Chapter 6

Fractions:
Understanding their Schoolbook Arithmetic and The Mathematical Truth

Contents

42. Fractions: Where are we?

We started the last chapter with an overview of the muddled story of fractions typically presented in most school curriculums. Various real-world explanations are brought at choice moments to "explain" the meaning of fractions and the operations on fractions, and it is never quite clear which real-world model is appropriate to bring in when (nor is it clear why one is allowed to keep switching one's imagery of what a fraction is).

We did identify one real-world model of fractions that seems quite robust: the pies-per-student sharing model. It's a concrete scenario that seems to unify the story of fractions to some degree.

But we are now wise enough to know that no one model of a piece of mathematics can be used to explain all aspects of that mathematics: mathematics sits at a higher plane than any one concrete scenario. Although a topic of mathematics can be applied, when appropriate, to a wide variety of real-world instances, no one real-world instance can "see" and explain all the mathematics of that topic.

So, we worked to let go of the pie-per-student model. We identified a small set of basic properties that arose from the model that seemed to "carry the goods" of all fraction arithmetic. And we saw in Section 41 that they did.

We moved from a real-world story of fractions to a purely mathematical story.

That mathematical story can still be tightened up—and we will indeed wrap it up into an exceptionally tight logical bundle in this chapter—but we still have the weight of schoolbook version of fraction arithmetic on our shoulders.

Can we explain the schoolbook versions of fraction arithmetic as well?

The answer is that we can, and our job this chapter is to do that too.

This chapter is technically irrelevant: you can now do all the mathematics of fractions!

But this chapter is important as it will clear the all the hazy thinking of the past and help you see all you were taught in a clear, logical light. There is mathematical validity to it all.

Were We Are at Right Now

We've been playing with a system of numbers that contains all the integers $-3, -2, -1, 0, 1, 2, 3, \ldots$ (the counting numbers and all their opposites) and includes all the answers to division problems ("fractions"):

> For two numbers a and b in our system of numbers (with b not zero), we write $\frac{a}{b}$ as the answer to the division problem $a \div b$.

$$\frac{a}{b} = a \div b$$

Everything we developed in the last chapter boiled down to accepting just four properties of fractions.

- Every number can be written as a fraction with a denominator of 1. $$a = \frac{a}{1}$$
- Multiplying the top and bottom of a fraction by the same number does not change the value of the fraction. $$\frac{a}{b} = \frac{k \times a}{k \times b}$$
- Multiplying a fraction by its denominator "cancels" the denominator. $$\cancel{b} \times \frac{a}{\cancel{b}} = a$$
- We can "pull apart" a fraction. $$\frac{a}{b} = a \times \frac{1}{b}$$

(We'll see later in this chapter that these four properties all follow from <u>one</u>, even simpler, belief!)

Like the "star page" of Chapter 5, this page shows all the fractions goods needed to make everything work.

MECHANICS PRACTICE

Practice 42.1 Which of the following expressions has a value equal to a counting number? Which counting number?

a) $\frac{36}{12}$ b) $3 \times \frac{20}{3}$ c) $\frac{1002}{1}$ d) $\frac{107}{107}$ e) $4 \times \frac{5}{4}$ f) $\frac{0}{8}$ g) $10 \times 6 \times \frac{7}{5}$

43. Fractions the School Way: “Parts of a Whole”

As we saw, the story of fractions for students typically begins with developing familiarity with the concept of a **half**, **third**, **quarter** (fourth), **fifth**, **sixth**, and so on, of a given fixed object (a given “whole”).

A “half” of a given object is loosely defined as a portion of that object such that two copies of that portion together reconstitute the whole.

A ”third” of a given object is loosely defined as a portion of that object such that three copies of that portion together reconstitute the whole.

And so on.

half of a pie

a third of a set of kittens

a quarter of a line segment

Students are introduced to the notation $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \ldots$ for these **basic fractions**.

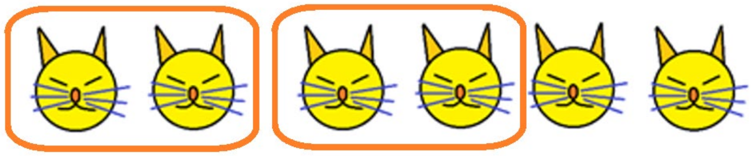
They are also taught to denote the whole in any given context as “1” and the statement “two copies of a half make the whole” is thus written $2 \times \frac{1}{2} = 1$, “three copies of a third make a whole” is written $3 \times \frac{1}{3} = 1$, and so on.

This feels strange and confusing, but the curriculum is trying to help students accept that for each counting number b it is useful to posit the existence of another number, denoted $\frac{1}{b}$, with the property:

$$b \times \frac{1}{b} = 1$$

This statement is aligned with one of our recognized properties of fractions.

Students are also taught that "two thirds," for example, is precisely what it reads: two copies of a third.



two thirds of a set of kittens

Similarly, "four fifths" is four copies of a fifth, and so on.

The notation $\frac{a}{b}$, with a a counting number is thus being introduced to mean "a copies of $\frac{1}{b}$."
That is, $\frac{a}{b}$ is shorthand for $a \times \frac{1}{b}$.

$$\frac{a}{b} = a \times \frac{1}{b}$$

This too is one of our recognized properties of fractions.

Just to be clear: In the early grades multiplication is repeated addition and so $\frac{2}{3} = 2 \times \frac{1}{3}$ is interpreted as $\frac{1}{3} + \frac{1}{3}$, just as the picture of kittens above suggests: we have one third of the kittens <u>and</u> one third of the kittens, side by side.

In the same way, $\frac{4}{5} = 4 \times \frac{1}{5} = \frac{1}{5} + \frac{1}{5} + \frac{1}{5} + \frac{1}{5}$, for copies of a fifth of a certain "whole" put side by side.

Mathematicians' Start to Fractions

Our mathematical journey back in Chapter 1 started with the **counting numbers** 0, 1, 2, 3, …. And in that chapter, we showed that all the arithmetic of the counting numbers logically follows from just eight general rules.

> **Rule 1:** For any two numbers a and b we have $a + b = b + a$.
>
> **Rule 2:** For any number a we have $a + 0 = a$ and $0 + a = a$.
>
> **Rule 3:** In a string of additions, it does not matter in which order one conducts individual additions.
>
> **Rule 4:** For any two numbers a and b we have $a \times b = b \times a$
>
> **Rule 5:** For any number a we have $a \times 1 = a$ and $1 \times a = a$.
>
> **Rule 6:** In a string of multiplications, it does not matter in which order one conducts individual multiplications.
>
> **Rule 7:** For any number a we have $a \times 0 = 0$ and $0 \times a = 0$.
>
> **Rule 8:** "We can chop up rectangles from multiplication and add up the pieces."

These eight rules pinpoint the behavior of addition and multiplication as it applies to the counting numbers. Identifying this behavior then allowed us to extend these operations to a larger class of numbers, the **integers**, and we did that in Chapter 3 with the addition of just one more rule.

> **Rule 9:** For each number a, there is one other number "$-a$" such that $a + -a = 0$.

We wanted "opposite numbers," so we made them happen.

It seems with fractions we again want a new type of opposite numbers—not opposite in the sense of addition, but opposite in a multiplicative sense. We want the basic fractions, numbers that match the statements “two halves make a whole,” “three thirds make a whole,” and so on, to exist in our world of numbers too.

So, let’s make them happen!

Rule 10: For each number a different from zero, there is one other number “$\frac{1}{a}$” such that $a \times \frac{1}{a} = 1$.

(This rule agrees that saying “zero zeroths makes a whole” is meaningless!)

This is how mathematicians start their story of fractions, by expanding the worldview of “number” by declaring the existence of the basic fractions.

And this is in line with the human start to fraction thinking—positing the existence of numbers deserved to be called a half, a third, a fourth, and so on.

We’ll explore the full magic of this Rule 10 at the end of the chapter.

MUSINGS

Musing 43.1 Schoolbooks typically define one third, $\frac{1}{3}$, for example as follows:

> *Choose your whole. (Usually a pie.)*
> *Divide that pie into three "equal" pieces.*
> *Then each of those pieces is called a third.*

We've just defined a third a different way.

> *Choose your whole.*
> *A "third" is a portion of that whole with the property that three copies of that portion match the whole.*

Are these two ways of thinking equivalent?

Musing 43.2 Here's our picture of two-thirds of a set of kittens.

From our study of fractions, we know that $3 \times \frac{2}{3} = 2$.

Is it possible to "justify" this statement using the picture of kittens?

44. Adding and Subtracting Fractions the School Way

In the early grades addition is interpreted as the physical act of "putting items together" and recounting.

One apple and two apples make for three apples.
One house and two houses make for three houses.

Consequently, one tenth and two tenths should likewise make for three tenths (especially if one is imagining portions of a given pie).

$$\frac{1}{10}+\frac{2}{10}=\frac{3}{10}$$

Schoolbooks use this physical thinking to "explain" how to add fractions sharing the same denominator (a **common denominator**): just add up the numerators and write their sum over that common denominator.

And this does align with the mathematics of fractions we've seen.

Example a) Evaluate $\frac{1}{10}+\frac{2}{10}$ mathematically.

b) Prove that $\frac{a}{N}+\frac{b}{N}+\frac{c}{N}+\frac{d}{N}$ is indeed $\frac{a+b+c+d}{N}$.

Answer: a) We have

$$\frac{1}{10}+\frac{2}{10}=\frac{\frac{1}{10}+\frac{2}{10}}{1}=\frac{10\times(\frac{1}{10}+\frac{2}{10})}{10\times 1}=\frac{1+2}{10}=\frac{3}{10}$$

b) And we have

$$\frac{a}{N}+\frac{b}{N}+\frac{c}{N}+\frac{d}{N}=\frac{\frac{a}{N}+\frac{b}{N}+\frac{c}{N}+\frac{d}{N}}{1}=\frac{N\times(\frac{a}{N}+\frac{b}{N}+\frac{c}{N}+\frac{d}{N})}{N\times 1}=\frac{a+b+c+d}{N}$$

Of course, it is much easier to think "one tenth plus two tenths equals three tenths," say, rather than go through the mathematical hoopla of putting over 1, multiplying top and bottom by ten, and so on.

But now that we know that school mathematics is right in what is says about adding fractions with a common denominator, we can make use of this result anytime too and avoid the spelling out the mathematical details each and every time.

ADDING FRACTIONS: Same Denominator

$$\frac{a}{N}+\frac{b}{N}=\frac{a+b}{N}$$

But this result now begs the question:

How do we add two fractions whose denominators do not match?
For example, what is the value of $\frac{2}{5}+\frac{4}{9}$?

We computed such sums without a problem in Section 41. (We placed the quantity over 1 and proceeded from there.)

Practice 44.1: Compute $\frac{2}{5}+\frac{4}{9}$ vis the technique of Section 41.

But this is not the approach schoolbooks take.

Curriculums have the opportunity, right here, to teach students a meta-lesson: to step back and learn a problem-solving strategy often used by mathematicians

When faced with a challenge, try engaging in **wishful thinking**.

Not knowing how to proceed (having not read the previous chapter), we might muse as follows:

We know how add two fractions with the same denominator.

We want to compute $\frac{2}{5}+\frac{4}{9}$.

They don't have a common denominator. Life would be easier if they did.

Hmm. How do we make that happen?

We know how to change the numerators and denominators of fractions without changing the value of the fraction. So, one approach might be to start listing all the equivalent fractions of the two fractions we are considering. Let's double their numerators and denominators, then triple them, then quadruple them, and so on.

$$\boxed{\mathbf{\frac{2}{5}}} \quad \frac{4}{10} \quad \frac{6}{15} \quad \frac{8}{20} \quad \frac{10}{25} \quad \frac{12}{30} \quad \frac{14}{35} \quad \frac{16}{40} \quad \boxed{\frac{18}{45}} \quad \frac{20}{50} \quad \frac{22}{55} \ldots$$

$$\boxed{\mathbf{\frac{4}{9}}} \quad \frac{8}{18} \quad \frac{12}{27} \quad \frac{16}{36} \quad \boxed{\frac{20}{45}} \quad \frac{24}{54} \quad \frac{28}{63} \quad \frac{32}{72} \quad \frac{36}{81} \quad \frac{40}{90} \quad \frac{44}{99} \ldots$$

Lo and behold. We see that $\frac{2}{5}$ is the same as $\frac{18}{45}$, and that $\frac{4}{9}$ is the same as $\frac{20}{45}$.

To compute $\frac{2}{5}+\frac{4}{9}$ we can just compute $\frac{18}{45}+\frac{20}{45}$ instead. And that has value $\frac{38}{45}$.

$$\frac{2}{5}+\frac{4}{9}=\frac{38}{45}$$

Question: Is this the answer you obtained?

Thank you wishful thinking!

School students are taught …

ADDING FRACTIONS: Different Denominators
Rewrite the fractions to have a common denominator and then add as before.

Comment: Students aren't typically encouraged to write out long lists of equivalent fractions to find two fractions with the same denominator. For example, realizing that multiplying the numerator and denominator of $\frac{2}{5}$ each by 9, and multiplying the numerator and denominator of $\frac{4}{9}$ each by 5, will yield two fractions with the same denominator of 45. Of course, feel free to employ such swiftness anytime you want to too.

Example: Evaluate $\frac{3}{4}+\frac{11}{10}$ the schoolbook way.

Answer: Let's create fractions with the same denominators by multiplying the tops and bottoms of the fractions with each other's denominators.

$$\frac{3}{4}+\frac{11}{10}=\frac{10\times 3}{10\times 4}+\frac{4\times 11}{4\times 10}=\frac{30}{40}+\frac{44}{40}$$

We now see that the sum is $\frac{74}{40}$.

Some school courses won't accept this as a final answer as we can "reduce" this fraction. Those curricula often insist that we must reduce if we can.

$$\frac{74}{40}=\frac{2\times 37}{2\times 20}=\frac{37}{20}$$

Some hyper-fussy school curricula will insist that students be as efficient as possible from the get-go.

They would want them to realize that we could rewrite $\frac{3}{4}$ and $\frac{11}{10}$ with a common denominator smaller than 40, and say that they should! (I don't know why.)

$$\frac{3}{4}+\frac{11}{10}=\frac{5\times 3}{5\times 4}+\frac{2\times 11}{2\times 10}=\frac{15}{20}+\frac{22}{20}$$

The answer $\frac{37}{20}$ pops out again.

Question: Does the phrase **least common multiple** ring a bell for you?

Here's the thing. There is no need to be hyper-efficient in mathematics work. All good and correct approaches and answers are good and correct!

If you want to work out the sum of two fractions the schoolbook way, great!
If you want to work it out by putting the sum over 1 and going from there, also great!

If you get an answer that you might later need to rewrite in a different form, rewrite it later.

If you don't ever need to rewrite an answer, then the answer you first obtain is fine as it is.
(Though test writers often object to this final comment.)

Example: What is $\frac{12}{17} - \frac{8}{17}$?

Thinking concretely of portions of pie, schoolbooks encourage students to think "12 slices take away 8 slices obviously leaves 4 slices." We have $\frac{12}{17} - \frac{8}{17} = \frac{4}{17}$.

And we know this aligns with solid mathematics.

$$\frac{12}{17} - \frac{8}{17} = \frac{12}{17} + \frac{-8}{17} = \frac{\frac{12}{17} + \frac{-8}{17}}{1}$$

$$= \frac{17 \times (\frac{12}{17} + \frac{-8}{17})}{17 \times 1}$$

$$= \frac{12 + \ \ -8}{17}$$

$$= \frac{4}{17}$$

Practice 44.2 Show, in general, that $\frac{a}{N} - \frac{b}{N}$ equals $\frac{a-b}{N}$.

School intuition and solid mathematics do align. We have, and can use anytime, …

SUBTRACTING FRACTIONS: Same Denominator

$$\frac{a}{N} - \frac{b}{N} = \frac{a-b}{N}$$

You can guess how schoolbooks advice students to subtract fractions whose denominators do not match.

For instance, most people would compute $\frac{2}{3} - \frac{1}{4}$ by saying:

> $\frac{2}{3}$ *is really* $\frac{8}{12}$.
>
> $\frac{1}{4}$ *is really* $\frac{3}{12}$.
>
> *Eight twelfths take away three twelfths is five twelfths.*
> *Done!*

$$\frac{2}{3} - \frac{1}{4} \quad = \quad \frac{8}{12} - \frac{3}{12} \quad = \quad \frac{5}{12}$$

Question What is the value of $\frac{5}{6} - \frac{1}{2}$?

a) $\frac{4}{12}$ b) $\frac{2}{6}$ c) $\frac{1}{3}$ d) All of these

We can see this problem as $\frac{10}{12} - \frac{6}{12} = \frac{4}{12}$ or as $\frac{5}{6} - \frac{3}{6} = \frac{2}{6}$, and both these values equal $\frac{1}{3}$.

Students are taught …

SUBTRACTING FRACTIONS: Different Denominators
Rewrite the fractions to have a common denominator and then subtract as before.

Of course, you also have the option to put the quantity over 1 and just work on from there.

Example: Compute $\frac{7}{12} - \frac{2}{5} + \frac{13}{6} - 1$ the schoolbook way.

Answer: With denominators of 12, 5, and 6, I am thinking 60ths might be good for this challenge.

Also, think of "1" as the fraction $\frac{60}{60}$. (1 is the answer to $60 \div 60$.)

$$\frac{5 \times 7}{5 \times 12} - \frac{12 \times 2}{12 \times 5} + \frac{10 \times 13}{10 \times 6} - \frac{60}{60} \quad = \quad \frac{35}{60} - \frac{24}{60} + \frac{130}{60} - \frac{60}{60} \quad = \quad \frac{81}{60}$$

MUSINGS

Musing 44.3 We showed that $\frac{2}{5}+\frac{4}{9}$ equals $\frac{38}{45}$. We did this purely by what mathematics led us to do.

But can we make sense of this with our pies-and-student model?

Is sharing 2 pies equally among 5 students together with sharing 4 pies equally among 9 students somehow connected with sharing 38 pies equally among 45 students?

I personally do not see how, but maybe you do?

> **Comment:** This question illustrates the challenge (absurdity, actually) of expecting one real-world experience to model absolutely everything mathematics has to offer. Mathematics sits at a different level to real-world applications and has relevance to many, many different scenarios. It is not the other way round.

Musing 44.4 In school, with our concrete thinking, we were taught that adding fractions was just a matter of bringing portions of pie together on plate. We have that $\frac{2}{5}$ of a pie and $\frac{4}{9}$ of a pie together make $\frac{38}{45}$ of a pie.

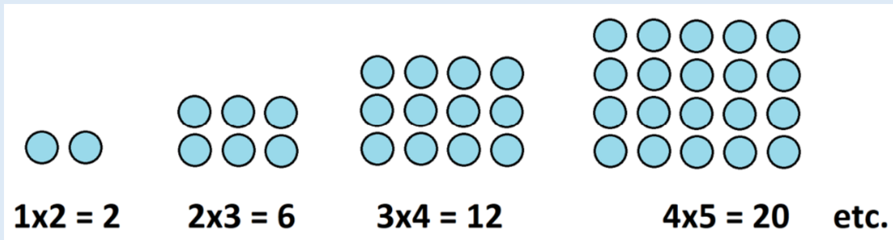
What picture would you have drawn back in your school days to verify this particular example?

Musing 44.5 Match each arithmetic statement on the left with its value on the right.

$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}$	$\frac{3}{16}$
$\frac{1}{4}-\frac{1}{5}$	$\frac{15}{16}$
$1-\frac{1}{20}$	$\frac{1}{20}$
$\frac{9}{32}-\frac{1}{8}+\frac{1}{32}$	$\frac{6}{20}$
$\frac{3/4}{5}+\frac{3}{20}$	$\frac{19}{20}$

Musing 44.6 The **oblong numbers** are the numbers that arise from arranging dots into rectangular arrangements with one side one unit longer than the other.

The first oblong number is $1 \times 2 = 2$. The second oblong number is $2 \times 3 = 6$. And so on.



a) What's the thirtieth oblong number?
b) What is the value of $\frac{1}{30} - \frac{1}{31}$?

c) What's the tenth oblong number and what is the value of $\frac{1}{10} - \frac{1}{11}$?

In the previous question you matched $\frac{1}{4} - \frac{1}{5}$ with $\frac{1}{20}$, and 20 is the fourth oblong number.

d) If you are feeling game and particularly mathy, perform some mathematics to show how $\frac{1}{n} - \frac{1}{n+1}$ is connected to the nth oblong number. (But do this only if this is fun for you!)

Musing 44.7
a) Compute $\frac{a}{b} + \frac{c}{d}$ by putting this quantity over 1 and working from there. Show that you get the (unfriendly) answer $\frac{ad+bc}{bd}$.
b) Compute $\frac{a}{b} + \frac{c}{d}$ again, but now the schoolbook way by working with a common denominator. Show that you again get the answer $\frac{ad+bc}{bd}$.

Once more, all correct approaches to mathematics match and are good and correct!

MECHANICS PRACTICE

Practice 44.8 Did you do the matching exercise in Musing 43.5?
(I guess that is technically a Yes/No question!)

$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}$	$\frac{3}{16}$
$\frac{1}{4}-\frac{1}{5}$	$\frac{15}{16}$
$1-\frac{1}{20}$	$\frac{1}{20}$
$\frac{9}{32}-\frac{1}{8}+\frac{1}{32}$	$\frac{6}{20}$
$\frac{3/4}{5}+\frac{3}{20}$	$\frac{19}{20}$

Practice 44.9 Compute each of the following.

a) $\frac{6}{11}+\frac{7}{10}$ b) $\frac{5}{14}-\frac{1}{70}$ c) $\frac{4}{3}-\frac{4}{9}$ d) $2+\frac{7}{5}-\frac{3}{7}$

Practice 44.10 Compute each of these.

a) $\frac{1}{19}-\frac{1}{21}$

b) $1+\frac{1}{2}+\frac{1}{4}-\frac{7}{2}$

c) $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}$

45. Multiplying Fractions the School Way: The Word "Of"

We saw how to multiply two fractions in Section 41.

MULTIPLYING FRACTIONS

$$\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d}$$

Practice 45.1 Do you remember how to obtain this result?

a) Use the technique of Section 41 to find the value of $\frac{1}{3} \times \frac{1}{7}$.

b) Find the value of $\frac{2}{3} \times \frac{4}{7}$.

c) Show that $\frac{a}{b} \times \frac{c}{d}$ does indeed equal $\frac{a \times c}{b \times d}$.

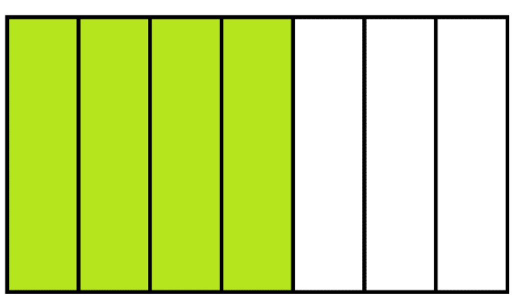
But schoolbooks do not typically show the approach of Section 41 to justify this multiplication rule. Instead, they appeal to a curious aphorism: "of means multiply" (or, perhaps it should be "multiply means of"?)

What exactly is the link between the everyday use of the word "of" and the multiplication of fractions?

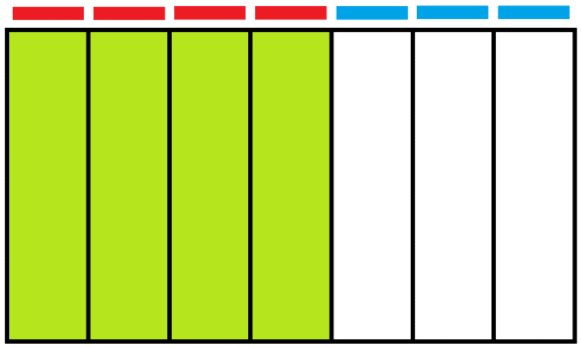
Let's start slowly by first asking …

What do we mean by "four sevenths of a pie"?

Well, everyday practice suggests we divide the pie into seven equal pieces and select four of them. For ease of drawing, let's draw rectangular pie.



But we let me point out that in dividing the rectangle into seven equal pieces and selecting four of them, we've also divided the top edge of the pie into seven equal pieces as well and selected four of those segments.



Or we could look at this picture and think of it as coming from having divided the top edge of the pie into seven equal pieces first, selecting four of them, and then using them to guide to how to divide the whole rectangular pie.

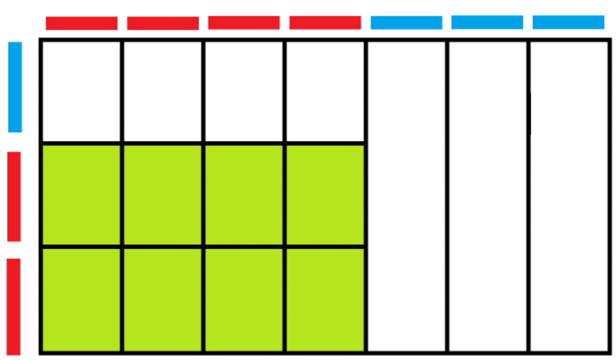
> There is a natural correspondence between portions of the side of a rectangular pie and portions of the whole pie itself.

Now let's ask:

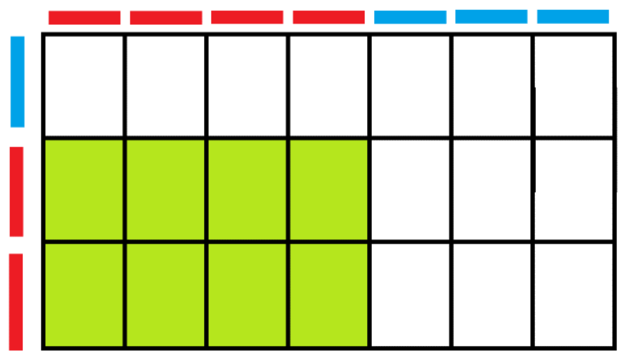
> What do we mean by "two thirds of four sevenths of a pie"?

The green part of the picture above, which represents four fifths of our pie, looks like a rectangular pie in-and-of itself. The question seems to be asking us to identify two-thirds of this green rectangular portion.

Here it is.



Drawing in extra lines helps makes sense of this picture.



Our "everyday thinking" here has led us to a picture of rectangular pie subdivided into 21 pieces with 8 of them highlighted. We are led to say:

Two thirds of four sevenths of a rectangular pie is eight twenty-firsts of the pie.

Practice 45.2 OPTIONAL Can you draw a picture of two thirds of four sevenths of a circular pie? Can you see eight twenty-firsts of the pie as a result of what you draw?
[Rectangular pies are so much friendlier to draw!]

This work suggests the following:

$$\frac{2}{3} \text{ of } \frac{4}{7} \text{ equals } \frac{8}{21}$$

Schoolbooks tend to suppress the words "of pie," which is naughty. The statement really should read: "$\frac{2}{3}$ of $\frac{4}{7}$ of a pie equals $\frac{8}{21}$ of the pie."

And one cannot help but notice that $8 = 2 \times 4$ and $21 = 3 \times 7$.

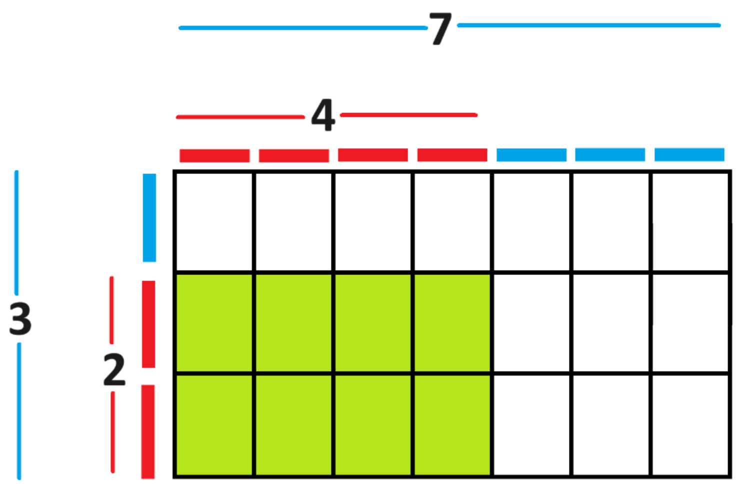
$$\frac{2}{3} \text{ of } \frac{4}{7} \text{ equals } \frac{2\times4}{3\times7}$$

Let's be clear.

The operation "of" in this context of slicing pie is a physical action.
It is not a mathematical operation.

Nonetheless, we can use mathematical thinking to understand why a numerator of 8 and a denominator of 21 appeared when answering this particular practical problem.

In computing $\frac{2}{3}$ of $\frac{4}{7}$ of a rectangular pie, we naturally see the numbers 2, 3, 4, and 7 arise in the picture we draw.



The shaded pieces form a 2-by-4 rectangle, and so there are $2 \times 4 = 8$ of them.

These shaded pieces sit inside a 3-by-7 rectangle. There are $3 \times 7 = 21$ pieces in total.

Our final answer is thus naturally $\frac{2\times4}{3\times7}$.

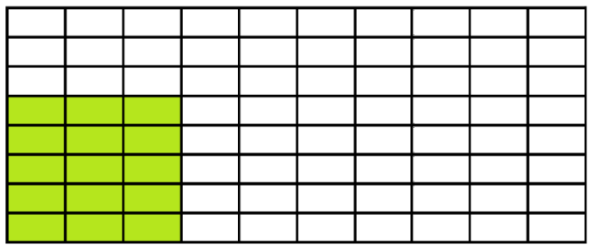
What is remarkable is that this answer happens to match the mathematical answer to $\frac{2}{3} \times \frac{4}{7}$.

What a lovely coincidence!

Practice 45.3 The picture shows the result of another "portion of a portion" problem.

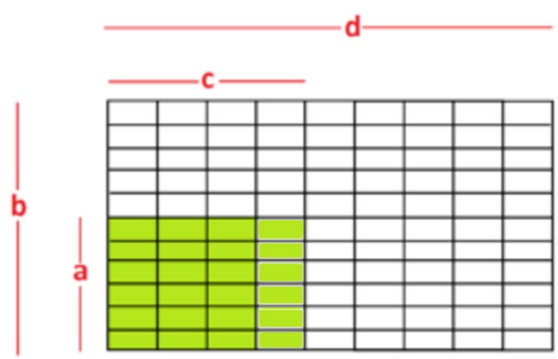
Fill in the blanks to show the coincidence continues.

$$\frac{\square}{8} \text{ of } \frac{\square}{\square} \text{ equals } \frac{\square \text{ x } 3}{8 \text{ x } \square}$$



Of course, this is not a coincidence. This general picture of a portion of a portion of a rectangular pie shows that the result will always be a fraction that happens to be given by the same formula we derived in Section 41 for the product of two fractions.

$\frac{a}{b}$ of $\frac{c}{d}$ of a pie is $\frac{a \times c}{b \times d}$ of the pie.



And $\frac{a \times c}{b \times d}$ is the mathematical answer to $\frac{a}{b} \times \frac{c}{d}$.

Schoolbooks lean into this alignment.

Since we often read $\mathbf{2} \times \mathbf{5}$, say, as **two groups <u>of</u> five** it feels reasonable to say that the word "of" is synonymous with the word "multiply."

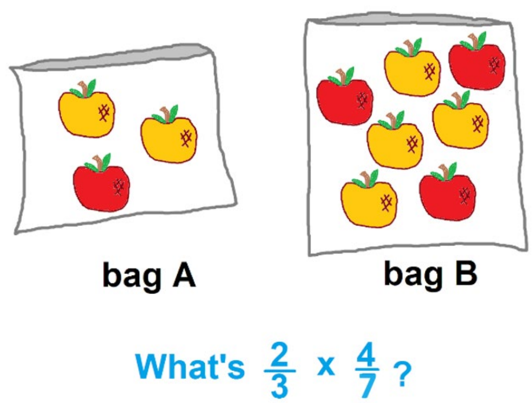
And drawing pictures of portions or portions of rectangular pie, which very much look like pictures of the area model for multiplication, gives the feeling of justifying why "of means multiply."

But the truth is that fractions have their own mathematics, and that mathematics tells us how to define fraction multiplication.

And, yes, it is absolutely lovely that this definition matches how people think of portions of portions of pie. But it is the math that comes first and the connection to real-world thinking that comes second.

Most schoolbooks muddle this up.

Not all real-world scenarios have a place or need for fraction multiplication. For example, in thinking about bags of yellow and red apples, trying to make sense of $\frac{2}{3} \times \frac{4}{5}$ is meaningless and silly.



Just because you and I know there is a mathematical answer to $\frac{2}{3} \times \frac{4}{5}$, it doesn't mean that the answer is meaningful for a particular scenario.

Math has the answers for all the places where it is meaningful and relevant to ask particular math questions. That's the beauty and power of mathematics.

It's always at-the-ready for us to use.

But it is up to us to recognize when a piece of mathematics does or doesn't have relevance to a particular scenario.

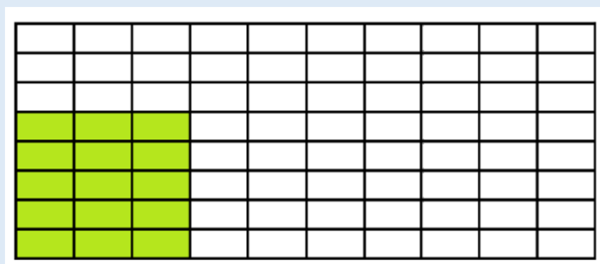
MUSINGS

Musing 45.4 In this section we showed that "$\frac{5}{8}$ of $\frac{3}{10}$" of a pie matches $\frac{5}{8} \times \frac{3}{7}$ of the pie.

But it is possible to switch multiplications around: $\frac{5}{8} \times \frac{3}{10}$ is the mathematically the same as $\frac{3}{10} \times \frac{5}{8}$. This means we should be able to switch the order of the fractions around in a portion of a portion problem and obtain the same result.

> Do "$\frac{3}{10}$ of $\frac{5}{8}$ of a pie" and "$\frac{5}{8}$ of $\frac{3}{10}$ of a pie" represent the same amount of pie?
> It is not *a priori* obvious that they should!

Look at this picture.



a) Explain how to view this picture as representing "$\frac{5}{8}$ of $\frac{3}{10}$ of a pie."

b) Explain how to also view the same picture as representing "$\frac{3}{10}$ of $\frac{5}{8}$ of a pie."

Conclude that yes, we can switch the order of the fractions in a "portion of a portion" of statement.

Musing 45.5 Is "two-thirds of three-fifths of seven-ninths of a pie" the same amount as "seven-ninths of two-thirds of three-fifths" of the same pie?

MECHANICS PRACTICE

Of course, one doesn't want to always want to think of the word "of" when multiplying fractions. Try these problems purely as arithmetic problems with no regard to pie!

Practice 45.6

a) Show that $\frac{407}{933} \times \frac{933}{407}$ equals 1.

b) More generally, show that multiplying a fraction $\frac{\mathrm{a}}{\mathrm{b}}$ by its **reciprocal** $\frac{\mathrm{b}}{\mathrm{a}}$ is sure to give 1.

$$\frac{a}{b} \times \frac{b}{a} = 1$$

Practice 45.7 If $n = p \times q$, what is the value of $n \times \frac{p}{q}$?

Practice 45.8 What is $\frac{-8}{9} \times \frac{2}{-5}$?

Practice 45.9 What do you need multiply $\frac{4}{7}$ by to get the answer $\frac{1}{2}$?

Practice 45.10 What is the value of this unfriendly quantity?

$$\frac{7}{12} \times \frac{2}{5} \times 90 \times \frac{10}{21} \times \frac{3}{19} \times \frac{1}{3} \times \frac{57}{9}$$

46. Dividing Fractions the School Way: “Keep, Change, Flip” and the word “Of” again!

Dividing fractions in Section 41 was a not really an issue.

Example: Compute $\frac{5}{3} \div \frac{4}{7}$.

Answer: A fraction is an answer to a division problem. So, the answer to this division problem is the fraction

$$\frac{\frac{5}{3}}{\frac{4}{7}}$$

Our job is to make this fraction look friendlier.

Multiplying top and bottom each by 3 and by 7 yields

$$\frac{3 \times 7 \times \frac{5}{3}}{3 \times 7 \times \frac{4}{7}} = \frac{7 \times 5}{3 \times 4} = \frac{35}{12}$$

That’s it,

$$\frac{5}{3} \div \frac{4}{7} = \frac{35}{12}$$

But let’s pause for a moment. At one point along the way we wrote:

$$\frac{5}{3} \div \frac{4}{7} = \frac{7 \times 5}{3 \times 4}$$

We can switch the order of multiplication in the numerator and write instead

$$\frac{5}{3} \div \frac{4}{7} = \frac{5 \times 7}{3 \times 4}$$

And this answer can be thought of as the result of multiplying the two fractions $\frac{5}{3}$, our first fraction, and $\frac{7}{4}$, our second fraction upside down!

$$\frac{5}{3} \div \frac{4}{7} = \frac{5}{3} \times \frac{7}{4}$$

To divide these two fractions we, in effect, **KEPT** the first fraction as it was, **CHANGED** the division symbol to a multiplication symbol, and **FLIPPED** the second fraction upside down, and computed that product instead.

Practice 46.1 Prove that the KCF trick is valid in the general setting.

a) Compute $\frac{a}{b} \div \frac{c}{d}$

b) Compute $\frac{a}{b} \times \frac{d}{c}$

and show these have the same answer.

This **KCF** trick from schoolbooks can indeed be mathematically justified. But it really is unnecessary to memorize such a trick. (Moreover, it is deeply sad and disturbing if this trick is the only thing offered to students in a discussion on the division of fractions.)

We simply looked at the expression $\frac{\frac{5}{3}}{\frac{4}{7}}$, balked at how confusing it looks, and followed our mathematical noses to make it appear more tractable. The answer $\frac{35}{12}$ naturally popped out. Moreover, if anyone would challenge our answer, we can explain why it is so!

Practice 46.2: Match each quantity on the left with its value on the right. (You see that there are more options than correct answers!)

$\frac{1}{2} \div \frac{1}{3}$	1
$\frac{4}{5} \div \frac{3}{7}$	2
$\frac{2}{3} \div 5$	5
$\frac{45}{45} \div \frac{902}{902}$	$\frac{2}{3}$
$\frac{10}{13} \div \frac{2}{13}$	$\frac{3}{2}$
	$\frac{5}{13}$
	$\frac{2}{15}$
	$\frac{28}{15}$
	$\frac{15}{28}$

Practice 46.3: Did you catch in the previous question that $\frac{45}{45} \div \frac{902}{902}$ is just $1 \div 1$ in disguise and so has answer 1? (I love it when you can just use common sense and cut through clunky work!)

The problems $\frac{10}{13} \div \frac{2}{13}$ in Practice 46.2 is also curious. Its answer happens to be the same as the answer to $10 \div 2$ from just ignoring the common denominator of 13.

Example: Show that $\frac{a}{N} \div \frac{b}{N}$ and $a \div b$ are sure to have the same answer.

Some textbooks call this observation the **common denominator division method.**

Answer: The answer to the division problem $\frac{a}{N} \div \frac{b}{N}$ is the fraction

$$\frac{\frac{a}{N}}{\frac{b}{N}}$$

The answer to the division problem $a \div b$ is the fraction $\frac{a}{b}$.
Are these the same fraction?

It seems natural to multiply top and bottom of the complicated fraction each by N. This gives

$$\frac{\frac{a}{N}}{\frac{b}{N}} = \frac{N \times \frac{a}{N}}{N \times \frac{b}{N}} = \frac{a}{b}$$

Yes! They are the same fraction!

Wow!

Here's an example of how schoolbooks suggest using this "common denominator method."

Example: Compute $\frac{3}{4} \div \frac{2}{3}$ via the common denominator method.

Answer: We need to express each fraction with a shared denominator.

Since we have fourth and thirds, it seems that working with twelfths could be good.

$$\frac{3}{4} \div \frac{2}{3} = \frac{9}{12} \div \frac{8}{12}$$

Via the common denominator method, this equals $9 \div 8$, which is $\frac{9}{8}$.

Practice 46.4: Follow your instincts and show that $\frac{\frac{3}{4}}{\frac{2}{3}}$ does indeed equal $\frac{9}{8}$.

Truly, there is no need to memorize any rule or method, including this "common denominator method." (But it is fun to try to explain why all these various curious methods actually work!)

Practice 46.5: Show that $\frac{12}{15} \div \frac{3}{5}$ equals $\frac{4}{3}$.

There is something extra strange about this example.

Notice that

$$12 \div 3 = 4$$

and

$$15 \div 5 = 3$$

and

$$\frac{12}{15} \div \frac{3}{5} = \frac{4}{3} = \frac{12 \div 3}{15 \div 5}$$

It is as though to compute $\frac{12}{15} \div \frac{3}{5}$ we just divided the numerators and divided the denominators separately.

Could it be true that $\frac{a}{b} \div \frac{c}{d}$ equals $\frac{a \div c}{b \div d}$? That is, to divide two fractions, just divide the numerators and divide the denominators individually?

That seems weird!

Practice 46.6: Show that $\frac{32}{35} \div \frac{8}{7}$ does equal $\frac{4}{5}$ (and again this is just $\frac{32 \div 8}{35 \div 7}$).

Practice 46.7:

a) The answer to $\frac{a}{b} \div \frac{c}{d}$ is $\frac{\frac{a}{b}}{\frac{c}{d}}$. Rewrite this expression to show that it equals $\frac{a \times d}{b \times c}$.

b) The quantity $\frac{a \div c}{b \div d}$ is $\frac{\frac{a}{c}}{\frac{b}{d}}$. Rewrite this expression to show that it also equals $\frac{a \times d}{b \times c}$.

This allows us to conclude that $\frac{a}{b} \div \frac{c}{d}$ does indeed equal $\frac{a \div c}{b \div d}$.

“Of” and Division

People also often use the word “of” when they mean, or perhaps are just thinking, division.

For example, someone might say

> “half of six is three,” while thinking $6 \div 2 = 3$
>
> “a third of twelve is four,” while thinking $12 \div 3 = 4$.

But if we’re think “of means multiply” from our schoolbook training, then

> “half of six is three” is saying $\frac{1}{2} \times 6 = 3$.
>
> “a third of twelve is four” is saying $\frac{1}{3} \times 12 = 4$.

Is everything the same here? That is,

> Is multiplication by a basic fraction the same as division?

To think through this, let's play with

$$\frac{1}{2} \times 6$$

a wee bit.

By the general rules of arithmetic, we can switch the order of multiplication.

$$6 \times \frac{1}{2}$$

Now we just see this as a fraction "pulled apart." It is

$$\frac{6}{2}$$

But a fraction is the answer to a division problem: $\frac{6}{2}$ is the answer to $6 \div 2$.

Putting this all together we get

$$\frac{1}{2} \times 6 = 6 \times \frac{1}{2} = \frac{6}{2} = 6 \div 2$$

Yes! "A half of six" interpreted as multiplication by a fraction is the same as interpreting the expression as dividing by 2.

> **Practice 46.8** Show that interpreting "a third of twelve" as a statement of multiplication by a fraction ($\frac{1}{3} \times 12$) is equivalent to interpreting it as a statement of division ($12 \div 3$).

In general,

> $a \div N$ is the fraction $\frac{a}{N}$
>
> $\frac{1}{N} \times a$ equals $a \times \frac{1}{N}$, which is also $\frac{a}{N}$

So

> $a \div N$ and $\frac{1}{N} \times a$ are the same!

Multiplying a quantity by $\frac{1}{N}$ is the same as dividing that quantity by N.

MUSINGS

Musing 46.9

a) Find the value of $\frac{1}{\frac{8}{7}}$.

b) In general, what is the value of $\frac{1}{\frac{a}{b}}$?

Musing 46.10 Mathematicians say that "Division does not exist: it is multiplication by the reciprocal." This comes from noting that

To "divide" a quantity by a number N is to multiply that quantity by $\frac{1}{N}$.

(We saw why this is so on the previous page.)

Consequently, mathematicians don't use the division symbol ÷ (the **obelus**) and will always write a quantity multiplied by a fraction instead.

Musing 17.2 had us consider the following ambiguous expression that regularly makes the rounds on the internet

$$8 \div 2(2 + 2)$$

a) Rewrite the expression in terms "multiplication by a fraction" that unambiguously evaluates to 1.
b) Rewrite the expression in terms "multiplication by a fraction" that unambiguously evaluates to 16.

MECHANICS PRACTICE

Practice 46.11 Compute each of the following.

a) $\frac{1}{2} \div \frac{1}{3}$ b) $\frac{4}{5} \div \frac{3}{7}$ c) $\frac{2}{3} \div \frac{1}{5}$

Practice 46.12 Make the following look much friendlier.

$$\frac{8/5}{2/\left(\frac{5}{4}\right)}$$

47. Mixed Numbers

We have in our mathematical world positive whole numbers, negative whole numbers, zero, and fractions.

And we know that every integer can be thought of as a fraction too: rewrite it by introducing a denominator of 1. For example,

3 is the fraction $\frac{3}{1}$.

-20 is the fraction $\frac{-20}{1} = -\frac{20}{1} = \frac{20}{-1}$.

Practice 47.1 What fraction is zero?

This means we can add together integers and fractions.

Example: What is the value of $3 + \frac{1}{2}$?

Answer: We have

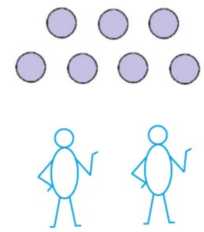
$$3 + \frac{1}{2} = \frac{3}{1} + \frac{1}{2}$$

The schoolbook approach has us find a common denominator.

$$3 + \frac{1}{2} \quad = \quad \frac{3}{1} + \frac{1}{2} \quad = \quad \frac{6}{2} + \frac{1}{2} \quad = \quad \frac{7}{2}$$

Practice 47.2 Show again that $3 + \frac{1}{2} = \frac{7}{2}$ by writing $\frac{3+\frac{1}{2}}{1}$ and doubling top and bottom.

Going back to the practical world for a moment …



Question: If 7 pies are shared equally among 2 students, does each student get three whole pies and half a pie?

The answer is yes.

I brought up this "real-world" example as most people would prefer to think of and work with the expression $3+\frac{1}{2}$ ("three and a half") rather than $\frac{7}{2}$ ("seven halves") in the real world. It's a common practice in society to mix counting numbers and fractions.

Let's pin this idea down.

The counting number 3 and the fraction $\frac{1}{2}$ added together give the "mixed number" $3+\frac{1}{2}$. People usually omit the plus sign in such an expression and just write $3\frac{1}{2}$, and still read it out loud as even "three **and** a half."

$5\frac{4}{17}$ means $5+\frac{4}{17}$ (five **and** four seventeenths).

$200\frac{1}{200}$ means $200+\frac{1}{200}$ (two hundred **and** one two-hundredth).

But society is fussy about what it will permit as a "valid" mixed number. (Are you surprised?)

Example: Show that mixed number $6\frac{4}{3}$ is the same as mixed number $7\frac{1}{3}$.

Answer: We have that $6\frac{4}{3}$ is

$$6+\frac{4}{3}$$

and $7\frac{1}{3}$ is

$$7+\frac{1}{3}$$

Are these the same?

Well, we can rewrite $7 + \frac{1}{3}$ as

$$6 + 1 + \frac{1}{3}$$

which is a little closer to making the two look alike.

The next thing is to realize that $\frac{4}{3}$ and $1 + \frac{1}{3}$ are the same number.

We can think of $\frac{4}{3}$ as $\frac{3+1}{3} = \frac{3}{3} + \frac{1}{3}$ by doing addition in reverse. And this is indeed $1 + \frac{1}{3}$.

Swiftly …

$$6\frac{4}{3} \;=\; 6 + \frac{4}{3} \;=\; 6 + \frac{3+1}{3} \;=\; 6 + \frac{3}{3} + \frac{1}{3} \;=\; 6 + 1 + \frac{1}{3} \;=\; 7 + \frac{1}{3} \;=\; 7\frac{1}{3}$$

Practice 47.3: Show that mixed number $9\frac{17}{5}$ is the same as mixed number $12\frac{2}{5}$.

Practice 47.4: Explain why the mixed number $2\frac{0}{9}$ is just the number 2.

Example: Show that the mixed number $4\frac{-2}{3}$ is the same as the mixed number $3\frac{1}{3}$.

Answer: Let's be swift from the get-go this time. Make sure you follow what is happening from one line to the next.

We have

$$4\frac{-2}{3} = 4 + \frac{-2}{3}$$

$$= 3 + 1 + \frac{-2}{3}$$

$$= 3 + \frac{3}{3} + \frac{-2}{3}$$

$$= 3 + \frac{3 + \ -2}{3} \;=\; 3 + \frac{1}{3} \;=\; 3\frac{1}{3}$$

Practice 47.5 Match each quantity on the left with a quantity on the right.

$\frac{20}{7}$	7
$0\frac{7}{20}$	$7\frac{1}{20}$
$7\frac{0}{10}$	$7\frac{1}{2}$
$8\frac{-1}{2}$	$\frac{7}{20}$
$7\frac{1/2}{10}$	$2\frac{6}{7}$

Each expression on the right is either a positive whole number, a fraction with numerator and denominator each a counting number with numerator smaller than denominator, or a counting number and such a fraction added together.

Society won't accept any of the expressions on the left as a "valid" mixed number, but it will accept each answer on the right as either a whole number, a "proper" fraction, or as a valid mixed number.

Here's the societal definition of a mixed number.

A **mixed number** is an expression of the form $a\frac{b}{c}$ with a, b, and c is each a non-zero counting number and with b smaller than c.

Such an expression is really the number $a + \frac{b}{c}$.

No zeros allowed within a mixed number expression.
No negative integers are allowed either.
And no fractional expressions with numerators larger than their denominators are permitted as well.

Demanding!

Let's practice doing some arithmetic with mixed numbers.

Example: What is $6\frac{2}{7}+3\frac{3}{4}$? Write your answer as a societally acceptable mixed number.

Answer: We must work with

$$6+\frac{2}{7}+3+\frac{3}{4}$$

Since we can conduct a string of summations in any order we like, this is

$$9+\frac{2}{7}+\frac{3}{4} \;=\; 9+\frac{8}{28}+\frac{21}{28} \;=\; 9+\frac{29}{28}$$

But we have a fraction with numerator larger than denominator. We must keep going!

$$9+\frac{29}{28}=9+\frac{28+1}{28}=9+\frac{28}{28}+\frac{1}{28}=10+\frac{1}{28}$$

So,

$$6\frac{2}{7}+3\frac{3}{4}=10\frac{1}{28}$$

(This is an absurd amount of work just to appease societal expectations!)

Practice 47.6: Compute $15\frac{1}{4}+5\frac{4}{5}$ and write your answer as a societally acceptable mixed number.

Example: Write $\frac{16}{3}$ as a societally acceptable mixed number.

Answer: We have

$$\frac{16}{3}=\frac{15+1}{3}=\frac{15}{3}+\frac{1}{3}=5+\frac{1}{3}=5\frac{1}{3}$$

(Make sure you followed each step here.)

Practice 47.7: Write $\frac{400}{99}$ as a societally acceptable mixed number.

Example: Write $7\frac{4}{9}$ as a single fraction.

Answer: We have

$$7+\frac{4}{9} = \frac{7}{1}+\frac{4}{9} = \frac{63}{9}+\frac{4}{9} = \frac{67}{9}$$

(Again, make sure you followed each step here.)

Alternatively …

$$7+\frac{4}{9} = \frac{7+\frac{4}{9}}{1} = \frac{9\times(7+\frac{4}{9})}{9\times 1} = \frac{63+4}{9} = \frac{67}{9}$$

(Does this make sense too?)

Practice 47.8: Write $200\frac{1}{200}$ as a single fraction.

Practice 47.9 What is $5\frac{3}{4}$ quadrupled?

Negative Mixed Numbers

Even though we can make sense of mixed numbers with negative numbers as entries, the textbook definition won't allow them.

Nonetheless, you will see in textbooks quantities such as the following.

$$-5\frac{3}{4}$$

Question: How do you think school students are expected to interpret this expression?

As …

a) $-5+\frac{3}{4}$ b) $5-\frac{3}{4}$ c) $-5-\frac{3}{4}$ d) $-5+\frac{-3}{4}$ e) Some other way

Matters are confusing.

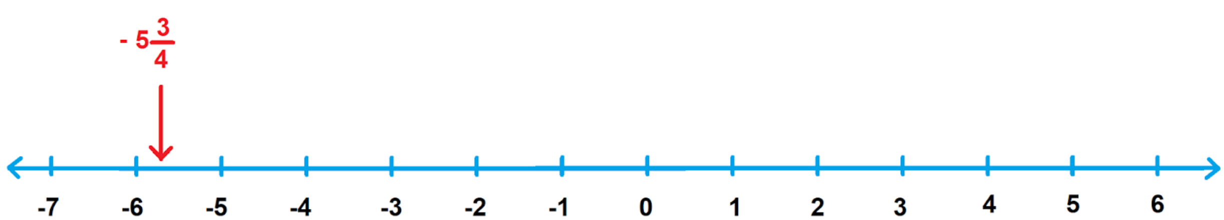
The convention is that, since mixed numbers are written using only positive whole numbers, a negative sign in front of a mixed number applies to the mixed number altogether, not just to part of the mixed number.

$-a\frac{b}{c}$ is taken to mean $-\left(a+\frac{b}{c}\right)$.

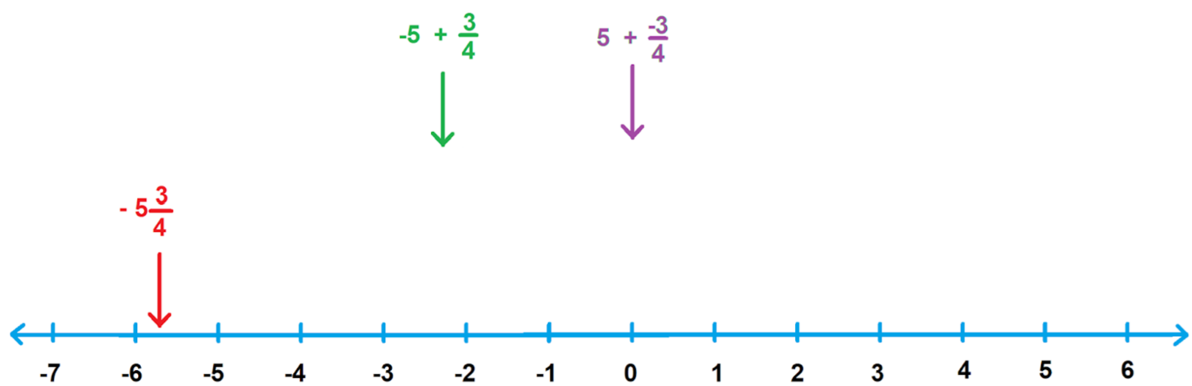
And by distributing the negative sign (Section 21), this means:

$$-a\frac{b}{c} = -a - \frac{b}{c}$$

So, $-5\frac{3}{4}$ means $-5-\frac{3}{4}$ (which happens to be the same as $-5+\frac{-3}{4}$, making options c) and d) both correct in the question above). Consequently, $-5\frac{3}{4}$ is value between -5 and -6 on the number line.



Practice 47.10: Where on the number line is $-5+\frac{3}{4}$? Where is $5+\frac{-3}{4}$? Can you imagine the correct placements of the green and purple arrows?



Subtracting Mixed Numbers

Here's a subtraction problem.

Compute $13\frac{3}{7} - 10\frac{5}{7}$.

Think for a moment about how you might work through this question.

This question represents a good moment **not** to believe in subtraction!
Regard subtraction as the <u>addition</u> of the opposite.

$$13\frac{3}{7} - 10\frac{5}{7} = 13\frac{3}{7} \;+\; -10\frac{5}{7}$$

And what is $-10\frac{5}{7}$? It's $-10 - \frac{5}{7}$.

So

$$13\frac{3}{7} - 10\frac{5}{7} \;=\; 13 + \frac{3}{7} \;-10 - \frac{5}{7}$$

This gives us

$$3 + \frac{-2}{7}$$

(Do you see this?)

Now let's fix up this answer for society. We have

$$3 + \frac{-2}{7} = 2 + \frac{7}{7} + \frac{-2}{7} = 2 + \frac{5}{7}$$

Thus

$$13\frac{3}{7} - 10\frac{5}{7} = 2\frac{5}{7}$$

Practice 47.11 Match time!

$2\frac{2}{5} + 3\frac{4}{5}$	$6\frac{1}{5}$
$10\frac{7}{10} - 4\frac{3}{10}$	$6\frac{2}{5}$
$8\frac{5}{12} - 1\frac{2}{3}$	$6\frac{1}{2}$
$102\frac{2}{5} - 95\frac{9}{10}$	$6\frac{3}{4}$

Of course, we can always sidestep all this mixed number work and convert each number into a single fraction if we like.

Example: Compute $5\frac{1}{3} - 1\frac{1}{2}$ by turning each mixed number into a single fraction first.

(Aside: It is sometimes fun—and good!—to get an estimate of the answer to a calculation first. This question is essentially "$5 - 1$,", which is 4, plus the difference of two fractions. One should obtain an answer close to 4.)

Answer: Writing each mixed number as a fraction gives

$$3\frac{1}{3} = \frac{15}{3} + \frac{1}{3} = \frac{16}{3}$$

and

$$1\frac{1}{2} = \frac{2}{2} + \frac{1}{2} = \frac{3}{2}$$

So, we need to compute $\frac{16}{3} - \frac{3}{2}$. Going for a common denominator, we get

$$\frac{16}{3} - \frac{3}{2} = \frac{32}{6} - \frac{9}{6} = \frac{23}{6}$$

Thus, $5\frac{1}{3} - 1\frac{1}{2} = \frac{23}{6}$.

There is no need to do anything further with this answer (the question had no further instructions), but if you like you can rewrite $\frac{23}{6}$ as the mixed number

$$3\frac{5}{6}$$

(And yes, this is an answer close to 4.)

Practice 47.12 What is $90\frac{1}{6} - 100\frac{2}{3}$. Write your answer as a negative mixed number.

Dividing Mixed Numbers

There is nothing new for us here. We just have to remember that the answer to a division problem is a fraction.

Back in Section 39 when we were playing with pies and students and, while being quirky, asked for the value of

$$2\frac{1}{2} \div 4\frac{1}{2}$$

That is, we were asking for the result of "sharing two-and-a-half pies equally among four-and-a-half students."

The answer is this fraction:

$$\frac{2\frac{1}{2}}{4\frac{1}{2}}$$

It feels compelling to double the numerator and denominator here. That should clear away those annoying halves.

$$\frac{2 \times (2 + \frac{1}{2})}{2 \times (4 + \frac{1}{2})} = \frac{2 \times 2 + 2 \times \frac{1}{2}}{2 \times 4 + 2 \times \frac{1}{2}} = \frac{4 + 1}{8 + 1} = \frac{5}{9}$$

And there we have it

$$2\frac{1}{2} \div 4\frac{1}{2} \quad = \quad \frac{5}{9}$$

Sharing two-and-a-half pies equally among four-and-a-half students is equivalent to sharing five pies among nine students. Each student gets five-ninths of a pie. That's much more manageable to envision.

It was hard to nut our way through this problem when we were working with the "real-world" scenario of sharing pies with students. But mathematics set free from real-world limitations is often much easier to do and is more universal: we can likely apply the result to more than one concrete scenario.

Practice 47.13 Rewrite $\frac{3\frac{1}{3}}{\frac{2}{3}}$ as a much simpler number. That is, compute $3\frac{1}{3} \div \frac{2}{3}$.

Example: Show that $\frac{1\frac{4}{5}}{2\frac{1}{6}}$ is equivalent to $\frac{54}{65}$.

Answer: We have

$$\frac{1\frac{4}{5}}{2\frac{1}{6}} = \frac{1+\frac{4}{5}}{2+\frac{1}{6}}$$

Let's contend with the fifths first by quintupling the numerator and the denominator.

$$\frac{5\times(1+\frac{4}{5})}{5\times(2+\frac{1}{6})} = \frac{5+5\times\frac{4}{5}}{10+5\times\frac{1}{6}} = \frac{5+4}{10+\frac{5}{6}}$$

Let's now multiply top and bottom each by 6.

$$\frac{6\times(5+4)}{6\times(10+\frac{5}{6})} = \frac{30+24}{60+6\times\frac{5}{6}} = \frac{54}{65}$$

And there it is.

Okay. Time for an extra juicy example.

Example: Show that $7\frac{2}{3} \div 5\frac{3}{4}$ is $1\frac{1}{3}$.

Answer: The answer to a division problem is a fraction. So, our goal is to show that

$$\frac{7\frac{2}{3}}{5\frac{3}{4}} = \frac{7+\frac{2}{3}}{5+\frac{3}{4}}$$

is $1\frac{1}{3}$ in disguise.

We have an expression involving thirds and fourths. Multiplying top and bottom of this expression each by 3 and then by 4 would be helpful.

Let's start by tripling top and bottom.

$$\frac{3 \times (7 + \frac{2}{3})}{3 \times (5 + \frac{3}{4})} = \frac{21 + 2}{15 + \frac{9}{4}}$$

Now quadruple top and bottom.

$$\frac{84 + 8}{60 + 9} = \frac{92}{69}$$

So, $7\frac{2}{3} \div 5\frac{3}{4}$ is $\frac{92}{69}$.

This doesn't look like $1\frac{1}{3}$. Hmm.

Well, we can write $\frac{92}{69}$ as a mixed number, at least.

$$\frac{92}{69} = \frac{69 + 23}{69} = 1 + \frac{23}{69}$$

Is $\frac{23}{69}$ just one-third in disguise?

Yes.

$$\frac{23}{69} = \frac{1 \times 23}{3 \times 23} = \frac{1}{3}$$

Great!

Putting it all together …

$$\frac{7\frac{2}{3}}{5\frac{3}{4}} = 1 + \frac{23}{69} = 1 + \frac{1}{3} = 1\frac{1}{3}$$

Phew!

The previous example involved a lot of arithmetic.

But the technique was simply to "follow your mathematical nose" all the way through it.

Practice 47.14 When faced with $\frac{7+\frac{2}{3}}{5+\frac{3}{4}}$, did it cross your mind to multiply top and bottom each by 3×4 right off the bat? If you do that, do you see that you get $\frac{84+8}{60+9} = \frac{92}{69}$ as we had earlier?

Practice 47.15: True or False? All three of these quantities have the same value.

$$\frac{3\frac{1}{3}}{12\frac{1}{2}} \qquad \frac{\frac{4}{3}}{5} \qquad \frac{2}{3} \div \frac{5}{2}$$

MUSINGS

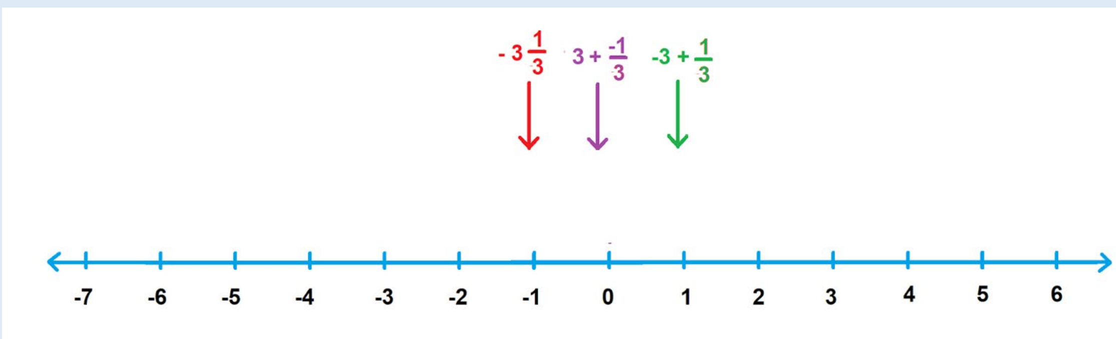
Musing 47.16

a) What number must we add to $2\frac{5}{13}$ to get the value 10?

b) What number must we add to $-2\frac{5}{13}$ to get the value 10?

c) What number must we subtract from -10 to get the value $2\frac{5}{13}$?

Musing 47.17 Place each of these arrows in their correct location on the number line.



Musing 47.18 MULTIPLYING MIXED NUMBERS

Consider $3\frac{1}{3} \times 4\frac{1}{2}$.

Without thinking too deeply, many students say that this product has value $12\frac{1}{6}$.

a) Do you see why this is a tempting answer to give?

We can check this answer by converting each mixed number into single fractions and then multiplying the fractions.

We have

$$3\frac{1}{3} = \frac{9}{3} + \frac{1}{3} = \frac{10}{3}$$

$$4\frac{1}{2} = \frac{8}{2} + \frac{1}{2} = \frac{9}{2}$$

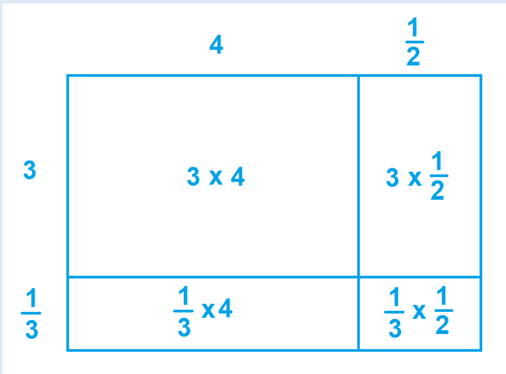
and so

$$3\frac{1}{3} \times 4\frac{1}{2} = \frac{10}{3} \times \frac{9}{2} = \frac{90}{6} = 15$$

This is not $12\frac{1}{6}$. What went wrong?

b) Let's compute $3\frac{1}{3} \times 4\frac{1}{2}$ via the area model, chopping a rectangle into four pieces.

Show that $12 + \frac{4}{3} + \frac{3}{2} + \frac{1}{6}$ does indeed sum to 15. (This is the correct value of $3\frac{1}{3} \times 4\frac{1}{2}$.)



c) When a student says that $3\frac{1}{3} \times 4\frac{1}{2}$ equals $12\frac{1}{6}$, which pieces of the chopped-up rectangle did the student fail to consider in their calculation?

d) Compute $1\frac{2}{3} \times 2\frac{1}{2}$ by

i) converting each mixed number into a fraction and then multiplying
ii) the area model

Do your answers agree? (They should!)

MECHANICS PRACTICE

Practice 47.19 Write each of these expression as a societally acceptable mixed number

a) $\frac{8}{5}$ b) $\frac{100}{13}$ c) $\frac{200}{199}$ d) $\frac{199\frac{1}{2}}{199}$

Practice 47.20 Write each expression as single fraction.

a) $7\frac{2}{9}$ b) $2\frac{3}{4}+5\frac{2}{7}$ c) $300\frac{299}{300}$ d) $2\frac{3}{4}-5\frac{2}{7}$ e) 873

Practice 47.21

a) What is $9\frac{1}{2}$ doubled?

b) What is $-8\frac{1}{3}$ tripled?

Practice 47.22 Compute as many of these as you have the patience for.

a) $2\frac{2}{5}+\frac{2}{3}$ b) $10\frac{1}{7}-9\frac{3}{7}$ c) $6\frac{3}{4}-5\frac{8}{9}$ d) $199\frac{1}{3}-198\frac{1}{2}$

Practice 47.23 Make each of the following look much friendlier.

a) $\frac{3\frac{1}{2}}{1\frac{1}{2}}$ b) $\frac{4\frac{2}{3}}{5\frac{1}{3}}$ c) $\frac{2\frac{1}{5}}{2\frac{1}{4}}$ d) $\frac{1\frac{4}{7}}{2\frac{3}{10}}$ e) $\frac{3/5}{4/7}$

48. Percentages

Let's start with a little piece of history.

Two thousand years ago, emperor Augustus (25 BCE - 14 CE) was cash-strapped. He needed funds to support the expansion of the Roman Empire.

As expansion was supposedly for the good of the Roman citizenry, he came up with the idea of levying a tax on Roman citizens to pay for this enterprise. He decreed that "one part per hundred" of all monetary transactions that occurred in markets were to be transferred to the Empire.

Consequently, if 100 **aureus** (gold coins) were passed between the hands of two citizens making a sale, one coin was given to the Empire. If they exchanged a thousand coins, ten were given to the Empire.

The Romans also used silver, bronze, and copper coins each worth fractional amounts of gold coins. For example, 25 **denarius** (silver coins) were equivalent to 1 aureus.

> **Question 48.1** If two Roman citizens exchanged 4 aureus, how many denarius were handed to the Empire?

From the Latin term *per centum* meaning "part per hundred" we obtained the word **percentage**.

We are still being charged taxes to this day and tax rates are still presented in terms of parts per one hundred (as are interest rates and store discounts and other transactional quantities).

The Mathematical Meaning of Percentage

Quite simply, a **percentage** is a fraction expressed as a part per hundred. That is, it is a fraction with denominator 100.

But annoyingly, we don't write percentages explicitly as fractions, and use the curious symbol % to mean "and the denominator is 100."

For example,

17% is the fraction $\frac{17}{100}$.

50% is the fraction $\frac{50}{100}$ (and this is the same as $\frac{50\times 1}{50\times 2} = \frac{1}{2}$).

300% is the fraction $\frac{300}{100}$ (and this is the same as the number 3).

In general, N% is just the fraction with numerator N and denominator 100.

$$N\% = \frac{N}{100}$$

Practice 48.2 Explain why 25% is the fraction $\frac{1}{4}$

Example: Explain why $\frac{1}{2}$% is the fraction $\frac{1}{200}$.

Answer: We have that $\frac{1}{2}$% is the fraction

$$\frac{\frac{1}{2}}{100}$$

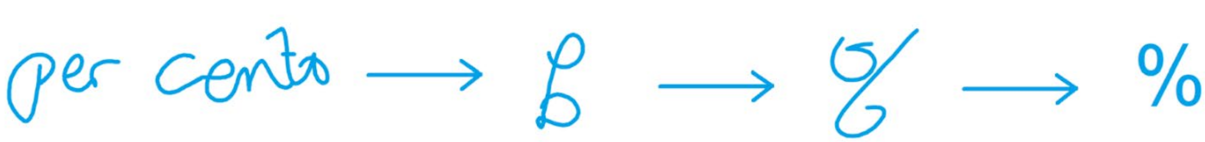
Multiplying top and bottom each by 2 shows that this is the fraction

$$\frac{1}{200}$$

Practice 48.3 What fraction is $\frac{1}{10}$%?

During the Renaissance in Europe, Italy developed robust financial institutions. Accountants and financiers, writing in Italian rather than Latin, wrote *per cento* in their legers for "one part per hundred."

But they started to abbreviate this to phrase to just the letter "p" with a curly tail. (Perhaps the tail represented the "c" or the "o" from the word *cento*.) Eventually this shorthand for *per cento* transmuted into the symbol % we use today.



Practice 48.4 The term *per mille* is denoted ‰ and is sometimes used today. Make a guess as to what this term means. (Hint: How many years are in a millennium?)

To convert a fraction into a percentage we need to rewrite the fraction to have a denominator of 100.

For example, $\frac{1}{5}$ can be rewritten as $\frac{1\times 20}{5\times 20}=\frac{20}{100}$ and so

$$\frac{1}{5}=20\%$$

And, $\frac{3}{2}$ can be rewritten as $\frac{3\times 50}{2\times 50}=\frac{150}{100}$ and so

$$\frac{3}{2}=150\%$$

Practice 48.5 What is $\frac{7}{10}$ as a percentage?

Of course, most fractions are not this friendly to work with.

Let's rewrite $\frac{15}{32}$ as a percentage. (I am not sure why we would want to in real life! We're just doing it now for practice.)

We can get a denominator <u>close</u> to 100 by tripling the top and bottom.

$$\frac{15\times 3}{32\times 3}=\frac{45}{96}$$

and so, as a percentage, the fraction is close to 45%. (Will the exact value be bigger or smaller than this?)

Let's find the exact percentage.

We want to convert the fraction $\frac{15}{32}$ to one with a denominator of 100. But that denominator of 32 is annoying! I wish we had a nicer denominator.

Well, we can always create a denominator of 1.

$$\frac{15}{32} = \frac{\frac{15}{32}}{1}$$

And to make a denominator of 100, let's multiply top and bottom by 100.

$$\frac{15}{32} = \frac{\frac{15}{32} \times 100}{1 \times 100} = \frac{\frac{1500}{32}}{100}$$

So,

$$\frac{15}{32} = \frac{1500}{32}\%$$

My calculator says that $\frac{1500}{32}$ is a value between 46 and 47. So ,yes, this is close-ish to 45%.

Practice 48.5 Write $\frac{1500}{32}$ as a mixed number. Show that $\frac{15}{32} = 46\frac{7}{8}\%$.

Practice 48.6 Many schoolbooks tell students a rule.

> *To convert a fraction $\frac{a}{b}$ into a percentage, multiply the fraction by 100 and slap on a percentage sign.*

Did we, in effect, do just that when we converted $\frac{15}{32}$ into a percentage?

My advice is to **not** memorize a rule like this. Memorizing items that feel random is just too hard!

Instead, simply understand that a percentage is just a fraction with denominator 100. And your task is thus to do whatever you can to create a denominator of 100! (You always have the option to start by creating a denominator of 1.)

Practice 48.7.6 What is $\frac{5}{6}$ as a percentage? Write your final answer as a mixed number.

Example: What fraction is $2\frac{1}{2}\%$?

Answer: This percentage is the fraction $\frac{2+\frac{1}{2}}{100}$. Doubling top and bottom we see that this is

$$\frac{4+1}{200} = \frac{5}{200} = \frac{5 \times 1}{5 \times 40} = \frac{1}{40}$$

This next question feels strange.

What is the number 1 as a percentage?

To answer this we need to write 1 as a fraction with denominator 100. Writing

$$1 = \frac{100}{100}$$

does the trick.

We see that

$$1 = 100\%$$

People say, “If you’ve got the whole of something, then you’ve got 100% of it,” and the number 1 represents “the whole” they have in mind in this statement.

Practice 48.8 What is the number 30 as a percentage?

a) 30% a) 300% a) 3,000% a) 30,000%

Example: What is $71\frac{3}{7}\%$ as a fraction?

Answer: It's

$$\frac{71\frac{3}{7}}{100}$$

To make this fraction look friendlier, let's multiply top and bottom each by 7. This gives

$$\frac{490+7+3}{700}=\frac{500}{700}=\frac{5}{7}$$

It's the fraction five sevenths!

Practice 48.9 Match each quantity on the left with its matching quantity on the right.

$\frac{13}{2}$	$\frac{1}{100}$
1000%	$\frac{3}{20}$
$\frac{4}{5}$	$116\frac{1}{3}\%$
1%	10
15%	80%
$\frac{7}{6}$	650%

MUSINGS

Musing 48.10 Speaking of the Romans …

The Romans of ancient times used words to describe fractions.

A twelfth of a unit of weight, $\frac{1}{12}$, for instance, was called the **uncia** (from which we obtained the word "ounce") and $\frac{11}{12}$ was called **deunx**, short for *de uncia* meaning "one twelfth taken away."

Two twelfths, $\frac{1}{6}$, was called **sextans**, and three twelfths, $\frac{1}{4}$, **quadrans**.

a) Makes some guesses: Which fraction is **dextans**? Which fraction is **dodrans**?

b) Make some more guesses: Which fraction is **quinque unciae**? Which is **septem unciae**?

Six twelfths, $\frac{1}{2}$, was called **semis.**

c) Make a guess: Which fraction was **semuncia**?

MECHANICS PRACTICE

Musing 48.10 Recall that the multiplication of fractions matches society's use of the word "of."

For example, "two-thirds of 90" can legitimately be calculated as

$$\frac{2}{3} \times 90$$

which equals $2 \times \frac{1}{3} \times 90 = 2 \times \frac{90}{3} = 2 \times 30 = 60$.

In the same way, computing "105% of 400" requires computing the product

$$\frac{105}{100} \times 400$$

It gives the answer 420.

a) What is 15% of 80?
b) What is 80% of 15?

c) What is 75% of 50?
d) What is 50% of 75?

e) What is 25% of 300?
f) What is 300% of 25?

Can you explain why "a% of b" and "b% of a" are sure to have the same value?

49. Tips, Percentage Increase, and Percentage Decrease

The Mechanics Practice problem of the previous page reminded us that the multiplication of fractions matches society's use of the word "of."

Example: What is $17\frac{1}{2}\%$ of 160?

To answer this, we need to compute

$$\frac{17\frac{1}{2}}{100} \times 160$$

which, to be frank, doesn't seem fun.

Practice 49.1 Try computing $17\frac{1}{2}\%$ of 160 before reading on.
(Or feel free to say "NO. I WON'T!")

Problems of this nature arise often in everyday life, most notably, when computing tips on services provided. And to handle such computation, people typically **don't** pull out pencil-and-paper and multiply awkward fractions.

Instead, they use the "anchor point' of 10% of the bill to help them do a mental calculation.

Example: You just had a very fancy dinner and a bill of \$160 has come your way. You want to add a 15% tip. How much is such a tip?

The tip is "15% of 160," that is, it's

$$\frac{15}{100} \times 160$$

This is tricky to work out in your head.

But working out "10% of 160" instead is fairly easy. It's $\frac{10}{100} \times 160$ and that's $\frac{1}{10} \times 160 = 16$.

10% of a quantity is that quantity divided by ten.

We have that 10% of \$160 is \$16.

If we halve this, we then deduce that 5% of $\$160$ is $\$8$.

10% ⟷ $16

5% ⟷ $8

Our 15% must thus be $16 + 8 = 24$ dollars.

Example: You are feeling generous and decide to instead leave a 20% tip instead on your $\$160$ bill. What's your new tip?

Let's just double our anchor point of 10%.

20% ⟷ $32

10% ⟷ $16

5% ⟷ $8

Your tip is now $32.

Example: You've changed your mind: a tip of 20% feels a bit too much and a tip of 15% too small. You decide to leave a tip of $17\frac{1}{2}\%$.

What is $17\frac{1}{2}\%$ of $\$160$?

Let's keep halving from our 10% anchor point.

20% ⟷ $32

10% ⟷ $16

5% ⟷ $8

$2\frac{1}{2}\%$ ⟷ $4

We can build the number $17\frac{1}{2}$ as $10 + 5 + 2\frac{1}{2}$.

This means that our $17\frac{1}{2}$% tip on \$160 must be $16 + 8 + 4 = 28$ dollars!

Practice 49.2 What $47\frac{1}{2}$% of \$160?

Keep in mind an "anchor point" of 10% of a bill
and use it to compute the tip amount for all standard percentages.

Practice 49.3 What is 18% of \$310? (Can you reason out an answer?)

Let's now consider another confusing matter, namely, statements like these you could read in a news article.

The average price of a home in the metropolitan area has increased 300% over the past four years.

What does this mean?

Let's say that the average home in the area cost \$200,000 four years ago.

"Three hundred percent" represents the fraction $\frac{300}{100}$. Consequently "300% of \$200,000" means

$$\frac{300}{100} \times 200{,}000 = 600{,}000$$

It's the figure, tripled.

But this is just the price increase!

So, the new price of the house is $\$200{,}000 + \$600{,}000 = \$800{,}000$.

The actual home price has quadrupled!

The language of “percentage increase” and “percentage decrease” is always horribly confusing.

But it is helpful to realize the percentage referred to is always (or should always be) **based on the initial figure before it changes.**

For instance, consider this problem.

> **Example:** A sofa normally costs \$600, but it is now on sale for \$420.
> What percentage decrease in cost is that?

Let’s work through this slowly.

We start by making sense of the term “percentage decrease.”

This term sounds like it wants the decrease in terms of a percentage. Great start!

What is the decrease?

Let’s just answer that in the obvious way. The price went from \$600 down to \$420. That's a decrease of \$180. But we want to express \$180 as a percentage.

A percentage of what?

We have two choices: of \$600 or of \$420.

A percentage increase or decrease should always refer to the initial figure.
So, we want \$180 as a percentage of \$600.

Now, a percentage is just a fraction.

What fraction is \$180 of \$600?

It’s $\frac{180}{600}$.

What’s $\frac{180}{600}$ written as a percentage?

It’s $\frac{180}{600} = \frac{\frac{1}{6}\times 180}{\frac{1}{6}\times 600} = \frac{30}{100} = 30\%$

A change of price from \$600 to \$420 represents a 30% decrease in price.

Example Continued: Even though you have a 30% discount on your \$600 sofa, you still have to pay a sales tax of 15%.

What will be your total payment for the sofa?

Answer: The sales tax is 15% of \$420.

Now,

10% of \$420 is \$42

and

5% of \$420 is half of this, \$21.

So, you need to pay $42 + 21 = 63$ dollars in sales tax.

The total cost of your sofa is \$483.

Example Still Continued: The shop clerk decides to apply 15% sales tax to the original price of \$600 first, and then reduce the total amount you owe by 30%.

Should you complain?

Answer: We have that 15% of \$600 is $60 + 30 = 90$ dollars.

Now, a 30% discount to this total amount, $600 + 90 = 690$ dollars, corresponds to a discount of

$$\frac{30}{100} \times 690 = \frac{30 \times 690}{100} = 3 \times 69 = 207$$

dollars.

Your final bill from the clerk is $690 - 207 = 483$ dollars again!

Her approach made no difference!

This is surprising! Adding 15% of value to a quantity and then reducing the total amount by 30% has the same effect as reducing the quantity by 30% first and then increasing that result by 15%.

Hmm!

Language really matters in percentage increase or percentage decrease statements.

Practice 49.4: Consider these two statements.

a) I got a raise! My salary has increased by 120%.
b) I got a raise! My salary has increased to 120% of what it was.

Which of these situations would you prefer to be in?

Let's end this Section with one more typical (and a bit annoying) textbook example.

Practice 49.5 A town's population has changed from 12,500 to 14,000 over the past three months.

a) What percentage increase is that?
b) What is the population of the town now as a percentage of its previous population figure?

The mathematics of percentage problems, in and of itself, isn't really the challenge. It's figuring out what mathematics one is expected to do that is the tricky part.

In the end, you really just have to follow your common sense of how people use language and go from there.

MUSINGS

Musing 49.6

a) Explain why applying a 25% discount to a price is the same as multiplying that price by $\frac{3}{4}$.

b) Explain why applying a 20% increase to a price is the same as multiplying that price by $\frac{6}{5}$.

Musing 49.7

a) Show that increasing 240 by 20% and then decreasing the result by 25% yields the same value as decreasing 240 by 25% first and then increasing the result by 20%.

b) **OPTIONAL CHALLENGE**: Can you explain why, in general, increasing a number N by $a\%$ and then decreasing the result by $b\%$ yields the same value as decreasing N by $b\%$ and then increasing the result by $a\%$?

MECHANICS PRACTICE

Practice 49.8 What is $7\frac{1}{2}\%$ of 480?

Practice 49.9 What's a $22\frac{1}{2}\%$ tip on \$20?

Practice 49.10

a) What is a $32\frac{3}{5}\%$ tip on \$100?

b) What is a $32\frac{3}{5}\%$ tip on \$200?

Practice 49.11 What is 70% of 12,000?

Practice 49.12 Consider this sad statement.

My salary has been reduced by 100%.

a) What is your salary now?

b) What if the statement was, instead “My salary has been reduced to 100% of what it was before”? Would you still panic?

50. Comparing Fractions

To get started, let's make sure we understand the mathematics of comparing numbers in general.

Practice 50.1 Which of these numbers is the smallest in value?

a) 917 b) 719 c) 197 d) 791 e) 179

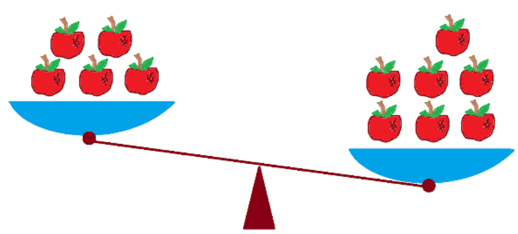
We say that a counting number a is **smaller than** or **less than** a counting number b, written $a < b$, if we can add a counting number to a to get the value b. (The symbol $<$ is called an **inequality sign**.)

$$a < b \quad \text{if } a + n = b \text{ for some (non-zero) counting number } n.$$

This is motivated by real-world thinking. For example, "5 is less than 7" because a set of five apples is less in count than a set of seven apples since we need to add two more apples to pile of 5 apples to make a pile of 7 apples.

Or, if we assume all the apples are identical, we notice that 5 apples together weigh less than 7 apples.

People often draw balance scales – well, unbalanced scales! – to visualize an inequality.



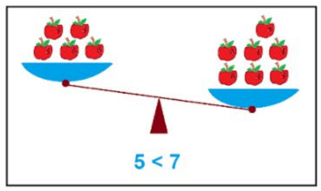
5 < 7

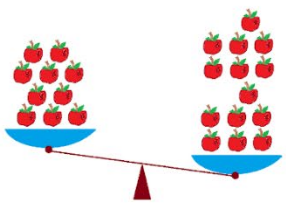
Something Cute: Imagine an equality sign with its lower horizontal bar titled down in the direction of the balance scale. That creates the inequality sign.

An **inequality** is any statement about one number being larger in value than another.

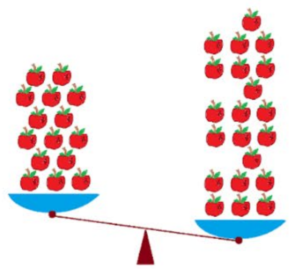
Each inequality can be written two ways. For instance, the statement $5 < 7$ can also be written $7 > 5$.

If we have a statement about inequality, then we like to believe we can double, triple, or halve all the numbers mentioned and not affect the inequality. This is motivated by real-world experience: double a light weight will still be less in weight than double a heavier weight, and so on.

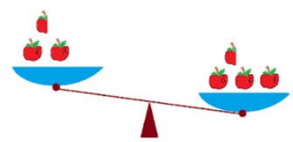




$5 \times 2 < 7 \times 2$



$5 \times 3 < 7 \times 3$



$5 \times \frac{1}{2} < 7 \times \frac{1}{2}$

This suggests a fundamental rule of inequalities: scaling two quantities by the same (positive) factor does not change the inequality.

If

$$a < b$$

then

$$k \times a \quad < \quad k \times b$$

for any positive number k.

And it is this fundamental belief about inequality that allows us to compare the sizes of fractions.

For example, consider the true statement $5 < 7$. Scale each number by an eleventh. We get

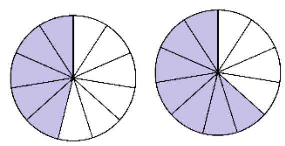
$$\frac{1}{11} \times 5 < \frac{1}{11} \times 7$$

which tells us that

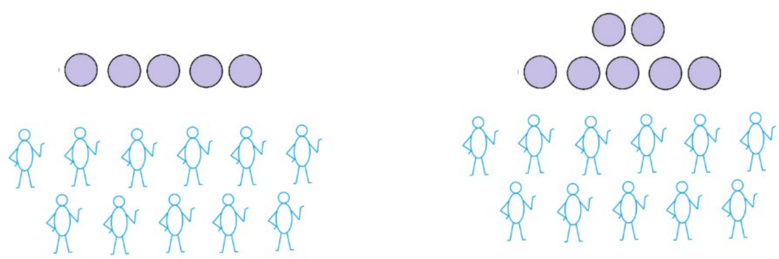
$$\frac{5}{11} < \frac{7}{11}$$

Practice 50.2

a) Does this fit with schoolbook intuition? Is five-elevenths of a pie less than seven elevenths of a pie?



b) Does this also fit with pies-per-student thinking? Which gives more pie per student: sharing 5 pies equally among 11 students or sharing 7 pies equally among 11 students?



In general, if $a < b$, then multiplying each term by $\frac{1}{n}$ gives

$$\frac{a}{n} < \frac{b}{n}$$

We have the schoolbook rule:

If two fractions have the same denominator, then the fraction with the smaller numerator is the smaller fraction.

Did you learn this rule?

This rule makes intuitive sense with our various "real-world" models of fractions. But it is good to see that it has a logical mathematical basis too.

Let's go back to the statement $5 < 7$ and this time let's scale each number by $\frac{1}{5}$ and then by $\frac{1}{7}$. This gives:

$$\frac{1}{7} \times \frac{1}{5} \times 5 \quad < \quad \frac{1}{7} \times \frac{1}{5} \times 7$$

Using that $\frac{1}{5} \times 5 = 1$ and $\frac{1}{7} \times 7 = 1$, this reads

$$\frac{1}{7} < \frac{1}{5}$$

So, one seventh is a smaller fraction than one fifth.

Practice 50.3 Show that one-thirteenth is smaller than one tenth. (Start with $10 < 13$ and scale the numbers in this inequality.)

Going back to $\frac{1}{7} < \frac{1}{5}$, we can now multiply through by 61, say, to obtain

$$\frac{61}{7} < \frac{61}{5}$$

(sixty-one sevenths is smaller than sixty-one fifths).

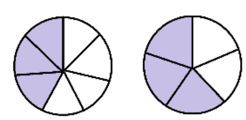
Or we can multiply instead through by 3 to obtain

$$\frac{3}{7} < \frac{3}{5}$$

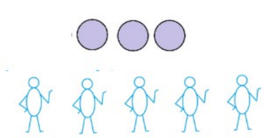
(three sevenths is less than three fifths).

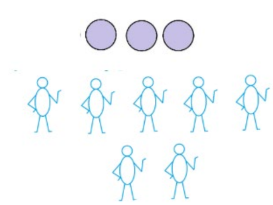
Practice 50.4

a) Does this too fit with schoolbook intuition? Is three-sevenths of a pie less pie than three-fifths of one?



b) Does this also fit with pies-per-student thinking? Which gives more pie per student: sharing 3 pies equally among 7 students or sharing 3 pies equally among 5 students?





In general, we have that $\frac{n}{b} < \frac{n}{a}$ if $b > a$.

This is the schoolbook rule:

If two fractions have the same numerator, then the fraction with the bigger denominator is the smaller fraction.

Is this something also familiar to you as a rule about fractions?

Practice 50.5 Arrange these three fractions in order from smallest to biggest: $\frac{7}{11}, \frac{9}{7}, \frac{7}{12}, \frac{9}{11}$.

Here's a more challenging question.

Which of these two fractions is the larger?

$$\frac{5}{9} \qquad \frac{6}{11}$$

The trouble—or delight—of this question is that the two fractions don't have the same numerator, nor do they have the same denominator. They are hard to compare!

One approach is to rewrite the fractions to have the same … err …. which?

Let's go for the same numerator. Why not?

We currently have numerators of 5 and 6. This suggests going for a common numerator of $5 \times 6 = 30$.

We have

$$\frac{5}{9} = \frac{6 \times 5}{6 \times 9} = \frac{30}{54}$$

and

$$\frac{6}{11} = \frac{5 \times 6}{5 \times 11} = \frac{30}{55}$$

We see that $\frac{5}{9} = \frac{30}{54}$ is the larger of the two fractions.

Practice 50.6 Rewrite $\frac{5}{9}$ and $\frac{6}{11}$ with a common denominator of 99. Deduce again that $\frac{5}{9}$ is the larger of the two fractions.

Practice 50.7 Arrange the fractions $\frac{5}{9}$, $\frac{6}{11}$, and $\frac{15}{28}$ from smaller to largest.

Young students “dream” of being able to add fractions just by adding together their numerators and their denominators.

For example, they would love $\frac{4}{5} + \frac{7}{10}$ to be $\frac{11}{15}$. But unfortunately, the addition of fractions does not work this way.

Practice 50.8 What is correct value of $\frac{4}{5} + \frac{7}{10}$?

Practice 50.9 How does the dream answer of $\frac{11}{15}$ compare to the two original fractions $\frac{4}{5}$ and $\frac{7}{10}$? Please arrange the three fraction is order from smallest to largest.

Practice 50.10 Adding $\frac{1}{2}$ and $\frac{1}{3}$ incorrectly gives $\frac{2}{5}$. Arrange these three fractions from smallest to largest.

If you are game, try to show why the incorrect addition of two fractions is sure to always produce a result that sits between the two original fractions.

$$\boxed{\frac{a}{b}} < \boxed{\frac{a+c}{b+d}} < \boxed{\frac{c}{d}}$$

MUSINGS

Musing 50.11 Recall that saying $a < b$ for two numbers a and b means that we can find a positive number n that makes $a + n$ equal to b. ("We need to add more to a to get b.")

$$a + n = b$$

Multiplying everything in this equation by two tells us that $2a + 2n = 2b$.
This is saying that we need to "add more to $2a$ to get $2b$." We conclude that $2a < 2b$.

a) From $a < b$, explain why we also have $3a < 3b$.
b) From $a < b$, explain why we also have $50a < 50b$.
c) From $a < b$, explain why we also have $k \times a < k \times b$, for any positive number k.

This means we don't have to rely on "real-world experience" to justify our rule for inequality. It too can be justified purely by mathematics.

Musing 50.12
a) Explain why a fraction $\frac{a}{b}$, with a and b both positive integers, is bigger than 1 if $a > b$.
b) Explain why a fraction $\frac{a}{b}$, with a and b both positive integers, is smaller than 1 if $a < b$.

Musing 50.13 *Throughout this question N represents a positive number.*

People say that multiplying a (positive) quantity by a number bigger than 1 gives an answer larger than the original quantity..

a) Show that $\frac{5}{4} \times 100$ is larger than 100.

b) Show that $\frac{5}{4} \times N$ is larger than N.

We can prove the general claim as follows:

Suppose k is a number bigger than 1.
Then we have $k > 1$.

Scale both sides of the inequality by N. This gives

$$k \times N > 1 \times N$$

which reads $k \times N > N$.

People also say that multiplying a (positive) quantity by a number smaller than 1 (but still positive) gives an answer smaller than the original quantity.

c) Show that $\frac{4}{5} \times 100$ is smaller than 100.

d) Show that $\frac{4}{5} \times N$ is smaller than N.

e) Suppose k is a positive number and $k < 1$. Explain what $k \times N$ is sure to be smaller than N.

What about dividing by quantities that are bigger or smaller than 1?

e) Show that $\frac{100}{5/4}$ is smaller than 100.

f) Show that $\frac{N}{5/4}$ is smaller than N.

g) Show that $\frac{100}{4/5}$ is bigger than 100.

h) Show that $\frac{N}{4/5}$ is bigger than N.

i) If you are game, try to prove that $\frac{N}{k}$ is sure to be smaller than N if $k > 1$, and bigger than N if $k < 1$ (and is still positive).

MECHANICS PRACTICE

Practice 50.14

a) Which is bigger: $\frac{2}{3}$ or $\frac{3}{4}$?

b) Which is bigger: $\frac{9}{11}$ or $\frac{11}{12}$?

c) Which is bigger: $\frac{15}{22}$ or $\frac{16}{23}$?

d) Which is bigger: $\frac{40}{9}$ or $\frac{41}{10}$?

51. Fractions and Rational Numbers: Some Schoolbook Fussiness

We've developed a system of arithmetic that is based on the **counting numbers** $0, 1, 2, 3, 4, \ldots$ and extended it to include numbers denoted "$-a$" and numbers denoted "$\frac{a}{b}$" with a and b other numbers within our system (b not zero).

People call each (non-zero) counting number, $1, 2, 3, 4, \ldots$, a **positive integer** and the opposite of each of these, $-1, -2, -3, -4, \ldots$, a **negative integer**. (Recall that zero is neither positive nor negative.)

We've also described fractions as positive and negative: $\frac{2}{3}$ is a positive fraction and $-\frac{2}{3}$ is a negative fraction, for instance. (And zero, thought of as a fraction, is neither positive or negative.) Speaking this way feels very natural.

But some schoolbooks are fussy. They like to use the term "fraction" only for what we've been thinking of as positive fractions and give a different name for the class of fractions as a whole.

> **(Some) Schoolbooks:**
> A **fraction** is a number that can be written in the form $\frac{a}{b}$ with a and b each a <u>positive</u> integer.
> A **rational number** is any number that can be written in the form $\frac{a}{b}$ with a and b each an integer, but necessarily positive. (We just need b to be non-zero).
>
> Every fraction is also a rational number.

For example, in schoolbook world:

$\frac{2}{3}$ is a fraction. It's a positive integer over a positive integer.

$-\frac{2}{3} = \frac{-2}{3}$ is not a fraction, but it is a rational number.

$\frac{-2}{-3}$ is equivalent to $- -\frac{2}{3} = \frac{2}{3}$, and so can be written as a positive integer over a positive integer. It is a fraction.

$\frac{7}{1/2}$ can be rewritten as $\frac{2\times 7}{2\times\frac{1}{2}} = \frac{14}{1}$, a positive integer over a positive integer. It is a fraction.

$\frac{1-\frac{4}{3}}{1+\frac{3}{5}}$ is equivalent to $\frac{15-20}{15+9} = \frac{-5}{24}$ is a rational number, but not a fraction.

Practice 51.1 According to schoolbook fussiness ...

a) Explain why 3 is allowed to be called a fraction.

b) Explain why 0 is a rational number but is not allowed to be called a fraction.

c) Explain why -3 is a rational number but is not allowed to be called a fraction

Mathematicians might think this schoolbook fussiness is too fussy to bother with, but they will agree to call any quantity that is equivalent to a number of the form $\frac{a}{b}$ with a and b each an integer (positive, negative, or zero—except b cannot be zero) a **rational number**.

Every integer is a rational number, because we learned that an integer a can be written as $\frac{a}{1}$.

The set of all rational numbers is denoted $\mathbb{Q}$ for "quotients."
We have:

$$\mathbb{N} = \{0, 1, 2, 3, 4, \ldots .\} \text{ or } \{1, 2, 3, 4, \ldots .\}$$

$$\mathbb{Z} = \{\ldots . -3, -2, -1, 0, 1, 2, 3, \ldots\}$$

$$\mathbb{Q} = \text{all the fractional quantities (which includes all the integers)}$$

Rational numbers can come in crazy guises. For example, can you see that this number

$$\frac{2 + \dfrac{1}{1 - \dfrac{1}{5}}}{1 + \dfrac{1}{3} - \dfrac{1 + \dfrac{1}{3}}{1 - \dfrac{6}{5}}}$$

is really $\frac{13}{32}$ in disguise and so is a rational number (a "positive fraction").

Aside: Mathematicians will call a fraction **positive** if it can be written in the form $\frac{a}{b}$ with a and b both positive integers, **negative** if it can be written instead in the form $-\frac{a}{b}$. (The number 0 is still neither positive nor negative!)

Some schoolbooks won't say "negative fraction."

Getting ahead of ourselves with decimals and square roots …

The quantity $\frac{1.2}{3}$ is a (positive) fraction. It is equivalent to $\frac{10\times1.2}{10\times3} = \frac{12}{30}$.

The quantity $\frac{2\sqrt{5}}{3\sqrt{5}}$ is also a fraction. It is equivalent to $\frac{2}{3}$.

So, even if the numerator and denominator of a fractional quantity are not themselves integers, it is possible that the quantity is still a fraction in disguise.

Example: Explain why the product of two rational numbers is sure to again be a rational number.

Answer: If the two numbers are rational, then they can be written as $\frac{a}{b}$ and $\frac{c}{d}$ for some integers a, b, c, and d (with b and d not zero).
|
Their product is

$$\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d}$$

which is "an integer over an integer," and so is a also rational number.

Right now, every number in our mathematical universe is a fraction/rational number.

It was a real shocker when some 2500 years ago scholars realized that not all numbers that arise in the "real world" are fractions!

We'll realize that too in the next chapter.

MUSINGS

Musing 51.2
a) Explain why the sum of two fractions is sure to be a fraction.
b) Explain why if we divide two fractions the answer is sure to be another fraction.

52. The Mathematical Truth about Fractions: Rule 10

We've been following the human progress of developing mathematics: using concrete, real-world scenarios to guide us as to what numbers exist (the counting numbers and zero, integers, and fractions) and how they should behave, and then pulling away from the concrete to let the logic of mathematics propel us to beyond that real-world thinking.

For example:

> *It is cumbersome (perhaps impossible) to give a convincing "real world" explanation of why negative times negative is positive, yet the logic of mathematics tells us this must be so.*

> *It makes no sense to multiply together portions of pie ("What's half a pie times a third of a pie?"), yet the logic of mathematics tells us how to multiply two fractions.*

Nonetheless, mathematics so often harks back to real-world contexts:

> *The product of two fractions matches the use of the word "of" in society's understanding of a fraction of a fraction.*

But in the end, the driving force behind all of our mathematics is mathematics itself, not the real-world examples that suggested it. (School mathematics tries to force everything to be "real world" justified and thus often gets muddled.)

We've made significant strides following the sheer logical power of math. And we can go a step further with this within this story of fractions.

As mentioned in Section 43, all our fraction work logically follows just by adding <u>one</u> additional rule to our list of general rules of arithmetic. This is the "Rule 10" we mentioned there.

So, let's now properly introduce Rule 10.

We've created universe of numbers that contains (and is in fact based on) the **counting numbers**:

$$0, 1, 2, 3, \ldots.$$

And within this universe there are two number operations—**addition** and **multiplication**. These two operations behave exactly as we expect them to when applied just to the counting numbers. But these operations go further and extend to all the numbers in our universe.

The two operations follow these nine rules of arithmetic.

Rule 1: For any two numbers a and b we have $a + b = b + a$.

Rule 2: For any number a we have $a + 0 = a$ and $0 + a = a$.

Rule 3: In a string of additions, it does not matter in which order one conducts individual additions.

Rule 4: For any two numbers a and b we have $a \times b = b \times a$

Rule 5: For any number a we have $a \times 1 = a$ and $1 \times a = a$.

Rule 6: In a string of multiplications, it does not matter in which order one conducts individual multiplications.

Rule 7: For any number a we have $a \times 0 = 0$ and $0 \times a = 0$.

Rule 8: "We can chop up rectangles from multiplication and add up the pieces."

Rule 9: For each number a, there is one other number "$-a$" such that $a + -a = 0$.

Rule 9 created the "opposite numbers" for us and shows how they behave with respect to addition.

In section 43 we saw it behooves us to create another kind of "opposite number," but opposite in a multiplicative sense this time.

So, let's add to our list Rule 10 that creates for us the basic fractions: $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ and shows how they behave with respect to multiplication.

> **Rule 10:** For each number a different from zero, there is one other number "$\frac{1}{a}$" such that $a \times \frac{1}{a} = 1$.

Question: Did we also just create $\frac{1}{1}$?

This rule tells us that there is a number called $\frac{1}{2}$ with the property that $2 \times \frac{1}{2} = 1$, and a number called $\frac{1}{3}$ with the property that $3 \times \frac{1}{3} = 1$, and so on.

Another way of stating Rule 10 is to say that each non-zero number a, there is a number that fills in the blank for this multiplication statement. We call that number $\frac{1}{a}$.

$$a \times \blacksquare = 1$$

For those who like the fancy language the quantity $\frac{1}{a}$ is called the **multiplicative inverse** of a.

(We called $-a$ the **additive inverse** of a).

Why a can't be zero in Rule 10?

Rule 10 steers clear of trying to create $\frac{1}{0}$, a quantity with denominator zero. The reason for this is that we would create a contradiction with Rule 7 which states that any quantity times zero is zero.

To explain …

If $\frac{1}{0}$ were to exist, then Rule 7 says that

$$0 \times \frac{1}{0} = 0$$

But Rule 10 says that

$$0 \times \frac{1}{0} = 1$$

The existence of $\frac{1}{0}$ would force us to conclude that zero equals one.

$$0 = 1$$

And then adding one to these numbers forces us to also conclude that one equals two.

$$1 = 2$$

And then adding one again has us conclude that two equals three.

$$2 = 3$$

And so on. All numbers would be equal, and mathematics would collapse to a universe of everything being the same!

To get the full picture of fractions, beyond just the basic fractions, we need an addendum to Rule 10. It takes the intuitive idea of being able to "pull fractions apart" and makes it fundamental.

> **Addendum:** We take an expression of the form $\frac{a}{b}$ to mean $a \times \frac{1}{b}$ (the basic fraction $\frac{1}{b}$ multiplied by a).

$$\frac{a}{b} = a \times \frac{1}{b}$$

So, for example, $\frac{2}{3}$ is understood to be shorthand for $2 \times \frac{1}{3}$, and $\frac{-7}{5}$ as shorthand for $(-7) \times \frac{1}{5}$.

This establishes for us the basic mathematical function we want fractions to have, namely, being answers to division problems.

$$\frac{a}{b} = a \div b$$

Here's how:

The mathematical definition of division is multiplication in reverse.

Specifically, for two numbers a and b, with b not zero, the number $a \div b$ is defined as the value that fills in the blank to the statement

$$b \times \blacksquare = a$$

And we can see that $\frac{a}{b} = a \times \frac{1}{b}$ fills in the blank. Using that $b \times \frac{1}{b} = 1$, as per Rule 10, we get

$$b \times \frac{a}{b} = b \times a \times \frac{1}{b} = a \times 1 = a$$

So, yes, $\frac{a}{b}$ is the answer to $a \div b$.

Since mathematicians tend to avoid using the division symbol ÷, let's rephrase what we have just established solely in terms of reverse multiplication.

We have established:

For two numbers a and b, with b not zero, $\frac{a}{b}$ is the number that fills in the blank to

$$b \times \blacksquare = a$$

That is, we have:

$$\cancel{b} \times \frac{a}{\cancel{b}} = a$$

In particular, $\frac{1}{1}$ is the number that fills the blank to

$$1 \times \blacksquare = 1$$

But 1 also fits that blank!

So $\frac{1}{1}$ must be 1.

And $\frac{1}{2\times 3}$ is the number that fills in the blank to

$$2 \times 3 \times \blacksquare = 1$$

But $\frac{1}{2} \times \frac{1}{3}$ also fills in this blank. So

$$\frac{1}{2} \times \frac{1}{3} = \frac{1}{2 \times 3}$$

All the properties and all the play of fractions we brought in over the past two chapters are starting to emerge from this Rule 10 and its addendum.

Let's keep going and show that <u>every</u> fraction property we highlighted these past two chapters follows from Rule 10 and its addendum.

Practice 52.1 To get a feel for how this will proceed, perhaps try each of these problems on your own before reading on.

a) According to the addendum, what is $\frac{20}{1}$? Explain why $\frac{20}{1}$ must be 20.

b) According to the addendum, what is $\frac{20}{20}$? Explain why $\frac{20}{20}$ must be 1.

c) What is the value of $20 \times \frac{1}{4} \times \frac{1}{5}$? Explain why $\frac{1}{4} \times \frac{1}{5}$ must be $\frac{1}{20}$.

d) Explain why $2 \times \frac{10}{3}$ and $\frac{20}{3}$ must have the same value. (What does the addendum have to say about each of these quantities?)

e) Explain why $\frac{12}{20}$ and $\frac{3}{5}$ must have the same value.

The Magic of Rule 10 and its Addendum

Example: For every number a we have $a = \frac{a}{1}$.

$$a = \frac{a}{1}$$

Reason:

The addendum says that $\frac{a}{1}$ is just shorthand for $a \times \frac{1}{1}$.

We just saw that $\frac{1}{1}$ is 1.

So, $\frac{a}{1}$ is really just $a \times 1$, which is a.

Example: For every non-zero number a the quantity $\frac{a}{a}$ is just 1.

$$\frac{a}{a} = 1$$

Reason:

Again, $\frac{a}{a}$ is shorthand for $a \times \frac{1}{a}$. By Rule 10, this is 1.

Alternatively:
$\frac{a}{a}$ is the number that fills in the blank to $a \times \blacksquare = a$.
Clearly 1 does the trick too!

Example: We have that $k \times \frac{a}{b}$ and $\frac{k \times a}{b}$ are the same for numbers a, b, and k with b non-zero.

$$k \times \frac{a}{b} = \frac{k \times a}{b}$$

Reason:

By the addendum...

$k \times \frac{a}{b}$ is really

$$k \times a \times \frac{1}{b}$$

and $\frac{k \times a}{b}$ is really

$$k \times a \times \frac{1}{b}$$

These are indeed the same!

Example: For non-zero numbers k and b, we have:

$$\frac{1}{k} \times \frac{1}{b} = \frac{1}{k \times b}$$

$$\frac{1}{k} \times \frac{1}{b} = \frac{1}{k \times b}$$

Reason:

By Rule 10, $\frac{1}{k \times b}$ is the number that fills in this blank.

$$k \times b \times \blacksquare = 1$$

But $\frac{1}{k} \times \frac{1}{b}$ fills in the blank too!

$$k \times b \times \frac{1}{k} \times \frac{1}{b} = 1 \times 1 = 1$$

This means that $\frac{1}{k} \times \frac{1}{b}$ is $\frac{1}{k \times b}$.

Example: We have that $\frac{k \times a}{k \times b}$ and $\frac{a}{b}$ are the same for numbers a, b, and k, with b and k nonzero.

$$\frac{a}{b} = \frac{k \times a}{k \times b}$$

Reason:

From the addendum, $\frac{k \times a}{k \times b}$ is shorthand for

$$k \times a \times \frac{1}{k \times b}$$

And we've just seen that $\frac{1}{k \times b}$ is $\frac{1}{k} \times \frac{1}{b}$. So, we this reads

$$k \times a \times \frac{1}{k} \times \frac{1}{b}$$

But $k \times \frac{1}{k} = 1$. So, we actually have

$$1 \times a \times \frac{1}{b}$$

which is

$$a \times \frac{1}{b}$$

And according to the addendum this as $\frac{a}{b}$.

So $\frac{k \times a}{k \times b}$ and $\frac{a}{b}$ are the same quantity in disguise!

Example: For numbers a and b, with b nonzero, we have

$$-\frac{a}{b} = \frac{-a}{b} = \frac{a}{-b}$$

Reason: We have from our properties of negative numbers that $-n = (-1) \times n$.

So, using this and the addendum,

$$-\frac{a}{b} = (-1) \times a \times \frac{1}{b}$$

Also,

$$\frac{-a}{b} = (-a) \times \frac{1}{b} = (-1) \times a \times \frac{1}{b}$$

So $-\frac{a}{b}$ and $\frac{-a}{b}$ are the same.

Also, by the previous example, we can multiply the top and bottom of a fraction by -1 and not affect its value.

$$\frac{a}{-b} = \frac{a}{(-1) \times b} = \frac{(-1) \times a}{(-1) \times (-1) \times b} = \frac{-a}{b}$$

(Recall we did prove in Chapter 3 that $(-1) \times (-1) = 1$.)
Thus $\frac{-a}{b}$ and $\frac{a}{-b}$ are also the same.

All three quantities are equivalent.

That's it!

Everything we did in the previous chapter (in particular in Section 41) and in this chapter follows logically and mathematically from all we've just now presented. And all that we just presented follows logically and mathematically from just that one additional rule, Rule 10, with the understanding that $\frac{a}{b}$ is shorthand for $a \times \frac{1}{b}$.

Fractions are logical and tight and mathematically meaningful. And the mathematics of fractions is beautifully aligned with each and every real-world context schoolbooks present to students.

But again, the mathematics of fractions sits at a higher plane to any one real-world context. No one real-world context "sees" the full mathematics of fractions, yet the mathematics of fractions has something to say about each real-world context.

One final example that looks out of place.

Example: Show if two numbers a and b have a product of zero,

$$a \cdot b = 0$$

then it is because at least one of the numbers is zero.

Schoolbooks usually cite this a special rule of arithmetic.

Reason: Suppose we do have two numbers a and b that do multiply together to make zero.

If the first number a is already zero, then we're set: one of the numbers is indeed zero!

But what if a is not zero?

Then look at the fraction $\frac{1}{a}$ and let's work out $\frac{1}{a} \times a \times b$.

We can look at this as

$$\frac{1}{a} \times a \times b = 1 \times b = b$$

and as

$$\frac{1}{a} \times a \times b = \frac{1}{a} \times 0 = 0$$

But we just looked at the same quantity two different ways. We have to conclude than that if a is not zero, then b is the same as 0!

MUSINGS

Musing 52.2 Mathematicians will likely quibble with how I phrased Rule 10.

For each number a different from zero, there is one other number "$\frac{1}{a}$" such that $a \times \frac{1}{a} = 1$.

They will say that "there is **one other** number" should be replaced with "there is **a** number" because logic dictates that there cannot be two or more numbers that fill in the blank to $a \times \blacksquare = 1$.
(So, if you've got "a number" that works, then it is the only "one number" that works.)

Here's the logic.

Consider a non-zero number a.

Suppose b and c are two different numbers that both fill in the blank to $a \times \blacksquare = 1$.
So,

$$a \times b = 1$$
$$a \times c = 1$$

Let's now work out $a \times b \times c$.

We can work out a string of products in any order we like, so we can think of this product as

$$(a \times b) \times c = 1 \times c = c$$

We can also think of it as

$$(a \times c) \times b = 1 \times b = b$$

So, this one product equals both b and c.

So b and c can't be two different numbers after all!

My questions:

i) Did that line of logical reasoning make sense to you?
ii) What do you think of this degree of logical fussiness?

53. All the Rules of Arithmetic in One Place

Here's absolutely everything we have learned about arithmetic all in one spot.

We have a mathematical universe of numbers, which includes the counting numbers

$$0, 1, 2, 3, \ldots$$

There are two operations—**addition** and **multiplication**—on these numbers which behave just as we expect them to when applied to the counting numbers. But they also apply to all numbers in our number universe and behave as follows:

Rule 1: For any two numbers a and b we have $a + b = b + a$.

Rule 2: For any number a we have $a + 0 = a$ and $0 + a = a$.

Rule 3: In a string of additions, it does not matter in which order one conducts individual additions.

Rule 4: For any two numbers a and b we have $a \times b = b \times a$

Rule 5: For any number a we have $a \times 1 = a$ and $1 \times a = a$.

Rule 6: In a string of multiplications, it does not matter in which order one conducts individual multiplications.

Rule 7: For any number a we have $a \times 0 = 0$ and $0 \times a = 0$.

Rule 8: "We can chop up rectangles from multiplication and add up the pieces."

Rule 9: For each number a, there is one other number "$-a$" such that $a + -a = 0$.

Some Logical Consequences of Rule 9: For any two numbers a and b

i) $-0 = 0$
("The opposite of zero is zero")

ii) $- - a = a$
("The opposite of the opposite is back to the original")

iii) $-(a + b) = -a + -b$
(We can "distribute a negative sign")

iv) $(-a) \times b$ and $a \times (-b)$ and $-(a \times b)$ all have the same value
(We can "pull out a negative sign")

v) $(-1) \times a = -a$
("Multiplying by -1 gives you the opposite number")

Rule 10: For each non-zero number a there is one number, $\frac{1}{a}$ such that $a \times \frac{1}{a} = 1$. (That is, we can fill in the blank to $a \times \blacksquare = 1$.)

Convention: For a number a and a non-zero number b, the notation $\frac{a}{b}$ is shorthand for $a \times \frac{1}{b}$.

Some Logical Consequences of Rule 10:

i) $\boldsymbol{\frac{a}{1} = a}$ ("We can put numbers over 1.")

ii) $\boldsymbol{\frac{a}{a} = 1}$ for a not zero ("We can write 1 in many forms.")

iii) $k \times \frac{a}{b} = \frac{ka}{b}$ for b not zero

iv) $\boldsymbol{b \times \frac{a}{b} = a}$ for b not zero ("We can cancel a denominator.")

v) $\boldsymbol{\frac{ka}{kb} = \frac{a}{b}}$ for k and b each not zero ("We can simplify fractions.")

vi) **ADDING/SUBTRACTING FRACTIONS**: $\frac{a}{N} \pm \frac{b}{N} = \frac{a \pm b}{N}$ for N not zero

vii) **MULTIPLYING FRACTIONS**: $\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d}$ for b and d each not zero

viii) **PULLING OUT NEGATIVE SIGNS:** $-\frac{a}{b} = \frac{-a}{b} = \frac{a}{-b}$ for b non-zero

ADDING/SUBTRACTING FRACTION WITH DIFFERENT DENOMINATORS
Either rewrite each fraction using v) to create a common denominator, or just put the quantity over 1 and use v).

DIVIDING FRACTIONS Rewrite $\frac{a}{b} \div \frac{c}{d}$ as $\frac{\frac{a}{b}}{\frac{d}{c}}$ and use v).

Finally …

ix) **$a \div b$ and $\frac{a}{b}$ are the same number**, assuming b is not zero, or course.

Point ix) is just a reinterpretation of iv).

Allow me to reiterate one more time …

Division does not really exist. Division is just multiplication by a fraction.

For any two numbers a and b, with b not zero, we have by Rule 10, consequence iv) that $\frac{a}{b}$ is the number that fills in the blank to

$$b \times \blacksquare = a$$

Schoolbooks call this number $a \div b$ (using the "reverse multiplication" understanding of division.)

But $\frac{a}{b}$ is $a \times \frac{1}{b}$, or, switching the order of the multiplication, $\frac{a}{b}$ is $\frac{1}{b} \times a$.

We have

To divide a quantity by a value b, just multiply that quantity by $\frac{1}{b}$.

$$a \div b = \frac{a}{b} = \frac{1}{b} \times a$$

So, in our mathematical universe, there are only two operations: addition and multiplication.
Subtraction is just the addition of the (additive) opposite.
Division is just multiplication by the (multiplicative) opposite.

And the full mathematical story of arithmetic follows from just ten basic rules.

Solutions

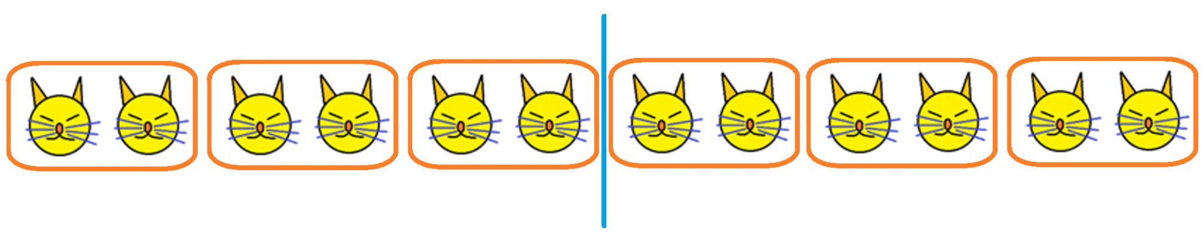
42.1 They all do.

a) $\frac{36}{12} = 3$ b) $3 \times \frac{20}{3} = 20$ c) $\frac{1002}{1} = 1002$ d) $\frac{107}{107} = 1$ e) $4 \times \frac{5}{4} = 5$ f) $\frac{0}{8} = 0$
g) $10 \times 6 \times \frac{7}{5} = 2 \times 5 \times 6 \times \frac{7}{5} = 2 \times 6 \times 7$

43.1 They are.
If a pie is divided into three “equal” pieces, then a picture of the divided pie shows how three copies of any one piece make the whole.
Conversely, if three copies of one piece of pie fit together to make the whole, then a picture showing that can be interpreted as a pie divided into three “equal” pieces.

43.2
Here's one whole set of kittens.

Here's three copies of two thirds of this set.



This matches two copies of the whole set of kittens.

44.1

$$\frac{2}{5} + \frac{4}{9} = \frac{5 \times 9 \times (\frac{2}{5} + \frac{4}{9})}{5 \times 9 \times 1} = \frac{18 + 20}{45} = \frac{38}{45}$$

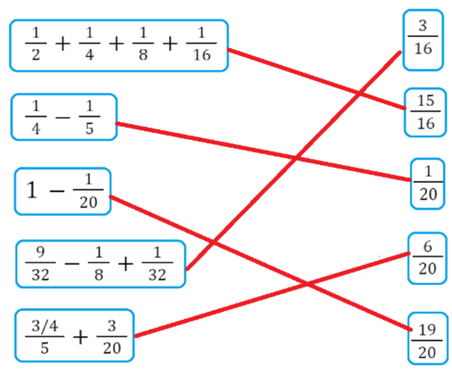
44.2

$$\frac{a}{N} - \frac{b}{N} = \frac{\frac{a}{N} + \frac{-b}{N}}{1} = \frac{N \times \left(\frac{a}{N} + \frac{-b}{N}\right)}{N \times 1} = \frac{a + -b}{N} = \frac{a - b}{N}$$

44.3 As I said, I personally don’t see a natural way to think through this.

44.4 You likely would have drawn $\frac{2}{5}$ of a pie with each slice divided into nine to see it as $\frac{18}{45}$ of a pie, and drawn $\frac{4}{9}$ of a pie with each slice divided into five to see it as $\frac{20}{45}$, and then combined1 bthose pictures.

44.5



44.6 a) $30 \times 31 = 930$ b) $\frac{1}{930}$ c) 110 and $\frac{1}{110}$

d) $\frac{1}{n} - \frac{1}{n+1} = \frac{(n+1)\times 1}{(n+1)\times n} - \frac{n\times 1}{n\times(n+1)} = \frac{n+1-n}{n\times(n+1)} = \frac{1}{n(n+1)}$

44.7 a) $\frac{a}{b} + \frac{c}{d} = \frac{\frac{a}{b}+\frac{c}{d}}{1} = \frac{b\times d\times(\frac{a}{b}+\frac{c}{d})}{b\times d\times 1} = \frac{da+bc}{bd}$ b) $\frac{a}{b} + \frac{c}{d} = \frac{ad}{bd} + \frac{bc}{bd} = \frac{ad+bc}{bd}$
These are the same.

44.8 Yes! (See above!)

44.9 a) $\frac{137}{110}$ b) $\frac{24}{70}$ c) $\frac{8}{9}$ d) $\frac{6}{35}$

44.10 a) $\frac{2}{399}$ b) $-\frac{7}{4}$ c) $\frac{47}{60}$

45.1 a) $\frac{1}{3} \times \frac{1}{7} = \frac{\frac{1}{3}\times\frac{1}{7}}{1} = \frac{3\times 7\times\frac{1}{3}\times\frac{1}{7}}{3\times 7\times 1} = \frac{1}{21}$

b) $\frac{2}{3} \times \frac{4}{7} = 2 \times \frac{1}{3} \times 4 \times \frac{1}{7} = 8 \times \frac{1}{21} = \frac{8}{21}$

c) Follow the approach of part a) to show that $\frac{1}{b} \times \frac{1}{d} = \frac{1}{bd}$
and then $\frac{a}{b} \times \frac{c}{d} = a \times \frac{1}{b} \times c \times \frac{1}{d} = ac \times \frac{1}{bd} = \frac{ad}{bc}$.

45.2 It is possible to do this, but it is not easy. Try it!

45.3 $\frac{5}{8}$ of $\frac{3}{10}$ of a pie is $\frac{5\times3}{8\times10}$ of the pie.

45.4 a) Look at ten equally spaced columns first. Identify three of them. Then highlight five eights of these three tenths by drawing in rows.

b) Look at eight equally spaced rows first. Identify five of them. Then highlight three tenths of these five eighths by drawing in columns.

[Alternatively, turn the picture 90 degrees and look at columns and then rows in that order.]

45.5 It is, but justifying this with real world pictures is ghastly and gets us into three-dimensional pictures. Math is so often so much easier than the "real world."

45.6 Part b) covers part a) and so I am going to be lazy and just answer part b)!

$$\frac{a}{b}\times\frac{b}{a}=a\times\frac{1}{b}\times b\times\frac{1}{a}=1\times1=1$$

45.7 $n\times\frac{p}{q}=p\times q\times\frac{p}{q}=p\times q\times p\times\frac{1}{q}=p\times p\times1=p^2$

45.8 $\frac{16}{45}$

45.9 Multiplying $\frac{4}{7}$ by $\frac{7}{8}$ gives $\frac{4\times7}{7\times8}=\frac{4}{8}=\frac{1}{2}$.

45.10 $\frac{10}{3}$

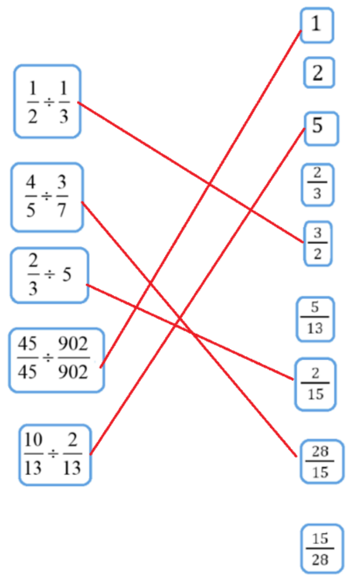
46.1

a) $\frac{a}{b}\div\frac{c}{d}=\frac{\frac{a}{b}}{\frac{c}{d}}=\frac{b\times d\times\frac{a}{b}}{b\times d\times\frac{c}{d}}=\frac{d\times a}{b\times c}$

b) $\frac{a}{b}\times\frac{d}{c}=\frac{a\times d}{b\times c}$

These are indeed the same.

46.2



46.3 Did you?

46.4 $\frac{\frac{3}{4}}{\frac{2}{3}} = \frac{4\times3\times\frac{3}{4}}{4\times3\times\frac{2}{3}} = \frac{3\times3}{4\times2} = \frac{9}{8}$

46.5 Please do so.

46.6 Do verify this.

46.7

a) The answer to 46.1a) shows this.

b) $\frac{a\div c}{b\div d} = \frac{\frac{a}{c}}{\frac{b}{d}} = \frac{c\times d\times\frac{a}{c}}{c\times d\times\frac{b}{d}} = \frac{d\times a}{c\times b}$

The answer are the same.

46.8 The lines of text that follow this question explain this question!

46.9 a) $\frac{7}{6}$ b) $\frac{b}{a}$

46.10 a) $\frac{1}{2(2+2)}\times 8$ b) $\frac{1}{2}\times 8\times(2+2)$

46.11 a) $\frac{3}{2}$ b) $\frac{28}{15}$ c) $\frac{10}{3}$

46.12 It's just 1.

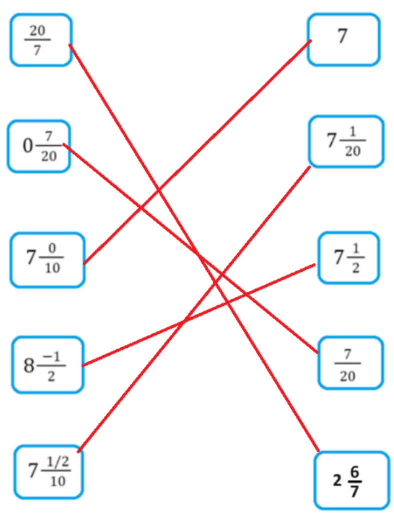
47.1 It's $\frac{0}{1}$ or eve n $\frac{0}{5}$ or $\frac{-0}{1745}$.

47.2 $3+\frac{1}{2}=\frac{3+\frac{1}{2}}{1}=\frac{2x(3+\frac{1}{2})}{2\times 1}=\frac{6+1}{2}=\frac{7}{2}$

47.3 $9\frac{17}{5}=9+\frac{15+2}{5}=9+\frac{15}{5}+\frac{2}{5}=9+3+\frac{2}{5}=12\frac{2}{5}$

47.4 $2\frac{0}{9}=2+\frac{0}{9}$. Since $\frac{0}{9}$ is just 0, we have $2+0$, which is 2.

47.5



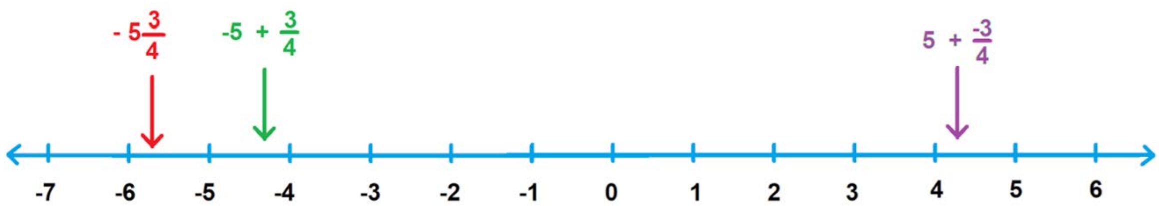
47.6 $21\frac{1}{20}$

47.7 $4\frac{4}{99}$

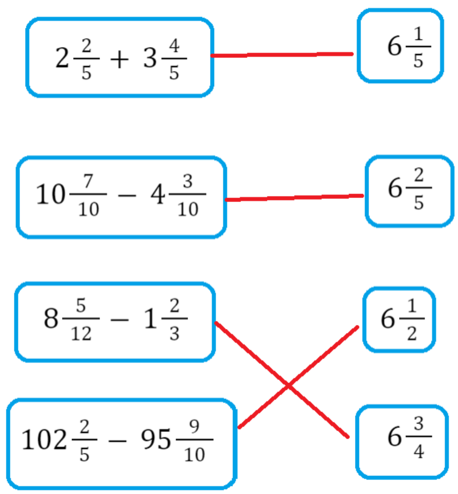
47.8 $\frac{40001}{200}$

47.9 $4\times\left(5+\frac{3}{4}\right)=20+3=23$

47.10



47.11



47.12 $-10\frac{1}{2}$

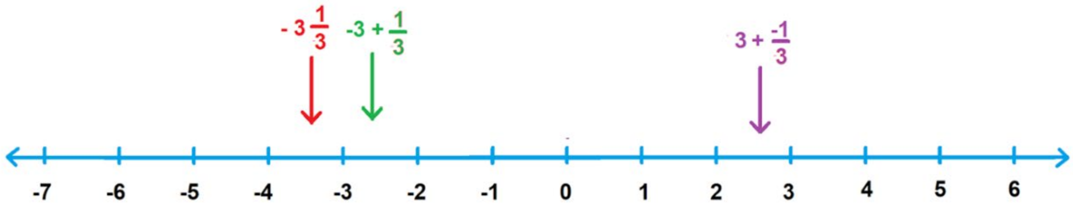
47.13 5

47.14 Try it.

47.15 True. They are each $\frac{4}{15}$.

47.16 a) $7\frac{7}{13}$ b) $12\frac{5}{13}$ c) $-12\frac{5}{13}$

47.17



47.18 a) $3 \times 4 = 12$ and $\frac{1}{3} \times \frac{1}{2} = \frac{1}{6}$. It seems natural to think this is all there is to it.

b) $12 + \frac{4}{3} + \frac{3}{2} + \frac{1}{6} = 12 + \frac{8}{6} + \frac{9}{9} + \frac{1}{6} = 12 + \frac{18}{6} = 15$

c) $\frac{1}{3} \times 4$ and $3 \times \frac{1}{2}$

d) One gets $\frac{25}{6} = 4\frac{1}{6}$ either way.

47.19 a) $1\frac{3}{5}$ b) $7\frac{9}{13}$ c) $1\frac{1}{199}$ d) $1\frac{}{398}$

47.20a) $\frac{65}{9}$ b) $\frac{225}{28}$ c) $\frac{90299}{300}$ d) $-\frac{25}{7}$ e) $\frac{873}{1}$

47.21 a) 19 b) -25

47.22 a) $3\frac{1}{15}$ b) $\frac{5}{7}$ c) $1\frac{7}{36}$ d) $\frac{5}{6}$

47.23 a) $\frac{7}{3}$ b) $\frac{7}{8}$ c) $\frac{44}{45}$ d) $\frac{110}{161}$ e) $\frac{21}{20}$

48.1 Four aureus equals one-hundred denarius. So 1 denarius is paid as tax.

48.2 $25\% = \frac{25}{100} = \frac{25 \times 1}{25 \times 4} = \frac{1}{4}$

48.3 $\frac{1}{1000}$

48.4 It means "one part per thousand."

48.5 70%

48.6 $\frac{1500}{32} = \frac{1472+28}{32} = 46 + \frac{28}{32} = 46 + \frac{7}{8}$

48.7 We did. To write a number N as a percentage we look at

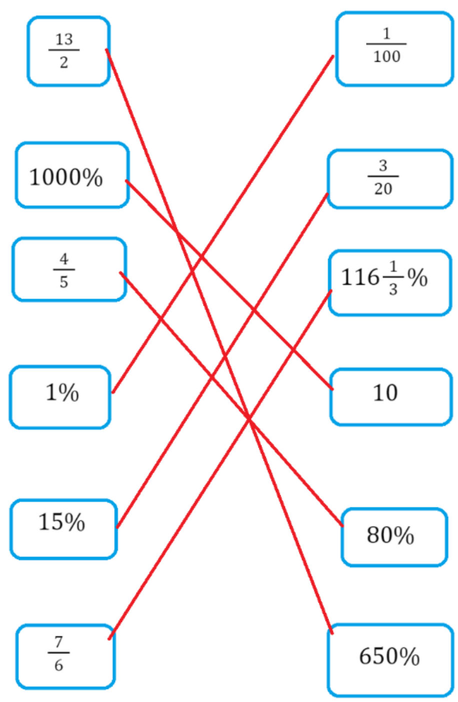
$$N = \frac{N}{1} = \frac{100 \times N}{100}$$

and read this as $100N$ %.

48.8 $83\frac{1}{3}\%$

48.9 3000%

48.10



48.11

a) *Dextans* comes from "do dextans." It is $1 - \frac{1}{6} = \frac{1}{6}$.

Dodrans comes from "do quadrans" and is $1 - \frac{1}{4} = \frac{3}{4}$

b) *Quinque unciae* is $\frac{5}{12}$ and this was shortened to *quincunx*.

Septem unciae is $\frac{7}{12}$ and this was shortened to *septunx*.

c) Half of one twelfth, $\frac{1}{24}$.

48.12 a) Each are 12. b) Each are $37\frac{1}{2}$ c) Each are 75.

c) $\boldsymbol{a}$% **of** $\boldsymbol{b}$ is $\frac{a}{100} \times b = \frac{ab}{100}$ and $\boldsymbol{b}$% **of** $\boldsymbol{a}$ is $\frac{b}{100} \times a = \frac{ab}{100}$. These are the same.

49.1 Good luck!

49.2 10% of \$160 is \$16.

So, 5% corresponds to \$8, and $2\frac{1}{2}\%$ to \$4.

Thus $47\frac{1}{2}\%$ of \$160 is $4 \times 16 + 8 + 4 = 76$ dollars.

(This is also 50% of \$160, take away \$4.)

49.3 10% of \$310 is \$31.
So, 1% of this is \$3.10 and 2% of it is \$6.20.

Thus 18% of \$310 is $31 + 31 - 6.20 = 55.80$ dollars.

49.4 I prefer a). My salary would have more than doubled!

49.5 a) The population increase is 1500 residents.
As a fraction of the (original) town's population, this is $\frac{1500}{12500} = \frac{3}{25}$.
Expressed as a percentage, this is a $\frac{3 \times 4}{25 \times 4} = \frac{12}{100} = 12\%$ percentage increase.

b) The new population compared to the original population is the fraction $\frac{14000}{12500} = \frac{28}{25} = \frac{112}{100}$.
It is 112% of what it was. (This fits with the answer to a).)

49.6 a) Taking one quarter off of a price of N dollars leaves the price $N - \frac{1}{4} \times N$.
Factoring this expression gives

$$N - \frac{1}{4} \times N = \left(1 - \frac{1}{4}\right) \times N = \frac{3}{4} \times N$$

b) Adding one fifth to a price of N dollars gives the price $N + \frac{1}{5} \times N$.
Factoring this expression gives

$$N + \frac{1}{5} \times N = \left(1 + \frac{1}{5}\right) \times N = \frac{6}{5} \times N$$

49.7 a) Following the previous question, we are looking at $\frac{3}{4} \times \frac{6}{5} \times N$ and $\frac{6}{5} \times \frac{3}{4} \times N$. These are the same.
b) In general, $\left(1 + \frac{a}{100}\right) \times \left(1 - \frac{b}{100}\right) \times N$ and $\left(1 - \frac{b}{100}\right) \times \left(1 + \frac{a}{100}\right) \times N$are sure to have the same value.

49.8 36

49.9 \$4.50

49.10 a) \$32.60 b) Double this: \$65.20

49.11 $8{,}400$

49.12 a) \$0 b) This means there was no change to your salary.
.
50.1 179

50.2 It does in both cases.

50.3 Scale both sides of $10 < 13$ as follows

$$\frac{1}{10} \times \frac{1}{13} \times 10 < \frac{1}{10} \times \frac{1}{13} \times 13$$

This is reads

$$\frac{1}{13} \times 1 < \frac{1}{10} \times 1$$

or

$$\frac{1}{13} < \frac{1}{10}$$

50.4 It all does.

50.5 $\frac{7}{12} < \frac{7}{11} < \frac{9}{11} < \frac{9}{7}$

50.6 $\frac{5}{9} = \frac{55}{99}$ and $\frac{6}{11} = \frac{54}{99}$. We see again that $\frac{5}{9}$ is larger.

50.7 $\frac{15}{28}=\frac{30}{56}$ and $\frac{6}{11}=\frac{30}{55}$. We see that $\frac{15}{28}$ is smaller than $\frac{6}{11}$. So we have $\frac{15}{28}<\frac{6}{11}<\frac{5}{9}$.

50.8 $\frac{3}{2}$

50.9 $\frac{7}{10}<\frac{11}{15}<\frac{4}{5}$

50.10 $\frac{1}{3}<\frac{2}{5}<\frac{1}{2}$

50.11 Part c) covers it all.

Since $a+n=b$, multiplying everything by k shows that $k\times(a+n)=k\times b$. That is,

$$ka+kn=kb$$

So, we must add something to ak to “get to” the value kb.

We have $ka<kb$.

50.12

a) If $a>b$, then $\frac{1}{b}\times a>\frac{1}{b}\times b$. That is, we have $\frac{a}{b}>1$.
b) If $a<b$, then $\frac{1}{b}\times a<\frac{1}{b}\times b$. That is, we have $\frac{a}{b}<1$.

50.13 a) $\frac{5}{4}\times 100=5\times\frac{1}{4}\times 100=5\times 25=125$.
b) $\frac{5}{4}\times N=\left(1+\frac{1}{4}\right)\times N=N+more$. We have something larger than N.
c) It’s 80.
d) From $4<5$ we get $\frac{1}{5}\times 4\times N<\frac{1}{5}\times 5\times N$, which gives $\frac{4}{5}\times N<N$.
e) It’s 80.
f) Multiplying top and bottom by 4 gives $\frac{4N}{5}$ which is the same as $\frac{4}{5}\times N$. By part d), this is smaller than N.
g) It’s 125.
h) Multiplying top and bottom by 5 gives $\frac{5N}{4}$ which is the same as $\frac{5}{4}\times N$. By part b), this is bigger than N.
i) Start with $k>1$ or with $k<1$ and scale by sides by N and by $\frac{1}{k}$.

50.14 a) $\frac{3}{4}$ b) $\frac{11}{12}$ c) $\frac{16}{23}$ d) $\frac{40}{9}$

51.1 a) We can write 3 as $\frac{3}{1}$ which fits the schoolbook definition of being a fraction.
b) We can write 0 as $\frac{0}{1}$, so it fits the definition of being a rational number. But we can’t write zero as $\frac{a}{b}$ with a and b both positive integers. So, zero is not considered a fraction in schoolbook world.

c) $-3 = \frac{-3}{1}$. It is a rational number, but we can't think of it as a schoolbook fraction.

51.2 If the two numbers are rational, then they can be written as $\frac{a}{b}$ and $\frac{c}{d}$ for some integers a, b, c, and d (with b and d not zero).

a) Their sum is

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$$

which is "an integer over an integer," and so is a also rational number.

b) Their quotient is

$$\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{ad}{bc}$$

which is "an integer over an integer," and so is a also rational number.

52.1 Try these.
The notes that follow establish each of these claims in the general context. So, if you are stuck, read on and try to repeat what is done in a relevant general example.

52.2 I wonder what you think.