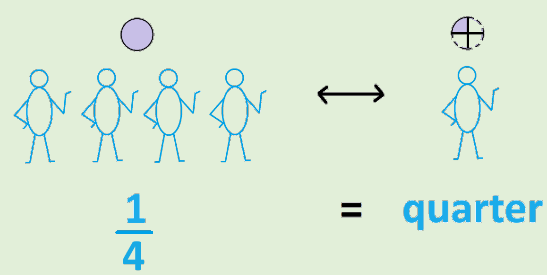


The answer matches what we call a *half* in our early understanding of fractions. This is good!



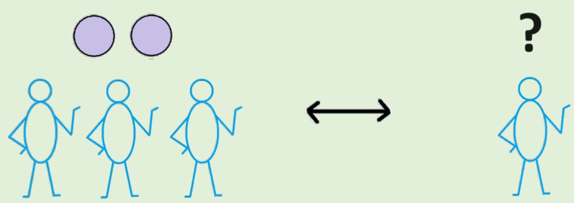
In the same way, $\frac{1}{3}$ matches what we earlier learned to call a *third*, and $\frac{1}{4}$ matches what we earlier learned to call a *quarter*, and so on.





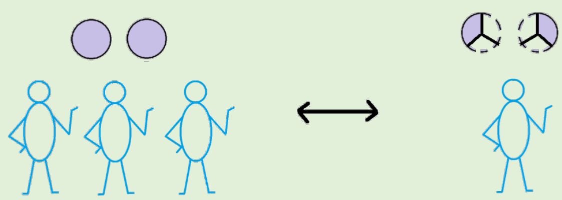
We also interpreted $\frac{2}{3}$ as “two copies of a third.”

Does this pies-per-student model interpretation of $\frac{2}{3}$ align with this?



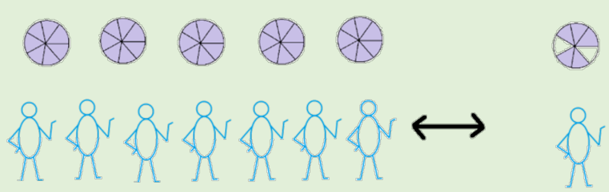
One way to physically share two pies equally among three students is to divide each pie into three equal pieces (thirds) and give each student a slice from each of the pies.

This does indeed give two thirds of a pie to each student.



$\frac{2}{3}$ = **two thirds**

In the same way, $\frac{5}{7}$ would also be “five copies of a seventh:” just divide each of five pies into seven equal-sized pieces (sevenths) and then give each student a slice from each pie. As there are five pies, each student gets five sevenths.

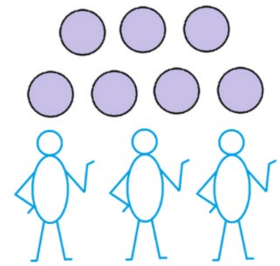


$\frac{5}{7}$ = **five sevenths**

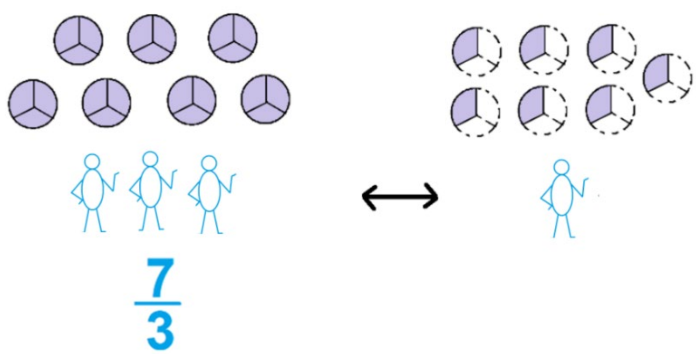
It’s comforting that our pies-per-student model is matching our real-world intuition for fractions thus far.

Question: How would you personally share 7 pies equally among 3 students?

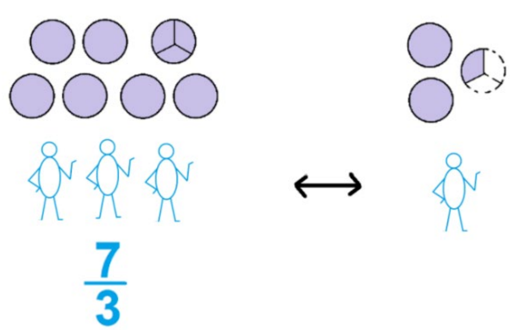
There are at least two ways to complete this task.



Approach 1: As per the intuition check, we could slice each pie into three equals parts and give each student a third from each pie. This means that $\frac{7}{3}$ corresponds to seven copes of a third.



Approach 2: Give each student two whole pies and then slice the one remaining pie into three equal pieces. This shows that $\frac{7}{3}$ also corresponds to $2 + \frac{1}{3}$.



This shows that answers to sharing problems can come in more than one guise.
This is both a good and an annoying feature of fractions!

INTUITION CHECK

In the early grades we learned to circle a third of a group of kittens, or half of a set of stars, or identify a quarter of a bag of apples.

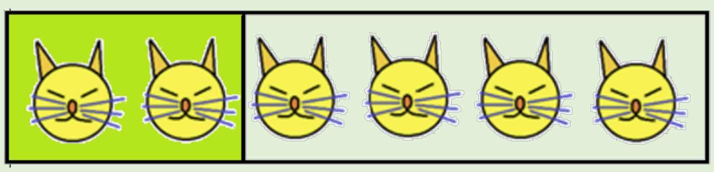
Does our pies-per-student model align with that work too?

Yes … if we regard everything as pie!

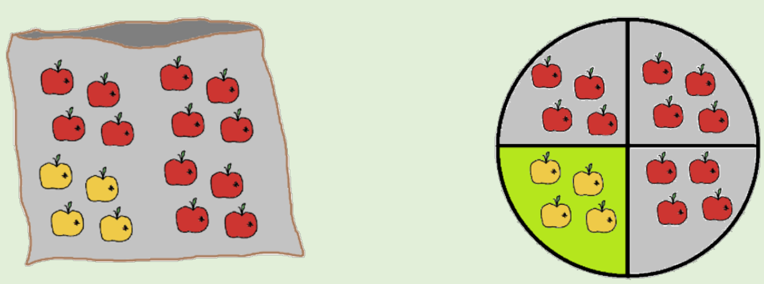
For example, here's a rectangular pie (perhaps cake) that happens to be decorated with six kitten faces.

If 1 of these cakes is shared equally among 3 students, how many kitten faces will each student receive?

Well, each students receives a slice of the cake we call a third, $\frac{1}{3}$. And each third has 2 kitten faces.

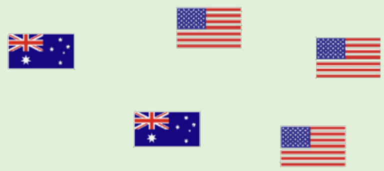


If we regard this bag of apples as a pie, then sharing the 1 pie equally among 4 students does show that the yellow apples match $\frac{1}{4}$ of the pie.



Let's push this thinking a tad further.

Here's a collection of flags.

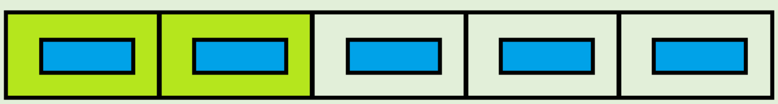


It is natural to say that "two fifths of the flags are Australian" and that "three fifths of them are American."

Does that language match our pie-sharing thinking?

Yes!

Imagine pies decorated with five generic flags. Now share 2 such pies equally among 5 students. This picture shows what each student will receive. It's a picture of $\frac{2}{5}$.



Philisophically, this is the same as picture of five flags with two of them identified as special, like the Australian flags in our original picture. (They are special in my mind!)

So yes, it is valid to say that "$\frac{2}{5}$ of the flags are Australian" in our original picture of flags.

And similarly, one can also legitimately say that "$\frac{3}{5}$ of the flags are American."

Our pie-sharing model—admittedly with some mental contortion—can be seen as aligned with the early-school approach to fractions.

ASIDE: Some Quirky Fun

Let's be a little bit naughty and get ahead of ourselves and put some non-whole numbers in unexpected places. It gives a hint of how powerful this sharing model is going to be.

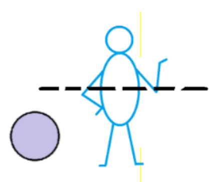
Here's the question.

Can you make sense of $\frac{1}{\frac{1}{2}}$?

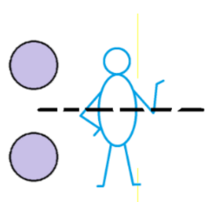
That is, if one pie is "shared equally among" half a student, how much pie does a whole student get?

It's a weird question, but maybe the thinking to go with it is something like the following.

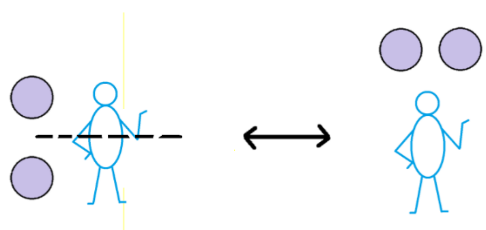
Half a student gets a pie. This picture shows the bottom half of a student getting a pie as dictated.



But there is a top half of the student too. And half a student gets a pie.



So, how much pie does a whole student get? Two pies!

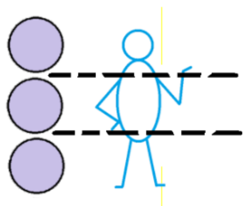


One pie for half a student results in two pies per whole student. Wild!

$$\frac{1}{\frac{1}{2}} = 2$$

What's the value of $\frac{1}{\frac{1}{3}}$, the result of giving one whole pie to each third of a student?

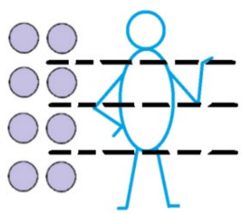
This picture shows it has value 3



Let's keep going.

What's the value of $\frac{2}{\frac{1}{4}}$?

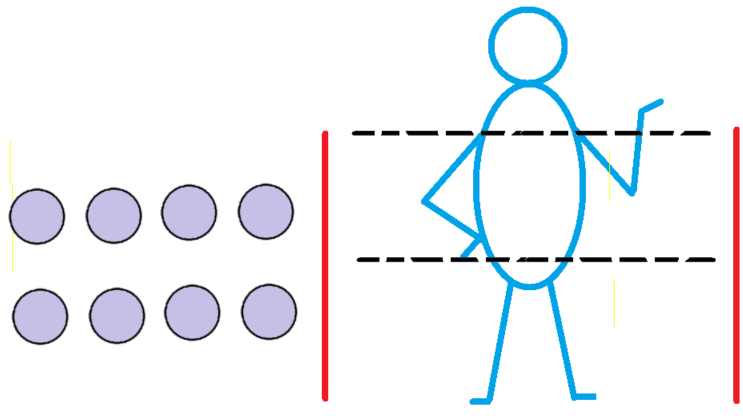
It's 8!



Here's a challenge question. Before turning the page can you figure out the answer to this question?

What's the value of $\frac{8}{\frac{2}{3}}$?

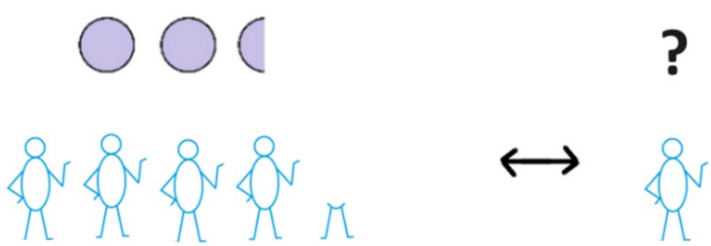
Well, if 8 pies are assigned to $\frac{2}{3}$ of a student, then there are 4 pies for each third of the student. This makes for 12 pies in total for one whole student!



This quirkiness is not serious stuff. But it shows we can have some fun with this pies-per-student model if we want to push it.

Question: Can you reason that $\frac{5}{1/7}$ must be 35?

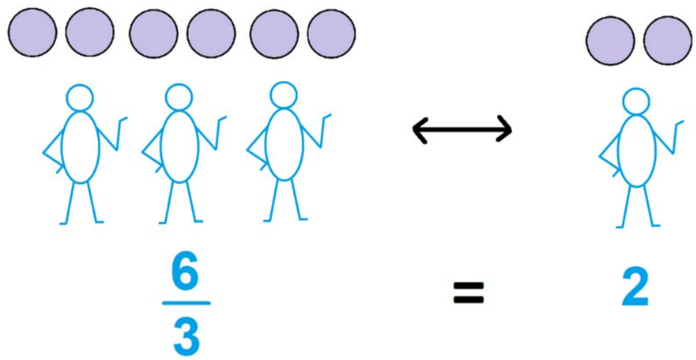
An Absurd Challenge not worth Considering:
Two-and-a-half pies are to be shared equally among four-and-a-half students! How much pie does an individual (whole) student receive?

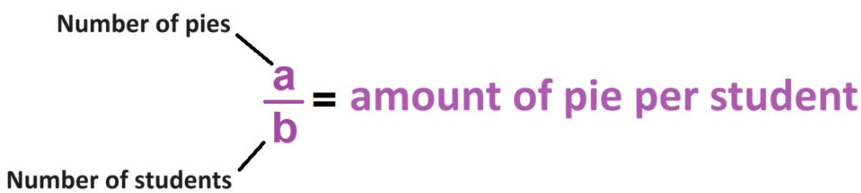


Going Backwards

We’ve been sharing pies equally among students.

And we’ve been using the notation of fractions to describe that. After all, fractions are answers to sharing problems in this model.





Here’s a puzzle.

Some pies were shared equally among some students.

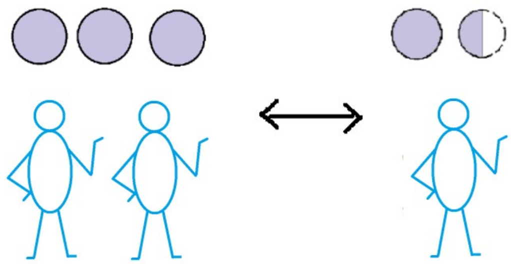
Each student received one full pie and one-half pie.

How many students were there initially? How many pies?

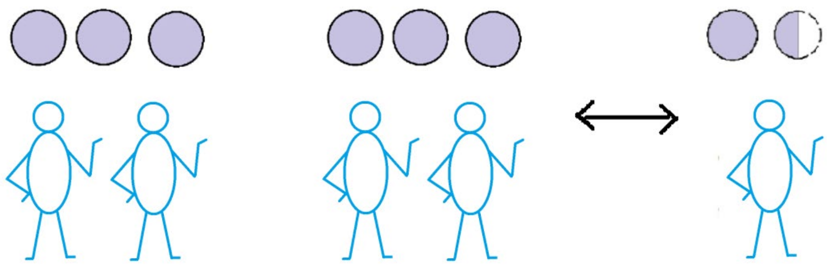
Give one possible answer.

We can reason this way.

Splitting one pie into two equal parts makes us think that there were 2 students. And if each student is getting a whole pie and a half a pie, it must be because there were 3 pies.



Practice 38.1: There are other possible scenarios that could lead to the same amount of pie per student. For example, sharing 6 pies equally among 4 students can give the same final result. (Do you see why?)



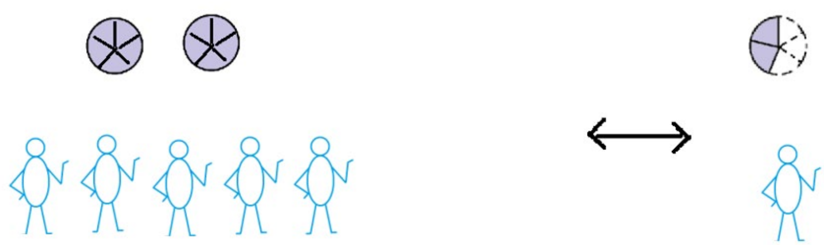
Give a third scenario that leads to the same result again.

Seeing a small portion of pie handed out to a student gives a hint as to how many students there could have been to begin with.

For example, here's a student receiving **two fifths** of a pie. This suggests some pie, or perhaps pies, were divided into five equal parts. Consequently, there could have been 5 students.

And if there were 5 students, how many pies must there have been to get this result?

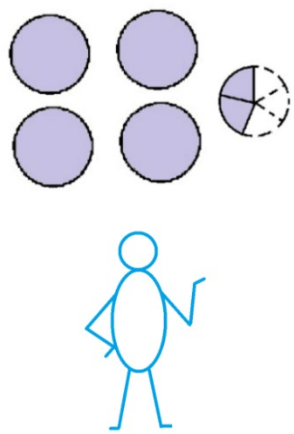
Each slice likely came from a pie. As the student received two slices, we deduce there must have been 2 pies.



Our mathematical notation presents "two fifths" as $\frac{2}{5}$, with a 2 and a 5. The notation itself is describing a scenario that yields the desired result: 2 pies being equally shared among 5 students!

A written fraction describes a working scenario!

Practice 38.2 Give a number of pies and a number of students that yields 4 whole pies and $\frac{2}{5}$ of a pie for each student when shared equally.

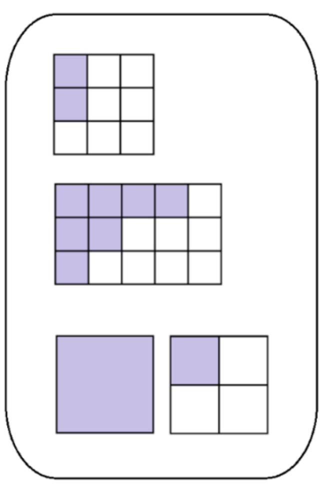


Practice 38.3 Here's a matching puzzle.

On the left we have some pictures of the amount of square/rectangular pie each student received in a sharing game, and on the right, we have some counts of pies and students.

Match each result on the left with the set-up on the right that produces it.

(Really try to imagine a picture of pies being shared among students each time.)



5 pies and 4 students

2 pies and 9 students

7 pies and 15 students

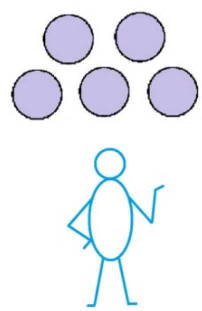
Practice 38.4: Is $\frac{10}{13}$ double the value of $\frac{5}{13}$?
If so, how would you justify this thinking about pies and students?

Some Properties of Fractions

Let's now play with this pies-per-student model to see what it tells us mathematically about fractions.

To start, suppose we have 5 pies to share among one (lucky) student. How many pies per student is that?

Clearly 5.



It is tautological, but we have just learned that $\frac{5}{1}$ (five pies for one student) is 5 (five pies per student).

In the same way, 20 pies for one (even luckier) student makes for $\frac{20}{1} = 20$ pies per student.
And $4{,}096$ pies for one (burdened?) student makes for $\frac{4096}{1} = 4096$ pies per student.

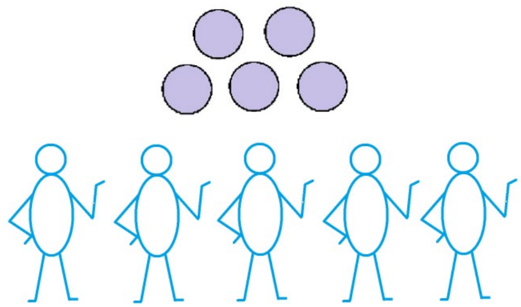
In general, we're seeing:

FRACTION PROPERTY 1

$\frac{a}{1} = a$ for each counting number a.

Practice 38.5: Does this property make sense if a is zero? Does saying that $\frac{0}{1} = 0$ feel right?

Suppose now I share 5 pies equally among 5 students. How many pies per student is that?
It's clearly 1 pie per student.



In the same way, 20 pies shared equally among 20 students makes for 1 pie per student, as does sharing 503 pies among 503 students.

$$\frac{20}{20} = 1$$

$$\frac{503}{503} = 1$$

In general, we're seeing that

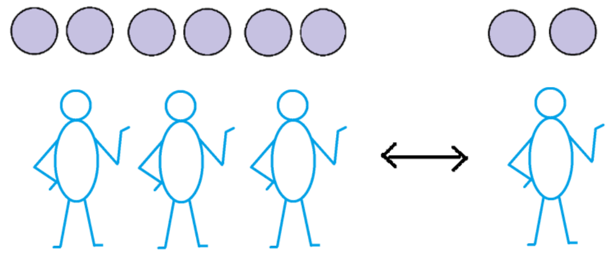
FRACTION PROPERTY 2
$\frac{a}{a} = 1$ for each counting number a different from zero.

We saw in Section 17 that dividing (sharing) by zero is problematic. Property 2 is avoiding this issue by attending only to nonzero numbers. (After all, if you were tasked to share zero pies equally among no students, what would you do and how much pie would each student get? (What student?).)

Let's now go back to an early example.

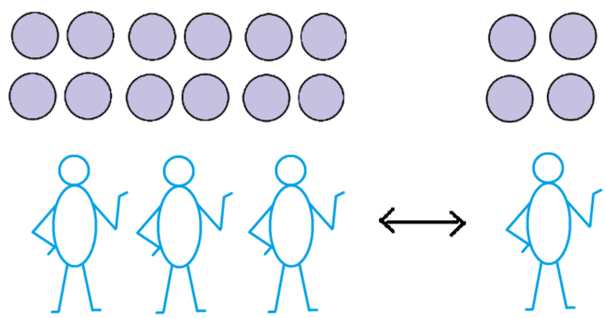
Sharing 6 pies equally among 3 students yields 2 pies per student.

$$\frac{6}{3} = 2$$



Suppose I am feeling generous and want to double the amount of pie each of the three students receives. How might I do that?

Well, I would have to double the count of pies I share. Make it 12 pies instead of 6.



Here's how to write that mathematically.

$$2 \times \frac{6}{3} = \frac{12}{3}$$

To double the amount of pie per student, double the number of pies.

Similarly, to triple the amount of pie each student receives, just triple the number of pies you give out. Or to centuple the amount of pie each student receives, just centuple the number of pies you share.

$$3 \times \frac{6}{3} = \frac{18}{3}$$

$$100 \times \frac{6}{3} = \frac{600}{3}$$

In general, $\frac{a}{b}$ is the amount of pie an individual student receives when a pies are shared equally among b students. To double, triple, quadruple, quintuple, or even centuple the amount of pie each student receives, change the number of pies we give out to $2a$ or $3a$ or $4a$ or $5a$ or $100a$, respectively.

To change the amount of pie each student receives by a factor k, change the number of pies by that factor.

We have:

FRACTION PROPERTY 3

$k \times \frac{a}{b} = \frac{k \times a}{b}$ for each number k and fraction $\frac{a}{b}$.

INTUITION CHECK

We saw earlier that $\frac{2}{3}$ is usually interpreted as "two copies of $\frac{1}{3}$," that is, as $2 \times \frac{1}{3}$.

Does the mathematics say this too?

It does!

Fraction Property 3 tells us

$$2 \times \frac{1}{3} = \frac{2 \times 1}{3}$$

And the numerator 2×1 is just 2:

$$2 \times \frac{1}{3} = \frac{2}{3}$$

So, yes, $\frac{2}{3}$ and $2 \times \frac{1}{3}$ are the same.

In the same way,

$$\frac{4}{5} = 4 \times \frac{1}{5}$$

("four fifths" is four copies of a fifth) and

$$\frac{13}{9} = 13 \times \frac{1}{9}$$

("thirteen ninths" is thirteen copies of a ninth).

The mathematics is aligned with our early intuition.

Let's put mathematics to another test.

Rule 3 from our general Rules of Arithmetic says that multiplying a number by zero is sure to give zero. So, for example,

$$0 \times \frac{2}{5} = 0$$

But what does Property 3 say about $0 \times \frac{2}{5}$?

Well,

$$0\times \frac{2}{5} = \frac{0\times 2}{5}$$

The numerator here is 0×2 which is 0. So, $0 \times \frac{2}{5}$ also equals $\frac{0}{5}$.

We now have two conclusions:

1. $0 \times \frac{2}{5}$ equals 0 (from Rule 6)
2. $0 \times \frac{2}{5}$ equals $\frac{0}{5}$ (from Property 3)

We conclude that 0 and $\frac{0}{5}$ must be the same value.

$$\frac{0}{5} = 0$$

Sharing no pies equally among 5 students yields zero pie per student. And that too matches our intuition.

Everything is hanging together!

We are learning that it is possible for a fraction to have a numerator of zero. Such a fraction is sure to have value 0.

$\frac{0}{b} = 0$ for each non-zero counting number b .

But what about a fraction with denominator zero?

Is $\frac{5}{0}$, for instance, a meaningful quantity?
(It doesn't make sense in our pie-sharing scenario.)

Of course, we addressed this issue in Section 17 where we showed that division by zero is undefined in mathematics (and sharing, as we are doing in this section, is an interpretation of division).

But let's have another Intuition Check and discuss denominators of zero.

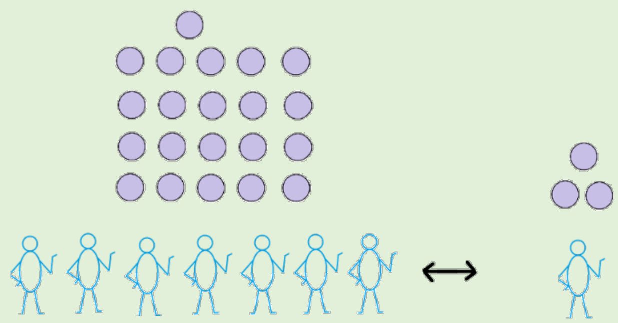
First, this warm-up problem.

Practice 38.6 In a sharing scenario, some pies were shared equally among 7 students. If each student received 3 pies, how many pies were there in total to begin with?
(That is, if $\frac{a}{7} = 3$, what is a?)

INTUITION CHECK

We know that $\frac{21}{7} = 3$.

If 21pies are shared equally among 7 students, each student will receive 3 pies.



We would then have 7 happy students each with 3 pies, and that accounts for $7 \times 3 = 21$ pies, all of them.

That is: **We can see that $\frac{21}{7} = 3$ is correct because $7 \times 3 = 21$.**

We can also deduce that $\frac{100}{5} = 20$ is correct because $5 \times 20 = 100$. (Yep, if five students each have 20 pies, that accounts for all 100 pies.)

We also deduce that $\frac{42}{6} = 8$ is <u>not</u> correct because 6×8 is not 42. (If six students each have 8 pies, that makes for 48 pies, not 42 of them.)

We have:

The statement $\frac{a}{b} = N$ is correct if $b \times N = a$.

Practice 38.7: Use multiplication to check the validity of each statement shown. Which of the statements are incorrect?

$\frac{103}{103} = 1$ $\qquad$ $\frac{1000}{125} = 10$ $\qquad$ $\frac{999}{1} = 998$ $\qquad$ $\frac{5}{0} = 3$

The fourth example, $\frac{5}{0}$, is particularly interesting.

$\frac{5}{0}$ can't be 3 because 0×3 is not 5.

Also

$\frac{5}{0}$ can't be 7 because 0×7 is not 5.

$\frac{5}{0}$ can't be $12{,}002$ because 0×12002 is not 5.

There is no value to assign to $\frac{5}{0}$, because zero times any value you care to choose will be 0, not 5.

If a is a counting number different from zero, then there is no possible value to assign to $\frac{a}{0}$.

As we saw in Section 17, the expression $\frac{0}{0}$ suffers from a different mathematical challenge.

$\frac{0}{0} = 3$ is seemingly correct because 0×3 does equal 0.

$\frac{0}{0} = 7$ is seemingly correct because 0×7 does equal 0.

$\frac{0}{0} = 12002$ is seemingly correct because 0×12002 does equal 0.

Every value passes the multiplication check $\frac{0}{0}$.

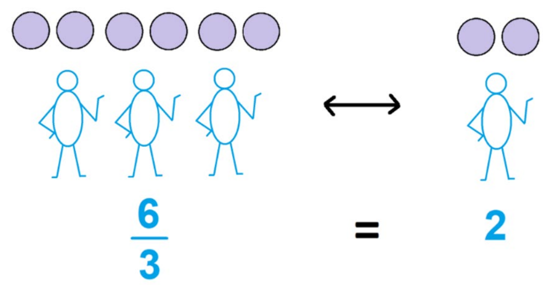
The expression $\frac{0}{0}$ is "indeterminant:" there is no consistent value to assign to it.

In all cases, fractional expressions with a denominator of zero are mathematically inconsistent. (And they are intuitively fraught too: how do you share pies among zero students?)

We are learning that our mathematical theory of fractions must be about quantities of the form $\frac{a}{b}$ with a and b numbers and with b not zero.

39. One More Fraction Property

Fractions are answers to division problems. In particular, they are the answers to problems about sharing pies equally among students in this chapter so far.



This model naturally leads to three observed properties of fractions.

FRACTION PROPERTY 1

$\frac{a}{1} = a$ for each counting number a.

FRACTION PROPERTY 2

$\frac{a}{a} = 1$ for each counting number a different from zero.

FRACTION PROPERTY 3

$k \times \frac{a}{b} = \frac{k \times a}{b}$ for each number k and fraction $\frac{a}{b}$.

For example,

- Sharing 7 pies among 1 student results in $\frac{7}{1} = 7$ pies for that student
- Sharing 23 pies among 23 students results in $\frac{23}{23} = 1$ pie per student
- To double the amount of pie each individual student receives in a sharing problem, just double the number of pies you share out.

And we've learned to avoid denominators of zero.

Fraction Property 3 allowed us to see "two thirds" as "two copies of a third"

$$\frac{2}{3} = 2 \times \frac{1}{3}$$

Similarly,

$$\frac{4}{5} = 4 \times \frac{1}{5}$$

and

$$\frac{13}{9} = 13 \times \frac{1}{9}$$

But let's rephrase this observation in a slightly different way. Let's say that we have "pulled apart" each fraction into an integer multiplied by a more basic fraction.

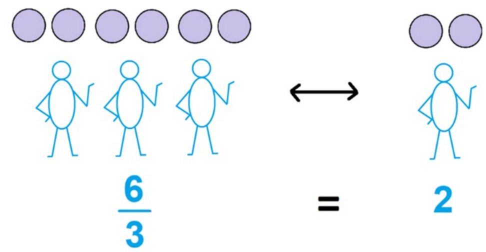
OBSERVATION 5

We can always "pull fractions apart."

$$\frac{a}{b} = a \times \frac{1}{b}$$

There is one more property of fractions to note.

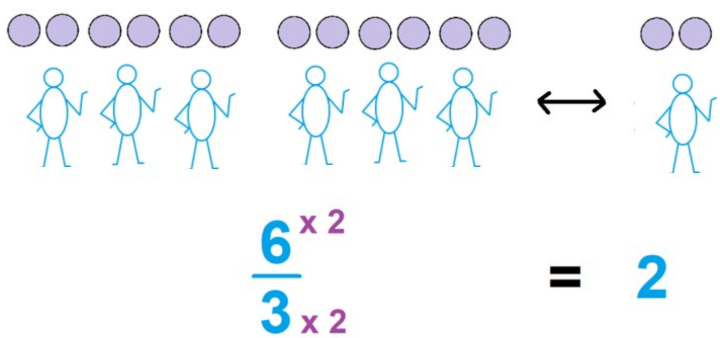
Consider again 6 pies being shared equally among 3 students.



What happens if we double the number of pies **and** double the number of students?

Nothing!

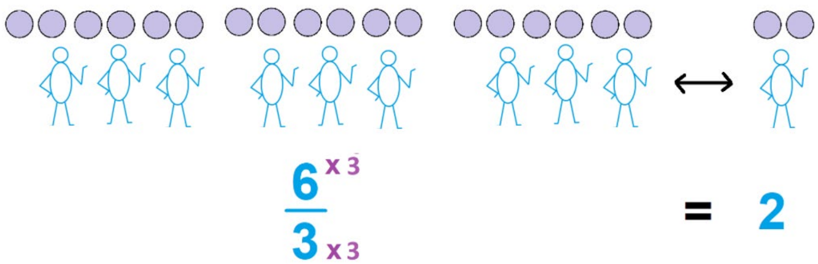
The amount of pie per student doesn't change.



Sharing 12 pies among 6 students, $\frac{12}{6}$, gives the same result as sharing 6 pies among 3 students, $\frac{6}{3}$. We're just doing the same work of giving 2 pies per student, twice over.

What happens if instead we triple the number of pies and triple the number of students?
Do we still get 2 pies per student?

You bet!



We are seeing that $\frac{18}{9}$ and $\frac{12}{6}$ and $\frac{6}{3}$ all give the same result.

Even if we centuple the number of pies and centuple the number of students, $\frac{600}{300}$, we're still handing out 2 pies per student. Nothing changes (except the amount of work we do to get to the same final result!).

Changing the number of pies we have by some factor and changing the number of students we are working with by the same factor changes nothing about the amount of pie each student receives.

$$\frac{a}{b} = \frac{2a}{2b} = \frac{3a}{3b} = \frac{4a}{4b} = \cdots = \frac{100a}{100b} = \cdots$$

This gives us our fourth property of fractions, which I'll add to the list with the other three.

FRACTION PROPERTY 1

$\frac{a}{1} = a$ for each counting number a.

FRACTION PROPERTY 2

$\frac{a}{a} = 1$ for each counting number a different from zero.

FRACTION PROPERTY 3

$k \times \frac{a}{b} = \frac{k \times a}{b}$ for each number k and fraction $\frac{a}{b}$.

FRACTION PROPERTY 4

$\frac{k \times a}{k \times b} = \frac{a}{b}$ for each number k and fraction $\frac{a}{b}$.

These rules look scary—visually. But the idea is to always "step back" to see through what each is saying.

For example, taking the number k to be 4 in the fourth rule, we read:

> *Quadrupling the number of pies and the number of students in a sharing problem does not change the final result: each student gets the same amount as pie if the numbers weren't quadrupled.*

Question: Which of the following fractions are equivalent to $\frac{3}{7}$?

a) $\frac{6}{14}$ b) $\frac{30}{70}$ c) $\frac{12}{28}$ d) $\frac{32}{77}$

The answer is that they all are. For example, $\frac{12}{28}$ is $\frac{4\times 3}{4\times 7}$, and so, by Property 4, has the same value as $\frac{3}{7}$.

We have in this question a set of **equivalent fractions**.

(To be clear, two fractional expressions are said to be **equivalent** if they represent the same value.)

Property 4 shows us how to recognize equivalent fractions.

For example,

$\frac{3}{5}$ is equivalent to $\frac{8\times 3}{8\times 5} = \frac{24}{40}$

$\frac{20}{32}$ is equivalent to $\frac{5}{8}$ (from noticing that $\frac{20}{32} = \frac{4\times 5}{4\times 8}$)

This second example shows that sharing 20 pies equally among 32 students is gives the same amount of pie per student as sharing just 5 pies among 8 students. (That second scenario seems more manageable!)

People say that we have **canceled a common factor** from within $\frac{20}{32}$ and, as such, we have just **simplified** the fraction.

$$\frac{20}{32} = \frac{\cancel{4} \times 5}{\cancel{4} \times 8} = \frac{5}{8}$$

Students are taught to, and are often required to, always simplify fractions this way. It becomes automatic.

Some people might say instead that we **reduced the fraction** $\frac{20}{32}$.

This term is a little misleading as the fraction $\frac{5}{8}$ is no smaller or bigger than $\frac{20}{32}$. What we have "reduced" is the number of pies and the number of students we are working with (that is, we have reduced the size of the numerator and the denominator). We haven't at all reduced the value of the outcome.

Practice 39.1: Select all the fractions that are equivalent to $\frac{140}{490}$.

a) $\frac{14}{49}$ b) $\frac{2}{7}$ c) $\frac{6}{21}$ d) $\frac{1400}{4900}$ e) $\frac{15}{35}$

Often students are expected to "simplify as far as possible."

Example: Reduce $\frac{280}{350}$ as far as possible.

Answer: We can certainly make this fraction look more manageable by noticing that there is a common factor of 10 in the numerator and denominator.

$$\frac{280}{350} = \frac{10 \times 28}{10 \times 35} = \frac{28}{35}$$

We can go further by noticing that 28 and 35 are both multiples of 7.

$$\frac{28}{35} = \frac{7 \times 4}{7 \times 5} = \frac{4}{5}$$

We're seeing that sharing 280 pies among 350 students gives the same result as sharing just 4 pies among 5 students. This is much easier to conceptualize.

$$\frac{280}{350} = \frac{4}{5}$$

As 4 and 5 share no common factors, this is as far as we can go with this example (while staying with whole numbers).

Question: Jennie has read this entire book and has leaned about "mixed numbers" and about being quirky with math. She says that $\frac{4}{5}$ does "simplify further" is you are willing to move away from whole numbers. She writes:

$$\frac{4}{5} = \frac{2 \times 2}{2 \times 2\frac{1}{2}} = \frac{2}{2\frac{1}{2}}$$

If you feel you understand what she is writing, what do you think? Is she right?
Does sharing 4 pies among 5 students yield the same result as sharing 2 pies among $2\frac{1}{2}$ students? (And is her answer "simpler"?)

Some Logical Consequence of our Fraction Properties

Here's a question:

> What is the value of $7 \times \frac{3}{7}$?

I bring this up as I personally have a knee-jerk response to this expression:

> *Just cancel the 7s to be left with 3!*

$$\cancel{7} \times \frac{3}{\cancel{7}} = 3$$

This is my school training kicking in. But is this maneuver valid?
Does it follow from one or some of our properties of fractions?

FRACTION PROPERTY 1

$\frac{a}{1} = a$ for each counting number a.

FRACTION PROPERTY 2

$\frac{a}{a} = 1$ for each counting number a different from zero.

FRACTION PROPERTY 3

$k \times \frac{a}{b} = \frac{k \times a}{b}$ for each number k and fraction $\frac{a}{b}$.

FRACTION PROPERTY 4

$\frac{k \times a}{k \times b} = \frac{a}{b}$ for each number k and fraction $\frac{a}{b}$.

and

OBSERVATION 5

$\frac{a}{b} = a \times \frac{1}{b}$ for each fraction $\frac{a}{b}$.

Property 3 looks relevant here. It tells us

$$7 \times \frac{3}{7} = \frac{7 \times 3}{7}$$

But what can we do next? What makes us want to "cancel the 7s"?

That would be **Property 4** if we rewrite the denominator as 7×1.

$$7 \times \frac{3}{7} = \frac{7 \times 3}{7} = \frac{7 \times 3}{7 \times 1} = \frac{3}{1}$$

And **Property 1** tells us that $\frac{3}{1}$ is just 3.

$$7 \times \frac{3}{7} = \frac{7 \times 3}{7} = \frac{7 \times 3}{7 \times 1} = \frac{3}{1} = 3$$

So, yes! We can just cancel the 7s from the get-go as we were trained in school to do.

$$\cancel{7} \times \frac{3}{\cancel{7}} = 3$$

Since this is such a common practice in working with fractions, let's take note of it and add it to our list of properties. We can then use this property whenever we want and not have to go through this mathematical reasoning every single time.

LOGICAL CONSEQUENCE 6

$b \times \frac{a}{b} = a$ for all counting numbers a and b with b not zero.

$$\cancel{b} \times \frac{a}{\cancel{b}} = a$$

INTUITION CHECK

Our real-world experience tells us that two halves make a whole, three thirds make a whole, four quarters make a whole, and so on.

Logical Consequence 6 is telling us this as well!

$2 \times \frac{1}{2} = 1 \qquad 3 \times \frac{1}{3} = 1 \qquad 4 \times \frac{1}{4} = 1 \qquad$ etc.

We're now developing a body of fraction properties we can use any time.
They are all aligned with what we were taught in school and with the intuition of fractions we developed from school and everyday life.

FRACTION PROPERTY 1

$\frac{a}{1} = a$ for each counting number a.

FRACTION PROPERTY 2

$\frac{a}{a} = 1$ for each counting number a different from zero.

FRACTION PROPERTY 3

$k \times \frac{a}{b} = \frac{k \times a}{b}$ for each number k and fraction $\frac{a}{b}$.

FRACTION PROPERTY 4

$\frac{k \times a}{k \times b} = \frac{a}{b}$ for each number k and fraction $\frac{a}{b}$.

OBSERVATION 5

$\frac{a}{b} = a \times \frac{1}{b}$ for each fraction $\frac{a}{b}$.

LOGICAL CONSEQUENCE 6

$b \times \frac{a}{b} = a$ for all counting numbers a and b with b not zero.

MUSINGS

Musing 39.2 Consider the expression $1 \times \frac{20}{1}$.

a) Andre says it has value 20 because you can apply Logical Consequence 6 directly to it.
Do you agree?
b) Andrea has it has value 20 because of Property 1 and the general rule of arithmetic that 1 times a number equals the number itself. Is her reasoning also valid?

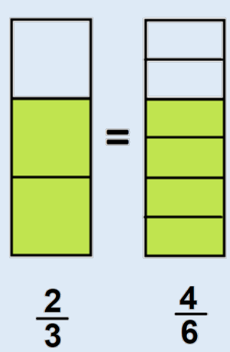
Musing 39.3 Carefully explain why $\frac{40}{4}$ has the value 10 using Properties 4 and 1.

Musing 39.4 For the following questions use the standard Rules of Arithmetic 1-9 and the Properties of Fractions we developed.

a) Explain why $\frac{39}{17} \times 17$ has value 39.
b) Explain why $\frac{39}{17} \times 34$ has value 78.
c) Explain why $3 \times \frac{7}{2} \times 12 \times \frac{3}{14} \times \frac{1}{3}$ has value 9.

Musing 39.5 Our Fraction Properties can be made logically tighter. We don't actually need to explicitly list Fraction Property 2. It follows as a logical consequence of Properties 4 and 1.
Do you see how?

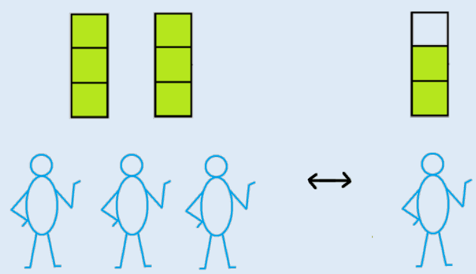
Musing 39.6 INTUITION CHECK
Students are typically asked to construct pictures like this to justify why $\frac{2}{3}$ and $\frac{4}{6}$, for instance, are equivalent fractions. We see that dividing each piece of the left picture in half produces the right picture.



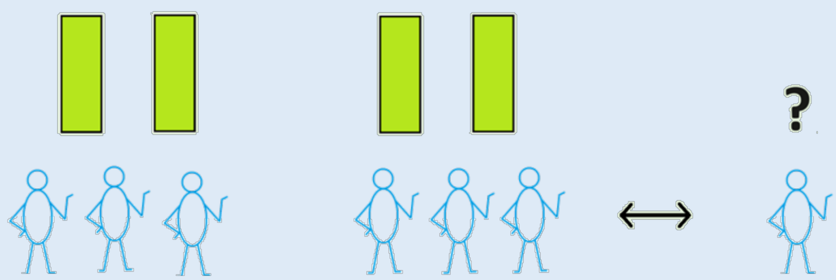
How does this fit with our pies-per-student thinking?

The trick is to draw rectangular pies.

The student picture of $\frac{2}{3}$ is the result of sharing 2 pies equally among 3 students. (Divide each pie into thirds and give a third from each pie to each student.)

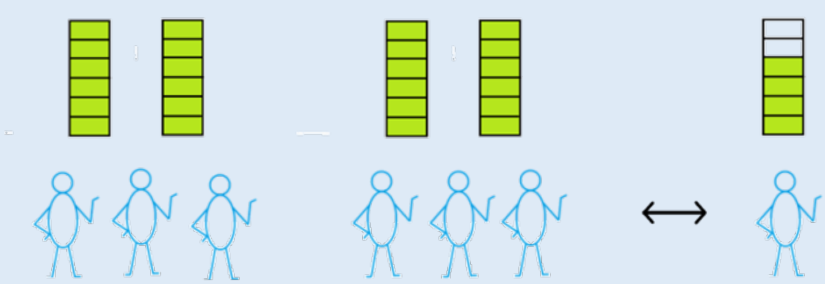


Now we draw a picture of $\frac{4}{6}$. This is the result of sharing 4 pies equally among 6 students.

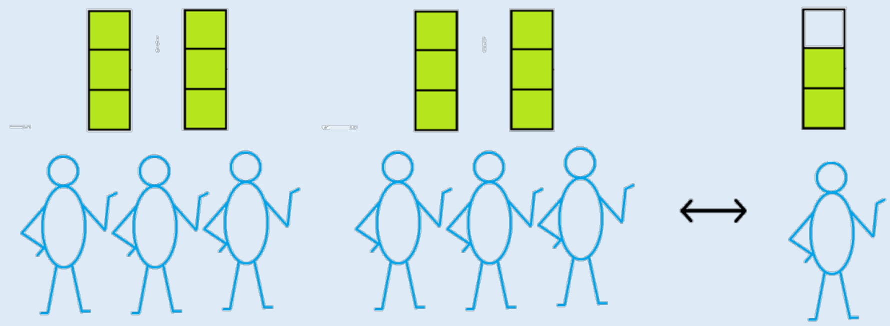


There are at least two ways to complete this task:

Approach 1: Divide each pie into sixths and give each student a sixth from each of these four pies. Each student gets four sixths of a pie.



Approach 2: Recognize that our task is just a matter of sharing 2 pies equally among 3 students—twice! In which case, each student receives a third from each of two pies, that is, two thirds of a pie.

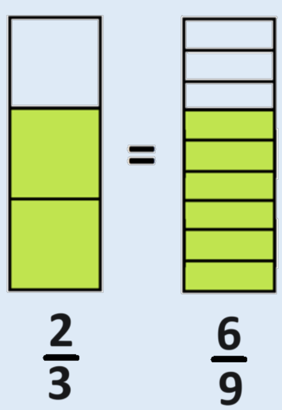


We are seeing $\frac{2}{3}$ and $\frac{4}{6}$ each as the answer to the same sharing problem. They represent the same amount of pie per student and so are equivalent fractions.

Our pies-per-student model for fractions really does yield the same pictures for the "equivalent fractions" pictures students are asked to draw in school.

Your turn:

Draw pictures to show two ways of sharing 6 pies equally among 9 students.
Deduce that $\frac{2}{3}$ and $\frac{6}{9}$ are equivalent fractions.



40. Opening Up Our Fraction Rules a Wee Bit

We have six observed properties of fractions from our pies-per-student model.

FRACTION PROPERTY 1

$\frac{a}{1} = a$ for each counting number a.

FRACTION PROPERTY 2

$\frac{a}{a} = 1$ for each counting number a different from zero.

FRACTION PROPERTY 3

$k \times \frac{a}{b} = \frac{k \times a}{b}$ for each number k and fraction $\frac{a}{b}$.

FRACTION PROPERTY 4

$\frac{k \times a}{k \times b} = \frac{a}{b}$ for each number k and fraction $\frac{a}{b}$.

OBSERVATION 5

$\frac{a}{b} = a \times \frac{1}{b}$ for each fraction $\frac{a}{b}$.

LOGICAL CONSEQUENCE 6

$b \times \frac{a}{b} = a$ for all counting numbers a and b with b not zero.

We started by stating that **a fraction is a number**, the answer to a division problem, and that we were thinking of division as sharing.

But Consequence 6 is saying something lovely. It brings us right back to division, but now thinking of division as multiplication in reverse.

Let me explain what I mean.

Consider $20 \div 5$.

If we are thinking of division as sharing (our pies-per-student), then $20 \div 5$ is the result of sharing 20 pies equally among 5 students. We called that $\frac{20}{5}$.

If we are thinking of division as multiplication in reverse, then $20 \div 5$ is the number that fills in the blank of this multiplication statement.

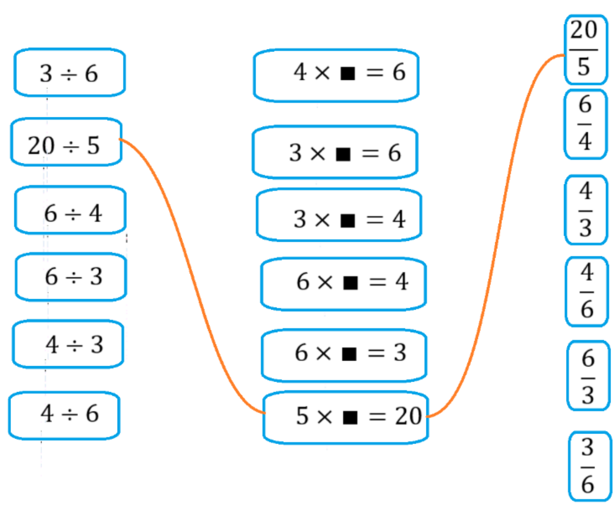
$$5 \times \blacksquare = 20$$

And look what Consequence 6 says. It tells us that $5 \times \frac{20}{5} = 20$. The fraction $\frac{20}{5}$ truly is the answer to $20 \div 5$.

In the same way, $\frac{2}{3}$ truly is the answer to $2 \div 3$, because, by Consequence 6, It is the number that fills in this blank to $3 \times ■ = 2$.

REALIZATION 7
The fraction $\frac{a}{b}$ truly is the answer to $a \div b$.
(It is the number that fills in the blank to $b \times ■ = a$.)

Practice 40.1: Complete this three-way match table. One triple match is already shown.



Back in Section 38 we were naughty and started putting fractions within fractions. We asked, for example, for the value of $\frac{1}{\frac{1}{2}}$, which we interpreted as the result of sharing one pie for each half of a student.

But now we can make valid mathematical sense of $\frac{1}{\frac{1}{2}}$. Copying what is written in parentheses in the box at this top of the page:

> $\frac{1}{\frac{1}{2}}$ is the answer to the division problem $1 \div \frac{1}{2}$.
>
> It the number that fills in the blank to $\frac{1}{2} \times \blacksquare = 1$.

Can we think of a number that fills the blank?

Well, we do know that

$$2 \times \frac{1}{2} = 1$$

This is just Consequence 6.

But the general rules of arithmetic tell us that we can switch the order of the product of two numbers without any harm. So, we can rewrite this statement as

$$\frac{1}{2} \times 2 = 1$$

This is now saying that the number 2 completes the statement $\frac{1}{2} \times \blacksquare = 1$.

So, $\frac{1}{\frac{1}{2}}$, the number that fills in the blank, is the number 2.

$$\frac{1}{\frac{1}{2}} = 2$$

Wow!

And this matches the answer we deduced when we were being quirky: sharing one pie for each half of a student results in two pies for a whole student.

Practice 40.3 Explain why the number 3 fills in the blank to $\frac{1}{3} \times \blacksquare = 1$.
What then is the value of $\frac{1}{\frac{1}{3}}$, the answer to $1 \div \frac{1}{3}$?

[**Aside**: If you switch your brain to thinking of division a "division by groups," how many thirds can you find in a whole? Are intuition and math again aligned?]

It looks like we really are allowed to be quirky with fractions and allow the numerators and denominators of fractions to be fractions too! The mathematics we have seems to be robust enough to handle this.

So, let's be bold and open up our five fraction properties to free our numerators and denominators from having to be counting numbers.

Here are our observations again, with each mention of "counting number" replaced with just "number."

FRACTION PROPERTY 1
$\frac{a}{1} = a$ for each number a.

FRACTION PROPERTY 2
$\frac{a}{a} = 1$ for each number a different from zero.

FRACTION PROPERTY 3
$k \times \frac{a}{b} = \frac{k \times a}{b}$ for each number k and fraction $\frac{a}{b}$.

FRACTION PROPERTY 4
$\frac{k \times a}{k \times b} = \frac{a}{b}$ for each number k and fraction $\frac{a}{b}$.

and

OBSERVATION 5
$\frac{a}{b} = a \times \frac{1}{b}$ for each fraction $\frac{a}{b}$.

LOGICAL CONSEQUENCE 6
$b \times \frac{a}{b} = a$ for all numbers a and b with b not zero.

We also have our nine general Rules of Arithmetic to play with too.

And we need to keep in mind what Consequence 6 is really saying for us:

REALIZATION 7
The fraction $\frac{a}{b}$ truly is the answer to $a \div b$.
(It is the number that fills in the blank to $b \times \blacksquare = a$.)

The power of Realization 7 is this:

If you happen to know the answer to a division problem, then you know the value of a fraction.

For example,

$$\frac{12}{3} = 12 \div 3 = 4$$

$$\frac{200}{10} = 200 \div 10 = 20$$

$$\frac{55}{5} = 55 \div 5 = 11$$

I don't the value of $33 \div 7$ off the top of my head. (Well, it's about four-and-a-half.) Consequently, I can't say too much about the value of $\frac{33}{7}$ (except that it is about four-and-a-half!)

As a final comment, we can point out that Fraction Rule 2 is now, for sure, obsolete:

Dividing a number by itself gives the answer 1.
Knowing this gives:

$$\frac{a}{a} = a \div a = 1$$

41. All Fraction Arithmetic in One Hit

Let's just get into it. Let's see how all the fraction arithmetic you were taught to do in school is actually mathematically correct and logically follows from the Fraction Properties we've identified.

Actually, we only need to focus on these properties and consequences.

Fraction Property 1:

Every number can be written as a fraction with denominator 1.

$$a = \frac{a}{1}$$

Fraction Property 4:

Multiplying the top and bottom of a fraction by the same number does not change the value of the fraction.

$$\frac{a}{b} = \frac{k \times a}{k \times b}$$

Observation 5:

We can always "pull apart" a fraction.

$$\frac{a}{b} = a \times \frac{1}{b}$$

Consequence 6:

Multiplying a fraction by its denominator "cancels that denominator."

$$\cancel{b} \times \frac{a}{\cancel{b}} = a$$

and **Realization 7:**

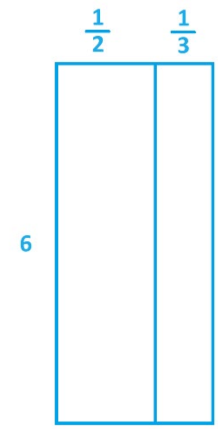
A fraction is the answer to a division problem. (So, if you happen to know the answer to a division problem, then you know the value of the matching fraction.)

$$\frac{a}{b} = a \div b$$

(We saw that Fraction Property 2 is obsolete. Fraction Property 3 led us to Observations 5, 6, and 7, but we won't ever need Property 4 itself directly.)

To develop a feel for how all this is going to work, consider these warm-up examples.

Example: Show that $6 \times \left(\frac{1}{2}+\frac{1}{3}\right)$ equals 5.



Answer: From the general arithmetic principle of "chopping up rectangles," $6 \times \left(\frac{1}{2}+\frac{1}{3}\right)$ is

$$6 \times \frac{1}{2} \;+\; 6 \times \frac{1}{3}$$

We see here two fractions that are pulled apart. We really have:

$$\frac{6}{2}+\frac{6}{3}$$

Each of these fractions matches a division problem whose answer we know. This means that we have

$$3+2$$

which equals 5, just as the question suggested.

Practice 41.1 Show that $5 \times (\frac{1}{5}+\frac{1}{5})$ equals 2.

Practice 41.2 Determine the value of $100 \times (\frac{1}{20}+\frac{1}{25}+\frac{2}{5})$.

Example: Show that $60 \times \frac{1}{3} \times \frac{1}{4}$ equals 5.

Answer: It helps to notice that $60 = 3 \times 4 \times 5$, so the product we are asked to look at is really

$$3 \times 4 \times 5 \times \frac{1}{3} \times \frac{1}{4}$$

The general Rules of Arithmetic tell us that it does not matter in which order one computes a string of products. Since we can see $3 \times \frac{1}{3} = 1$ and $4 \times \frac{1}{4} = 1$ within this product (Consequence 6), we can see that our product equals

$$1 \times 1 \times 5$$

which is just 5, as the question suggested.

Practice 41.3 Determine the value of $100 \times \frac{1}{2} \times \frac{1}{25}$.

Example: Find the value of $70 \times \frac{4}{5} \times \frac{2}{7} \times \frac{1}{8}$.

Answer: Let's tease each term this product apart. We have

$$70 = 2 \times 5 \times 7$$

$$\frac{4}{5} = 4 \times \frac{1}{5}$$

$$\frac{2}{7} = 2 \times \frac{1}{7}$$

So, the question is asking us to evaluate

$$2 \times 5 \times 7 \times 4 \times \frac{1}{5} \times 2 \times \frac{1}{7} \times \frac{1}{8}$$

The general rules of arithmetic tell us we can compute a string of products in any order we like. Noticing that $5 \times \frac{1}{5} = 1$ and $7 \times \frac{1}{7} = 1$ we see that the product is

$$2 \times 4 \times 1 \times 2 \times 1 \times \frac{1}{8}$$

But, we can read this as

$$8 \times 2 \times \frac{1}{8}$$

and see it as 2×1, which equals 2.

Practice 41.4 Determine the value of $360 \times \frac{5}{6} \times \frac{7}{12} \times \frac{2}{25}$.

Practice 41.5 What's $8 \times \frac{3}{2}$?

Now to the fraction arithmetic we all taught in school.

The general ideas we'll follow are these:

- **It never hurts to "put a number over 1."**
- **Multiplying the top and bottom of fraction by the same number won't change the value of the fraction.**
- **We can be clever about what to multiply the top and bottom by.**

$$a = \frac{a}{1} \qquad \frac{a}{b} = \frac{k \times a}{k \times b} \qquad \cancel{b} \times \frac{a}{\cancel{b}} = a$$

And let's also remember that if you know the answer to a particular division problem, then you know the value of a fraction. (For example, knowing that $6 \div 3 = 2$ tells me that $\frac{6}{3} = 2$.)

$$\frac{a}{b} = a \div b$$

Also remember we can "pull a fraction apart" if it helps make it easier to see what is going on. (For example, $8 \times \frac{3}{2} = 2 \times 4 \times 3 \times \frac{1}{2} = 4 \times 3 \times 1 = 12$.)

$$\frac{a}{b} = a \times \frac{1}{b}$$

THIS IS A STAR PAGE!

It summarizes all you need to actually know to make sense of all the mathematics of fractions.

Can you put a marker on this page?

Adding Fractions

Intuition, and school mathematics, tells us that one-fifth plus one-fifth equals two-fifths.

Mathematics agrees with this. (Thank heavens!)

Example: Evaluate $\frac{1}{5}+\frac{1}{5}$.

Answer: Let's "put this quantity over 1."

$$\frac{1}{5}+\frac{1}{5}=\frac{\frac{1}{5}+\frac{1}{5}}{1}$$

Having fifths in the numerator seems annoying. Multiplying top and bottom each by 5 will probably help.

$$\frac{1}{5}+\frac{1}{5}=\frac{5\times(\frac{1}{5}+\frac{1}{5})}{5\times 1}$$

The numerator is now $5\times\left(\frac{1}{5}+\frac{1}{5}\right)=5\times\frac{1}{5}\ +\ 5\times\frac{1}{5}$ which is $1+1=2$.
The denominator is $5\times 1=5$.

So, we have

$$\frac{1}{5}+\frac{1}{5}=\frac{5\times(\frac{1}{5}+\frac{1}{5})}{5\times 1}=\frac{1+1}{5}=\frac{2}{5}$$

Practice 41.6 Follow this technique to show that $\frac{2}{7}+\frac{3}{7}$ equals $\frac{5}{7}$.

Example: Evaluate $\frac{2}{9}+\frac{4}{7}$.

Answer: Let's "put this quantity over 1."

$$\frac{2}{9}+\frac{4}{7}=\frac{\frac{2}{9}+\frac{4}{7}}{1}$$

Having ninths in the numerator is annoying. Let's multiply top and bottom each by 9.

$$\frac{2}{9}+\frac{4}{7}=\frac{9\times(\frac{2}{9}+\frac{4}{7})}{9\times 1}$$

The numerator is $9 \times \frac{2}{9} \;+\; 9 \times \frac{4}{7}$, which is $2 + 9 \times \frac{4}{7}$, and will still contain sevenths, which are annoying.

Let's go back a step and try this: Let's multiply top and bottom by 9 and by 7 in one hit.

$$\frac{2}{9} + \frac{4}{7} = \frac{9 \times 7 \times (\frac{2}{9} + \frac{4}{7})}{9 \times 7 \times 1}$$

The numerator is $9 \times 7 \times \frac{2}{9} \;+\; 9 \times 7 \times \frac{4}{7}$, which we can see is $7 \times 2 \;+\; 9 \times 4 = 14 + 36 = 50$. The denominator is 63.

We've got it!

$$\frac{2}{9} + \frac{4}{7} = \frac{9 \times 7 \times (\frac{2}{9} + \frac{4}{7})}{9 \times 7 \times 1} = \frac{50}{63}$$

Practice 41.7 Show that $\frac{3}{4} + \frac{2}{11}$ equals $\frac{41}{44}$.

Example: Write $1 + \frac{1}{2}$ as a single fraction.

Answer: Well, putting the quantity over 1 does the trick!

$$1 + \frac{1}{2} = \frac{1 + \frac{1}{2}}{1}$$

To handle the annoying half in the numerator, let's double the top and the bottom.

$$1 + \frac{1}{2} = \frac{2 \times (1 + \frac{1}{2})}{2 \times 1}$$

We can now evaluate the numerator and the denominator.

$$1 + \frac{1}{2} = \frac{2 \times (1 + \frac{1}{2})}{2 \times 1} = \frac{2 \times 1 + 2 \times \frac{1}{2}}{2} = \frac{2 + 1}{2} = \frac{3}{2}$$

Read a long string of equal signs as "… equals … which equals … which equals … which equals …

(Now, also make sure you see how we got from one "which equals" to the next line the previous example.)

Practice 41.8 Show that $2 + \frac{4}{9}$ equals $\frac{22}{9}$.

Practice 41.9: What is the value of $\frac{3}{19} + \frac{7}{19} + \frac{9}{19}$?

Example: Show that $\frac{a}{N} + \frac{b}{N}$ equal $\frac{a+b}{N}$.
(We're assuming here that N is a number different than zero.)

Answer: Even though this question is dealing with unspecified numbers a, b, and N, we can follow exactly the same procedure as before.

Let's start by putting our given quantity over 1.

$$\frac{a}{N} + \frac{b}{N} = \frac{\frac{a}{N} + \frac{b}{N}}{1}$$

To manage the pesky Nths in the numerator, multiply top and bottom by N.

$$\frac{a}{N} + \frac{b}{N} = \frac{\frac{a}{N} + \frac{b}{N}}{1} = \frac{N \times (\frac{a}{N} + \frac{b}{N})}{N \times 1}$$

Now, let's just work it all work it all out, reading "which equals" in our minds as we go along.

$$\frac{a}{N} + \frac{b}{N} = \frac{\frac{a}{N} + \frac{b}{N}}{1} = \frac{N \times (\frac{a}{N} + \frac{b}{N})}{N \times 1} = \frac{N \times \frac{a}{N} + N \times \frac{b}{N}}{N} = \frac{a+b}{N}$$

We got the result the question expected!

Looks like we've got the addition of fractions licked!

Comment: The last example shows that if we add fractions that all happen to have the same denominator, then we can, in essence, just "add the numerators."

We'll see in the next chapter that this is the typical take followed by schoolbooks when first introducing the addition of fractions. What we are doing now is showing that all the mathematics of fraction arithmetic doesn't actually require memorizing special rules for special cases like these.

Just do the math and all will naturally follow!

Multiplying Fractions

Let's get straight into it!

Example: What does mathematics say is the value of $\frac{1}{2} \times \frac{1}{3}$?

Answer: Let's put the quantity over 1 and find out!

$$\frac{1}{2} \times \frac{1}{3} = \frac{\frac{1}{2} \times \frac{1}{3}}{1}$$

We have halves and thirds in the numerator. Let's multiply the top and bottom each by 2 and 3 to handle those.

$$\frac{1}{2} \times \frac{1}{3} = \frac{\frac{1}{2} \times \frac{1}{3}}{1} = \frac{2 \times 3 \times \frac{1}{2} \times \frac{1}{3}}{2 \times 3 \times 1}$$

We see that the numerator is $1 \times 1 = 1$ and that the denominator is 6.

$$\frac{1}{2} \times \frac{1}{3} = \frac{\frac{1}{2} \times \frac{1}{3}}{1} = \frac{2 \times 3 \times \frac{1}{2} \times \frac{1}{3}}{2 \times 3 \times 1} = \frac{1}{6}$$

Practice 41.10: Show that $\frac{1}{3} \times \frac{1}{5} \times \frac{1}{10}$ equals $\frac{1}{150}$.

Example: What is the value of $\frac{3}{7} \times \frac{5}{8}$?

Answer: Putting the quantity over 1 seems to do the magic.

$$\frac{3}{7} \times \frac{5}{8} = \frac{\frac{3}{7} \times \frac{5}{8}}{1} = \frac{7 \times 8 \times \frac{3}{7} \times \frac{5}{8}}{7 \times 8 \times 1}$$

The numerator is $3 \times 5 = 15$ (do you see that?) and the denominator is $7 \times 8 = 56$. So,

$$\frac{3}{7} \times \frac{5}{8} = \frac{15}{56}$$

Practice 41.11: Find the value of $\frac{17}{21} \times \frac{2}{5}$.

Practice 41.12: Show that $\frac{10}{64} \times \frac{16}{50}$ equals $\frac{1}{20}$.

Practice 41.13: Show, in general, that $\frac{a}{b} \times \frac{c}{d}$ equals $\frac{a \times c}{b \times d}$.

Actually, we can approach these practice problems in a different way if we want.

To set things up, let's first give a name to the $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$ with a numerator of 1. We'll call them the **basic fractions** as they, well, the basic fractions of everyday life: halves, thirds, quarters, and so on.

Our first example of multiplying fractions was that with two basic fractions. We saw

$$\frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$$

It looks as though we just multiplied together the denominators of two basic fractions to make another basic fraction.

We can prove that multiplying basic fractions this way is valid.

Multiplying Basic Fractions: We have that $\frac{1}{n} \times \frac{1}{m}$ equals $\frac{1}{n \times m}$.

Reason: Let's put the quantity we're looking at over 1.

$$\frac{1}{n} \times \frac{1}{m} = \frac{\frac{1}{n} \times \frac{1}{m}}{1}$$

Multiply top and bottom each by n and by m.

$$\frac{1}{n} \times \frac{1}{m} = \frac{\frac{1}{n} \times \frac{1}{m}}{1} = \frac{n \times m \times \frac{1}{n} \times \frac{1}{m}}{n \times m \times 1}$$

The numerator is $1 \times 1 = 1$ and the denominator is $n \times m$.

We've got it:

$$\frac{1}{n} \times \frac{1}{m} = \frac{\frac{1}{n} \times \frac{1}{m}}{1} = \frac{n \times m \times \frac{1}{n} \times \frac{1}{m}}{n \times m \times 1} = \frac{1}{n \times m}$$

Now we're set to multiply all fractions just by teasing everything apart.

Example: Find the value of $\frac{3}{7} \times \frac{5}{8}$ again.

Answer: We can write

$$\frac{3}{7} \times \frac{5}{8} = 3 \times \frac{1}{7} \times 5 \times \frac{1}{8}$$

We can work out a string of products in any order we like.

Noticing that $3 \times 5 = 15$ and $\frac{1}{7} \times \frac{1}{8} = \frac{1}{56}$, we have

$$\frac{3}{7} \times \frac{5}{8} = 3 \times \frac{1}{7} \times 5 \times \frac{1}{8} = 15 \times \frac{1}{56}$$

And $15 \times \frac{1}{56}$ is just the fraction $\frac{15}{56}$ pulled apart

If you are tired of putting a product of fractions over 1, you now have the option of just pulling the product completely apart and computing parts of the product in turn.

Practice 41.14: Compute $\frac{10}{64} \times \frac{16}{50}$ again by "pulling apart" the terms of the product. See the answer $\frac{1}{20}$ again.

Example: Show that $\frac{37}{97} \times \frac{97}{37}$ is just 1.

Answer: We have

$$\frac{37}{97} \times \frac{97}{37} = 37 \times \frac{1}{97} \times 97 \times \frac{1}{37}$$

Since $37 \times \frac{1}{37} = 1$ and $97 \times \frac{1}{97} = 1$, this product is

$$\frac{37}{97} \times \frac{97}{37} = 37 \times \frac{1}{97} \times 97 \times \frac{1}{37} = 1 \times 1 = 1$$

Example: Show again that $\frac{a}{b} \times \frac{c}{d}$ equals $\frac{a \times c}{b \times d}$.

Answer: We have

$$\frac{a}{b} \times \frac{c}{d} = a \times \frac{1}{b} \times c \times \frac{1}{d}$$

We have in this product $a \times c$ and we have $\frac{1}{b} \times \frac{1}{d}$, which equals $\frac{1}{b \times d}$. We can thus think of the product as

$$\frac{a}{b} \times \frac{c}{d} = (a \times c) \times \frac{1}{b \times d}$$

And this is just a fraction pulled apart. It is the fraction

$$\frac{a \times c}{b \times d}$$

Practice 41.15 What do you need to multiply $\frac{4}{7}$ by to get the answer $\frac{20}{21}$?

Example: Ibrahim was asked to compute $\frac{39}{7} \times \frac{14}{13}$ and within three seconds he said that the answer was 6. How did he see this so quickly?

Answer: Well, we can't say for sure what the fellow saw, but we do have

$$\frac{39}{7} \times \frac{14}{13} = 39 \times \frac{1}{7} \times 14 \times \frac{1}{13}$$

Since we can multiply a string of products in any order we like, perhaps Ibrahim saw this as

$$39 \times \frac{1}{13} \times 14 \times \frac{1}{7} = \frac{39}{13} \times \frac{14}{7} = 3 \times 2 = 6$$

Practice 41.16 What is the value of $\frac{51}{35} \times \frac{14}{17}$?

Practice 41.17 Show that $\frac{1}{2} \times \frac{2}{3} \times \frac{3}{4} \times \frac{4}{5}$ equals $\frac{1}{5}$.

Dividing Fractions

Remember, a fraction is the answer to a division problem.

$$\frac{a}{b} = a \div b$$

Or, saying this another way, the answer to a division problem is a fraction.

$$a \div b = \frac{a}{b}$$

Example: Find the value of $1 \div \frac{1}{7}$.

Answer: The answer is the fraction

$$\frac{1}{\frac{1}{7}}$$

To make this look friendlier, let's multiply the top and bottom by 7.

$$\frac{1}{\frac{1}{7}} = \frac{7 \times 1}{7 \times \frac{1}{7}} = \frac{7}{1}$$

And this equals 7.

Example: Compute $\frac{3}{5} \div \frac{4}{7}$.

Answer: The answer is the fraction.

$$\frac{\frac{3}{5}}{\frac{4}{7}}$$

To make this more tractable, let's handle the awkward fifths and sevenths by multiplying the top and bottom each by 5 and by 7.

We get

$$\frac{5 \times 7 \times \frac{3}{5}}{5 \times 7 \times \frac{4}{7}}$$

Since $5 \times \frac{3}{5} = 3$ and $7 \times \frac{4}{7} = 4$ we really have

$$\frac{7 \times 3}{5 \times 4}$$

giving our final answer

$$\frac{21}{20}$$

So, $\frac{3}{5} \div \frac{4}{7} = \frac{21}{20}$.

Practice 41.18: Show that $\frac{2}{3} \div \frac{5}{7}$ equals $\frac{14}{15}$.

Practice 41.19: Compute $\frac{3}{4} \div \frac{2}{3}$.

Practice 41.20: Show that $1 \div \frac{1}{N}$ equals N.

In Section 39, we tried to share "8 pies equally among $\frac{2}{3}$ of a student" (which makes little sense) and concluded that somehow doing so results in giving a whole student 12 pies.

Practice 41.21: Show that $8 \div \frac{2}{3}$ does indeed equal 12.

A Comment on Notation

Fractions are written with a horizontal bar (once called a **virga**) to separate numerator from denominator. But writing fractions within fractions is hard and looks confusing.

For example, $\frac{1}{\frac{2}{3}}$, intended to be read as "one over two-thirds," could easily be misread as "one-half over three."

To avoid confusion, a slanted line / (called a **solidus**) is often used instead in fraction notation. For example, writing

$$\frac{1}{2/3}$$

makes clear that we have a fraction as the denominator of a fraction.

Rewriting $\frac{\frac{3}{4}}{\frac{2}{3}}$ as

$$\frac{3/4}{2/3}$$

makes matters clearer, as does rewriting $\frac{\frac{4}{3}}{5}$ as

$$\frac{4/3}{5}$$

Example: Figure out $\frac{3}{4} \div \frac{2}{3/5}$.

Here's my approach.

Answer:

First of all, I don't like the number $\frac{2}{\frac{3}{5}}$.

I am going to make it look friendlier by multiplying the top and bottom each by 5. Doing so turns it into $\frac{10}{3}$.

This means that we're really being asked to compute $\frac{3}{4} \div \frac{10}{3}$. The answer to a division problem is a fraction. So, the answer is

$$\frac{\frac{3}{4}}{\frac{10}{3}}$$

To make this tractable, multiply the top and bottom each by 4 and 3.

$$\frac{\frac{3}{4}}{\frac{10}{3}} = \frac{3 \times 4 \times \frac{3}{4}}{3 \times 4 \times \frac{10}{3}} = \frac{3 \times 3}{4 \times 10} = \frac{9}{40}$$

And we're done!

$$\frac{3}{4} \div \frac{2}{3/5} = \frac{9}{40}$$

Practice 41.22: Show that $1 \div \frac{1}{a/b}$ is $\frac{a}{b}$.

Fractions and Negative Numbers

You have no doubt noticed that I have skipped over the arithmetic of subtracting fractions. This is because, as you know, I do not believe subtraction exists.

Subtraction is the addition of the opposite.

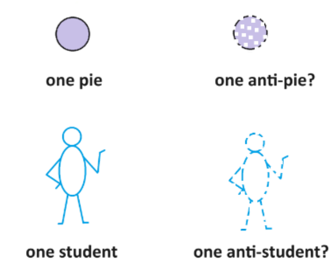
So, we need to be sure we understand "the opposite of a fraction" and how to make sense of fraction that might incorporate negative signs.

Question: Are $\frac{-2}{3}$ and $\frac{2}{-3}$ and $-\frac{2}{3}$ the same fraction in different guises, or are they different?

(If you want some fun … Is there any way to make sense of these quantities in the pies-per-student model?

$\frac{-2}{3}$ is result of sharing two anti-pies equally among three actual students.

$\frac{2}{-3}$ is result of sharing two actual pies equally among three anti-students.



$-\frac{2}{3}$ is the opposite of the result of sharing two pies equally among three students.

Do any of these interpretations make sense? If so, do they represent the same final amount of pie (or anti-pie) per student (or is it anti-student?)

Don't take this question seriously!

To remind ourselves, here are the properties of negative numbers we deduced back in Sections 24 and 25.

Negative Numbers

- Adding together a number and its opposite produces zero: $a + -a = 0$
- The opposite of the opposite of a number is the original number: $-(-a) = a$
- Multiplying by negative one produces the opposite number: $(-1) \times a = -a$
- The opposite of zero is zero: $-0 = 0$
- We can "pull out" negative signs from products: $(-a) \times b = a \times (-b) = -ab$
- Negative times negative is positive: $(-a) \times (-b) = ab$

Let's see what math has to say about the three quantities $\frac{-2}{3}$ and $\frac{2}{-3}$ and $-\frac{2}{3}$.

For starters, we know

$$-2 = (-1) \times 2$$

and

$$-3 = (-1) \times 3$$

and

$$-\frac{2}{3} = (-1) \times \frac{2}{3}$$

Let's pull apart $\frac{-2}{3}$:

$$\frac{-2}{3} = (-2) \times \frac{1}{3} = (-1) \times 2 \times \frac{1}{3}$$

Let's pull apart $-\frac{2}{3}$:

$$-\frac{2}{3} = (-1) \times \frac{2}{3} = (-1) \times 2 \times \frac{1}{3}$$

They are the same: $\frac{-2}{3}$ and $-\frac{2}{3}$ are the same quantity in different guises.

What about $\frac{2}{-3}$?

$$\frac{2}{-3} = \frac{2}{(-1) \times 3}$$

Multiply the top and bottom each by -1.

$$\frac{2}{-3} = \frac{2}{(-1) \times 3} = \frac{(-1) \times 2}{(-1) \times (-1) \times 3}$$

Since $(-1) \times (-1) = 1$ (negative times negative is positive) and $(-1) \times 2 = -2$, this is the same as $\frac{-2}{3}$.

$$\frac{2}{-3} = \frac{2}{(-1) \times 3} = \frac{(-1) \times 2}{(-1) \times (-1) \times 3} = \frac{-2}{3}$$

So, $-\frac{2}{3}$ is the same as $\frac{-2}{3}$, and $\frac{2}{-3}$ is the same as $\frac{-2}{3}$. All three quantities are thus the same!

I don't know what that any one of these quantities means in the real world, but that is not the point. We have just proven that they are the same mathematically.

Now we know that we are free to "pull out" a negative sign from a fraction any time.

$$-\frac{a}{b} = \frac{-a}{b} = \frac{a}{-b}$$

Practice 41.23 Show that $\frac{-a}{-b}$ is the same as $\frac{a}{b}$.

(I can think of two approaches you could take here.

1. Try multiplying the top of bottom of $\frac{-a}{-b}$ each by -1.
2. Pull out a negative sign, twice!)

Practice 41.24 What is the value of $\frac{5}{-2} \times \frac{-3}{10}$?

Practice 41.25 What is the value of $\frac{2}{9} \div \frac{-2}{3}$?

Subtracting Fractions

Let's now practice adding the opposite.

Example: What is $\frac{5}{7}-\frac{3}{7}$?

Intuition says that "five-sevenths take away three-sevenths is two-sevenths."

Answer: Subtraction is the addition of the opposite. We must compute

$$\frac{5}{7}+-\frac{3}{7}$$

This is.

$$\frac{5}{7}+\frac{-3}{7}$$

Now, "putting over 1" does the trick for us.

$$\frac{5}{7}-\frac{3}{7}=\frac{\frac{5}{7}+\frac{-3}{7}}{1}=\frac{7\times\left(\frac{5}{7}+\frac{-3}{7}\right)}{7\times 1}=\frac{5+\ -3}{7}=\frac{2}{7}$$

Practice 41.26: Show that $\frac{3}{7}-\frac{5}{7}$ is $-\frac{2}{7}$.

Example: Compute $\frac{2}{3}-\frac{1}{4}$.

Answer: First, it is really an addition problem.

$$\frac{2}{3}-\frac{1}{4}\quad=\quad\frac{2}{3}+-\frac{1}{4}\quad=\quad\frac{2}{3}+\frac{-1}{4}$$

Now we're set to go

$$\frac{2}{3}+\frac{-1}{4}=\frac{3\times 4\times\left(\frac{2}{3}+\frac{-1}{4}\right)}{3\times 4\times 1}=\frac{4\times 2+3\times(-1)}{12}=\frac{8+\ -3}{12}=\frac{5}{12}$$

Example: Compute $1 - \frac{1}{20}$.

Answer: Here it is.

$$1 + \frac{-1}{20} = \frac{1 + \frac{-1}{20}}{1} = \frac{20 \times \left(1 + \frac{-1}{20}\right)}{20 \times 1} = \frac{20 + \ -1}{20} = \frac{19}{20}$$

Practice 41.27

a) Compute $\frac{12}{11} - \frac{8}{11}$.

b) Show, in general, that $\frac{a}{N} - \frac{b}{N} = \frac{a-b}{N}$.

MUSINGS

Musing 41.28 We just went through the basic arithmetic operations of fractions without reference to the usual schoolbook approaches to them.

Just turn any quantity you have into a fraction (perhaps with a denominator of 1) and multiply top and bottom by what you need to make that fraction more manageable.

What is your reaction to this?
Are matters clearer and less cluttered? More confusing? Strange and weird? Delightful and freeing?

MECHANICS PRACTICE

Practice 41.29 Did you try all 27 practice problems throughout this section?

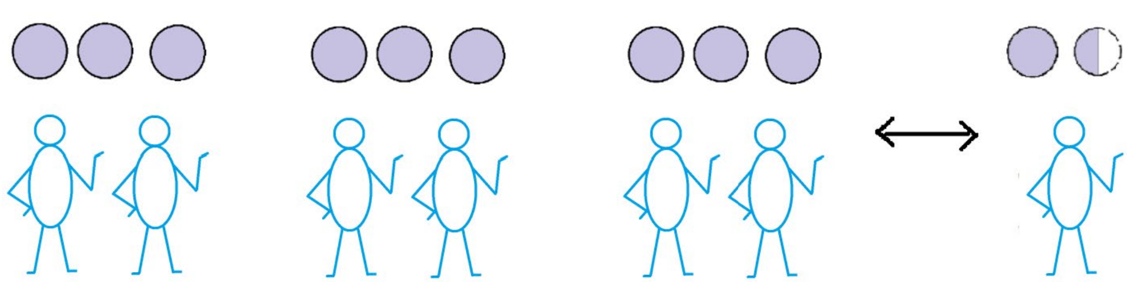
SOLUTIONS

37.1 Do research this. It seems tricky to track down properly. Folk China and in India, it seems, millennia ago were expressing fractions as two numbers, one written above the other, but without a horizontal line to separate them. Arab scholar al-Hassar around the year 1200 is often cited as being first to introduce the fraction bar, and it is said that Italian scholar Fibonacci of that time called it a *virga*, which is Latin for "twig."

37.2
i) $2\frac{1}{2} \div \frac{3}{4} = 3\frac{1}{3}$.
Three copies of $\frac{3}{4}$ get us to $2\frac{1}{4}$ and we need an extra quarter length, which is a third of $\frac{3}{4}$, to get us all the way.

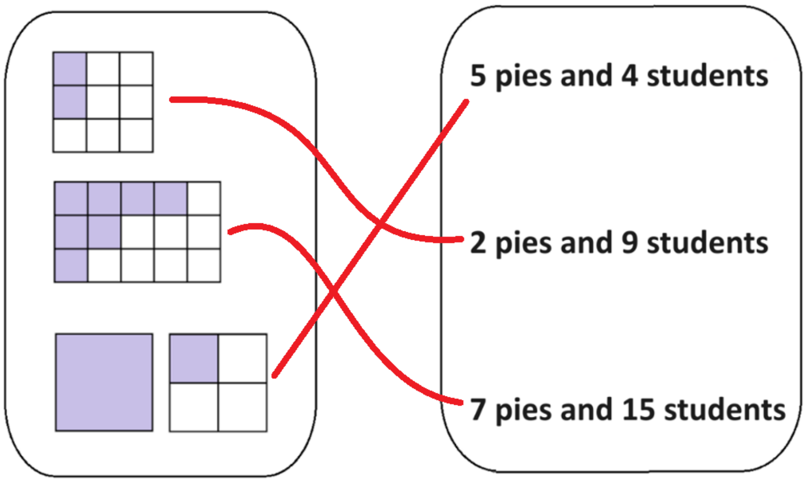
ii) $\frac{1}{2} \div 3 = \frac{1}{6}$
One sixth of "three" fits into a half.

38.1 Sharing 9 pies equally among 6 students produces the same result of one whole pie and half a pie per student.



38.2 Sharing 22 pies equally among 5 students works.

38.3



38.4 Yes! Doubling the number of pies you share out (equally) doubles the amount of pie each student receives.

38.5 Yes. Giving out no pies to one student results in 0 pie for that student.

38.6 21

38.7

a) $\frac{103}{103} = 1$ is correct because $103 \times 1 = 103$. (If each of 103 students has 1 pie, then all 103 pies are accounted for.)
b) $\frac{1000}{125} = 10$ is not correct because 125×10 means there were 1250 pies, not 1000.
c) $\frac{999}{1} = 998$ is not correct because 1×998 means there were only 998 pies, not 999.
d) Read on!

39.1 All are.

39.2
a) Yes. We have $1 \times \frac{20}{1} = 1$ just be "canceling the 1s" as per Consequence 6.
b) Yes. We have $1 \times \frac{20}{1} = 1 \times 20 = 20$ by Property 1 and the usual rules of arithmetic.

39.3 We have

$$\frac{40}{4} = \frac{4 \times 10}{4 \times 1} = \frac{10}{1}$$

using Property 4, and

$$\frac{10}{1} = 10$$

by Property 1.

39.4

a) Swap the order of multiplication to write $17 \times \frac{39}{17}$ and then use Consequence 5 to get 39.

b) Again swap the order of multiplication to write $34 \times \frac{39}{17}$. Think of this as $2 \times 17 \times \frac{39}{17}$. By Consequence 5 this is 2×39, giving 78.

c) Let's pull everything apart (Consequence 6) to write

$$3 \times 7 \times \frac{1}{2} \times 12 \times 3 \times \frac{1}{14} \times \frac{1}{3}$$

Let's tease the twelve apart too.

$$3 \times 7 \times \frac{1}{2} \times 2 \times 2 \times 3 \times 3 \times \frac{1}{14} \times \frac{1}{3}$$

We can compute this product in any order we like. Let's compute $7 \times 2 = 14$.

$$3 \times \frac{1}{2} \times 14 \times 2 \times 3 \times 3 \times \frac{1}{14} \times \frac{1}{3}$$

Now, I also see within the product $2 \times \frac{1}{2} = 1$ and $14 \times \frac{1}{14} = 1$ and $1 \times \frac{1}{3} = 1$ (Consequence 6), so we have

$$3 \times 1 \times 1 \times 3 \times 1$$

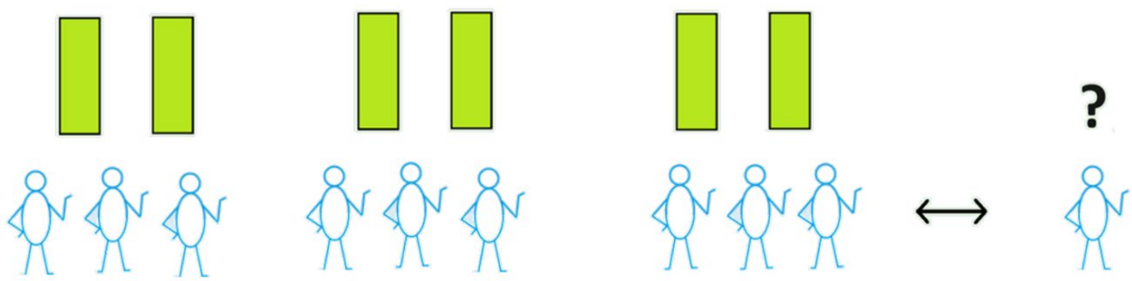
and this is 9.

39.5 For a non-zero counting number a we have

$$\frac{a}{a} = \frac{a \times 1}{a \times 1} = \frac{1}{1}$$

by Property 4, and $\frac{1}{1} = 1$ by Property 1. So, we've just logically deduced that $\frac{a}{a} = 1$.

39.6

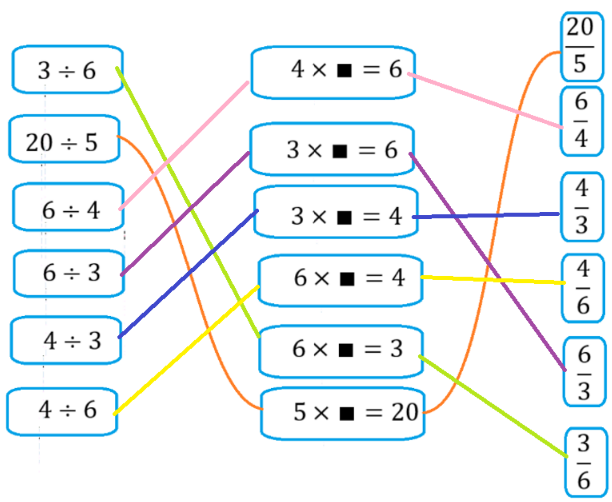


We can see this scenario as doing the work of sharing 2pies equally among 3 students, three times(!), then each student receives two thirds of a pie.

Or, we could divide each of the six pies into ninths and give each student a ninth from each pie. Then each student has six ninths of a pie.

The result must be the same.

40.1



40.2 Consequence 6 gives $3 \times \frac{1}{3} = 1$. Changing the order of multiplication, this reads

$$\frac{1}{3} \times 3 = 1$$

Thus, 3 is the number that fills in the blank to $\frac{1}{3} \times ■ = 1$. That is, 3 is the answer to $1 \div \frac{1}{3}$ when we think in terms of reverse multiplication.

41.1 By looking at a rectangle, we see

$$5 \times \left(\frac{1}{5} + \frac{1}{5}\right) = 5 \times \frac{1}{5} + 5 \times \frac{1}{5}$$

By Consequence 6, this is

$$1 + 1 = 2$$

41.2 By looking at a rectangle, we see

$$100 \times \left(\frac{1}{20} + \frac{1}{25} + \frac{2}{5}\right) = 100 \times \frac{1}{20} + 100 \times \frac{1}{25} + 100 \times \frac{2}{5}$$

By Consequence 5, $100 \times \frac{1}{20} = \frac{100}{20} = 5$, since $100 \div 20 = 5$.

By Consequence 5, $100 \times \frac{1}{25} = \frac{100}{25} = 4$, since $100 \div 25 = 4$.
By Consequence 5 used twice

$$100 \times \frac{2}{5} = 100 \times 2 \times \frac{1}{5} = 2 \times \frac{100}{5}$$

and this is $2 \times 20 = 40$.

So we have

$$100 \times \left(\frac{1}{20} + \frac{1}{25} + \frac{2}{5}\right) = 5 + 4 + 40 = 49$$

41.3 We have

$$100 \times \frac{1}{2} \times \frac{1}{25} = 2 \times 2 \times 25 \times \frac{1}{2} \times \frac{1}{25} = 2 \times 1 \times 1 = 2$$

41.4 It's 14.

41.5 It's 12.

41.6 We have

$$\frac{2}{7} + \frac{3}{7} = \frac{\frac{2}{7} + \frac{3}{7}}{1}$$

Multiply top and bottom by 7 to get

$$\frac{2 + 3}{7}$$

which is $\frac{5}{7}$.

41.7

$$\frac{3}{4}+\frac{2}{11}=\frac{4\times 11\times(\frac{3}{4}+\frac{2}{11})}{4\times 11\times 1}=\frac{11\times 3+4\times 2}{44}=\frac{33+8}{44}=\frac{41}{44}$$

41.8

$$2+\frac{4}{9}=\frac{9\times(2+\frac{4}{9})}{9\times 1}=\frac{18+4}{9}=\frac{22}{9}$$

41.9

$$\frac{3}{19}+\frac{7}{19}+\frac{9}{19}=\frac{19\times(\frac{3}{19}+\frac{7}{19}+\frac{9}{19})}{19\times 1}=\frac{3+7+9}{19}=\frac{19}{19}=1$$

41.10

$$\frac{1}{3}\times\frac{1}{5}\times\frac{1}{10}=\frac{\frac{1}{3}\times\frac{1}{5}\times\frac{1}{10}}{1}=\frac{3\times 5\times 10\times\frac{1}{3}\times\frac{1}{5}\times\frac{1}{10}}{3\times 5\times\times 10\times 1}=\frac{1\times 1\times 1}{150}=\frac{1}{150}$$

41.11

$$\frac{17}{21}\times\frac{2}{5}=\frac{\frac{17}{21}\times\frac{2}{5}}{1}=\frac{21\times 5\times\frac{17}{21}\times\frac{2}{5}}{21\times 5\times 1}=\frac{34}{105}$$

41.12

$$\frac{10}{64}\times\frac{16}{50}=\frac{64\times 50\times\frac{10}{64}\times\frac{16}{50}}{64\times 50\times 1}=\frac{10\times 16}{64\times 50}=\frac{10\times 16\times 1}{16\times 4\times 5\times 10}=\frac{1}{4\times 5}=\frac{1}{20}$$

41.13

$$\frac{a}{b}\times\frac{c}{d}=\frac{\frac{a}{b}\times\frac{c}{d}}{1}=\frac{b\times d\times\frac{a}{b}\times\frac{c}{d}}{b\times d\times 1}=\frac{a\times c}{b\times d}$$

41.14

$$\frac{10}{64}\times\frac{16}{50}=10\times\frac{1}{64}\times 16\times\frac{1}{50}$$

Now $10\times\frac{1}{50}=\frac{10}{50}=\frac{1}{5}$ and $6\times\frac{1}{64}=\frac{16}{64}=\frac{16\times 1}{16\times 4}=\frac{1}{4}$, so we really have

$$\frac{1}{5}\times\frac{1}{4}$$

which is $\frac{1}{20}$.

41.15 $\frac{5}{3}$

41.16 $\frac{6}{5}$

41.17

$$\frac{1}{2}\times\frac{2}{3}\times\frac{3}{4}\times\frac{4}{5}=\frac{1}{2}\times 2\times\frac{1}{3}\times 3\times\frac{1}{4}\times 4\times\frac{1}{5}=1\times 1\times 1\times\frac{1}{5}=\frac{1}{5}$$

41.18

$$\frac{2}{3}\div\frac{5}{7}=\frac{\frac{2}{3}}{\frac{5}{7}}=\frac{3\times 7\times\frac{2}{3}}{3\times 7\times\frac{5}{7}}=\frac{7\times 2}{3\times 5}=\frac{14}{15}$$

41.19

$$\frac{3}{4}\div\frac{2}{3}=\frac{\frac{3}{4}}{\frac{2}{3}}=\frac{4\times 3\times\frac{3}{4}}{4\times 3\times\frac{2}{3}}=\frac{3\times 3}{4\times 2}=\frac{9}{8}$$

41.20

$$1\div\frac{1}{N}=\frac{1}{\frac{1}{N}}=\frac{N\times 1}{N\times\frac{1}{N}}=\frac{N}{1}=N$$

41.21

$$8\div\frac{2}{3}=\frac{8}{\frac{2}{3}}=\frac{3\times 8}{3\times\frac{2}{3}}=\frac{24}{2}=12$$

41.22 First of all …

$$\frac{1}{a/b}=\frac{1}{\frac{a}{b}}=\frac{b\times 1}{b\times\frac{a}{b}}=\frac{b}{a}$$

So,

$$1\div\frac{1}{a/b}=1\div\frac{b}{a}=\frac{1}{\frac{b}{a}}=\frac{a\times 1}{a\times\frac{b}{a}}=\frac{a}{b}$$

41.23 “Pull out” a negative sign, twice.

$$\frac{-a}{-b} = (-1) \times \frac{-a}{b} = (-1) \times (-1) \times \frac{a}{b} = 1 \times \frac{a}{b} = \frac{a}{b}$$

41.24 $\frac{15}{20} = \frac{3}{4}$

41.25 $-\frac{1}{3}$

41.26

$$\frac{3}{7} - \frac{5}{7} = \frac{\frac{3}{7} + \frac{-5}{7}}{1} = \frac{7 \times \left(\frac{3}{7} + \frac{-5}{7}\right)}{7 \times 1} = \frac{3 + \ -5}{7} = \frac{-2}{7} = -\frac{2}{7}$$

41.27 We might as well do just part b) as it covers part a).

$$\frac{a}{N} - \frac{b}{N} = \frac{\frac{a}{N} + \frac{-b}{N}}{1} = \frac{N \times \left(\frac{a}{N} + \frac{-b}{N}\right)}{N \times 1} = \frac{a + \ -b}{N} = \frac{a - b}{N}$$

41.28 I am curious as to what your reaction is here.

41.29 No need to have done everything. In fact, this whole book is optional!