Chapter 7

Decimals
and Scientific Notation,
Rounding, and Significant Figures

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53. Discovering Decimals

Let's go back to the machines in Chapter 4 and their *Exploding Dots*. Something about them has been bothering me all this time.

Recall, no matter the machine, we had boxes going to the left as far as we pleased.

But that seems awfully lopsided! Why can't we have boxes going infinitely far to the right as well?

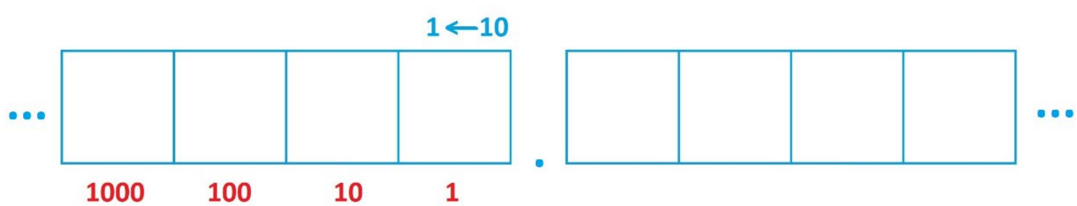
Mathematicians like symmetry and so let's follow suit and now make all our machines symmetrical. Let's have boxes going to the left and to the right.

But the challenge now is to figure out what those boxes to the right mean.

Focusing on the $1 \leftarrow 10$ machine.

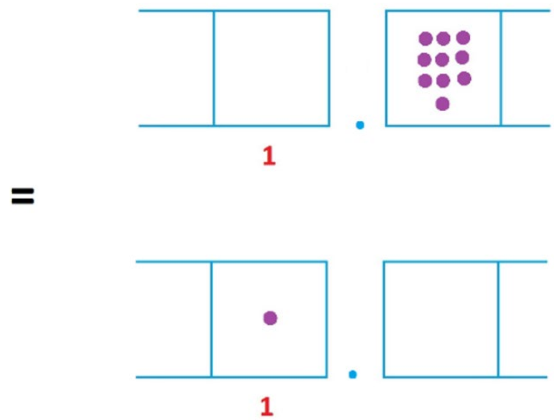
Let's focus on a $1 \leftarrow 10$ machine and see what boxes to the right could mean for that machine.

To keep the left and right boxes visibly distinct, we'll separate them with a point. (Society calls this point—for base ten, at least—a **decimal point**.)

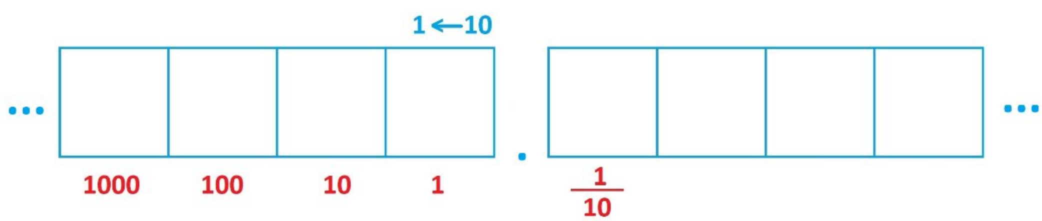


So, what does it mean to have dots in the right boxes? What are the values of dots in those boxes?

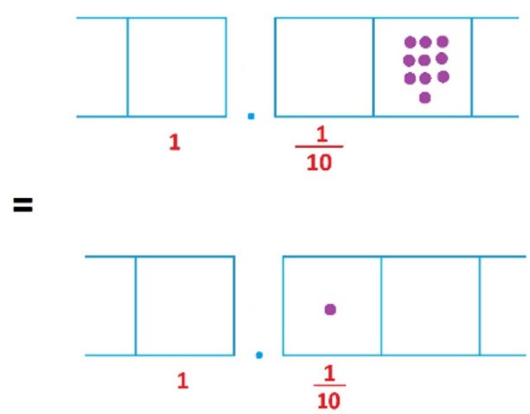
Since this is a $1 \leftarrow 10$ machine, we do know that ten dots in any one box explode to make one dot one place to the left. So, ten dots in the box just to the right of the decimal point are equivalent to one dot in the 1s box. Each dot in that box must be worth one tenth. (And yes, ten tenths is one: $10 \times \frac{1}{10} = 1$.)



We have our first place-value to the right of the decimal point.

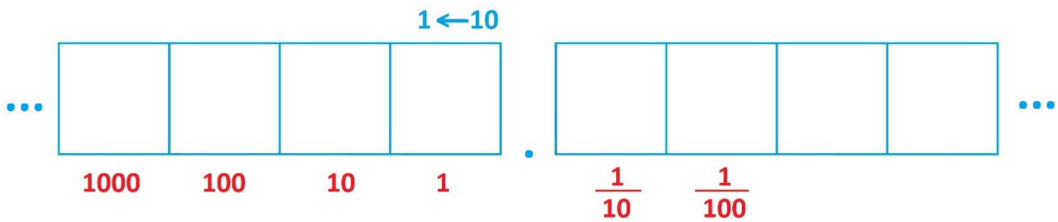


In the same way, ten dots in the next box over are worth one dot in the one tenth place.

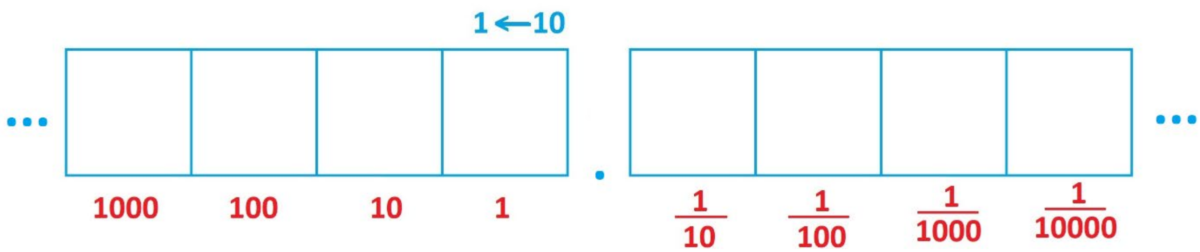


And so, each dot in that next box over must be worth one tenth of one tenth. That's one hundredth. (Yes, ten one hundredths do make one tenth: $10 \times \frac{1}{100} = \frac{10}{100} = \frac{1}{10}$.)

We now have two place values to the right of the decimal point.



And we can keep going: ten one-thousandths make a hundredth, and ten ten-thousandths make a thousandth, and so on.



Practice 53.1 Show that $10 \times \frac{1}{1,000} = \frac{1}{100}$ and $10 \times \frac{1}{10,000} = \frac{1}{1,000}$.

We see that the boxes to the left of the decimal point represent place values given by tens multiplied together and boxes to the right of the decimal point represent place values given by tenths multiplied together.

Practice 53.2 Are you clear on why $\frac{1}{10} \times \frac{1}{10} \times \frac{1}{10} = \frac{1}{1,000}$ and $\frac{1}{10} \times \frac{1}{10} \times \frac{1}{10} \times \frac{1}{10} = \frac{1}{10,000}$?

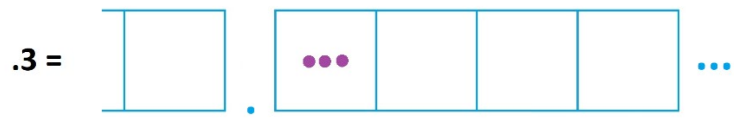
We have just discovered the decimal places!

Practice 53.3 What does the prefix deci- or deca- mean in English?
(How many years are in a decade? In geometry, how many sides does a decagon have?)

Practice 53.4 What might we call the point that separates left and right boxes if we were doing this work in a $1 \leftarrow 2$ machine instead? (There isn't an official name for one, but can you see why it shouldn't be called a decimal point?)

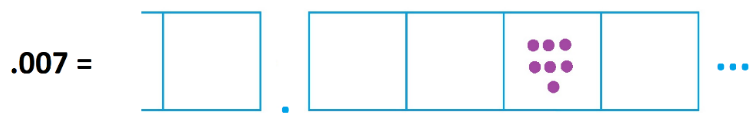
Practice 53.5 Do all cultures use a point to separate boxes?

When people write 0.3, for example, in base ten, they mean the value of three dots placed in the first box after the decimal point.



We see that 0.3 equals three tenths: $0.3 = \frac{3}{10}$.

Seven dots in the third box after the decimal point is seven thousandths: $0.007 = \frac{7}{1000}$.

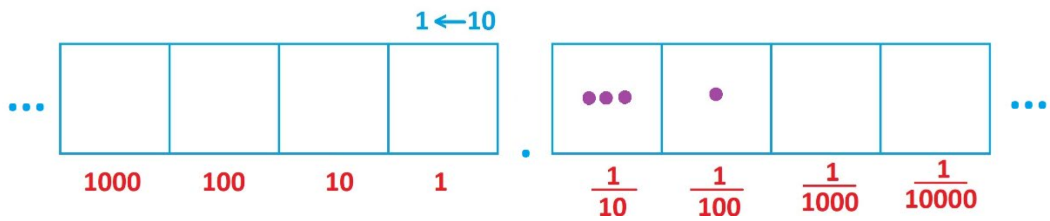


Practice 53.6 What fraction is 0.00008?

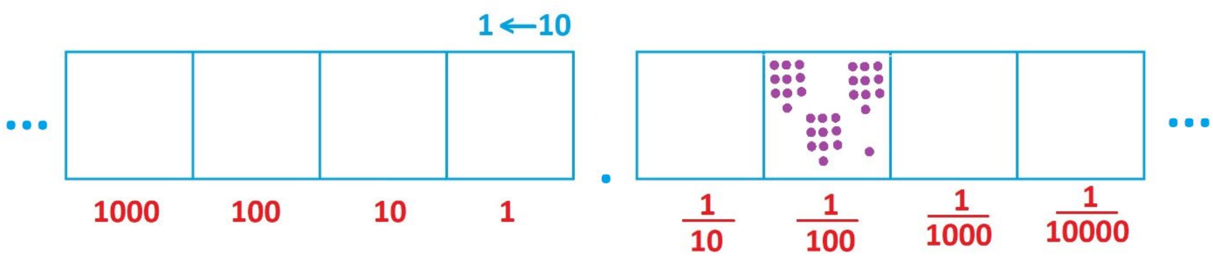
Comment: Some people might leave off the beginning zero and just write $.007$ rather than 0.007. This is just a matter of personal taste. I've already used both styles of presentation on this page.

Question: Some people read 0.6 out loud as "point six" and others read it out loud as "six tenths." Which is more helpful for understanding what the number really is?

There is a possible source of confusion with a decimal such as 0.31. This is technically three tenths and one hundredth: $0.31 = \frac{3}{10} + \frac{1}{100}$.



But some people read 0.31 out loud as "thirty-one hundredths," which looks like this.



Are these the same thing?

Well, yes! With three explosions we see that thirty-one hundredths becomes three tenths and one hundredth.

Some Language

People are a little loose in how they describe a number written with a decimal point and some digits to the right of the decimal point.

They might say that the number has been written in **decimal notation** or that it has been expressed simply as a **decimal**. One might call for the **decimal representation** of a number, meaning that one is meant to express a given number in decimal notation.

The term **decimal number** means any number that is expressed via decimal notation.

One final thought.

Question: What does -0.31 mean?

Answer: Well, 0.31 is the fraction $\frac{31}{100}$ and so -0.31 is this fraction made negative. It's $-\frac{31}{100}$.

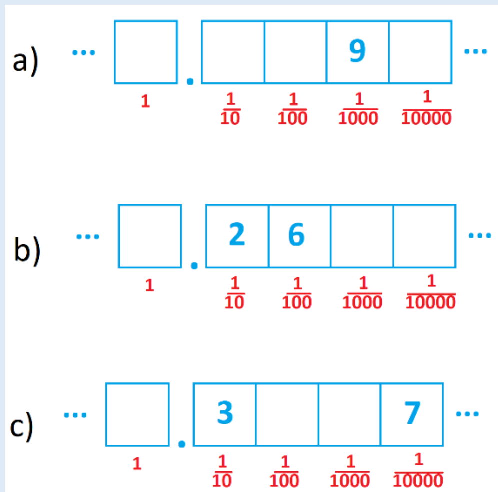
People like to think of negative sign in front of decimal as applying to all of the decimal number. For example, if you are thinking of 0.31 as $\frac{3}{10}+\frac{1}{100}$, then -0.31 is

$$-\left(\frac{3}{10}+\frac{1}{100}\right)=\frac{-3}{10}+\frac{-1}{100}$$

Question: Can you see that $\frac{-3}{10}+\frac{-1}{100}$ is the same as $\frac{-31}{100}$, which is $-\frac{31}{100}$?

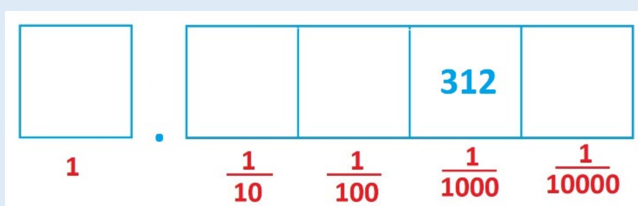
MUSINGS

Musing 53.7 For each picture, write the decimal number the picture represents and the fraction that that decimal equals. (For example, anwer to part a is 0.009, which is $\frac{9}{1000}$.)

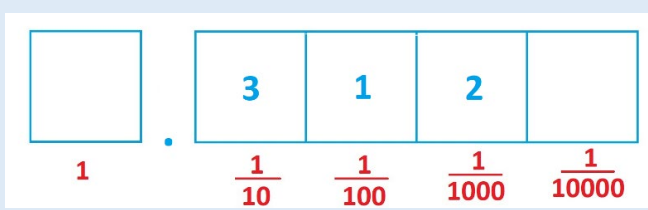


Musing 53.8 A schoolteacher asked his students to each draw a $1 \leftarrow 10$ machine picture of the fraction $\frac{312}{1000}$.

JinJin drew:



Subra drew:



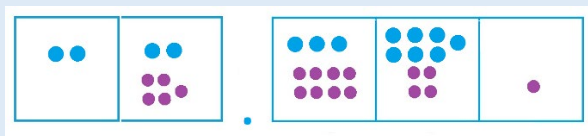
The teacher marked both students as correct.

Are both of these responses indeed valid? Explain your thinking.

Musing 53.9 Aparna was asked to compute $22.37 + 5.841$. She wrote this answer for her professor.

$$\begin{array}{rcccccc} & 2\,2 & . & 3 & & 7 & \\ + & 5 & . & 8 & & 4 & 1 \\ \hline & 2|7 & . & 11 & | & 11 & |1 \end{array}$$

Her professor was confused, so she added this picture to her page.



Her professor was still a bit puzzled, but she had an idea now as to what Aparna might be thinking. The professor said "I was expecting to see the answer 28.211. Does your work lead to that answer?"

What could Aparna do next to show her professor that $2|7.11|11|1$ is indeed the number 28.211 in disguise?

Musing 53.10 Tijana said that 23.56×11 equals $22|33.55|66$.
Can you explain what she is thinking and how to fix her answer to one that society understands?

Musing 53.11 Can you explain using dots and boxes in a $1 \leftarrow 10$ machine why 22.37×10 equals 223.7? (It looks like the decimal point shifted one place. Did it really?)

Musing 53.12 James, feeling naughty, wrote the decimal number $3|-4|0\,.\,5|-7|0|-1$.
What is a societally acceptable version of this decimal number?

MECHANICS PRACTICE

Musing 53.13

a) The decimal 0.23 equals which of the following?

a) $\frac{23}{10}$ b) $\frac{23}{100}$ c) $\frac{23}{1000}$ d) $\frac{23}{10000}$

b) The decimal 0.0409 equal which of the following?

a) $\frac{409}{100}$ b) $\frac{409}{1000}$ c) $\frac{409}{10000}$ d) $\frac{409}{100000}$

Musing 53.14 Draw $1 \leftarrow 10$ machine pictures of each of these quantities.

a) $\frac{66}{100}$ b) $\frac{6}{10} + \frac{6}{100}$ c) $1\frac{66}{100}$ d) $\frac{66}{50}$

Musing 53.15

a) This picture for a $1 \leftarrow 10$ machine is equivalent to which of the following numbers?

a) 31.2 b) 3.12 c) $.312$ d) $.0312$

b) Can you see that this picture is the decimal 34.32in disguise?

(This is technically a YES/NO question. It would be nice if the answer were YES.)

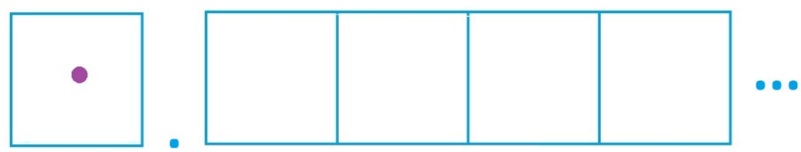
54. Fractions as Decimals

We saw in chapters 5 and 6 that fractions are numbers that match answers to division problems.

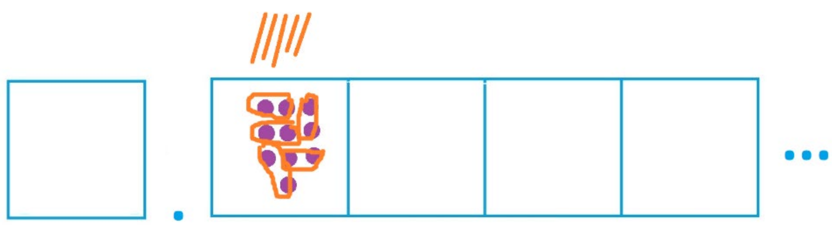
For example, $\frac{2}{3}$ is the answer to $2 \div 3$, and $\frac{1}{2}$ is the answer to the division problem $1 \div 2$.

Moreover, as we saw in Chapter 4, we can compute answers to division problems in a $1 \leftarrow 10$ machine, even a division problems like $1 \div 2$. All we have to do is make use of the boxes to the right of the decimal point. The division process is exactly the same.

For instance, to compute $1 \div 2$ we need to identify groups of two in this picture with just one dot.



No groups of two can be seen at present, so let's unexplode. Doing so reveals five groups of two at the tenths level.



We have that $\frac{1}{2}$ is 0.5 as a decimal. (And as a check, $\frac{5}{10}$ does indeed equal $\frac{1}{2}$.)

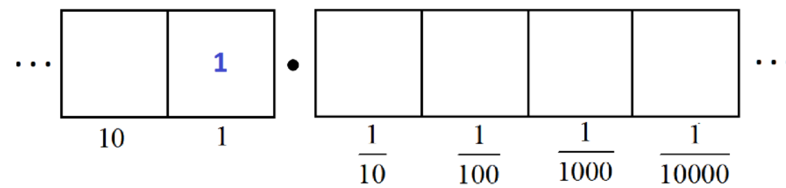
Practice 54.1: Write $\frac{1}{4}$ as a decimal by computing $1 \div 4$. Do you get 0.25?

Practice 54.2: Write $\frac{1}{5} = 1 \div 5$ as a decimal.

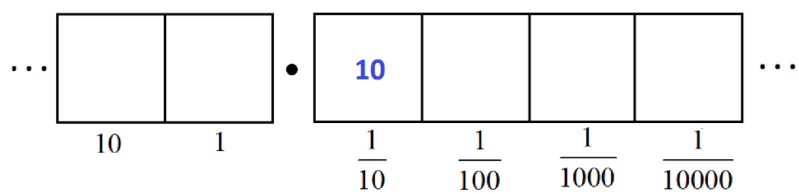
Practice 54.3: Write $\frac{1}{10} = 1 \div 10$ as a decimal. (Why should you get the answer 0.1?)

Another example: Let's write $\frac{1}{8}$ as a decimal. We need to compute $1 \div 8$ in a $1 \leftarrow 10$ machine.

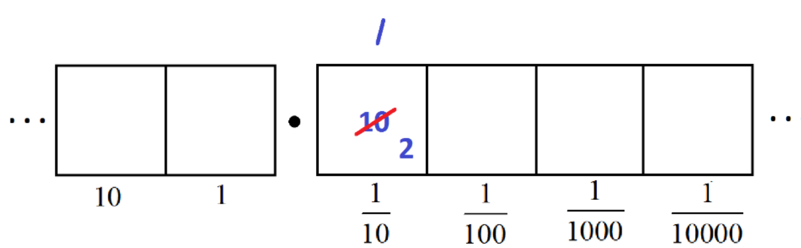
We seek groups of eight in the following picture. (I won't draw dots this time and just write numbers.)



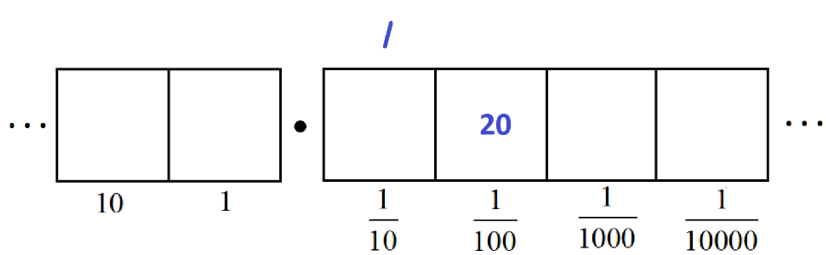
None are to be found right away, so let's unexplode.



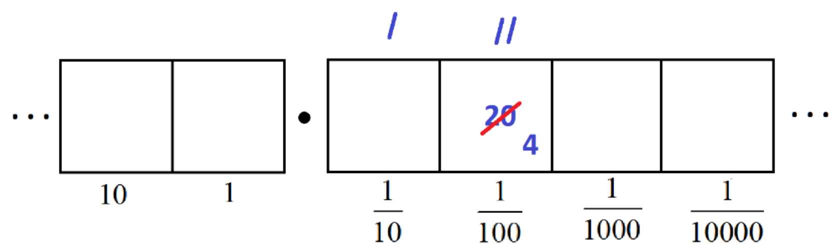
We have one group of 8, leaving two behind.



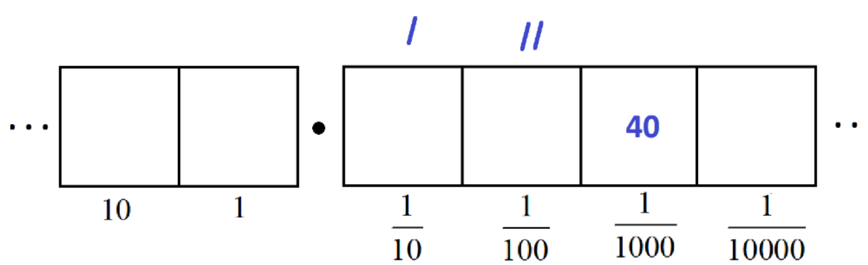
Two more unexplosions.



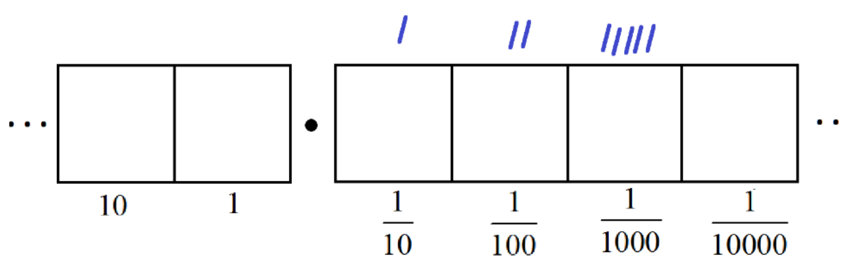
This gives two more groups of 8 leaving four behind.



Unexploding again



reveals five more groups of 8 leaving no remainders.



We see that, as a decimal, $\frac{1}{8}$ turns out to be 0.125. And as a check we have

$$0.125 = \frac{125}{1000} = \frac{25}{200} = \frac{5}{40} = \frac{1}{8}$$

Super!

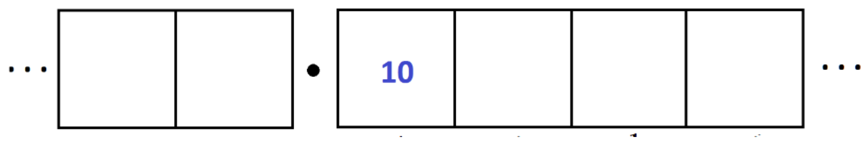
(Did you follow all the steps on this and the previous pages?)

Practice 54.4: Write $\frac{1}{40} = 1 \div 40$ as a decimal.
(Can you compute this in a $1 \leftarrow 10$, writing numbers instead of drawing dots?)

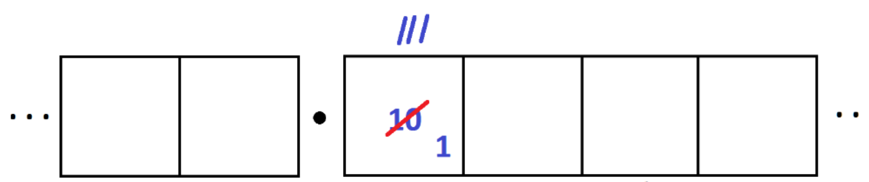
Not all fractions lead to simple decimal representations. For example, consider the fraction $\frac{1}{3}$.

To compute it, we seek groups of three in the following picture.

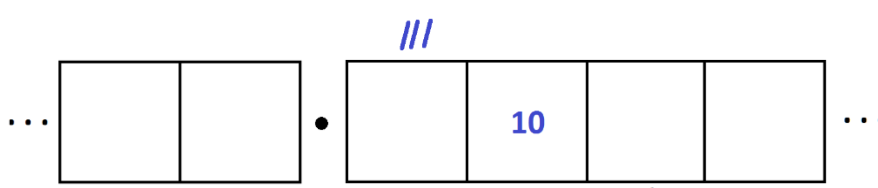
Let's unexplode.



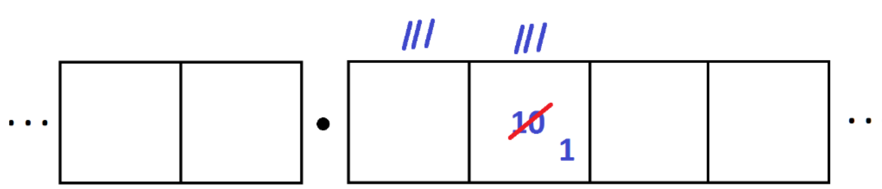
We see three groups of 3 leaving one behind.



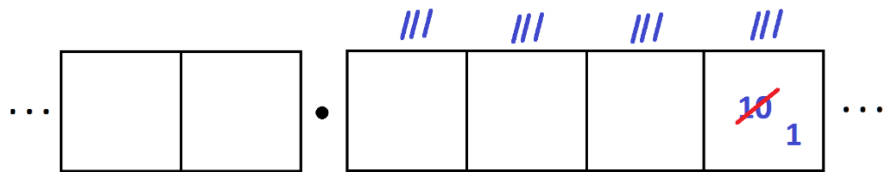
Unexploding gives another ten dots to examine.



We find another three groups of 3 leaving one behind.



And so on. We are caught in an infinite repeating cycle.



This puts us in a philosophically interesting position. As human beings we cannot conduct this, or any, activity for an infinite amount of time. But it seems very tempting to write

$$\frac{1}{3} = 0.33333333\ \ldots$$

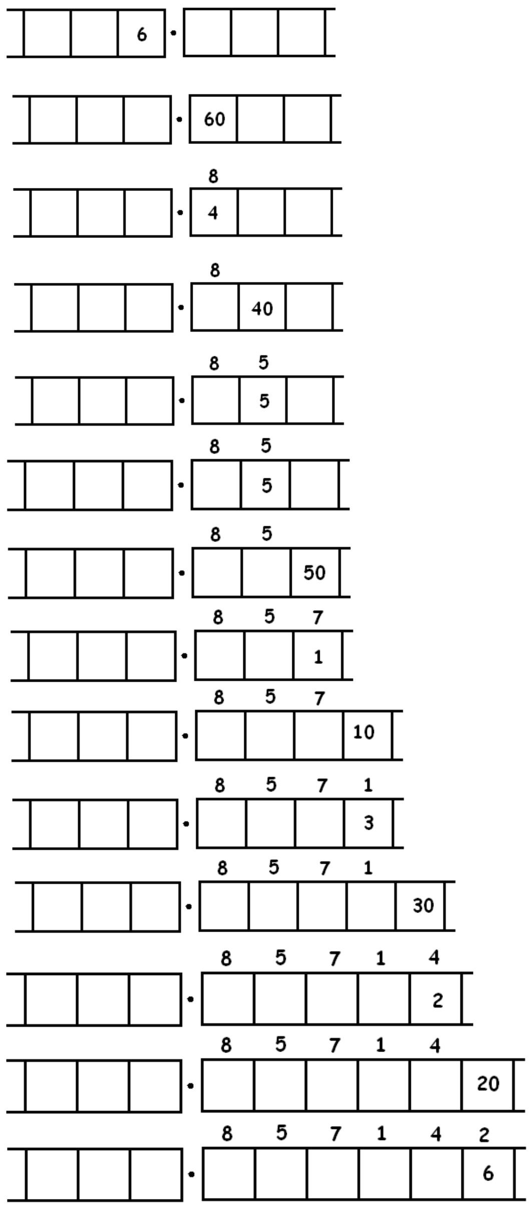
with the ellipsis representing the instruction "keep going with this pattern forever."

In our minds we can almost imagine what this means. But as a practical human being it is beyond our abilities: one cannot actually write down those infinitely many 3s represented by the ellipsis.

Nonetheless, many people choose not to contemplate what an infinite statement like this means and just carry on and say that some decimals are infinitely long and not worry about it. The fraction $\frac{1}{3}$ is one of those fractions whose decimal expansion goes on forever.

Practice 54.5: Write $\frac{1}{9}$ as an infinitely long decimal.

Here's a complicated example. Work through it if you are game! Here we convert the fraction $\frac{6}{7}$ to an infinitely long decimal.



Do you see with this 6 in the final rightmost box that we have returned to the very beginning of the problem? This means that we shall simply repeat the work we have done and obtain the same sequence "857142" of answers again, and then again, and then again.
We have

$$\frac{6}{7} = 0.857142\ \ 857142\ \ 857142\ \ 857142\ \ldots$$

Practice 54.6: Write $\frac{2}{15} = 2 \div 15$ as a decimal. Does it too fall into an infinitely repeating pattern?

Practice 54.7: Write $\frac{1}{6} = 1 \div 6$ as a decimal. Does it too fall into an infinitely repeating pattern? What about $\frac{2}{6}$?

Challenge 54.8: Which of the following fractions give infinitely long decimal expansions? (We've done some of these already.)

$$\frac{1}{2} \quad \frac{1}{3} \quad \frac{1}{4} \quad \frac{1}{5} \quad \frac{1}{6} \quad \frac{1}{7} \quad \frac{1}{8} \quad \frac{1}{9} \quad \frac{1}{10} \quad \frac{1}{11} \quad \frac{1}{12}$$

MUSINGS

Musing 54.9 We saw that $\frac{1}{3} = 0.333333\ldots$ is an infinitely long decimal.

a) What must $\frac{2}{3}$ be as a decimal?

b) Compute $\frac{4}{3}$ as a decimal. Is what you get the same as $0.12\,|\,12\,|\,12\,|\,12\,|\,12\,|\ldots$?

c) Sona says that every fraction of the from $\frac{N}{3}$ is sure to be an infinitely long decimal. (Here N is some number for the numerator.) Is Sona right?

Musing 54.10 We have that $\frac{1}{3} + \frac{1}{6} = \frac{1}{2}$. (Check this fraction arithmetic.)

If you haven't already (but surely you have by now!) work out $\frac{1}{3} = 1 \div 3$ and $\frac{1}{6} = 1 \div 6$ as decimals.

Add your two answers together. Do you get 0.5?

(You don't actually! This example shows that there is something philosophically deep we need to attend to in this non-human play of infinitely long decimal representations.)

Musing 54.11

a) Compare the decimal representations of $\frac{1}{2}$ and $\frac{1}{20}$. What do you notice?
b) Compare the decimal representations of $\frac{1}{5}$ and $\frac{1}{50}$. What do you notice?
c) Compare the decimal representations of $\frac{1}{3}$ and $\frac{1}{30}$. What do you notice?

MECHANICS PRACTICE

Musing 54.12 Performing the division in a $1 \leftarrow 10$ machine show that $\frac{3}{5}$ is 0.6 as a decimal.

Musing 54.13 Compute $\frac{4}{7}$ as an infinitely long repeating decimal.

Musing 54.14 If you haven't already, compute $\frac{1}{11}$ as an infinitely long repeating decimal.

BONUS (completely optional) MUSING

(as if anything in this book is compulsory!)

Musing 54.15 Here's a very strange way to divide a number by 9.
We'll illustrate it with a specific example.

To divide 312 by 9, write out the partial sums of its digits, computed from left to right

$$\begin{array}{lcl} 3 & = & 3 \\ 3+1 & = & 4 \\ 3+1+2 & = & 6 \end{array}$$

and then read off the answer:

$$312 \div 9 = 34\ R\ 6$$

(And indeed, $312 = 9 \times 34 + 6$*.)*

In the same way:

For the number 1221 we get the sums $1 = \mathbf{1}$, $1 + 2 = \mathbf{3}$, $1 + 2 + 2 = \mathbf{5}$ and $1 + 2 + 2 + 1 = \mathbf{6}$. This bizarre method suggests

$$1221 \div 9 = 135\ R\ 6$$

which turns out to be correct.

and

For the number 20,000 we get the sums $2 = \mathbf{2}$, $2+0 = \mathbf{2}$, $2+0+0 = \mathbf{2}$, $2+0+0+0 = \mathbf{2}$ and $2+0+0+0+0 = \mathbf{2}$. This method suggests

$$20{,}000 \div 9 = 2222\ R\ 2$$

which also turns out to be correct.

One might have to perform some explosions along the way and deal with extra-large remainders.

For instance, the method suggests that $5623 \div 9 = 5|11|13$ with a remainder of 16. (Do you see this?) With explosions, this gives

$$5623 \div 9 = 623\ \ R\ \ 16$$

But a remainder of "16" corresponds to one extra group of 9 and a remainder of 7. So, we really have

$$5623 \div 9 = 624\ \ R\ \ 7$$

a) Before reading on, can you explain why this strange method works?

One way to explain the puzzle is to write $\frac{1}{9}$ as an infinitely long decimal.

b) If you haven't done so already, show that $\dfrac{1}{9} = 0.111111......$

$\frac{1}{9}$ =			.	1	1	1	1	1	1	...

If we double this expression, and triple it, and so forth, we get

$$\frac{1}{9} = 0.1111\ldots \quad \frac{2}{9} = 0.2222\ldots \quad \frac{3}{9} = 0.3333\ldots \quad \frac{4}{9} = 0.44444\ldots$$

and so on. In general, $\frac{N}{9} = 0.N|N|N|N|N$ ... for any given number N for a numerator.

To compute $312 \div 9$ we need to compute $\frac{312}{9}$. This is a decimal with 312 dots in each box to the right of the decimal point.

$312 \times \frac{1}{9}$ =			.	312	312	312	312	312	312	···

We have a lot of explosions to do!

c) Perform the explosions, working left to right.

Do you see the numbers 3 and $3+1$ and $3+1+2$ playing a key role along the way?

Do you see the answer $34.666666\cdots$ appear?

d) What is $0.66666\cdots$ as a fraction?

e) Can you see that you have just established that $312 \div 9$ is 34 with a remainder of 6?

Repeat this argument to show that

f) $214 \div 9$ equals $23 + \frac{7}{9}$, with the numbers 2, $2+1$, and $2+1+4$ playing a key role along the way.

g) $432 \div 9$ equals $47 + \frac{9}{9}$ (also known as 48), with the numbers 4, 4+3, and $4+3+2$ playing a key role.

ANOTHER BONUS!

Let's attend to an age-old question.

Is $0.9999999\ldots$ equal to 1or is it not?

Many people argue that this quantity must equal 1 because of what we observed in the previous Bonus Musing. There we saw

$$\frac{1}{9} = 0.1111\ldots \quad \frac{2}{9} = 0.2222\ldots \quad \frac{3}{9} = 0.3333\ldots \quad \frac{4}{9} = 0.44444\ldots$$

It follows that

$$\frac{9}{9} = 0.99999999\ldots$$

and we know the value of $\frac{9}{9}$. It's one!

Others argue that it cannot be 1.

$0.9 = \frac{9}{10}$ is smaller than 1
$0.99 = \frac{99}{100}$ is also smaller than 1
$0.999 = \frac{999}{1000}$ still is smaller than 1

even

$0.9999999999 = \frac{9999999999}{10000000000}$ is smaller than 1

We humans can only ever write down a finite number of 9s and every time we do so we see a value smaller than 1. Surely, with an infinite number of 9s (if we could write them down) we still have a value smaller than 1?

The issue is that we are playing a mind game. With that ellipsis, we are never actually writing down the number we are talking about!

We humans will only ever be able to experience a finite number of 9s (to give a value smaller than 1), but no experience is about the number 0.9999 …. itself.

The jury has to remain "out" on any conclusions about 0.9999 … via this argument.

Most people feel like they can imagine the quantity $0.9999\ldots$ and feel like it should have a meaningful value.

If you are one of those people, then mathematics is suggesting it has value 1 (look at $\frac{9}{9}$ again).

Some people are more cautious about assuming a mind-game quantity like $0.9999\ldots$ really “exists” in the first place and will argue that asking for its value is a moot: $0.9999\ldots$ doesn’t exist!

Mathematicians take a different approach. They interpret an ellipses as indicating “journey,” not a destination.

The numbers we humans can write down—0.9 and 0.99 and 0.999 and 0.9999 and 0.99999 and so on—are certainly getting closer and closer to the value 1. So, let’s interpret an ellipsis as

the value that the decimals we humans can experience, as indicated by the decimal number, seem to approach.

For example, $0.3=\frac{3}{10}$ and $0.33=\frac{33}{100}$ and $0.333=\frac{333}{1000}$ and so on, are getting closer and closer to the value one third. So, mathematicians are happy to write

$$0.3333333\ldots=\frac{1}{3}$$

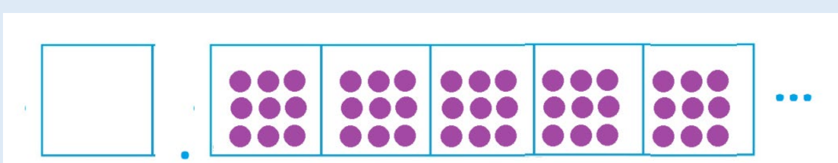
Mathematicians have checked that this line of thinking is consistent with all our rules of arithmetic, and so no surprises and contradictions will result if you play with infinite decimals via this mindset.

Question: What is $3\times 0.333333\ldots$? Do we get an answer consistent with $0.99999\ldots$ having value 1?

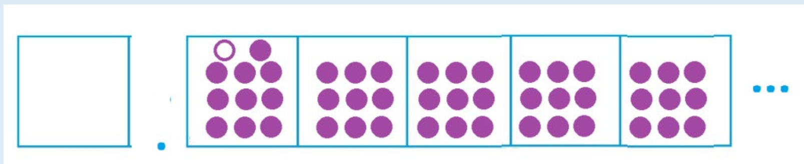
So, yes, $0.9999\ldots$ exists and “has” value 1 if you interpret the infinite decimal as a journey: Where do the numbers 0.9, 0.99, 0.999 , 0.9999, and so on, seem to be taking you?

Musing 54.16 Even Exploding Dots shows that $0.9999\ldots$ suggests a journey to the number 1.

Here’s a picture of $0.99999\ldots$ in a $1 \leftarrow 10$ machine.



Place a dot and an antidot in the first decimal place. This is technically adding nothing to the picture.



Perform an explosion. What picture do you get?

Now place a dot and antidot in the second decimal place and perform another explosion.

Do the same for the third decimal place, the fourth one, the fifth one.

What “final” picture do we seem to be approaching if we keep going?

55. Finite Decimals

The fractions $\frac{3}{10}$, $\frac{47}{100}$, and $\frac{813}{10000}$ have what are called **finite decimal expressions**: one only needs a finite number of digits to express each of them as a decimal number.

$$\frac{3}{10} = 0.3$$

$$\frac{47}{100} = 0.47$$

$$\frac{813}{10{,}000} = 0.0813$$

(Do you agree with these representations?)

In general, any fraction with denominator of either 10, 100, 1000, … has a decimal representation that stops after a finite number of places to the right of the decimal point.

Actually, any fraction that is equivalent to a fraction with denominator of either 10 or 100 or 1000 or 10000, and so on, has a finite decimal expression. For instance,

$\frac{7}{20}$ is equivalent to $\frac{7\times5}{20\times5} = \frac{35}{100}$ and so, as a decimal, is 0.35,

$\frac{131}{500}$ is equivalent to $\frac{131\times2}{500\times2} = \frac{262}{1000} = 0.262$,

and

$\frac{1}{2} = \frac{1\times5}{2\times5} = \frac{5}{10} = 0.5$.

Practice 55.1: Write $\frac{3}{5}$, $\frac{21}{250}$, and $\frac{3}{125}$ each as finite decimals.

Practice 55.2: Write $\frac{1}{2\times2\times2\times2\times2\times5\times5\times5}$ as a finite decimal.

MUSINGS

Musing 55.3 Do you think the reverse is true? If a number can be written as a decimal with only a finite number of non-zero digits to the right of the decimal point, must that number be a fraction with denominator 10 or 100 or 1000, and so on?

Musing 55.4 Do you think 0.7 and $0.70000\ldots$ represent the same number?

56. Decimal Arithmetic

Decimals have two advantages when thinking about doing arithmetic with them.

- Decimals are numbers written as codes from a $1 \leftarrow 10$ machine and so one can use all the tricks and tools of place value to conduct computations.
- Decimals can often be rewritten as fractions and so one can use all the tools of fraction arithmetic to help you out too.

For example, let's add, subtract, multiply, and divide the two numbers 0.05 and 0.006.

Addition, via place value, is straightforward.

$$\begin{array}{rl} & 0.\,0\,5 \\ + & 0.\,0\,0\,6 \\ \hline = & 0.\,0\,5\,6 \end{array}$$

Practice 56.1: As fractions, 0.05 is $\frac{5}{100}$ and 0.006 is $\frac{6}{1000}$. Check that $\frac{5}{100} + \frac{6}{1000}$ is indeed $\frac{56}{1000}$.

Subtracting these two numbers requires an unexplosion.

$$\begin{array}{rl} & 0.\,0\,5 \\ - & 0.\,0\,0\,6 \\ \hline = & 0.\,0\,5\,{-6} \end{array} \quad = \; 0.\,0\,4\,\overset{10}{-6}$$

$$= \; 0.\,0\,4\,\;4$$

Practice 56.2: Check that $\frac{5}{100} - \frac{6}{1000}$ is indeed $\frac{44}{1000}$.

Question: Which is easier for you here: place-value thinking or fraction thinking?

Multiplication this way, however, seems awkward. Hmm.

$$\begin{array}{r} 0.05 \\ \times\ 0.006 \\ \hline = \quad ??? \end{array}$$

But fraction-thinking makes it fairly straightforward.

$$0.05 \times 0.006 = \frac{5}{100} \times \frac{6}{1000}$$

$$= \frac{30}{100000}$$

$$= \frac{3}{10000} \qquad = 0.0003$$

Practice 56.3:
a) What is 23×37?

b) What is 0.023 as a fraction? What is 0.37 as a fraction?

c) Your work in parts a) and b) show that 0.023×0.37 equals $\frac{851}{100{,}000}$. Do you see how? What is 0.023×0.37 as a decimal?

d) What is 2.3×3.7 and 230×0.037?

If you want to multiply two decimal numbers by hand, it does seem like a good move to convert the numbers into fractions first.

One typically doesn't want to do such work by hand. But every now and then you might find yourself doing more tractable problems this way.

Practice 56.4: What is 0.7×0.004? (Do you see "28" in your head right away?)

Dividing our two decimal numbers directly seems awful.

$$0.006\overline{)\,0.05}$$

ICK!

But doing it via fraction thinking is fine!

$$0.05 \div 0.006 = \frac{0.05}{0.006} \quad = \frac{\frac{5}{100}}{\frac{6}{1000}}$$

$$= \frac{\frac{5}{100} \times 1000}{\frac{6}{1000} \times 1000}$$

$$= \frac{50}{6} \quad = \frac{25}{3} \quad = 8\frac{1}{3}$$

And we can write the final answer as $8.33333\ldots$. if we wish.

It's a good move too to convert decimal numbers into fractions first if you want to divide them.

(Of course, the best move of all most of the time is to just use a calculator!)

Practice 56.5

a) What is $0.21 \div 0.003$?

b) What is $0.021 \div 0.3$?

c) What is $2.1 \div 3$?

Some more practice.

Practice 56.6: Compute $13.276 + 5.94$ and $13.276 - 5.94$.

Practice 56.7:

a) Agatha says that computing $0.0348 + 0.0057$ is essentially a matter of adding 348 and 57. What does she mean by this? Is she right?

b) Percy says that computing $0.0852 + 0.037$ is essentially a matter of adding 852 and 37. He is not right. What is wrong with Percy's thinking?

Practice 56.8:

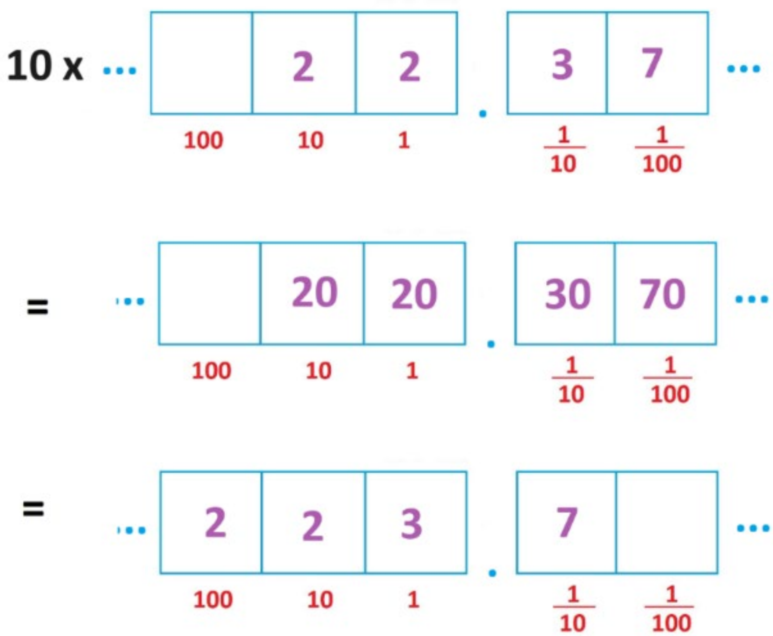
a) What is $\frac{1}{5} \times 0.02$?

b) What is $\frac{1}{5} \div 0.02$?

Multiplying and Dividing Decimals by 10.

Musing 53.11 had us multiply 22.37 by 10. We got the answer 223.7, making it appear that the decimal point magically shifted place.

Of course, it is really the explosions of dots in a $1 \leftarrow 10$ machine that give this illusion.



We can also see this if we write out the number 22.37 in full "expanded form" (as schoolbooks call it).

$$22.37 \times 10 = \left(20 + 2 + \frac{3}{10} + \frac{7}{100}\right) \times 10 = 200 + 20 + 3 + \frac{7}{10} = 223.7.$$

We can also readily divide 22.37 by ten (that is, multiply it by $\frac{1}{10}$) this way too.

$$22.37 \div 10 = \left(20 + 2 + \frac{3}{10} + \frac{7}{100}\right) \times \frac{1}{10} = 2 + \frac{2}{10} + \frac{3}{100} + \frac{7}{1000} = 2.237.$$

Both answers involve the same digits 2-2-3-7 in order.

To multiply or divide decimal numbers by 10 and 100 and 1000 and so on,
it's always good write out the decimals as fractions.

Example: Compute $0.9 \div 100$.

Answer: This is $\frac{0.9}{100} = \frac{\frac{9}{10}}{100} = \frac{\frac{9}{10}\times 10}{100\times 10} = \frac{9}{1000} = 0.009$.

Practice 56.9: Consider the "abstract" decimal $0.abc$.

a) Be clear on why $0.abc \div 10$ equals $0.0abc$.
b) Be clear on why $0.abc \div 100$ equals $0.00abc$.
c) Be clear on why $0.abc \div 0.1$ equals $a.bc$.
d) Compute $0.abc \div 0.01$.
e) Compute $0.abc \times 100$.
f) Compute $0.abc \times 0.01$.

Multiplying a number by 10 should make it ten times as big. (That's one of those "duh" comments!)

Dividing a number by 10 (that is, multiplying it by $\frac{1}{10}$) will make it ten times as small.

So …

A number in the 200s multiplied by ten will be in the 2000s.

A number in the 400s divided by ten will be in the 40s.

A number close to 15 multiplied by ten will be close to 150.

A number close to 15 divided by ten will be close to 1.5.

Also, we've seen in a $1 \leftarrow 10$ machine that multiplying a decimal number by ten produces an answer with the same digits—the digits "shifted" because of explosions. (Actually, we can do unexplosions too in order to divide by ten and see a shift of digits again.)

We can see this to by working directly with fractions. For example,

$$5.67 \times 10 = \left(5 + \frac{6}{10} + \frac{7}{100}\right) \times 10$$

$$= 50 + 6 + \frac{7}{10} = 56.7$$

$$5.67 \div 10 = \left(5 + \frac{6}{10} + \frac{7}{100}\right) \times \frac{1}{10}$$

$$= \frac{5}{10} + \frac{6}{100} + \frac{7}{1000} = 0.567$$

These observations allow us to deduce the values of decimal numbers multiplied or divided by ten quite swiftly.

Example: What is $307.231 \times 10 \times 10$? What is $307.231 \times \frac{1}{10}$?

Answer: Both problems will produce answers that involve the digits 3-0-7-2-3-1 in order.

We are starting with a number in the 300s.

Multiplying by ten twice should give us a number in the 30,000s. We deduce

$$307.231 \times 10 \times 10 = 30{,}723.1$$

Dividing by ten once should give an answer the 30s. We deduce

$$307.231 \times \frac{1}{10} = 30.7231$$

Example: What is $307.231 \times 10 \times 10 \times 10 \times 10 \times 10$? (That is, what is $307.231 \times 100{,}000$?)

Answer: We have a number in the 300s being multiply by ten five times.

This means we'll have and answer in the ~~3,000~~ ~~30,000~~ ~~300,000~~ ~~3,000,000~~ 30,000,000s.

The answer must be 30,723,100.

Practice 56.10: Write down the values of each of these computations.

a) 483.014×10
b) $483.014 \times 10 \times 10$
c) $483.014 \times 10{,}000$
d) $483.014 \times \frac{1}{10}$
e) $483.014 \times \frac{1}{10} \times \frac{1}{10}$
f) $483.014 \times \frac{1}{10{,}000}$

Let's end this Section with something quirky.

Example: Find 99 fractions that lie between $\frac{1}{11}$ and $\frac{1}{12}$.

Answer: Here are some that work!

$$\frac{1}{11.01} \quad \frac{1}{11.02} \quad \frac{1}{11.03} \quad \cdots \quad \frac{1}{11.99}$$

(Are these in increasing or decreasing in size when reading left to right?)

If you don't like how these look, you can always rewrite them as more traditional fractions. For instance,

$$\frac{1}{11.01} = \frac{1}{11 + \frac{1}{100}} = \frac{1 \times 100}{(11 + \frac{1}{100}) \times 100} = \frac{100}{1100 + 1} = \frac{100}{1101}$$

Practice 56.11: Write down 999 fractions that lie between $\frac{1}{11}$ and $\frac{1}{12}$. List them in increasing order.

Practice 56.12: Compute each the following. (Or not! These each look very ugly!)

a) $0.3 \times (5.37 - 2.07) + \frac{0.75}{2.5}$

b) $\frac{0.1+(1.01-0.1)}{0.11+0.09}$

c) $\frac{(0.002+0.2\times 2.02)(0.22-0.02)}{2.22-0.22}$

Actually, this final practice problem brings up a good point.

The fraction bar looks like a vinculum, but, historically, it isn't.
Nonetheless, we follow the convention of treating it like one, as its own symbol of grouping.

When presented with a complicated expression in the form of a fraction, treat the numerator as a quantity to be computed in its own right and treat the denominator as a quantity to be computed in its own right.

You probably naturally did this if you tried the practice problem.

MECHANICS PRACTICE

Practice 56.13 Ask Siri or Alexa or some virtual friend to compute each of the following.

$$11.9 + 1.7 \qquad 11.9 - 1.7 \qquad 11.9 \times 1.7 \qquad 11.9 \div 1.7$$

Feel free to ty them by hand too to see if you get the same answers. (Actually, can you do the first two in your head?)

Practice 56.14 Compute each of the following.
(Before you commence, make an estimate of each answer you should get.)

$$\begin{array}{r} 33.156 \\ +\ 8.814 \\ \hline = \quad\quad\quad \end{array} \qquad\qquad \begin{array}{r} 2.03 \\ -\ 0.777 \\ \hline = \quad\quad\quad \end{array}$$

Practice 56.15 What is a natural way to see that $1.45 - 0.96$ is 0.49?

Practice 56.16
a) What is 0.41 as a percentage?
b) What is 58% as a decimal?
c) What is 0.001 as a percentage?
d) What fraction is $33\frac{1}{3}\%$?
e) What fraction is 0.12%?

Practice 56.17 Compute each of the following

a) 1.5×1.5
b) 1.4×1.4
c) 1.45×1.45
d) 1.42×1.42

What I am trying to do here is come up with a decimal number with the property that if I square it, it will give the answer 2.

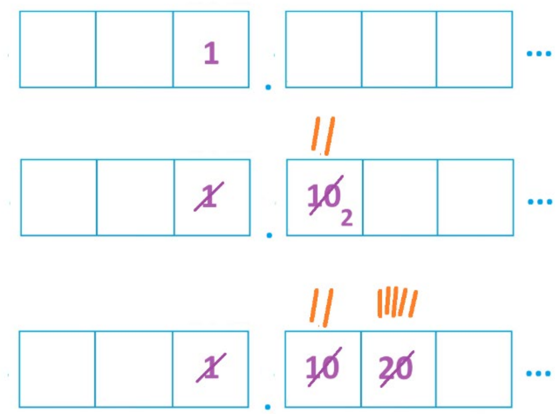
e) Do you see that 1.4 is too small a number for this and 1.42 is too big? What about 1.41? Is that too big or too small?

f) Care to pin down what the digit in the third decimal place for such a number must be?

57. Every Fraction is a Repeating Decimal

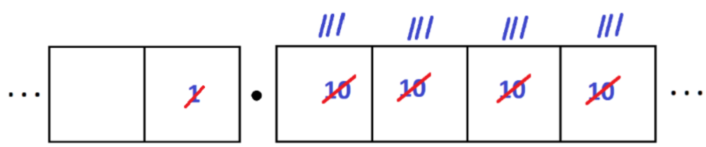
We've used division in a $1 \leftarrow 10$ machine to rewrite fractions as decimals.

For example, we saw that $\frac{1}{4}$, computed as $1 \div 4$, has decimal representation 0.25.



Other fractions have infinitely long decimal expansions. For example, we computed $\frac{1}{3}$ as $1 \div 3$ and saw

$$\frac{1}{3} = 0.33333\ \ldots$$



And we saw too that

$$\frac{6}{7} = 0.857142\ \ 857142\ \ 857142\ \ \ldots$$

with a have a repeating pattern of "857142."

Back in Section 7 we got ahead of ourselves and talked about the use of the **vinculum** (a horizontal bar) throughout mathematics. We mentioned its use in decimals with repeating patterns.

Rather than write out blocks of digits that repeat a few times and slapping on an ellipsis to mean "keep this pattern going," people sometimes put a vinculum over the repeating group with the understanding that group of digits is repeating indefinitely.

For example, folk write

$$\frac{6}{7} = 0.\overline{857142}$$

and

$$\frac{1}{3} = 0.\overline{3}$$

for the two fractions we just considered.

An expression such as $0.38\overline{142}$ means "repeat the group 142 indefinitely after the beginning hiccup of 38."

$$0.38\overline{142} = 0.38\ \ 142\ \ 142\ \ 142\ \ 142\ \ldots$$

All the examples of fractions with infinitely long decimal expansions we've seen so far fall into a repeating pattern. This is curious.

We can even say this is the case too for our finite decimal examples: they fall into a repeating pattern of zeros after an initial start.

$$\frac{1}{4} = 0.2500000\ldots = 0.25\overline{0}$$

$$\frac{1}{2} = 0.50000\ldots = 0.5\overline{0}$$

(After all, one quarter is 2 tenths and 5 hundredths and 0 of every other decimal place-value thereafter, and one half is indeed 5 tenths and 0 of every other decimal place-value.)

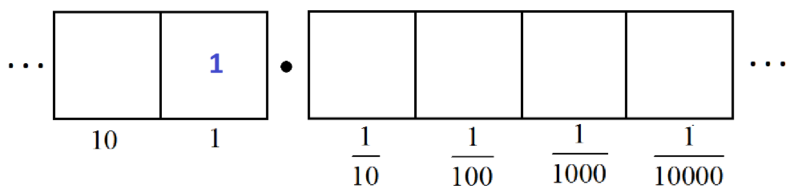
This begs the question:

> Does every fraction have a decimal representation that eventually repeats (allowing repeating zeros)?

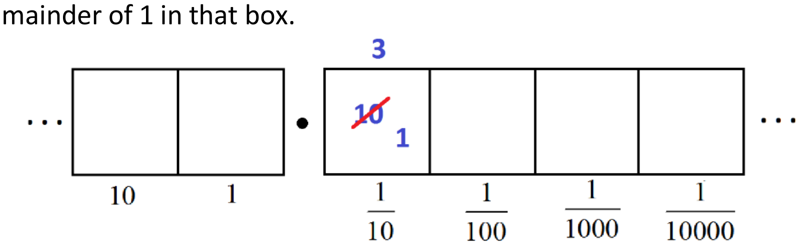
The answer to this question, surprisingly, is yes, and our method of division explains why.

Let's go through the division process again, slowly, first with a familiar example. Let's compute the decimal expansion of $\frac{1}{3}$ again in a $1 \leftarrow 10$ machine.

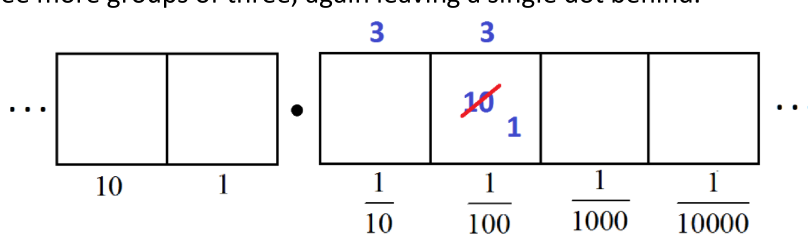
We think of $\frac{1}{3}$ as the answer to the division problem $1 \div 3$, and so we need to find groups of three within a diagram of one dot.



We unexplode the single dot to make ten dots in the tenths position. There we find three groups of three leaving a remainder of 1 in that box.



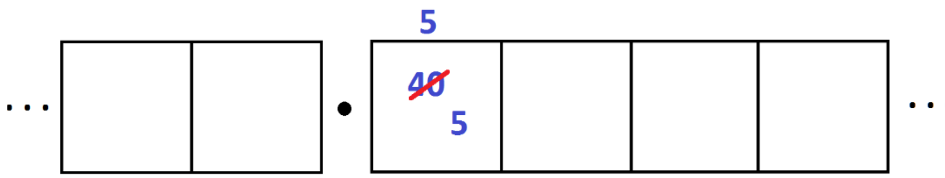
Now we can unexploded that single dot in the tenths box and write ten dots in the hundredths box. There we find three more groups of three, again leaving a single dot behind.



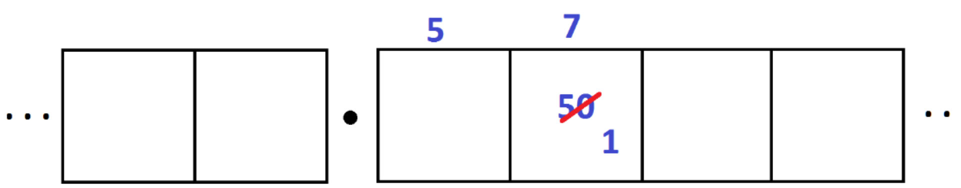
And so on. We are caught in a cycle of having the same remainder of one dot from cell to cell, meaning that the same pattern repeats. Thus, we conclude $\frac{1}{3} = 0.333\ldots$ The key point is that the same remainder of a single dot kept appearing.

Let's compute the decimal expansion of $\frac{4}{7}$ in the $1 \leftarrow 10$ machine. That is, let's compute $4 \div 7$ and be sure to take note of the remainders that occur.

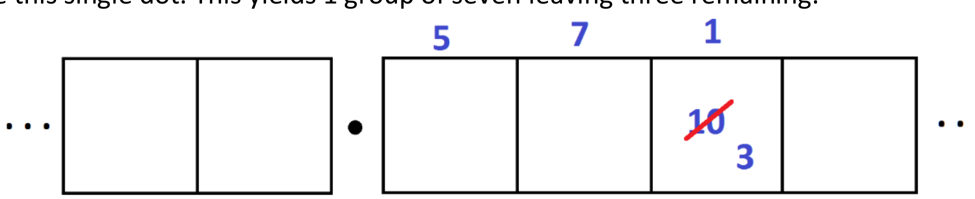
We start by unexploding the four dots to give 40 dots in the tenths cell. There we find 5 groups of seven, leaving five dots over.



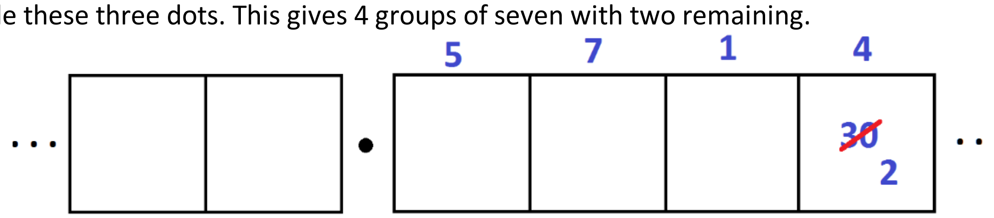
Now unexplode those five dots to make 50 dots in the hundredths position. There we find 7 groups of seven, leaving one dot over.



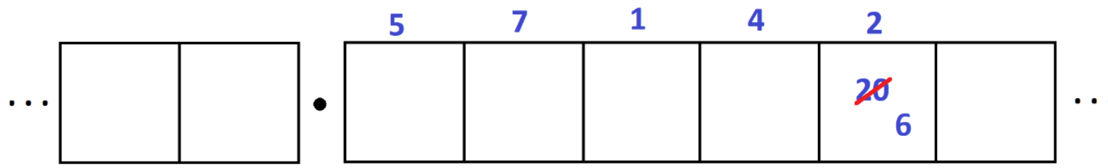
Unexplode this single dot. This yields 1 group of seven leaving three remaining.



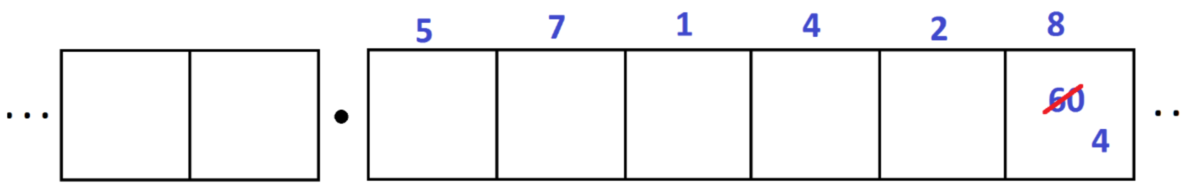
Unexplode these three dots. This gives 4 groups of seven with two remaining.



Unexplode the two dots. This gives 2 groups of seven with six remaining.



Unexplode the six dots. This gives 8 groups of seven with four remaining.



But this is the predicament we started with: four dots in a box!

So now we are going to repeat the pattern and produce a cycle in the decimal representation. We have

$$\frac{4}{7} = 0.571428\ \ 571428\ \ 571428\ \ \ldots$$

Stepping back from the specifics of this problem, it is clear now that one must be forced into a repeating pattern. In dividing a quantity by seven, there are only seven possible values for a remainder number of dots in a cell—0, 1, 2, 3, 4, 5, or 6—and there is no option but to eventually repeat a remainder and so enter a cycle.

In the same way, the decimal expansion of $\frac{18}{37}$ must also cycle. In doing the division, there are only thirty-seven possible remainders for dots in a cell (0, 1, 2, …, 36). As we conduct the division computation, we must eventually repeat a remainder and again fall into a cycle.

We have just established a very interesting fact.

Every fraction has a decimal representation that falls into a repeating pattern. (A pattern of repeating zeros is allowed.)

Practice 57.1 As a check, conduct the division procedure for the fraction $\frac{1}{4}$. Make sure to understand where the cycle of repeated remainders commences.

MUSINGS

Musing 57.2 Find the decimal representation of $\frac{23}{45}$. (After a "hiccup," its decimal representation repeats just one digit over and over again. Which digit?)

Musing 57.3 The fraction $\frac{1}{7}$ has a repeating decimal representation with a repeating block of six digits.

$$\frac{1}{7} = 0.\overline{142857}$$

Do you think it is possible for a fraction of the form $\frac{b}{7}$ (with b a counting number) to have a decimal representation with a repeating block of digits ten digits long? Eight digits long? Seven digits long?

Musing 57.4 BACKWARDS: Is Every Repeating Decimal a Fraction?

Consider the repeating decimal $0.6363636....$. Is this number a fraction? If so, which one?

A popular technique for attending to this issue starts by giving the quantity a name and to repeatedly multiply the quantity by ten. Let's call the decimal Cecile. (Why not?).

We have

$$\begin{aligned} C &= 0.63636363\ldots \\ 10 \times C &= 6.36363636\ldots \\ 100 \times C &= 63.63636363\ldots \end{aligned}$$

Let's stop here since the infinite parts of C and $100 \times C$ align perfectly.
Let's subtract them.

$$\begin{array}{rl} 100 \times C = & 63.\,6\,3\,6\,3\,6\,3\,6\,3\,\ldots \\ C = & 0.\,6\,3\,6\,3\,6\,3\,6\,3\,\ldots \\ \hline 99 \times C = & 63.\,0\,0\,0\,0\,0\,0\,0\,0\,\ldots \end{array}$$

We see that one hundred Cs take away one C, that's ninety nine Cs, must equal 63.

$$99 \times C = 63$$

Ahh! C must be the fraction $\frac{63}{99} = \frac{9\times7}{9\times11} = \frac{7}{11}$.

a) Use this technique to show that 0.111111 .... is the fraction $\frac{1}{9}$.

b) Show that 0.213213213213213 ... is the fraction $\frac{213}{999}$

c) What fraction is 0.2111111 ... ?
(Keep multiplying this number by 10 until you have two decimal parts that align.)

d) What fraction is 2.8213213213213213 ...?

This technique shows that if a decimal number (eventually) has a repeating pattern, then we can keep multiplying that number by ten and until we find two multiples whose decimal parts align perfectly. Subtraction then allows us to identify that decimal number as a fraction.

58. A Decimal that Does Not Repeat is not a Fraction

We established in the previous section

Every fraction has a decimal representation that falls into a repeating pattern (perhaps a repeating pattern of zeros).

People don't usually bother writing out a repeating pattern of zeros: writing $\frac{1}{2} = 0.5$, for instance, rather than $\frac{1}{2} = 0.50000 \ldots$ or $\frac{1}{2} = 0.5\overline{0}$.

People call decimals that have repeating zeros **finite decimals** because they can be expressed with only a finite number of digits.

For example, $276\frac{1}{2}$ and $0.0000000000000000000000000000001$ are finite decimals.

Practice 58.1 Do you think the number 3 could be called a finite decimal?

This now opens up a curious idea.

A quantity given by a decimal expansion that does not repeat cannot be a fraction.

Pause! Do you get the logic here?

Question: Consider the statement: "Every crow is a black bird."
Does it logically follow that if a bird is not black, it is not a crow? (Are there albino crows?)

If the statement "Every Australian is cheery" is true, then what can you say about a non-cheery person you meet?

If "Every fraction has a repeating decimal expansion" is true (and it is!), what can you say about a number that has a decimal expansion that never repeats?

Consider this decimal number

$$0.1011001110001111000011111000000\ldots$$

Even though we see a pattern to its decimal expansion (which allows us to figure out any particular decimal digit we want just by writing out the pattern far enough), it is not a repeating pattern.

This means that this number cannot be a fraction!

Question: Whoa! Pause again! Take this in.

The quantity $0.10110011100011110000\ldots$ is a bit bigger than 0.1, which is $\frac{1}{10}$, and so is just to the right of one tenth on the number line. It's a number!

Yet it's a number that cannot be a fraction: it doesn't have a repeating decimal representation.

Do you find this freaky?

We can invent all sorts of numbers that can't be fractions.

For example,

$$0.10203040506070809010011012013014015\ldots$$

and

$$0.30300300030000300000300000030000003\ldots$$

are numbers that are not fractions.

(Do you see a pattern in each of these examples? Do you see that neither is a repeating pattern?)

Recall that people call numbers that are either positive fractions or negative fractions **rational numbers**. Any number that cannot be a fraction, like the ones we are creating now, are called **irrational numbers**.

Irrational numbers can be positive and be negative. For example, $-0.10110011100011110000\ldots$ is a negative irrational number.

MUSINGS

Musing 58.2 Write down two infinitely long decimal expansions that you personally know cannot be rational numbers.

Musing 58.3 Write down a number slightly larger than $\frac{1}{3}$ that is not a fraction.

Musing 58.4 Could a number slightly larger than $\frac{1}{3}$ that is not a fraction and number slightly smaller than $\frac{1}{3}$ that is also not a fraction add to a number that is a fraction?

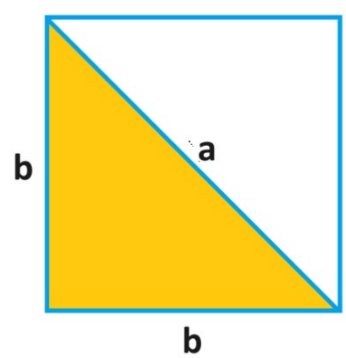
59. VERY OPTIONAL ASIDE: A Historically Famous Example a Number that is not a Fraction

The Pythagoreans of some 2500 years ago believed that all that is good and harmonious in the world can be expressed mathematically via a counting number or via a comparison of two counting numbers.

For example, simultaneously plucking two identical strings under the same tension, but with one sting twice the length of the other produces two notes that sound harmonious to the ear. (They make the musical interval of one *octave*.) Two identical strings, one 3 units long and the other 2 units long plucked simultaneously produce a pleasing *perfect fifth*, and one 5 units long the other 4 units long a pleasing *major third*, and so on.

Mathematics was central to the worldview of the Pythagoreans. They sought to describe the universe in terms of number (counting numbers) and geometry to such a degree that many people today regard their academic pursuits as tied to a religious cult.

A fundamental shape in geometry is a square. Surely, the two fundamental lengths in a square—the length of any one of its sides and the length of its diagonal—are "in harmony"? That is, if you choose the right basic unit of length, surely you can say the diagonal of the square is a units long while its sides are b units long with a and b both counting numbers?



The Pythagoreans were utterly shocked to eventually learn that this is not so!

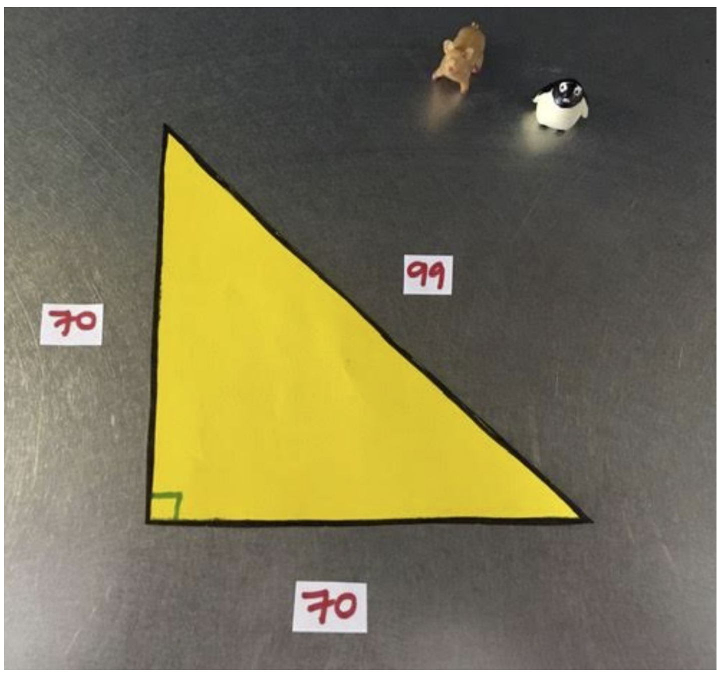
Their worldview was truly shattered.

There are a number of ways to show that there cannot be two counting numbers that describe the two fundamental lengths in a square.

My personal favorite approach is a physical one—cutting out a square in paper, folding it, and thinking about what the folding means.

I did this as an activity and took photos of my work. I started with paper square 70 centimeters in side length, which I then cut diagonally in half. (Two figurines, a pig and a penguin, assisted me.)

What's lovely about this choice of side length is that is diagonal is close, very close, to being 99 centimeters long, so close that my human eye cannot tell that it is not.

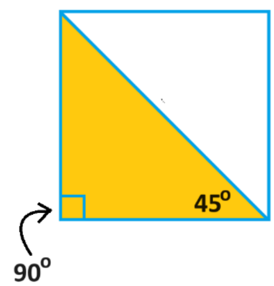


Our goal is to show that believing the diagonal has length a counting number too, such as 99, just cannot be so.

We need to show that something is terribly wrong with this picture.

Comment: As we proceed, we will need to one fact from geometry class, namely:

> Any triangle that comes from cutting a square in half diagonally has a 90-degree angle and a 45-degree angle. And, in reverse, any triangle that has a 90-degree angle and a 45-degree angle is half a square.



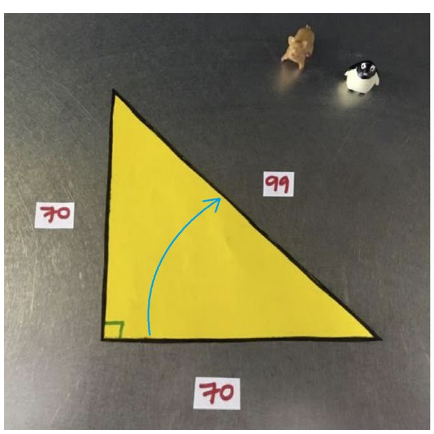
I hope this at least feels intuitively right to you.

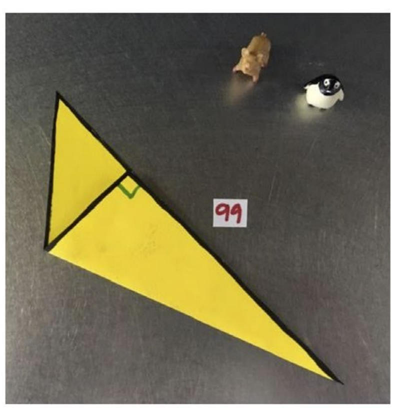
Okay, back to our triangle, which is half a square with sides allegedly of lengths 70, and 70, and 99 centimeters.

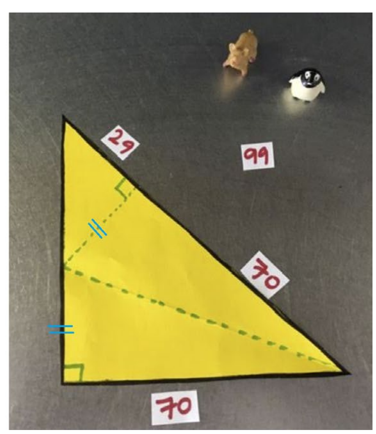
Fold the bottom edge of the triangle up to the hypotenuse of the triangle and draw in the lines to show the edges we created when doing that.

Notice two things:

- The two lengths marked with the double dashes in the third photo below are the same length. (We see that from the folding in the middle photograph.)
- We can identify an edge of length 29 centimeters because we folded an edge of length 70 centimeters up against an edge of length of 99 centimeters. (Again, look at the middle photograph.)



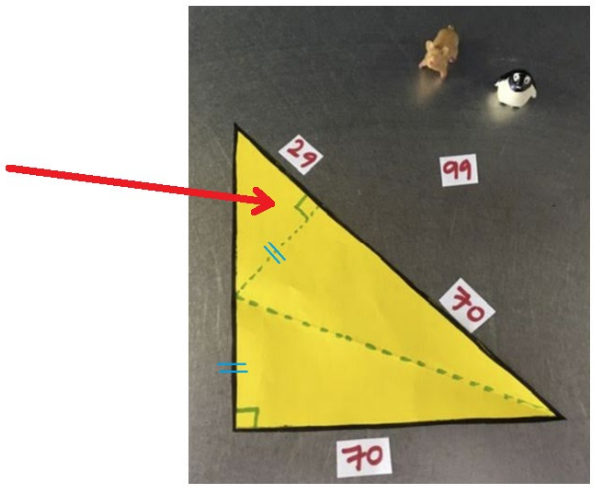




Actually, there is a third important thing to notice.

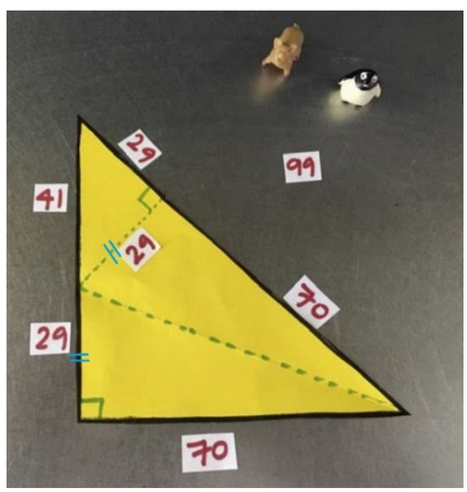
We see in our third photograph a little triangle with top angle 45 degrees (from the original big triangle that is half a square) and with a 90-degree angle (from folding a 90-degree angle up to the hypotenuse of the big triangle.)

This little triangle at the top must be half of its own square.



One side of the smaller square is 29 centimeters long. So, the other side of the square is 29 centimeters long as well.

And since the two lengths marked with a double dash are the same, we have three lengths of 29 centimeters in our picture!

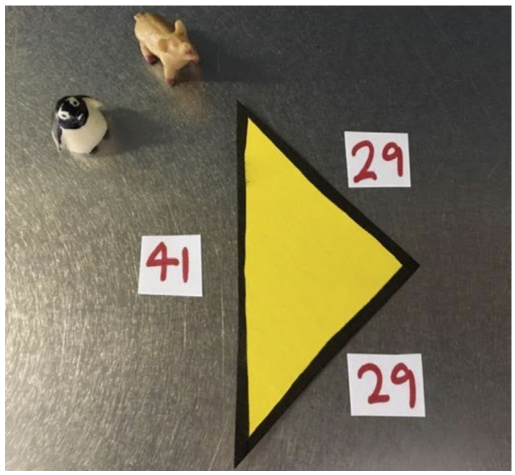


And we can mark a length of 41 centimeters as well: the left edge of the big triangle is 70 centimeters long, and $70 - 29 = 41$.

Everything we just did came from combining the two counting numbers 99 and 70 we started with via subtraction to produce new counting numbers.

$$29 = 99 - 70$$
$$41 = 70 - 29$$

And these new, smaller counting numbers are the side-lengths of another triangle that is half a square. And we see from the paper, this new triangle is much smaller than the original triangle.



Your Turn:

Cut out a 7 inch by 7 inch square from a piece of paper.
Cut your square in half diagonally to make a triangle.

With a ruler, verfiy that the length of the long edge of your triangle is very close to 10 inches.

Using the numbers 7, 7, and 10, follow the folding and thinking outlined above to create a smaller traingle that must also be half a square. If we believe those counting numbers are accurate what are the side lengths of your smaller half square? (You should be reasoning that they are 3, 3, and 4 inches long.)

What we are doing is worrisome.

We are seeing that if you have a triangle that is half a square with side lengths each a counting number, then you can fold that triangle to create a smaller half square also counting numbers as side lengths.

And each new half-square triangle we create is demonstrably smaller than the triangle we started with.

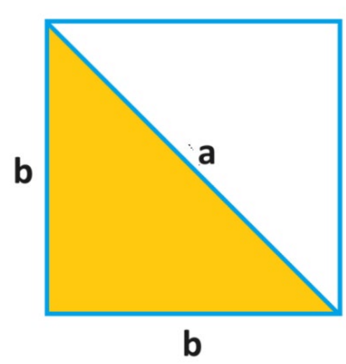
> **Practice 59.1** Right now, in my photos, we allegedly have a half square with side lengths 29, 29, and 41 centimeters—all counting numbers.
>
> If we apply the folding procedure on this triangle, what are the (alleged) dimensions of the even smaller half square we create?

If we do this folding process over and over and over and over and over again will eventually obtain a half-square smaller than an atom, all the while giving us counting number side lengths. The side-lengths can't ever be zero—we do have a triangle of some size—but no triangle smaller than an atom can have counting number side lengths!

Believing that we had counting number side lengths to begin with puts us into a logical pickle. Something is indeed terribly wrong!

The only way out of this pickle is to conclude that believing we had nothing but counting numbers to begin with is wrong. (Even though we assumed we had the counting numbers 70, 70, and 99, any set of beginning counting numbers will lead to this pickle. We'd again create a triangle smaller than an atom but still with counting number side lengths, allegedly!)

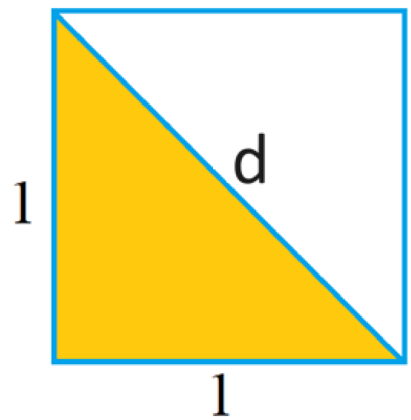
There are no counting numbers a and b that describe the lengths in this picture.



The side length of a square and the diagonal length of a square are in discord!

Let's rephrase what we just concluded in our modern setting.

Consider a square with side length 1 unit. Then the diagonal has some length. Call it d units.



Now, d is a number. It has some value.

Question: Getting ahead of ourselves again …
Do you remember the Pythagorean Theorem from geometry class?

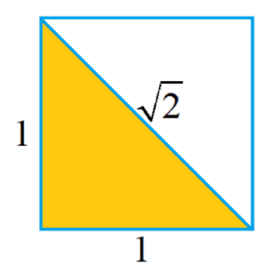
We have a right triangle in our picture, and we see by the Pythagorean Theorem that

$$1^2 + 1^2 = d^2$$

This tells us that $d^2 = 2$.

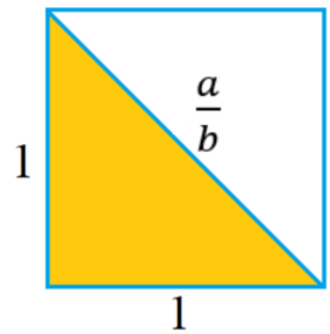
Thus, d is a number that multiplies by itself to give the value 2.
People call that the "square root" of 2.

$$d = \sqrt{2}$$

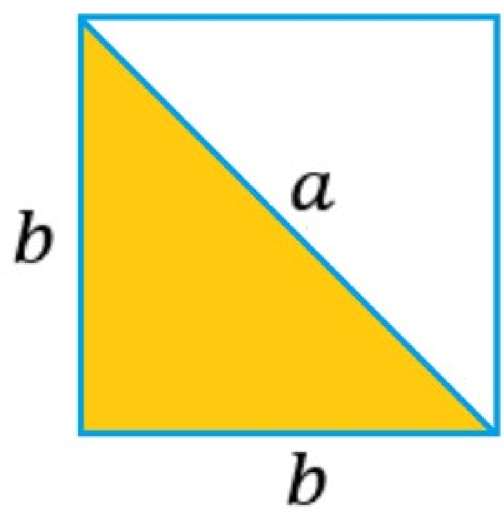


We have essentially just demonstrated that the number d, whatever it is, cannot be a fraction.

For if $d = \frac{a}{b}$ for two counting numbers a and b,



then we can scale our picture up by a factor b (have all the lengths grow to b times as big: $b \times 1 = b$ and $b \times \frac{a}{b} = a$) and obtain a half square with counting numbers lengths.



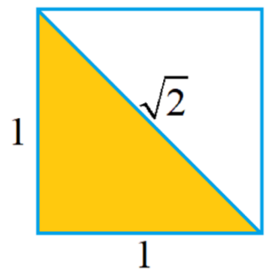
And we just proved that that cannot be!

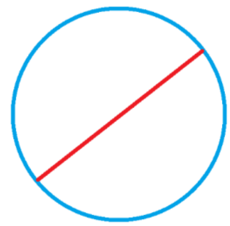
The number d, the length of the diagonal of a square with side length 1 unit, whatever that value is, must be an irrational number.

Irrational numbers exist in the real world!

Side Comment: Schoolbooks want students to "know" that two famous numbers in mathematics are irrational. These numbers are:

- $\sqrt{2}$, the length of the diagonal of a square with side length one unit
- π (pi) the number that arises if you take the circumference of a circle and divide it by its width.





$$\pi = \frac{\text{circumference}}{\text{diameter}}$$

We just went to an awful lot of effort to demonstrate that $\sqrt{2}$ is an irrational number. It is not at all "obvious" that $\sqrt{2}$ is not a fraction.

Matters are more challenging for the number π.

In fact, scholars wondered for millennia whether or not π is a fraction. They calculated its decimal expansion to many hundreds of counts of digits and saw no pattern or structure to those digits. They found fractions that approximated the value of π very closely, and developed methods for creating more fractions that would approximate it as closely as one pleases. But whether or not π itself is a fraction (with some gigantically large numerator and some gigantically large denominator) remained a frustrating mystery for centuries and centuries.

It wasn't until around the year 1761 that Swiss mathematician Johann Lambert was finally able to establish, once and for all, that π is an irrational number. It is not a fraction.

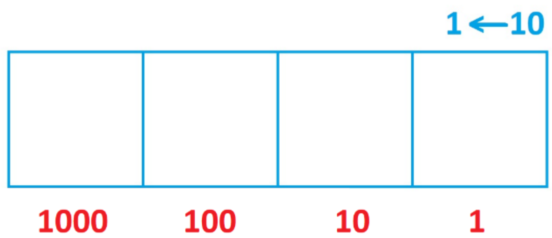
The proof of this is very hard, and well beyond the school curriculum. It is extremely far from "obvious" that π is not a fraction.

So, when school curriculums say "Students should know that $\sqrt{2}$ and π are examples of irrational number" they really mean, "Students should <u>be told</u> that $\sqrt{2}$ and π are examples of irrational number."

Question: Did your schoolbooks ever explain <u>why</u> $\sqrt{2}$ is an irrational number?

60. The Powers of Ten

We saw that dots in a $1 \leftarrow 10$ machine are worth 1 in the rightmost box and then have values 10, 100, 1000, and so on, as we move through the places to the left. These values grow by a factor of ten from one box to the next.



We have

$$
\begin{aligned}
&1 = 1\\
&10 = 1 \times 10\\
&100 = 1 \times 10 \times 10\\
&1000 = 1 \times 10 \times 10 \times 10\\
&10000 = 1 \times 10 \times 10 \times 10 \times 10
\end{aligned}
$$

and so on.

Rather than repeatedly write out products of ten, the mathematics community has settled on using superscripts to denote the result of repeatedly multiplying the number 1 by a fixed value.

If n is a counting number, then for any number a the notation a^n means

$$1 \times \overbrace{a \times a \times \cdots \times a \times a}^{n \text{ of these}}$$

We read a^n as a **raised to the nth power**.

For example, two raised to the third power is the number 1 doubled three times,

$$2^3 = 1 \times 2 \times 2 \times 2 = 8$$

and ten raised to the sixth power is the number 1 increased by a factor of ten, six times

$$10^6 = 1 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10 = 1{,}000{,}000$$

This gives a million.

We saw this notation way back in Section 10 when we were chopping up squares and cubes.

We had

$$5^2 = 5 \times 5$$

saying "five squared," and

$$5^3 = 5 \times 5 \times 5$$

saying "five cubed." (And we didn't talk about 5^4 and 5^5, and so on because we don't have everyday language for ideas that go beyond the third dimension!)

Question: Do we have a discrepancy here?

Shouldn't we have $5^2 = 1 \times 5 \times 5$ and $5^3 = 1 \times 5 \times 5 \times 5$?

Does it matter if we ignore the 1 that is meant to be up front?

You might see in some schoolbook authors defining a^n as

$$\overbrace{a \times a \times \cdots \times a \times a}^{n \text{ of these}}$$

without a 1 up front. This makes no difference if n is a counting number different from zero.

Practice 60.1

a) Does defining a^n as $1 \times \overbrace{a \times a \times \cdots \times a \times a}^{n \text{ of these}}$ make sense if n happens to be zero? (If so, what is the value of a^0?)

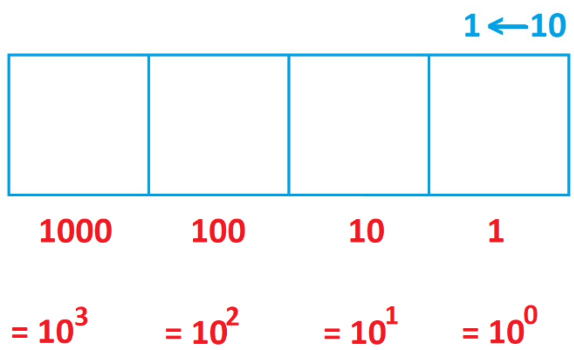
a) Does defining a^n as $\overbrace{a \times a \times \cdots \times a \times a}^{n \text{ of these}}$ make sense if n happens to be zero? (If so, what is the value of a^0?)

Thinking of 10^n as

$$1 \times \overbrace{10 \times 10 \times \cdots \times 10 \times 10}^{n \text{ of these}}$$

it is clear, for us, that 10^0 is 1.

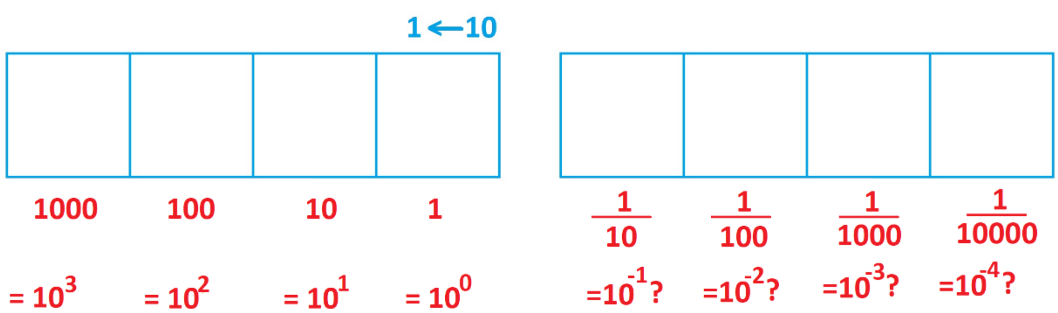
And this is nice as we can now say that all the place values in our $1 \leftarrow 10$ machine are powers of ten.



But what about the boxes to the right, the decimal places?

It seems irresistible then to keep the powers-of-ten pattern going: from 10^3, 10^2, 10^1, and 10^0 down into 10^{-1}, 10^{-2}, 10^{-3}, and so on.

Can we say this?



Mathematicians have settled on a second piece of convenient notation.

If n is a counting number, then for any number a (not zero) the notation a^{-n} means

$$1 \times \overbrace{\frac{1}{a} \times \frac{1}{a} \times \cdots \times \frac{1}{a} \times \frac{1}{a}}^{n \text{ of these}}$$

We read a^{-n} as a **raised to the negative nth power**.

This notation is motivated by the following idea:

Since 10^1 means (for us) "multiply the number 1 by ten," it feels like 10^{-1} should be the opposite of this, which would be: "divide the number 1 by ten." And dividing by ten, as we know, is the same as multiply by $\frac{1}{10}$.

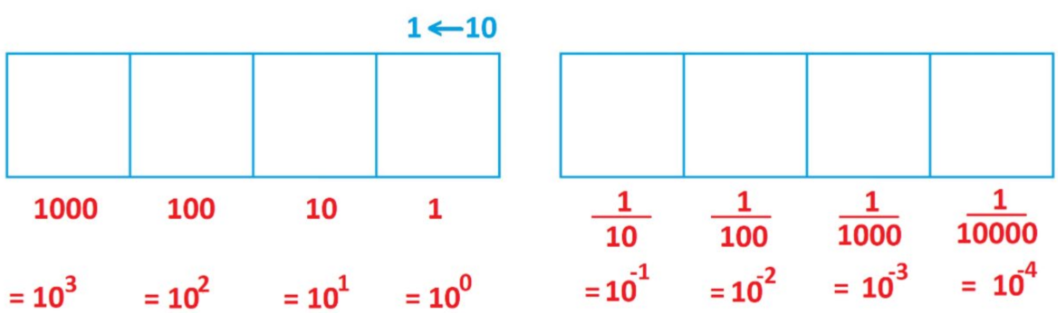
$$10^{-1} = 1 \times \frac{1}{10} = \frac{1}{10}$$

And since 10^2 means "multiply the number 1 by ten, twice," it feels like 10^{-2} should be the opposite of this, "divide the number 1 by ten, twice." That would be multiplying the number 1 by $\frac{1}{10}$, twice.

$$10^{-2} = 1 \times \frac{1}{10} \times \frac{1}{10} = \frac{1}{100}$$

And so on.

This notation is convenient as we can now represent each place value in our $1 \leftarrow 10$ machine using this notation.



Comment: We are not doing any mathematics with these powers 10^n and 10^{-n}. Both are just shorthand for writing out repeated multiplications

$$10^n = 1 \times \overbrace{10 \times 10 \times \cdots \times 10 \times 10}^{n \text{ of these}}$$

$$10^{-n} = 1 \times \overbrace{\frac{1}{10} \times \frac{1}{10} \times \cdots \times \frac{1}{10} \times \frac{1}{10}}^{n \text{ of these}}$$

But there is a slew of mathematics one can explore with such repeated products, and that mathematics agrees with the notation we happen to be using here. (We'll talk about the mathematics of "powers," for sure, in our next volume of chapters.)

For us now, this notation will just be notation, nothing more.

And this notation will help us write really really big numbers and really really small numbers in a $1 \leftarrow 10$ machine with ease.

MUSINGS

Musing 60.2 Recall that nine-year-old Milton Sirotta in 1938 coined the term *googol* for the number 1 with one-hundred zeros after it ($10000 \cdots 00$) and the term *googolplex* for the number 1 followed by a googol zeros.

a) Write down a googol as a power of ten.
b) Write down a googolplex as a power of ten.

Musing 60.3
Draw a $1 \leftarrow 10$ machine picture of each of these quantities. What number do they each represent?

a) $3 \times 10^5 + 2 \times 10^4 + 7 \times 10 + 5 \times 10^{-2}$

b) $17 \times 10^3 + 82 \times 10^2 + 90 \times 10 + 76 \times 1 + 23 \times 10^{-1} + 48 \times 10^{-2}$

Musing 60.4 We have:

$10^3 = 1{,}000$ is called a **thousand**.
$10^6 = 1{,}000{,}000$ is called a **million** (it's a thousand thousands).
$10^9 = 1{,}000{,}000{,}000$ is called a **billion** (it's a thousand millions).

a) What number is a **trillion**? A **quadrillion**?

b) In the past, a million million was called *billion.* What number is that as a power of ten?

In the past, a million billion was called *trillion.* What number is that as a power of ten?

In the past, a million trillion was called *quadillion.* What number is that as a power of ten?

c) The prefixes "bi," "tri," and "quad" mean *two*, *three*, and *four*, respectively.
Do these prefixes make sense for names of the numbers *billion*, *trillion*, and *quadrillion*?

d) What is a *milliard* ?

Do you care to look up the history of the names for these big numbers?
When did their meaning change? What instigated the change?

Musing 60.5 How many bytes is a gigabyte? Express your answer as a power of ten.

Musing 60.6 Are you a billion seconds old?

MECHANICS PRACTICE

Practice 60.7 What number do each of these quantities represent?

a) 2^2 b) 2^{-2} c) 3^4 d) 1^{506} e) 1^{-2} f) 0^{20}

Practice 60.8

a) What is 0.00001 as a power of ten?
b) What is 64 has a power of four?
c) What is $\frac{1}{8}$ as a power of two?
d) If $a^3 = \frac{8}{125}$, what is a?

Practice 60.9

a) Can you see that $10^7 \times 10^8$ has to be 10^{15}?
b) What is $10^3 \times 10^5 \times 10^6$ as a power of ten?
c) What is $10^5 \times 10^{-2}$ as a power of ten?
d) What is $10^{-3} \times 10^{-4}$ as a power of ten?

61. Scientific Notation

As we noted in Section 56,

- Multiplying a number by 10 gives an answer ten times as big.
- Dividing a number by 10 (that is, multiplying it by $\frac{1}{10}$) gives an answer ten times as small.

And multiplying a number written in base ten either by 10 or $\frac{1}{10}$ gives an answer with the same digits as the original number, in the same order. (Well, you might introduce some zeros.)

For example,

$$5.67 \times 10 \times 10 \times 10 = \left(5 + \frac{6}{10} + \frac{7}{100}\right) \times 10 \times 10 \times 10 = 5000 + 600 + 70 = 5670$$

$$5.67 \times \frac{1}{10} \times \frac{1}{10} \times \frac{1}{10} = \left(5 + \frac{6}{10} + \frac{7}{100}\right) \times \frac{1}{10} \times \frac{1}{10} \times \frac{1}{10} = \frac{5}{1000} + \frac{6}{10000} + \frac{7}{100000} = 0.00567$$

These observations allow us to deduce the values of decimal numbers multiplied or divided by ten quite swiftly.

Example: What is 7.03×10^6? What is 7.03×10^{-2}?

Answer: Both problems will produce answers that involve the digits 7-0-3.
And we are starting with a number that is close to 7.

Now, 10^6 is a million, so 7.03×10^6 must be a value close to seven million.
We must have

$$7.03 \times 10^6 = 7{,}030{,}000$$

Also, 10^{-2} is one hundredth, so 7.03×10^{-2} must be a value close to seven hundredths.
We must have

$$7.03 \times 10^{-2} = 0.00703$$

Example: What is 307.231×10^5?

Answer: We have a number in the 300s being multiply by ten five times. (Do you see that?)

This means we'll have and answer in the $\cancel{3{,}000}$ $\cancel{30{,}000}$ $\cancel{300{,}000}$ $\cancel{3{,}000{,}000}$ $30{,}000{,}000$s.

The answer must be $30{,}723{,}100$.

(Did this feel like *déjà vu*?)

Practice 61.1: Write down the values of each of these computations.

a) 52.004×10
b) $52.004 \times 10 \times 10$
c) 52.004×10^6
d) $52.004 \times \frac{1}{10}$
e) $52.004 \times \frac{1}{10} \times \frac{1}{10}$
f) 52.004×10^{-4}

We humans are not at all good at comprehending very very big numbers nor very very small numbers.

For example,

> Do you have a sense for how much time has passes in 78840000 seconds?
>
> Do you have a feel for the length 0.0000000892306 kilometers?

Even just saying these numbers is hard!

The first number has a lot of digits. Folks in the western world often use commas to separate long numbers into sets of three digits to help us think of the numbers in terms of thousands, millions, billions, and so on. Our count of seconds reads

$$78{,}840{,}000$$

We see immediately that we are talking about millions of seconds. We can even say that we're talking about roughly 79 million second, or maybe saying 80 million seconds is good enough.

Saying out loud just the first one or two digits of a large number—as millions or thousands and such—makes the number feel more manageable. (Though I still don't have a sense of how long a time about eighty million seconds actually is!)

Practice 61.2: In India the noun *lakh* is used for one-hundred thousand.
About how many lakh is the number 78840000?

Practice 61.3: Show that 78840000 seconds is about 2.5 years.

A number with a large number of decimal places is equally hard for us to wrap our brains around.

Rounding to just one or two digits usually makes matters feel better:

> 0.0000000892306 kilometers is approximately 0.00000009 kilometers.

And we can get a feel for this number if we change the units.

With a thousand meters in a kilometer, multiplying by a thousand gives the count of meters in this measurement.

$$0.00000009 \times 1000 = 0.00000009 \times 10 \times 10 \times 10 = 0.00009 \text{ meters}$$

Since there are 100 centimeters in a meter, multiplying this value by one hundred gives the count of centimeters in this length.

$0.00009 \times 100 = 0.00009 \times 10 \times 10 = 0.009$ centimeters

There are 10 millimeters in a centimeter, so this is

$0.009 \times 10 = 0.09$ millimeters.

And this looks and feels more manageable. (And just so you have it, 0.09 mm is a typical width of a human hair.)

Adjusting very big numbers and very small numbers by powers of ten is a common practice to get a manageable sense of the number. For instance, we see that

$$\begin{aligned} 78840000 &= 7884 \times 10{,}000 \\ &= 788.4 \times 10 \times 10{,}000 \\ &= 78.84 \times 10 \times 10 \times 10{,}000 \\ &= 7.884 \times 10 \times 10 \times 10 \times 10{,}000 \\ &= 7.884 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10 \\ &= 7.884 \times 10^7 \end{aligned}$$

and 7.884×10^7 is about 7.9×10^7 or 8×10^7, depending on how much you want to round.

Also,

$$0.0000000892306 = 8.92306 \times \frac{1}{10} \times \frac{1}{10} \times \frac{1}{10} \times \frac{1}{10} \times \frac{1}{10} \times \frac{1}{10} \times \frac{1}{10} \times \frac{1}{10} = 8.92306 \times 10^{-8}$$

which we might round to 8.9×10^{-8}or 9×10^{-8}.

Writing numbers like 7.9×10^7 and 9×10^{-8} makes measurements look much more tractable.

A number rewritten to be a single non-zero digit followed by some decimal places and then multiplied by a power of ten,

$$a.bcd\ldots \times 10^m$$

is said to be written in **scientific notation**.

Often humans find it convenient to round a number written in scientific notation one with either zero, or just one or two digits after the decimal point.

Also, people tend to have certain “powers of ten landmarks” in their heads:

10^3 corresponds to thousands
10^6 corresponds to millions
10^9 corresponds to billions

and

10^{-2} corresponds to hundredths (a hundredth of a meter is a centimeter)
10^{-3} corresponds to thousandths (a thousandth of a kilometer is a meter)

and so on.

This helps with developing an intuitive feel for the number.

MECHANICS PRACTICE

Practice 61.4

a) What is the approximate value of the number 8.02×10^6 in words?
b) What is the approximate value of the number 7.983×10^3 in words?
c) Approximate 2.01×10^{-2} as a fraction.

Practice 61.5
The average distance to the Moon is 384400 km. Write this number in scientific notation.

Practice 61.6
a) Write each of these numbers in scientific notation.

$$6539 \quad 750000000000 \quad 0.0004 \quad 212.872$$

b) Write each of the following as ordinary decimal numbers.

$$7.27 \times 10^2 \quad 7.27 \times 10^{-2} \quad 7.27 \times 10^5 \quad 7.27 \times 10^{-5}$$

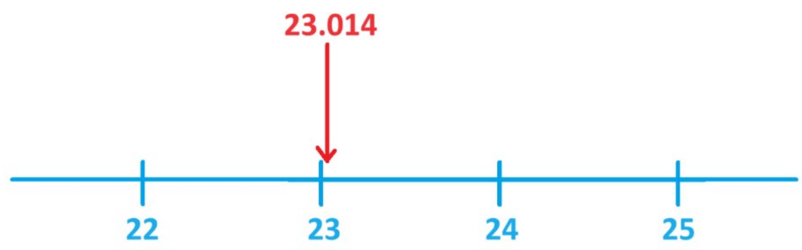
Practice 61.7 Write the answer to each of these computations in scientific notation.

a) $4.4 \times 10^6 \ + \ 2.2 \times 10^6$
b) $4.4 \times 10^6 \ + \ 2.2 \times 10^7$
c) $3 \times (2.2 \times 10^6)$
d) $5 \times (2.2 \times 10^6)$
e) $(2 \times 10^{18}) \times (5.5 \times 10^3)$

(Doing arithmetic with numbers presented in scientific notation can be annoying.
How did you handle part b)?)

62. Rounding

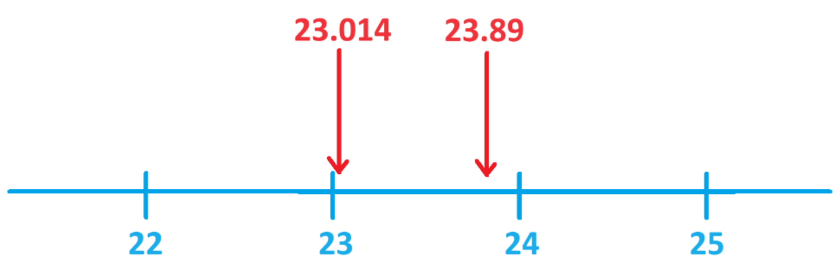
The decimal number 23.014, we all would agree, is close to the whole number 23 on the number line.



So too are the decimal numbers 23.09 and 23.1 and 23.26.

Practice 62.1 What about 23.4789? Which whole number is it closest to on the number line?

But 23.89 is closer to 24 on the number line than it is to 23.



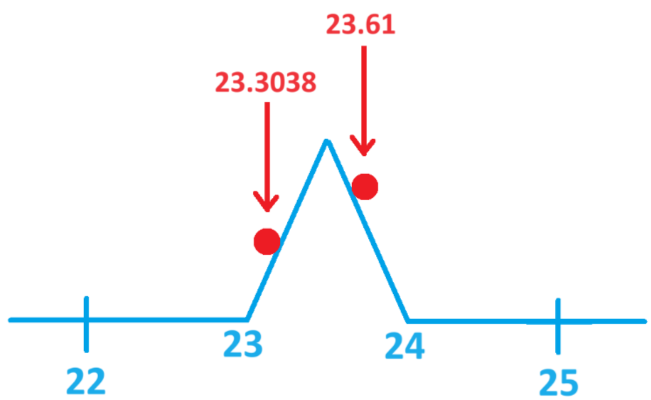
The number 23.5, on the other hand, is equally distant from the whole numbers 23 and 24.

We've learned in school that to **round** a decimal number composed of a whole number and some decimal digits. We look for the whole number (without any decimal places) that is closest to it on the number line.

For example,

> The decimal number 23.3038 rounds (down) to the whole number 23.
> The decimal number 23.61 rounds (up) to the whole number 24.

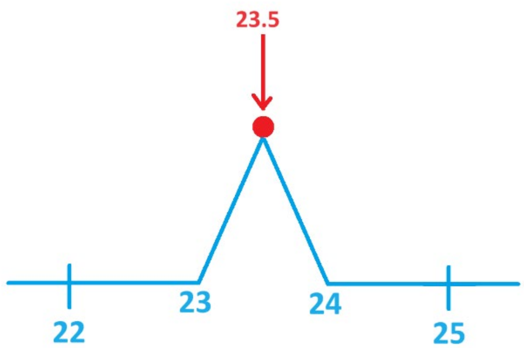
A lovely visual for this is to imagine a number line with a kink in it. Place a ball at the location of the decimal number in question and see to which whole number the ball rolls!



This leads to the schoolbook rule:

> To round decimal number to a whole number …
>
> Look at the first decimal place digit.
>
> If that digit is a 0, 1, 2, 3, or 4, then the ball will roll to the left. You will thus "round down" to a whole number.
>
> If, on the other hand, that digit is a 5, 6, 7, 8, or 9, then the ball will roll to the right. You will thus "round up" to a whole number.

The only ambiguous number in our specific example is 23.5. A ball sitting at this position on the number line is directly on the apex of the kink – and it could roll either direction!



Just to make the rounding rule consistent ("round up" if the first decimal digit is 5, 6, 7, 8, or 9), the world has decided to have the ball roll to right in this ambiguous case.

Convention: **23.5 rounds (up) to 24**

Practice 62.2 Which of these decimal numbers round to 8045?

8045.3909725555

8045.7097

8044.49999

8045.50001

8044.60986654

8045.060986654

8044.5

8045.5

8045.0

Rounding to hundreds, thousands, tenths, and more

Let's start with an example.

> **Example:** The average distance to the Moon is 384400 kilometers.
> Round that number to the nearest thousand kilometers.

Let's try to make sense of what is being asked of us.

For starters, we apparently should be thinking in terms of "thousands of kilometers."
So, let's try rewriting the figure given in terms of thousands. (Recall: A thousand can be written as 10^3.)

We have

$$384400 = 384.4 \times 10 \times 10 \times 10 = 384.4 \times 10^3$$

So, the distance to the Moon is 384.4 thousands of kilometers.

We're asked to round this number. Well, 384.4 rounds to 384 and so our answer must be: 384 thousands of kilometers. And this measurement is

$$384 \times 10^3 = 384{,}000$$

kilometers.

> **Example:** Round the measurement of the distance to the Moon to the nearest hundred thousand kilometers.
>
> **Answer:** No worries!
>
> One hundred thousand is 10^5 and so let's work with
>
> $$384400 = 3.844 \times 10^5$$
>
> So, we have 3.844 hundreds of thousands of kilometers.
>
> Rounding 3.844 gives 4.
>
> And so, the distance to the Moon rounded to the nearest hundred thousand kilometers is
>
> $$4 \times 10^5 = 400{,}000$$
>
> kilometers.

Another example.

Example: Round 0.5670764 to the nearest hundredth.

Answer: A hundredth is 10^{-2} so let's rewrite this number to make hundredths explicit. We have

$$0.5670764 = 56.70764 \times \frac{1}{10} \times \frac{1}{10} = 56.70764 \times 10^{-2}$$

Okay. We have 56.70764 hundredths, which rounds to 57 of them.

Our appropriately rounded number is

$$57 \times 10^{-2} = 0.57$$

We've got a procedure here!

If asked to round a measurement the nearest hundred or million or thousandth …

Rewrite the given figure in terms of a decimal number multiplied by the appropriate power of ten. (This tells you how many hundreds or millions or thousandths you actually have.)

Round that decimal number.

Work out the value of that rounded value times the power of ten you have.

Done!

Practice 62.3 Round $24{,}506.089$ to

a) the nearest whole number.
b) the nearest hundred
c) the nearest ten thousand
d) the nearest hundredth
e) the nearest tenth

Some schoolbooks students use the following technique for rounding.

> *To round the number in this question, say, to the nearest thousand, underline all the place values for thousands and up and then round according to the digit to the right of them.*
>
> $$\underline{24},506.089$$
>
> *The digit to the right is a five, so we round up and get* 25,000 *as our rounded value.*

Here are the answers to Practice 62.3 in turn:

$$\underline{24{,}506}.089 \rightarrow 24{,}506$$

$$\underline{24{,}5}06.089 \rightarrow 24{,}500$$

$$\underline{2}4{,}506.089 \rightarrow 20{,}000$$

$$\underline{24{,}506}.\underline{08}9 \rightarrow 24{,}506.09$$

$$\underline{24{,}506}.\underline{0}89 \rightarrow 24{,}506.1$$

Do you see that this is the same technique we've been following without mention of scientific notation?

MUSINGS

Musing 62.4 Write down a number smaller than $27{,}000$ that, when rounded to the nearest thousand, rounds to $27{,}000$.

Musing 62.5 What is the largest whole number which, when rounded to the nearest hundred, gives $50{,}000$?

Musing 62.6
a) Which number on the number line is equally distant from $5{,}700$ and from $5{,}800$.

b) A number N between $5{,}700$ and $5{,}800$ on the number line lies to the right of the number you gave in part a). What is N rounded to the nearest hundred? How do you know?

Musing 62.7 What, do you think, is $-35{,}483$ rounded to the nearest hundred?

MECHANICS PRACTICE

Musing 62.8 What is the average distance to Moon rounded to the nearest tens of kilometers?

Musing 62.9 Round $8{,}383{,}838.3838$ to

a) the nearest million
b) the nearest thousand
c) the nearest hundredth
d) the nearest thousandth

63. Significant Figures

One meter is divided into one hundred **centimeters**.
Each centimeter is divided into ten **millimeters** (and so a full meter is divided into one thousand millimeters).

> **Question:** Do the prefixes *centi*- and *milli*- make sense in this context?

If I gave you a length of string to measure with a meter stick just marked with centimeters, you would probably round the length you measure to the nearest centimeter.

> "The string is about 78 centimeters long" you might say.

The true length of the string might be a little less than 78 centimeters, or a little more, but it will be a value within a centimeter range of 78 centimeters.

If I then asked you to repeat the task, but this time using a meter stick with each centimeter length on it divided into millimeters, you would then likely give me an answer rounded to the nearest millimeter (tenth of a centimeter).

> "The string is about 78.2 centimeters long" you might say this time.

In science, the level of precision marked on an instrument tends to determine the level of precision to which we make measurements.

But this idea can lead to a mathematical curiosity.

For example, suppose a botanist measured the length of a reed stalk and wrote in her paper that it was 0.190 meters tall.

What is she telling us?

By giving us the values of three digits after the decimal point, she is saying that she measured the length as 1 tenth of a meter and 9 hundredths of a meter and 0 thousandths of a meter, thereby informing the readers of her paper that her measuring tool went to the thousandths (millimeters, in this case).

Her measurement is thus a number rounded to the nearest millimeter. So, the true height of the stalk is close to 19.0 centimeters with no more than a millimeter of inaccuracy.

Now, of course, mathematically the number 0.190 is no different than the number 0.19.

If the botanist decided to write in her paper that her measurement was 0.19 meters (mathematically the same number), then readers will presume something else, that she measured the length of the stalk only to the nearest hundredth of a meter (that is to the nearest centimeter).

We still conclude that the stalk is about 19 centimeters tall, but we'd now think this value could have up to a centimeter of inaccuracy to it.

Even though the expressions 0.190 and 0.19 represent the exact same number mathematically, to a scientist, the expressions "0.190 meters" and "0.19 meters" tell us more than just a numerical value. They indicate the level of accuracy associated with the numerical value in a practical context.

> **Practice 63.1:** A scientist records in an experiment a temperature of 713.020 degrees Celsius. What was the accuracy of his measuring instrument?

The botanist who measured the stalk length as 0.190 meters published this number in scientific notation.

1.90×10^{-1} meters

Again, she included the decimal digit of zero to indicate the level of precision she conducted her measurement. (She measured zero thousandths of a meter.) Every digit she wrote down in scientific notation was deliberate and significant.

> **Significant Figures:** If scientific measurements are recorded in scientific notation, then all of the digits written to the left of the power of ten are considered "significant." They were measured by an instrument that has a certain degree of accuracy. (So, if some of the digits written down are zero, that is deliberate—they were measured to be zero!)
>
> The **number of significant figures** in a recorded measurement is the number of digits written to the left of the power of ten.

For example, the recorded measurement 1.90×10^{-1} meters has three significant figures.

Another botanist measuring the same stalk with a different instrument writes its height as 1.89701×10^{-1}meters, giving a recorded result with six significant figures. She measured to the nearest millionth of a meter.

A third botanist writing the height as 2×10^{-1} meters is recording the measurement with just one significant figure. He measured to the to the nearest tenth of a meter. (His meter stick must have been divided into just ten equally spaced marks.)

Writing measurements in scientific notation avoids confusion.

Example: Using estimation techniques, I determined the population of a certain town to be $34{,}000$ residents.

Which of the five digits in the measurement are significant? (That is, which of the five digits did I actually measure?)

(Non) Answer:

I recorded the digit 3, so I was certainly measuring to the accuracy of ten-thousand residents.

I recorded the digit 4, so I was actually doing better, measuring at least to the nearest thousand residents.

Did I record that middle zero? Was I able to estimate to the nearest 100 residents?
You can't tell.

Maybe I was able to measure to the nearest 10 residents and two of the zeros I recorded are "genuine."

Or maybe I counted every single resident and got the figure $34{,}000$ on the nose? (All five digits are valid.)

It is impossible for you to say what exactly I mean by the zeros in my recorded result.

To obviate confusion, let me present my measurement in scientific notation instead.

Example Continued: I intend to publish this measurement as

$$3.400 \times 10^4$$

residents.

Now you can see that I was measuring to the nearest ten residents. The town population might well be $34{,}003$ residents, or $33{,}998$ residents, for instance. But we can be assured that the town population is $34{,}000$ with an accuracy range of 10 residents.

For such clarity, scientists record their results in scientific notation.

MECHANICS PRACTICE

Practice 63.2
a) A geographer determines the population of a county to be 310,000 people. He says he counted to the nearest thousand people. Rewrite his count in scientific notation, displaying the appropriate number of significant figures.

b) Later, he said he mis-spoke: actually measured to the nearest 100 people. Adjust your answer to part a) appropriately.

Practice 63.3: As part of a cleanliness study, a scientist measured the width of a dust particle to be 0.000100 millimeters. The scientific journal in which he wants to publish his results wants all measurements to be given in scientific notation.

a) How does his measurement appear when written in scientific notation?
b) He later learns that the journal wants all measurements to be given in units of kilometers, not millimeters (and still be in scientific notation). Now how will his measurement appear?

64. Order of Magnitude

If you earn a six-digit salary (lucky you!), you are earning an annual salary in the hundreds of thousands of dollars. Perhaps it is $\$240{,}000$ dollars a year, or $\$598{,}764$ dollars a year. (Salaries are usually given in units of dollars, not to the level of cents.)

Someone earning a seven-digit salary is even luckier and has an annual income in the millions.

> **Practice 64.1:** I am currently earning a six-digit salary. But if I earned just one more dollar per year, I'd be earning a seven-digit salary. What is my current salary?
>
> (By the way … the premise of this question is not true!)

We often speak in terms of the "order of magnitude" of a number, giving a sense of the size of a number without specifying the number directly. There are two typical ways this is done in society. They aren't quite the same mathematically.

Order of Magnitude 1: If the number is a counting number, then just say the number of digits of the number.

For example, most everyone earns a five-digit salary.

Order of Magnitude 2: Write the number in scientific notation: $a.bcd\ldots \times 10^m$. Then m, the power of ten mentioned, is the order of the magnitude of the number. This gives a natural means to talk about the order of magnitude of decimal numbers too.

For example, in this second setting

7.12×10^3 has order of magnitude **three**. (It is a number in the thousands.)

4.8×10^{-5} has order of magnitude -5. (It's a number "talking about" one-hundred thousandths.)

0.0074 has order of magnitude -3. (We're talking about thousands, essentially.)

6.9872×10^6 has order of magnitude 6. (It's a number in the millions.)

Practice 64.2:
a) How many digits does the number 7.12×10^3 have when written out as a counting number?

What is its order of magnitude according to the first definition and what is its order of magnitude according to the second?

b) How many digits does the number 6.9872×10^6 have when written out as a counting number?

We see that these two ways of expressing an order of magnitude of a large number are slightly different.

Order of Magnitude as Scales

Many natural phenomena, such as earthquake vibrations and sound intensities, occur in a vast array of strengths.

For example, the sound intensity of a NASA Saturn V rocket launch is about $100{,}000{,}000{,}000{,}000{,}000{,}000 = 10^{20}$ times more intense than the sound of a pin drop.

It would be very annoying to have a measurement scale for sound intensity that starts close to zero (for a pin drop) and goes to such huge multi-digit numbers (for rocket launches). For this reason, the **Bel scale** for sound intensity is based on order of magnitude rather than a direct intensity measure. (To be clear, it is the second definition of "order of magnitude" being used here.)

The sound of a pin drop measures 1 Bel and the sound of the rocket launch measures 20 Bel.

> **Aside:** Actually, it has become standard to speak in terms tenths of Bels—**decibels**. A pin drop is 10 dB and a rocket launch 200 dB.

The **Richter scale** for earthquake intensity is also a scale based on the second definition of order of magnitude. An earthquake of measure 6 on the Richter scale (it's 10^6 fundamental units of measurement) is one unit of magnitude up from one of measure 5 (which is 10^5 fundamental units of measurement). Consequently, the earthquake of magnitude 6 is ten times as strong as the one of magnitude 5.

MUSINGS

Musing 64.3: Two earthquakes measure 4 and 7, respectively, on the Richter scale. By what factor is the second quake stronger than the first?

Musing 64.4 The number 4.1×10^9 is a number in the billions. What is its order of magnitude according to the first definition? According to the second?

Musing 64.5:
a) Do the numbers $999{,}999$ and $1{,}000{,}000$ seem significantly different to you?
b) What is the order of magnitude of each of these numbers according to the first definition?
c) What is the order of magnitude of each of these numbers according to the second definition?

> **Comment:** We're seeing that a concept of the "order of magnitude" of a number is just a quick and rough attempt to give an intuitive sense of the size of the number. There are always going to be examples that show our estimations are a tad questionable. But that's not a real concern. We really are just going for rough-and-ready intuitive sense of the size of numbers.

Musing 64.6 **VERY OPTIONAL**

Scientists (in particular, astronomers) consistently working with very big numbers often represent *all* their numbers solely in terms of powers of ten. This means that they are willing to work with strange values for the powers.

We'll make sense of this when we discuss powers in proper detail in the next volume. But for now, let's just play with our calculators, even if we are not sure we understand what our calculators mean in what they are showing.

We know, for example, that

$$10^6 = 1{,}000{,}000$$

and

$$10^7 = 10{,}000{,}000$$

Now $3{,}300{,}00$, for instance, is a value between one million and ten million. It is possible to imagine that there is a power of ten, between 6 and 7, that gives this value.

Experimentation with a calculator shows that $10^{6.519}$ is about this value. (This is weird, I know. But try entering $10^{6.519}$ on a calculator.)

In the same way, we see $3{,}100{,}000$ seems to be about $10^{6.491}$.

Here's a third convention for stating what the "order of magnitude" of a number is.

Order of Magnitude 3: Write the given number solely as a power of ten, most likely with a decimal as the power: 10^r. Then round r up or down to the nearest integer. Call that rounded value the order of magnitude of the number.

For example,

$3{,}300{,}000 \approx 10^{6.519}$ and 6.519 rounds to 7. So $3{,}300{,}000$ has order of magnitude 7 in this third definition.

$3{,}100{,}000 \approx 10^{6.491}$ and 6.491rounds to 6. So $3{,}100{,}000$ has order of magnitude 6 in this third definition.

Confusing!

a) What do you think? Do 3.1million and 3.3 million seem like they should have different orders of magnitude to you?

b) What is the order of magnitude of the number 254 according to this third definition?

c) What is the order of magnitude of the number 0.0033 according to this third definition?

Solutions

53.1 $10 \times \frac{1}{1,000} = \frac{10}{1000} = \frac{10\times1}{10\times100} = \frac{1}{100}$ and $10 \times \frac{1}{10,000} = \frac{10}{10,000} = \frac{10\times1}{10\times1000} = \frac{1}{1000}$.

53. 2 $\frac{1}{10} \times \frac{1}{10} \times \frac{1}{10} = \frac{1}{10\times10\times10} = \frac{1}{1,000}$ and $\frac{1}{10} \times \frac{1}{10} \times \frac{1}{10} \times \frac{1}{10} = \frac{1}{10\times10\times10\times10} = \frac{1}{10,000}$

53.3 Ten

53.4 We need a name that represents two-ness. A "bimal" point? A "dimal" point?

53.5 In Europe, for instance, it is popular to use a comma to as a decimal "point."

53.6 $\frac{8}{100,000} = \frac{1}{12,500}$

53.7 a) $0.009 = \frac{9}{1,000}$ b) $0.26 = \frac{26}{100}$ c) $0.3007 = \frac{3,007}{10,000}$

53.8 Both are correct. One just needs to conduct some explosions or unexplosions to convert one to the other.

53.9 She should do some explosions.

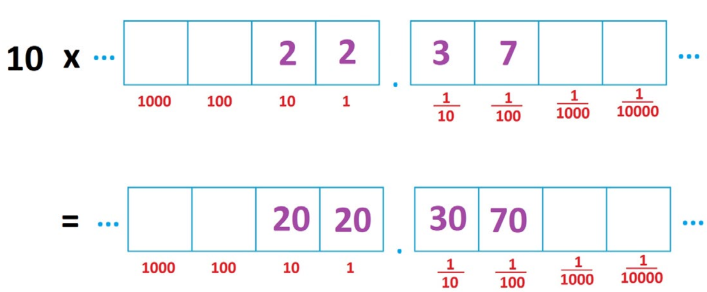
$$2.7|11|11|1 = 2.8|1|11|1 = 2.8|2|1|1$$

53.10 Yes. Eleven times two dots in a box makes 22 dots in the box. And eleven times three dots in a box makes 33 dots in the box. And so on.

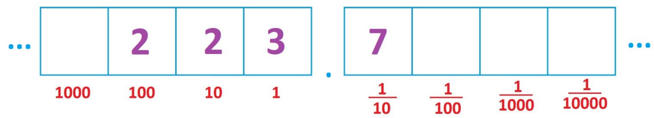
With explosions we get 259.16.

$$22|33.55|66 = 2|2|33.55|66 = 2|5|3.55|66 = 2|5|8.5|66 = 2|5|8.11|6 = 2|5|9.1|6$$

53.11



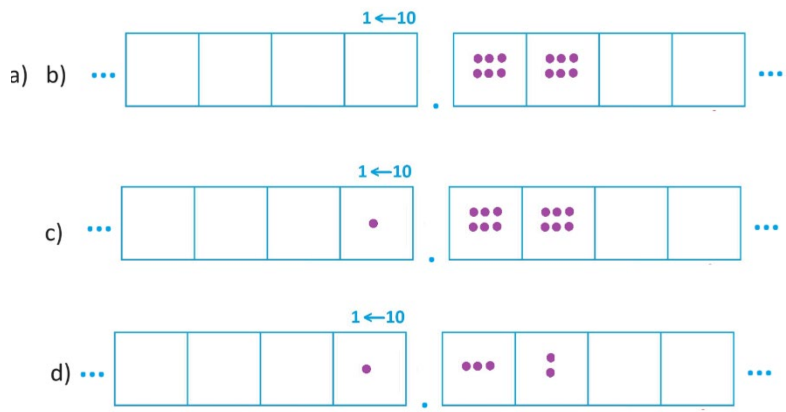
After explosions, this is



53.12 260.4299

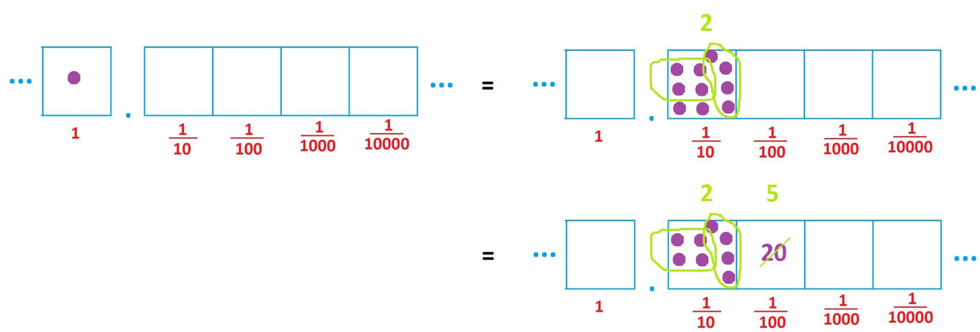
53.13 a) $\frac{23}{100}$ b) $\frac{409}{10000}$

53.14

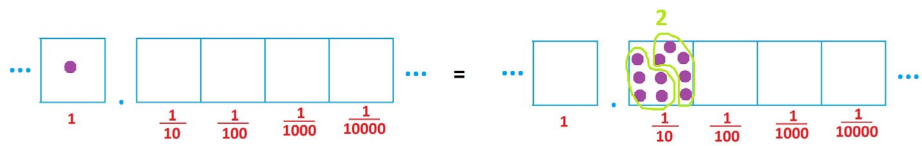


53.15 a) 31.2 b) Can you?

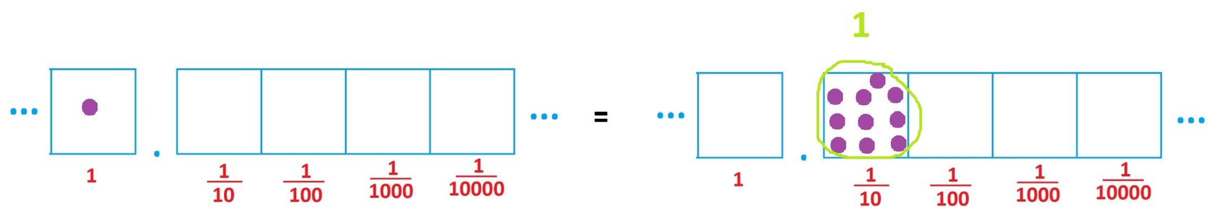
54. 1 $\frac{1}{4} = 0.25$



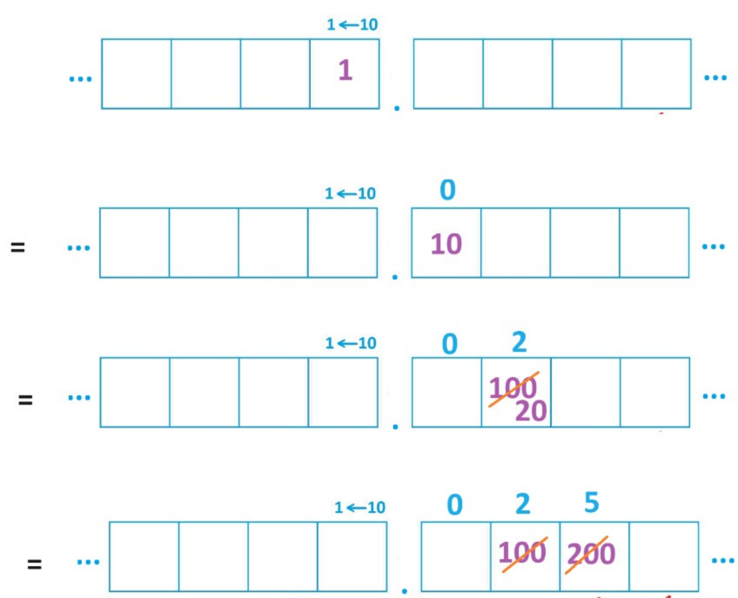
54.2 $\frac{1}{5} = 0.2$



54.3 The decimal 0.1 is $\frac{1}{10}$. So, of course, $\frac{1}{10} = 1 \div 10$, has to give 0.1. (And it does!)



54.4 $\frac{1}{40} = 0.025$



54.5 $\frac{1}{9} = 0.1111111\ ...$

54.6 $\frac{2}{15} = 0.1333333\ ...$

54.7 $\frac{1}{6} = 0.16666666\ ...$

54.8 $\frac{1}{3}$ and $\frac{1}{6}$ and $\frac{1}{7}$ and $\frac{1}{9}$ and $\frac{1}{11}$ and $\frac{1}{12}$ have infinitely long decimal expansions.

54.9 a) Double this, which is $0.66666\ ...$

b) It is! Both are $1.333333\ ....$

c) No. $\frac{3}{3}$, for example, is just 1 !

54.10

$$\frac{1}{3} = 0.33333\ ...$$

$$\frac{1}{6} = 0.166666\ ...$$

Add these and you get $0.49999999\ ....$
This looks different than 0.5.

54.11 a) We get 0.5 and 0.05. b) We get 0.2 and 0.02. c) We get $0.33333\ ...$ and $0.033333\ .....$

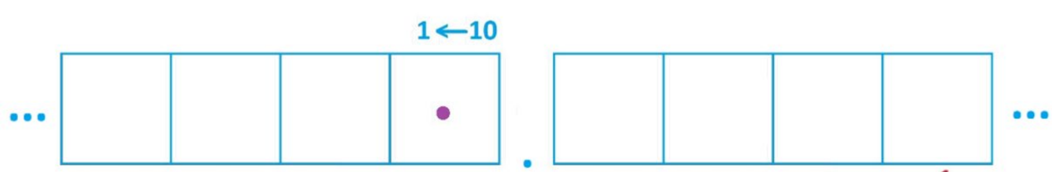
It looks like that multiplying a decimal by $\frac{1}{10}$ has the effect of inserting a zero right after the decimal point.

54.12 Do this.

54.13 $\frac{4}{7} = 0.571428\ \ 572418\ \ 572418\ ...$

54.14 $\frac{1}{11} = 0.09090909\ ...$
54.15 Can you verify everything you are asked to verify?

54.16 This process wants to "settle to" this picture of the number 1.



55.1 $\frac{3}{5} = \frac{6}{10} = 0.6,\ \frac{21}{250} = \frac{84}{1000} = 0.084,\ \frac{3}{125} = \frac{8}{1000} = 0.008$

55.2 $\frac{1}{2\times2\times2\times2\times2\times5\times5\times5} = \frac{5\times5}{2\times2\times2\times2\times2\times5\times5\times5\times5\times5} = \frac{25}{10\times10\times10\times10\times10} = 0.00025$

55.3 Yes. A decimal of the form $0.a$ is a certain number of tenths, a decimal of the form $0.ab$ is a certain number of hundredths, a decimal of the form $0.abc$ is a certain number of thousandths, and so on.

55.4 I personally think so. "Seven tenths" and "seven tenths + no hundredths + no thousandths + no ten-thousandths + …." feel the same to me.

56.1 $\frac{5}{100}+\frac{6}{1000}=\frac{50}{1000}+\frac{6}{1000}=\frac{56}{1000}$

56.2 $\frac{5}{100}-\frac{6}{1000}=\frac{50}{1000}+\frac{-6}{1000}=\frac{44}{1000}$

56.3 a) 851

b) $0.023=\frac{23}{1000}$ and $0.37=\frac{37}{100}$

c) $0.023\times 0.37=\frac{23}{1000}\times\frac{37}{100}=\frac{851}{100{,}000}=0.00851$

d) $2.3\times 3.7=\frac{23}{10}\times\frac{37}{10}=\frac{851}{100}=8.51$ and $230\times 0.037=23\times 10\times\frac{37}{100}=\frac{851}{10}=8.51$

56.4 $0.7\times 0.004=\frac{7}{10}\times\frac{4}{1000}=\frac{28}{10{,}000}=0.0028$

56.5 a) 70 b) 0.07 c) 0.7

56.6 $13.276+5.94=19.216$ and $13.276-5.94=7.336$.

56.7 a) Because of the alignment of the decimal places, this work is essentially the same.

$$\begin{array}{r} 0.0348 \\ +\ 0.0057 \\ \hline = \end{array} \qquad\qquad \begin{array}{r} 348 \\ +\ 57 \\ \hline = \end{array}$$

b) We don't have matching alignments of the decimal places this time.

$$\begin{array}{r} 0.0852 \\ +\ 0.037 \\ \hline = \end{array} \qquad\qquad \begin{array}{r} 852 \\ +\ 37 \\ \hline = \end{array}$$

56.8
a) $\frac{1}{5} \times 0.02 = \frac{1}{5} \times \frac{2}{100} = \frac{2}{500} = \frac{1}{125}$ or, as a decimal, $\frac{1}{5} \times 0.02 = \frac{2}{500} = \frac{4}{1000} = 0.004$
b) $\frac{1/5}{0.02} = \frac{100\times(\frac{1}{5})}{100\times 0.02} = \frac{20}{2} = 10$

56.9 Thinking of "*abc*" as a three-digit number. Then $0.abc = \frac{abc}{1{,}000}$.
a) $0.abc \div 10 = \frac{1}{10} \times 0.abc = \frac{1}{10} \times \frac{abc}{1{,}000} = \frac{abc}{10{,}000} = 0.0abc$
b) $0.abc \div 100 = \frac{1}{100} \times 0.abc = \frac{1}{10} \times \frac{abc}{1{,}000} = \frac{abc}{100{,}000} = 0.00abc$
c) $0.abc \div 0.1 = \frac{0.abc}{0.1} = \frac{\frac{abc}{1000}}{\frac{1}{10}} = \frac{10\times\frac{abc}{1000}}{10\times\frac{1}{10}} = \frac{\frac{abc}{100}}{1} = \frac{abc}{100} = a.bc$

d) $ab.c$ e) $ab.c$ f) $0.00abc$

56.10 Notice that 483.014 is a number in the hundreds.

a) We should get a number in the thousands: $4{,}830.14$
b) We should get a number in the ten thousands: $48{,}301.4$
c) We should get a number in the millions: $4{,}830{,}140$
d) We should get a number in the tens: 48.3014
e) We should get a number in the ones: 4.83014
f) We should get a number in the hundredths: 0.0483014

56.11 $\frac{1}{11.99} < \frac{1}{11.98} < \frac{1}{11.97} < \cdots < \frac{1}{11.01}$, for example.

56.12 a) 1.29 b) 5.05 c) 0.0812

56.13 Do use technology. (That is what any sane person would do!)

56.14 a) This is roughly $33 - 8$, so something under 25? Exact answer: 24.342
This is roughly $2 - 0.8$, so about 1.2? Exact answer: 1.253

56.15 It's $1.45 - 1 + 0.04 = 0.49$

56.16 a) 41% b) 0.58 c) 0.1% d) $\frac{1}{3}$ e) $\frac{12}{10{,}000}$

56.17 a) 2.25 b) 1.96 c) 2.1025 d) 2.0164 e) 1.9881, too small. f) 1.414

57.1 In the second box after the decimal place we have a remainder of 0 dots. Unexploding no dots gives zero dots in the next box. There are zero groups of 4 there with a remainder of 0, again. We are in a pattern of repeating a remainder of zero.

57.2 $0.5111111 \ldots$

57.3 As there are only seven possible remainders when you conduct the division process, one must cycle through at most seven remainders. Thus, one cannot have a cycle in a decimal expansion for sevenths longer than seven.
A cycle of length seven is also not possible. In this case one must be cycling through all seven remainders, including 0. But as soon as you hit 0 as a remainder, you are in a pattern of repeating zeros and so have cycles of length one, not seven.

57.4 a) Do it. b) Do it. c) $\frac{19}{90}$ d) $\frac{281850}{99{,}900}$

58.1 People do consider each integer to be a finite decimal.

58.2 Jut write an infinitely long decimal that fails to have a repeating pattern.

58.3 $0.3\ 4\ 33\ 4\ 333\ 4\ 3333\ 4\ 33333\ 4\ 333333\ 4\ 33 \ldots$, for instance.

58.4 Add $0.3\ 2\ 33\ 2\ 333\ 2\ 3333\ 2\ 33333\ 2\ 333333\ 2\ 33 \ldots$ to the previous example.
They sum to $0.66666..$ which is the fraction $\frac{2}{3}$.

59.1 12, 12, and 17 centimeters. And then again to 5, 5, and 7 centimeters.

60.1 a) It does. We'd have $a^0 = 1$. b) a^0 does not make sense with this definition.

60.2 a) 10^{100} b) 10^{googol}

60.3 a) $3|2|0|0|7|0.0|5 = 320{,}070.05$
b) $17|82|90|76.23|48 = 26{,}178.78$

60.4 a) Trillion $= 10^{12}$, a thousand billion; Quadrillion $= 10^{15}$, a thousand trillion.

b) Former billion $= 10^{12}$, Former trillion $= 10^{18}$, Former quadrillion $= 10^{24}$.

c) It does in the former system. The prefix tells you how many millions are multiplied together to get the number.

d) A thousand million. (That's now called a billion.)

60.5 One billion bits (10^9).

60.6 One billion seconds is about 31.7 years.

60.7 a) 4 b) $\frac{1}{4}$ c) 81 d) 1 e) 1 f) 0

60.8 a) 10^{-5} b) 4^3 c) 2^{-3} d) $a = \frac{2}{5}$

60.9 a) Seven 10s multiplied by eight 10s gives 15 tens all multiplied together. (Any 1s involved in the product have no effect.)
b) 10^{14} c) 10^3 d) 10^{-7}

61.1 a) 520.04 b) 5,200.4 c) 52,004,000 d) 5.2004 e) 0.52004 f) 0.0052004

61.2 788.4 lakh

61.3 Divide by 60 to obtain the number of minutes. Divide by 60 again to obtain the number of hours. Divide by 24 to obtain the number of days. Divide by 365.25 (or just 365) to obtain the number of years.

61.4 a) Eight million b) Eight thousand c) $\frac{2}{100}$

61.5 3.84400×10^5 km

61.6 a) 6.539×10^3 ; 7.5×10^{10} ; 4×10^{-4} ; c

b) 727 ; 0.0727 ; 72,700 ; 0.0000727

61.7 a) 6.6×10^6 b) 2.64×10^7 c) 6.6×10^6 d) 1.1×10^7 e) 1.1×10^{22}

62.1 24

62.2 These ones

8045.3909725555

~~8045.7097~~

~~8044.49999~~

~~8045.50001~~

8044.60986654

8045.060986654

8044.5

~~8045.5~~

8045.0

62.3 Read on.

62.4 $26{,}501$ works.

62.5 $50{,}049$

62.6 a) $5{,}750$ b) $5{,}800$

62.7 $-35{,}500$

62.8 $384{,}400$ km

62.9 a) $8{,}000{,}000$ b) $8{,}384{,}000$ c) $8{,}383{,}838.38$ d) $8{,}383{,}838.384$

63.1 To a thousandth of a degree.

63.2 a) 3.10×10^5 b) 3.100×10^5

63.3 a) 1.00×10^{-4} mm b) 1.00×10^{-9} km

64.1 $\$999{,}999$

64.2 a) Four digits. It has order of magnitude 4 according to the first definition, 3 according to the second definition.
b) Seven

64.3 A thousand times stronger.

64.4 10 and 9, respectively.

64.5 b) 6 and 7, respectively. c) 5 and 6, respectively.

65.5 b) $254 \approx 10^{2.4}$ has order of magnitude 2.
c) $0.0033 \approx 10^{-2.48}$ has order of magnitude -2.