Chapter 8

Beyond Base 10
All Bases, All at Once

Contents

65. A Famous Mystery About Prime Numbers

Throughout the millennia, scholars and enthusiasts have loved playing with mathematics. What seems "incidental" and frivolous at first can often lead to deep joy, curious mathematical mystery, and even to immense practical use. A prime example of this (a pun is intended) is some play with the **doubling numbers,** play that started some 400 years ago.

We met the doubling numbers via a mind-reading trick at the start of Chapter 4. They are the numbers that result if you repeated double the number 1.

1 2 4 8 16 32 64 128 256 ...

These are, of course, also the **powers of two**.

$$1 = 2^0$$
$$2 = 1 \times 2 = 2^1$$
$$4 = 1 \times 2 \times 2 = 2^2$$
$$8 = 1 \times 2 \times 2 \times 2 = 2^3$$
$$16 = 1 \times 2 \times 2 \times 2 \times 2 = 2^4$$

and so on.

These numbers get big quickly. For example, 2^{10} equals $1{,}024$ and 2^{300} is number 91 digits long. (Its value is a little over two *novemvigintillion*.)

A French monk by the name Merin Mersenne (1588-1648) noticed something curious if you subtract 1 from each doubling number. Doing so gives this list of numbers.

0 1 3 7 15 31 63 127 255 ...

Actually, Mersenne ignored the zero at the start and focused on this list of numbers instead.

1 [3] [7] 15 [31] 63 [127] 255 ...

Here observed:

3 is a prime number and 3 is the **second** number in this list.
7 is a prime number and 7 is the **third** number in this list.
31 is a prime number and 31 is the **fifth** number in this list.
127 is a prime number and 127 is the **seventh** number in this list.

It looks like prime numbers are appearing in every prime position!

Practice 65.1 Show, on the other hand, that 15, 63, and 255 are each a composite number. (Also remember, the number 1 is deemed neither prime nor composite.)

We can rewrite this observation in terms of the powers of two.

2 is prime and $2^2 - 1 = 3$ is prime
3 is prime and $2^3 - 1 = 7$ is prime
5 is prime and $2^5 - 1 = 31$ is prime
7 is prime and $2^7 - 1 = 127$ is prime

Of course, Mersenne wondered

Is $2^{prime} - 1$ always a prime number?

Practice 65.2 The next prime value to consider is 11 and $2^{11} - 1 = 2047$. Is 2047 a prime number? (Ask Siri or Alexa or some virtual friend?)

It turns out that the pattern we, and Mersenne, are observing is a false one.

> 2047 is composite. We have $2047 = 23 \times 89$.

But Mersenne wondered:

> When is $2^{prime} - 1$ a prime number?
> Is it often a prime number?
> There are infinitely many prime numbers, so maybe $2^{prime} - 1$ also prime infinitely often?

He could only find eleven examples of primes that arise this way (using only pencil and paper back in the early 1600s!),

2 is a prime number and $2^2 - 1 = 3$ is prime
3 is a prime number and $2^3 - 1 = 7$ is prime
5 is a prime and number $2^5 - 1 = 31$ is prime
7 is a prime number and $2^7 - 1 = 127$ is prime
13 is a prime number and $2^{13} - 1 = 8191$ is prime
17 is a prime number and $2^{17} - 1 = 13107$ is prime
19 is a prime number and $2^{19} - 1 = 524287$ is prime
31 is a prime number and $2^{31} - 1 = 2147483647$ is prime
67 is a prime number and $2^{67} - 1 = 2147483647$ is prime
127 is a prime number and $2^{127} - 1 = 2305843009213693951$ is prime
257 is a prime number and $2^{257} - 1 = 618970019642690137449562111$ is prime

and people have been looking for more examples ever since the time of Mersenne.

Prime numbers that happen to be of the form $2^{prime} - 1$ are today called **Mersenne primes**.

A FAMOUS UNSOLVED QUESTION

No one on this planet currently knows whether or not there are an infinite number of Mersenne primes to be found. Perhaps there is just a finite count of them?

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As of the time of writing this Section (April 2024), only 51 examples of Mersenne primes are known, the most recent discovered being $2^{82,589,933} - 1$, a number over 24 million digits long. (And 85,589,933 is a prime number, by the way.) It was discovered in October 2020.

Is there a 52nd prime of this form to be found? No one knows!

> **Optional Challenge:** For world fame, find a 52nd prime number that is one less than a power of two. (Or prove that no other Mersenne primes exist.)

Comment: You can join the *Giant Internet Mersenne Prime Search* (GIMPS) and have your computer hunt for a 52nd Mersenne prime!

Computers work in binary (base 2) and are thus quite adept at looking for prime numbers connected to the powers of 2. In fact, all the exceptionally large prime numbers we know today come from playing with the powers of 2.

And why do we care about finding larger and larger prime numbers?

Well, it turns out that all our computer encryption codes for financial services, military services, and the like, are based on knowing prime numbers and the larger the prime numbers used, the much harder those code are to crack. (It is thus likely that the financial institutions and the military might be aware of some large prime numbers not known to the public.)

But what is remarkable about this story is that Mersenne's casual play with the doubling numbers about 400 years ago led to a practical application of vital relevance to the 21st century!

Mathematicians, and the institutions that support them, thoroughly value the art of mathematical play and general investigation. You never know what curious questions might open up and what immensely practical applications might result. A new result might be "frivolous," but the new mathematical tools and ideas that led to it often turn out to be of immense value.

But let's move on.

We have seen that a number of the form $2^{prime} - 1$ could be prime or could be composite.

But what about numbers of the form $2^{composite} - 1$? Could they ever be prime?

The three examples we have in our list of numbers are each composite.

$2^4 - 1 = 15$ is composite
$2^6 - 1 = 63$ is composite
$2^8 - 1 = 255$ is composite

Practice 65.3 Evaluate $2^9 - 1$, $2^{10} - 1$, and $2^{12} - 1$ and show they too are each composite.

Mersenne was able to prove that a number of the form $2^{composite} - 1$ will never prime: such numbers will always have a proper factor.

That's a goal of this chapter: to develop the mathematics that will allow us to prove that too.

As a practice challenge:

Find a proper factor of $2^{300} - 1$ to show that this number is not prime.

(By the way, it is not possible to work with this number in a calculator. It is 91 digits long, after all!)

MUSINGS

Musing 65.4 Look at the doubling numbers again:

1 2 4 8 16 32 64 128 256 …

The first is one more than a multiple of 3. The second is one less. The third is one more. The fourth is one less. The fifth is one more.

It seems that this pattern persists.

a) If this pattern is truly valid (and is not a false pattern), would 2^{300} be one more or one less than a multiple of 3?

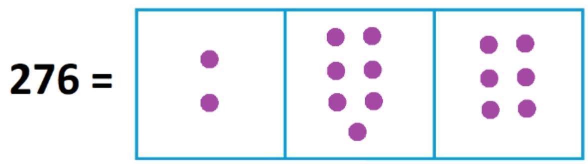
b) If you trust this pattern, deduce that $2^{300} - 1$ is divisible by 3 and so is not prime.

c) Should you trust the pattern?

66. Revisiting Division

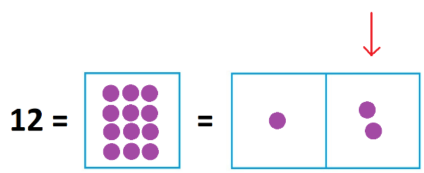
We learned how to conduct long division in a $1 \leftarrow 10$ machine back in Section 35.

For instance, to compute $276 \div 12$ we started by drawing a picture of the number 276.

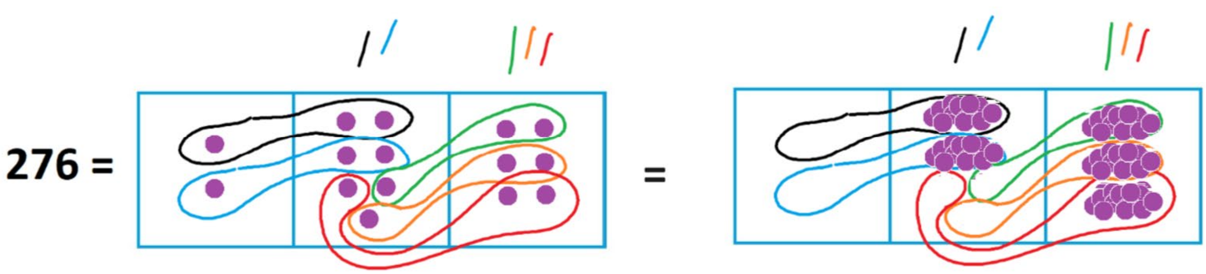


Our job is to then work with this diagram to see "what got multiplied by 12" to produce it.

A group of 12 appears as one dot next to two dots, but it is really twelve dots in one box. (Society insists on conducting the explosions!)



And we can identify groups of 12 in the picture: two at the tens level and three at the ones level.



We conclude that the number 23 was multiplied by 12 to give our picture of 276.

Thus,

$$276 \div 12 = 23$$

Question: Do you recall this process?
Would you like to practice it be showing that $27999 \div 132$ equals 212 with remainder of 15?

Division in Any Base

Let's now live up to the promise of Section 36 where we claimed that this long division is high-school algebra in disguise.

The only thing to realize is that there is nothing special about a $1 \leftarrow 10$ machine.

We could be doing all our arithmetic in a $1 \leftarrow 2$ machine if we desired, or a $1 \leftarrow 5$ machine, or even a $1 \leftarrow 37$ machine. The math doesn't care in which machine we conduct it. It is only us humans with a predilection for the number ten that draws us to the $1 \leftarrow 10$ machine.

So, let's now be bold and do our work in <u>all</u> possible machines, all at once!

That sounds crazy, but it is surprisingly straightforward.

What I am going to do is draw the a picture of a machine, but I am not going to tell you which machine it is. It could be a $1 \leftarrow 10$ machine again, I am just not going to say. Maybe it will be a $1 \leftarrow 2$ machine, or a $1 \leftarrow 4$ machine or a $1 \leftarrow 13$ machine. You just won't know as I am not telling. It's the mood I am in!

Now, in school algebra there seems to be a favorite letter of the alphabet to use for a quantity whose value you do not know. It's the letter x. Always the letter x. (It's a weird obsession.)

> **Question:** What can you find on the internet about why the letter x is the favored letter to represent an unknown quantity in mathematics?
>
> (Watch out! There are multiple thoughts, theories, and combinations of details. Don't fully believe the first explanation you encounter.)

So, let's work with an $1 \leftarrow x$ machine with the letter x representing some number whose actual value we do not know.

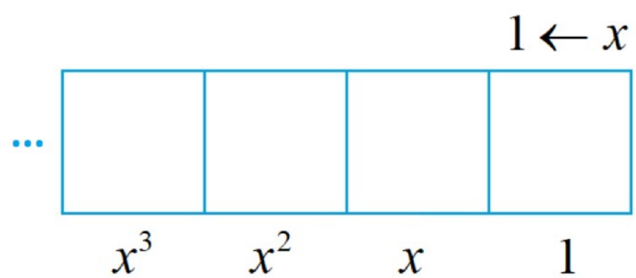
In Section 60, we saw that the place values of the boxes in a $1 \leftarrow 10$ machine are powers of ten.

$$\mathbf{1} = 10^0 \quad \mathbf{10} = 10^1 \quad \mathbf{100} = 10^2 \quad \mathbf{1,000} = 10^3 \quad \ldots$$

The place values of the boxes in a $1 \leftarrow 2$ machine are powers of two.

$$\mathbf{1} = 2^0 \quad \mathbf{2} = 2^1 \quad \mathbf{4} = 2^2 \quad \mathbf{8} = 2^3 \quad \mathbf{16} = 2^4 \quad \ldots$$

In an $1 \leftarrow x$ machine, the place values of the boxes will be the powers of x.



This $1 \leftarrow x$ machine represents all machines, all at once!

If I tell you that I am thinking of x as 10 in this picture, then you can see the picture as a $1 \leftarrow 10$ machine.

If I change my mind and tell you that I am actually thinking of x as 2 in this picture, then you will see a $1 \leftarrow 2$ machine.

(Or x could be 3 to give a $1 \leftarrow 3$, machine, or 7 for a $1 \leftarrow 7$ machine, and so on.)

Question 66.1: In a $1 \leftarrow 10$ machine …

Ten ones give ten: $10 \times 1 = 10$
Ten tens give a hundred: $10 \times 10 = 100$
Ten hundreds give a thousand: $10 \times 100 = 1000$

and so on.

In a $1 \leftarrow 2$ machine …

Two ones give two: $2 \times 1 = 2$
Two twos give four: $2 \times 2 = 4$
Two fours give eight: $2 \times 4 = 8$

and so on.

For an unspecified number x, can we still say …

$$x \times 1 = x$$

and

$$x \times x = x^2$$

and

$$x \times x^2 = x^3$$

and so forth?

The answer to Question 67.1 is yes (can you explain why?), which affirms that an $1 \leftarrow x$ machine really does match a base machine whenever you think of x as an actual specific number.

Okay. Out of the blue; with everything now set up; here's an advanced algebra problem.

Exercise: Compute

$$(2x^2 + 7x + 6) \div (x + 2)$$

Go!

Well, actually, let's pause and not "Go!" just yet. I don't know you about you, but I am having a bit of an emotional reaction right now.

Moments ago we were doing schoolbook arithmetic and now we're suddenly being thrust into something that looks very strange and very scary.

Deep breath!

When faced with a challenge in math—and in life—there are two fundamental steps to problem solving one must start with, yet no one seems to talk about.

Step 1: Be human and acknowledge your honest human reaction to the challenge.

If the problem looks weird, say "This is weird!"
If it looks scary, acknowledge that you are nervous.
If the challenge looks curiously quirky, acknowledge you are intrigued.

Whatever your human reaction is to the problem, take note of it.

Then, take a deep breath, and move to …

Step 2: Do something! ANYTHING!

When faced with an emotional reaction, many people shut down. But doing something—anything—no matter how tiny or indirect it might feel helps you get past an emotional impasse.

Could you draw a picture? Could you draw a picture perhaps relevant to the problem?

Could you underline some words in the problem statement – the scary words, or perhaps all the words that begin with a vowel?

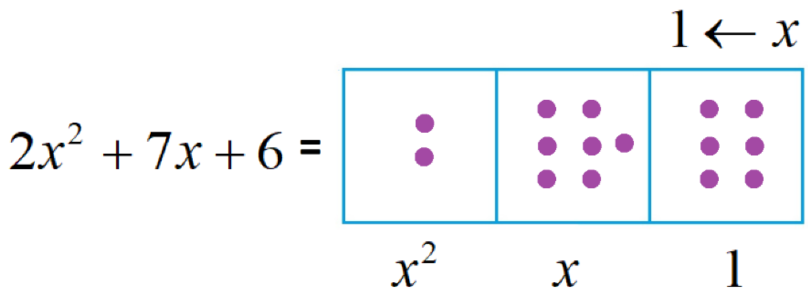
Could you reread the problem statement three times fast and then go for a short walk and not think about the problem—but with the promise you'll read it a fourth time, slowly, upon your return?

Okay. Now that I've acknowledged that I am nervous and have taken a deep breath, let me attempt to DO SOMETHING for the problem. We need to make sense of

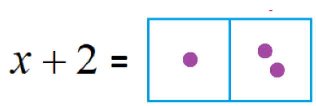
$$(2x^2 + 7x + 6) \div (x + 2)$$

Well. I can at least draw a picture for the challenge in an $1 \leftarrow x$ machine.

We have that $2x^2 + 7x + 6$ is two x^2, seven xs, and six ones.



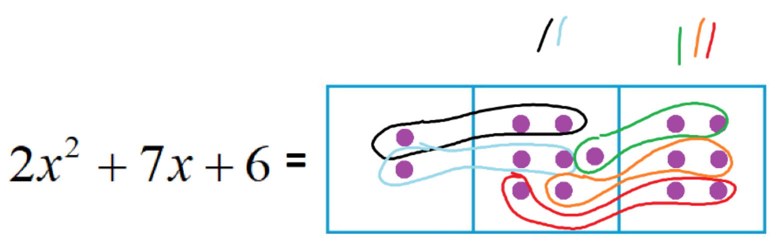
Here's what $x + 2$ looks like.



Great. I've done something!

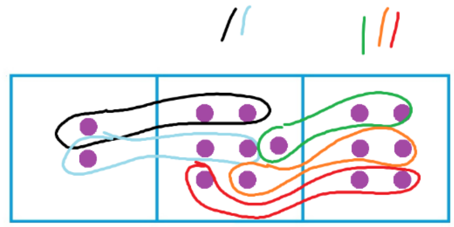
And I now feel like I know what to do next.

The division problem $(2x^2 + 7x + 6) \div (x + 2)$ is asking us to find copies of $x + 2$ in the picture of $2x^2 + 7x + 6$.

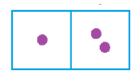


I see two copies of $x + 2$ at the x level and three copies at the 1 level.

$2x^2 + 7x + 6$ =



$x + 2$ =



The picture is showing me that answer must be $2x + 3$!

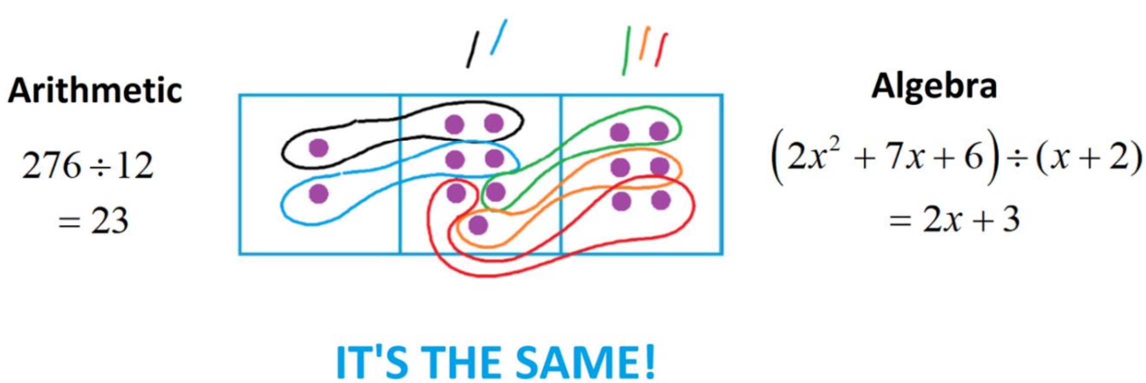
$$(\mathbf{2x^2 + 7x + 6}) \div (x + 2) = 2x + 3$$

We did it!

Now …

Stare at the picture above showing $(2x^2 + 7x + 6) \div (x + 2) = 2x + 3$.
Does it look familiar?

Look back at our picture for $276 \div 12$ we created on the first page of this section.
We have identical pictures!



We've just completed an advanced algebra problem as though it is nothing more than an early-grade school arithmetic problem.

Whoa! What's going on?

Suppose I told you that x really was 10 in my head all along though this work. Then

$\mathbf{2x^2 + 7x + 6}$ is the number $2 \times 10^2 + 7 \times 10 + 6$, which is $200 + 70 + 6 = \mathbf{276}$.

$\mathbf{x + 2}$ is the number $10 + 2 = \mathbf{12}$.

$\mathbf{2x + 3}$ is the number $20 + 3 = \mathbf{23}$.

and

$(2x^2 + 7x + 6) \div (x + 2) = 2x + 3$ is the statement $\mathbf{276} \div \mathbf{12} = \mathbf{23}$, which is exactly what we did on the first page of this Section!

Indeed, we really have just repeated a school arithmetic problem if I happen to declare that x was the number 10 in my head all along.

But here's the wonderful thing.

The statement

$$(2x^2 + 7x + 6) \div (x + 2) = 2x + 3$$

is really an infinitude of school arithmetic problems completed all in one hit!

For instance, suppose I tell you that x is actually the number 2 (not ten). Then

$2x^2 + 7x + 6$ is $\quad 2 \times 4 + 7 \times 2 + 6 = \mathbf{28}$

$x + 2$ is $\quad 2 + 2 = \mathbf{4}$

$2x + 3$ is $\quad 2 \times 2 + 3 = \mathbf{7}$

and we've just ascertained that **28** ÷ **4 = 7**, which is correct!

Or suppose I tell you that x represents the number 5. Then

$2x^2 + 7x + 6$ is $\quad 2 \times 25 + 7 \times 5 + 6 = \mathbf{91}$

$x + 2$ is $\quad 5 + 2 = \mathbf{7}$

$2x + 3$ is $\quad 2 \times 5 + 3 = \mathbf{13}$

and we've just ascertained that **91** ÷ **7 = 13**, which is correct!

Algebra really is the art of doing an infinite number of arithmetic problems all in one hit.

Practice 66.2 What is the statement $(2x^2+7x+6) \div (x+2) = 2x+3$ saying if x represents the number 3?

Practice 66.3:
a) Compute

$$(2x^3+5x^2+5x+6) \div (x+2)$$

in an $1 \leftarrow x$ machine. (You should get the answer $2x^2+x+3$.)

b) If I tell you that x is the number 10 in my mind, what school arithmetic problem have you just answered?

Quantities expressed as codes in an $1 \leftarrow x$ machine are called **polynomials**.

These codes are just like numbers expressed in base 10, except now they are "numbers" expressed in base x. (And if someone tells you x is actually 10, then they really are base-ten numbers!)

The work one learns to do in high-school "polynomial algebra" is essentially just a repeat of early-grade base-ten school arithmetic.

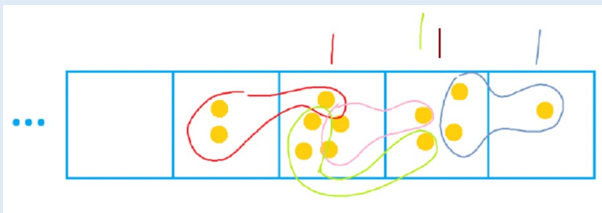
MUSINGS

Musing 66.4
a) Draw a picture of $x^3 + 2x^2 + 3x + 1$ in an $1 \leftarrow x$ machine.
Add $3x^3 + 7x^2 + 4x + 1$ to your picture.
What then is $(x^3 + 2x^2 + 3x + 1) + (3x^3 + 7x^2 + 4x + 1)$ according to your picture?

b) If x is the number 10 throughout part a), what ordinary arithemetic problem have you conducted?

Musing 66.5 What do you think $(5x^2 + 4x + 9) - (2x^2 + x + 6)$ equals?
What does this translate to if x is the number 10?

Musing 66.6 Kennedy drew this picture when she was working on a division problem.



a) If this is a picture of a division problem in a $1 \leftarrow 10$ machine, what division problem is she conducting and what is its answer?
b) If this is a picture of a division problem in an $1 \leftarrow x$ machine, what division problem is she conducting and what is its answer?

Musing 66.7
a) Compute $(2x^4 + 3x^3 + 5x^2 + 4x + 1) \div (2x + 1)$.
b) Compute $(x^4 + 3x^3 + 6x^2 + 5x + 3) \div (x^2 + x + 1)$.

If x is the number 10 in both these problems, what two division problems in ordinary arithmetic have you just computed?

Musing 66.8 Here's a polynomial division problem written in fraction notation. Can you compute its value? (Is there something tricky to watch out for?)

$$\frac{x^4 + 2x^3 + 4x^2 + 6x + 3}{x^2 + 3}$$

Musing 66.9

a) Show that $(x^4 + 4x^3 + 6x^2 + 4x + 1) \div (x + 1)$ equals $x^3 + 3x^2 + 3x + 1$.

b) What is this saying for $\mathrm{x} = 10$?

c) What is this saying for $\mathrm{x} = 2$?

d) What is this saying for x equal to each of 3, 4, 5, 6, 7, 8, 9, and 11?

e) What is this saying for $\mathrm{x} = 0$?

67. A Problem!

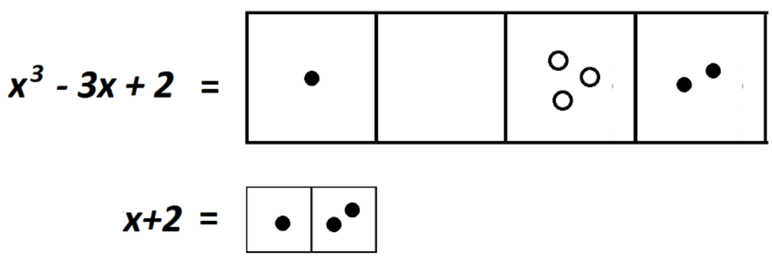
Okay. Now that we are feeling really good about doing advanced algebra, I have a confession to make. I've been hiding a problem. A serious problem!

I've been choosing examples that only use dots. What about antidots?

Consider, for example,

$$\frac{x^3 - 3x + 2}{x + 2}$$

Here's what we have as a picture in an $1 \leftarrow x$ machine.



We seek sets of one-dot-next-to-two-dots in this picture of $x^3 - 3x + 2$.

Do you see any single dots right next to a pair of double dots? I don't!

When we were in a predicament like this back in Section 35, we thought to unexplode dots. Perhaps we can take that leftmost dot in the picture and unexplode it into … umm … how many dots?

That's the snag!
Since we don't know the value of x, we don't know how many dots to draw when we unexplode.

Bother!

It seems like we are stuck.

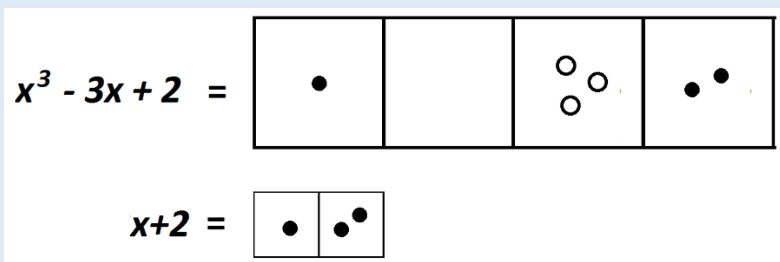
Either we need to conclude that any polynomial division that involves negative numbers just can't be done, or we need some amazing flash of insight that allows us to move forward.

MUSINGS

Musing 67.1 What's your take on this?

Are we at an impasse? Or do you think there is a way to move forward with a problem like this?

$$\frac{x^3 - 3x + 2}{x + 2}$$

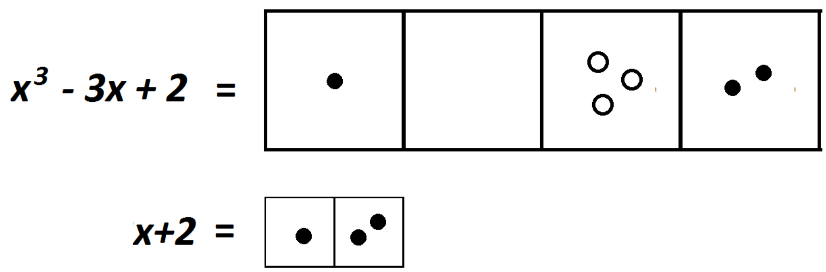


68. Resolution

We are stuck computing this division problem.

$$\frac{x^3 - 3x + 2}{x + 2}$$

Here's a picture of the problem in an $1 \leftarrow x$ machine.



We seek copies of $x + 2$—one dot right next to two dots—anywhere in the picture of $x^3 - 3x + 2$. But there are none!

And we can't unexplode dots to help us out as we don't know the value of x. (We don't know how many dots to draw when we unexplode.)

The situation seems hopeless at present.

But I have a piece of advice for you, a general life lesson in fact. It's this:

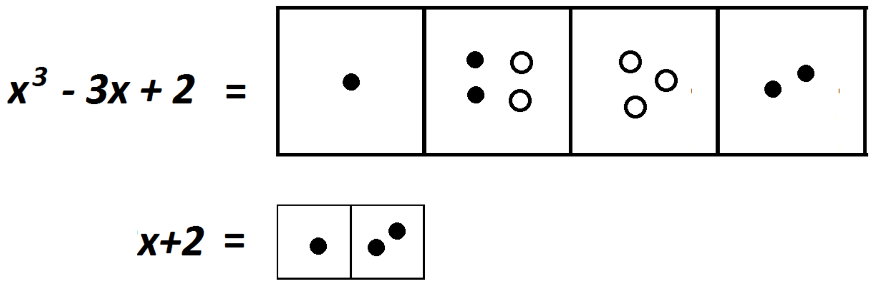
> **IF THERE IS SOMETHING IN LIFE YOU WANT … MAKE IT HAPPEN!**
> **(And deal with the consequences.)**

Right now, is there anything in life we want?

Look at that single dot way out to the left. Wouldn't it be nice if we had two dots in the box next to it, to make a copy of $x + 2$?

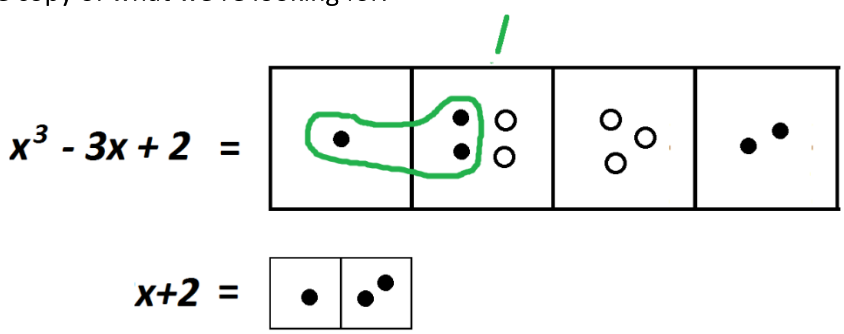
Well, if there is something you want … Make it happen!
Let's just put two dots into that empty box!

But we have to deal with the consequences. That box is meant to be empty and we can't just willy-nilly change it. So, in order to keep it empty, let's put in two antidots as well.



Brilliant!

We now have one copy of what we're looking for.

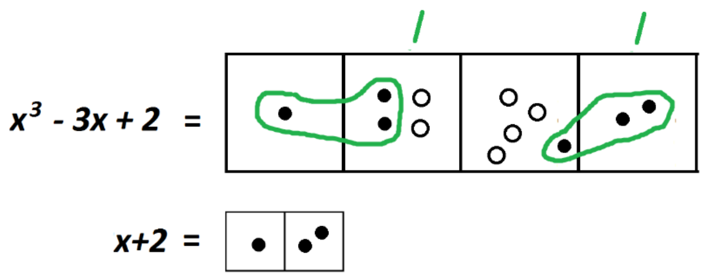


But there is still the question: Is this brilliant idea actually helpful?

Hmm.

Well. Is there anything else in life you want right now? Can you create another copy of $x + 2$ anywhere?

I'd personally like a dot to the left of the pair dots in the rightmost box. I am going to make it happen! I am going to insert a dot and antidot pair. Doing so finds me another copy of $x + 2$.

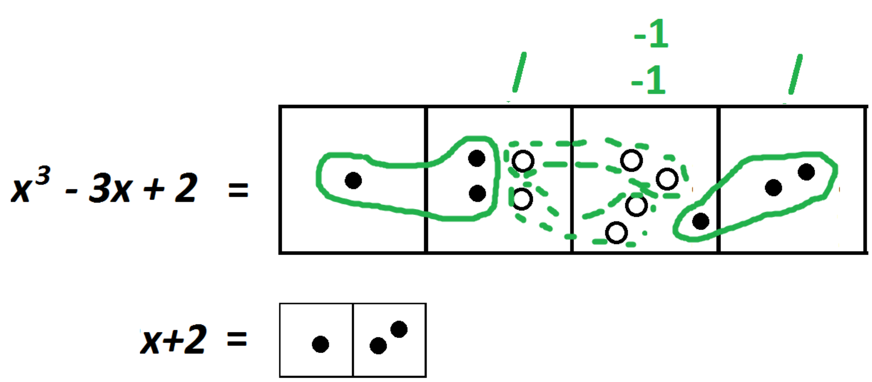


This is feeling good!

But … err … are we now stuck?
Maybe this brilliant idea isn't helpful.

> Stare at this picture.
> Do you notice anything?

Do you see copies of the exact opposite of what we're looking for? Instead of one dot next to two dots, there are copies of one antidot next to two antidots! We have two anti-copies!



Whoa! It looks like we did it.

How do we read the answer?
We see that

$$(x^3 - 3x + 2) \div (x + 2) = x^2 - 2x + 1$$

Fabulous!

So, actually, we <u>can</u> do all polynomial division problems with this dots-and-boxes method, even problems that involve negative numbers.

Practice 68.1 Try computing

$$\frac{x^4 - 1}{x - 1}$$

Can you get the answer $x^3 + x^2 + x + 1$?

Practice 68.2 Show that

$$\frac{x^6 + 1}{x^2 + 1}$$

equals $x^4 - x^2 + 1$.

Practice 68.3: Play with

$$(4x^4 - 7x^3 + 10x^2 - 4x + 2) \div (x^2 - x + 1)$$

to see that it equals $4x^2 - 3x + 3$ with a remainder of $2x - 1$ yet to be divided by $x^2 - x + 1$.

(People typically write this answer as $\frac{4x^4-7x^3+10x^2-4x+2}{x^2-x+1} = 4x^2 - 3x + 3 + \frac{2x-1}{x^2-x+1}$.)

MUSINGS

Musing 68.4 In this Section we showed

$$(x^3 - 3x + 2) \div (x + 2) = x^2 - 2x + 1$$

If x is actually the number 10, what ordinary arithmetic problem does this represent?

Musing 68.5 Compute $\frac{x^3-3x^2+3x-1}{x-1}$.

Musing 68.6 Try computing $\frac{4x^3-14x^2+14x-3}{2x-3}$.

Musing 68.7 If you can compute this problem, you can probably do any problem!

$$\frac{4x^5 - 2x^4 + 7x^3 - 4x^2 + 6x - 1}{x^2 - x + 1}$$

Musing 68.8 We compute $(2x^2 + 7x + 6) \div (x + 2)$ to be $2x + 3$.

Can you predict what the answer to $(2x^2 + 7x + 7) \div (x + 2)$ will be?

Musing 68.9 Compute $\frac{x^4}{x^2-3}$.

Musing 68.10 Try this crazy one: $\frac{5x^5-2x^4+x^3-x^2+7}{x^3-4x+1}$.

If you do this one with paper and pencil, you will find yourself trying to draw 84 dots at some point. Is it swift and easy just to write the number "84"? In fact, how about just writing numbers and not bother drawing dots at all?

In general, is there a swift way to conduct polynomial division with ease on paper? Rather than draw boxes and dots, maybe work with tables that keep track of numbers?

(The word **synthetic** is often used for algorithms one creates that are a step or two removed from that actual process at hand. Some school curriculums teach students a process called **synthetic division** which is really just our dots-and-boxes method in disguise.)

Musing 68.11 What do you think are the answers to each of these polynomial algebra questions?

$$(x^3 - 3x^2 + 3x - 1) + (x^2 - 2x + 1)$$

$$(x^3 - 3x^2 + 3x - 1) - (x^2 - 2x + 1)$$

$$(x^3 - 3x^2 + 3x - 1) \times (x^2 - 2x + 1)$$

$$(x^3 - 3x^2 + 3x - 1) \div (x^2 - 2x + 1)$$

Musing 68.12

a) Lara was asked to compute $(x^2 + 7x + 6) \div (x + 6)$ and reasoned as follows:

> "If $x = 10$, then this reads $176 \div 16$, which has the answer 11.
> So $(x^2 + 7x + 6) \div (x + 6)$ must be $x + 1$."

Is it?

b) Lara was also asked to compute $(3x^2 + x + 2) \div (x + 3)$. She reasoned that since $312 \div 13 = 24$ the answer must be $2x + 4$.

It's not!

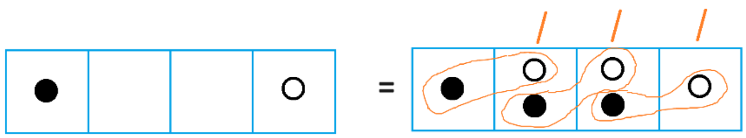
What is the correct answer and what went wrong?

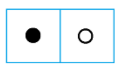
69. The Opening Mystery

This picture shows that

$$(x^3 - 1) \div (x - 1)$$

equals $x^2 + x + 1$.

$x^3 - 1$ = 

$x - 1$ = 

Remember that a division problem like $(x^3 - 1) \div (x - 1)$ is asking:

What is multiplied by $x - 1$ to give $x^3 - 1$?

And we see that $x^2 + x + 1$ is what must have been multiplied by $x - 1$ to give $x^3 - 1$

$$x^3 - 1 = (x - 1) \times (x^2 + x + 1)$$

But let's lose some detail and just write

$$x^3 - 1 = (x - 1) \times (something)$$

Practice 69.1

a) Show that $(x^4 - 1) = (x - 1) \times (something)$.
b) Show that $(x^7 - 1) = (x - 1) \times (something)$.
c) Show that $(x^{100} - 1) = (x - 1) \times (something)$.

The algebra is showing us that

$$\mathrm{x}^{\text{counting number}} - 1 = (\mathrm{x} - 1) \times (\text{something}).$$

and the "something" is always friendly: It's $x^3 + x^2 + x + 1$ or $x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$ or $x^{99} + x^{98} + \cdots + x^2 + x + 1$, or some such. If x represents a counting number, then the "something" will be a counting number too (a sum of counting numbers, in fact.)

Let's have some fun with this observation.

Example: Show that $17^6 - 1$ is sure to be a multiple of 16.

Answer: We know that

$$x^7 - 1 = (x - 1) \times (something)$$

If x represents the number 17, then this is saying

$$17^6 - 1 = (17 - 1) \times (something)$$

That is, $17 = 16 \times (something)$ and so is a multiple of 16.

Practice 69.2 Explain why $637^{52} - 1$ must be a multiple of 636.

Practice 69.3 Explain why $8^{100} - 1$ must be a multiple of 7.

Can we use this technique to show that $2^{300} - 1$ is a composite number?
This was a challenge from earlier on.

Let's try!

Our work gives

$$\begin{aligned} 2^{300} - 1 &= (2-1) \times (something) \\ &= 1 \times (something) \end{aligned}$$

Hmm. Not very helpful. We knew that already!

Here's something very sneaky.

Rather than think of 2^{300} as a product of three hundred 2s, think of it as a product of one hundred-and-fifty 4s by noticing that $2 \times 2 = 4$.

$$\begin{aligned} 2^{300} = 1 &\times (2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2) \\ &\times (2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2) \\ &\times (2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2) \\ &\times (2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2) \\ &\times (2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2) \\ &\times (2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2) \\ &\times (2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2) \\ &\times (2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2) \\ &\times (2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2) \\ &\times (2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2) \\ &\times (2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2) \\ &\times (2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2) \\ &\times (2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2) \\ &\times (2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2) \\ &\times (2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2) \end{aligned}$$

$$\begin{aligned} = 1 &\times (4)\times(4)\times(4)\times(4)\times(4)\times(4)\times(4)\times(4)\times(4)\times(4) \\ &\times (4)\times(4)\times(4)\times(4)\times(4)\times(4)\times(4)\times(4)\times(4)\times(4) \\ &\times (4)\times(4)\times(4)\times(4)\times(4)\times(4)\times(4)\times(4)\times(4)\times(4) \\ &\times (4)\times(4)\times(4)\times(4)\times(4)\times(4)\times(4)\times(4)\times(4)\times(4) \\ &\times (4)\times(4)\times(4)\times(4)\times(4)\times(4)\times(4)\times(4)\times(4)\times(4) \\ &\times (4)\times(4)\times(4)\times(4)\times(4)\times(4)\times(4)\times(4)\times(4)\times(4) \\ &\times (4)\times(4)\times(4)\times(4)\times(4)\times(4)\times(4)\times(4)\times(4)\times(4) \\ &\times (4)\times(4)\times(4)\times(4)\times(4)\times(4)\times(4)\times(4)\times(4)\times(4) \\ &\times (4)\times(4)\times(4)\times(4)\times(4)\times(4)\times(4)\times(4)\times(4)\times(4) \\ &\times (4)\times(4)\times(4)\times(4)\times(4)\times(4)\times(4)\times(4)\times(4)\times(4) \\ &\times (4)\times(4)\times(4)\times(4)\times(4)\times(4)\times(4)\times(4)\times(4)\times(4) \\ &\times (4)\times(4)\times(4)\times(4)\times(4)\times(4)\times(4)\times(4)\times(4)\times(4) \\ &\times (4)\times(4)\times(4)\times(4)\times(4)\times(4)\times(4)\times(4)\times(4)\times(4) \\ &\times (4)\times(4)\times(4)\times(4)\times(4)\times(4)\times(4)\times(4)\times(4)\times(4) \\ &\times (4)\times(4)\times(4)\times(4)\times(4)\times(4)\times(4)\times(4)\times(4)\times(4) \end{aligned}$$

We have

$$4^{150} - 1 = (4 - 1) \times (something)$$

$$= 3 \times (something)$$

which is really saying that $2^{300} - 1$ equals $3 \times (something)$ and so is a multiple of three.

It is a composite number!

Practice 69.4
a) Explain why 2^{300} and why 8^{100} are the same number.
b) Explain why $2^{300} - 1$ is also a multiple of 7.

Practice 69.5
a) Explain why 2^{300} and why 16^{75} are the same number.
b) Explain why $2^{300} - 1$ is also a multiple of 15.

Practice 69.6
a) Explain why $2^{300} - 1$ is also a multiple of 31.
b) Explain why $2^{300} - 1$ is also a multiple of 63.

Okay. I think we have well and truly established that the 91-digit long number $2^{300} - 1$ is not a prime number!

MUSINGS

Musing 69.7

a) Why must $2^{44} - 1$ be multiple of 15?
b) Why must $2^{55} - 1$ be multiple of 31?
c) Why must $2^{2222} - 1$ be multiple of 3?
d) Why must $2^{n} - 1$ be a multiple of 3 if n is an even number?
e) Why must $2^{n} - 1$ be a multiple of 7 if n is multiple of three?

Musing 69.8

a) Compute $\frac{x^3+1}{x+1}$ and $\frac{x^5+1}{x+1}$ and $\frac{x^7+1}{x+1}$.

b) Make a guess as to what $\frac{x^{107}+1}{x+1}$ equals.

c) Why must $78^{107} + 1$ be a multiple of 79?

Musing 69.9 CHALLENGE

Let's complete Mersenne's work. Let's see if we can show that if n is a composite number, then $2^{n} - 1$ is sure to be a composite number as well.

Since n is composite, we have $n = ab$ for two counting numbers a and b, each different from 1.

This means, $2^{n} - 1 = 2^{ab} - 1$.

a) Can you explain why 2^{ab} is the same as M^{b}, with M being the number 2^{a}?

We know that $M^{b} - 1$ is a multiple o $M - 1$.

b) Why then is $M - 1$ a factor of $2^{n} - 1$?

This shows that $2^{n} - 1$ has a proper factor and so is composite.

70. “Infinite Polynomials”

We’ve seen that the fraction $\frac{1}{9}$ has an infinite (repeating) decimal expansion.

$$\frac{1}{9} = 0.111111\ldots.$$

Practice 70.1 Feel free to compute the division problem $1 \div 9$ in a $1 \leftarrow 10$ machine to remind yourself of this.

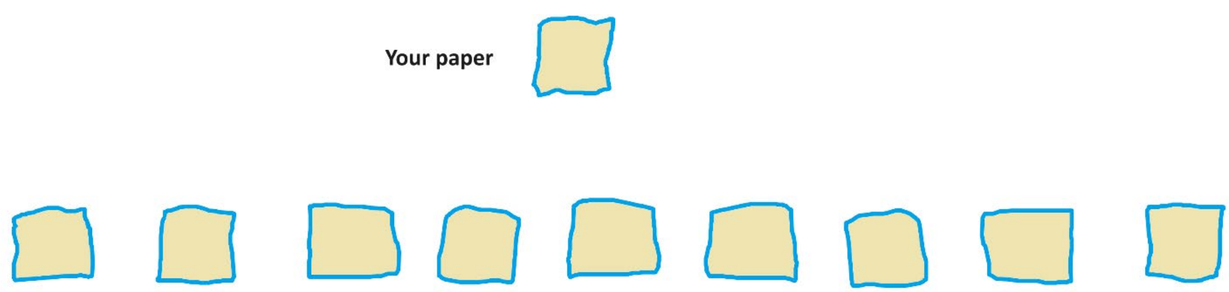
.

Given that the decimal places represent tenths, hundredths, thousandths, and so forth, this is really a statement about an infinite sum. It is saying

$$\frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + \frac{1}{10000} + \cdots = \frac{1}{9}$$

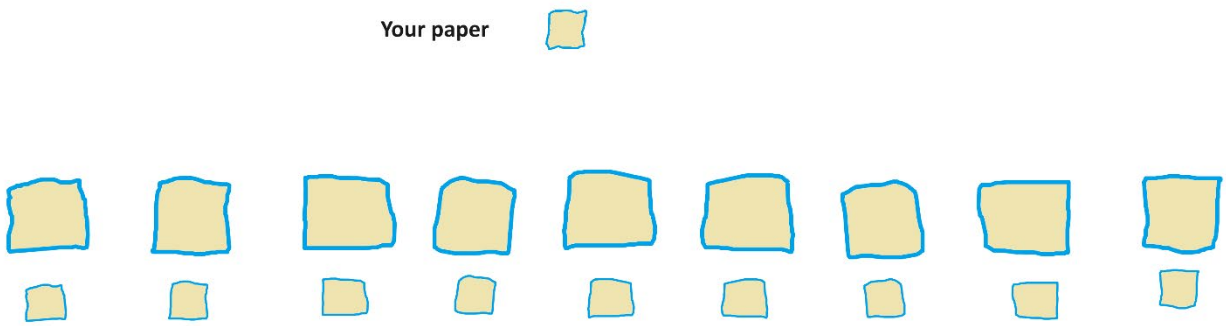
Here’s a fun, “practical” way to see this infinite sum in action.

Imagine sharing a piece of paper with nine of your friends. You divide the paper into tenths and give each of your friends a tenth of the paper ($\frac{1}{10}$) and keep a tenth for yourself.

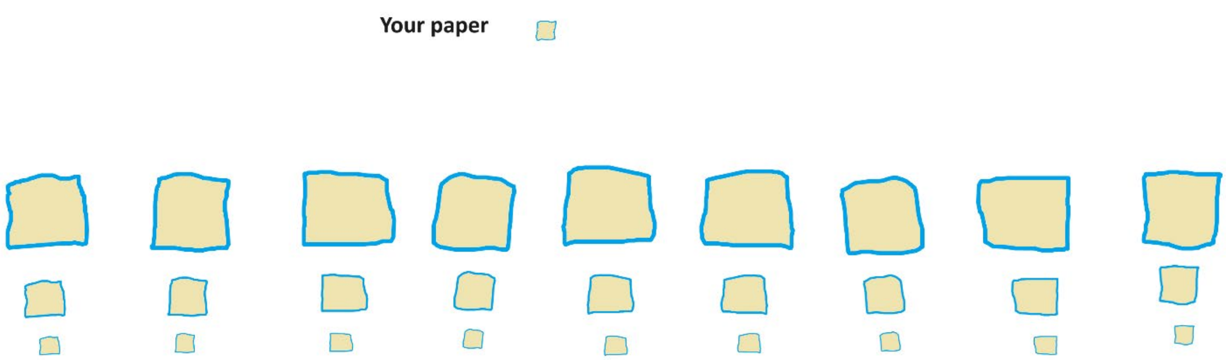


But you’re feeling generous and decide to share your tenth as well.

You divide it into ten parts and give each of your friends a tenth of that tenth (that’s $\frac{1}{100}$ of the original sheet) and keep one part for yourself, a hundredth.



You share again! Dividing your hundredth of the paper into ten parts, you give each of your friends a tenth of a hundredth ($\frac{1}{1000}$ of the original sheet) and keep a thousandth for yourself.



And you keep doing this … forever!

Once you've reached the end of time … how much paper will you hold?

None of it. You'll have given it all away.

And where did the paper go?

The sheet of paper was equally distributed among your nine friends.

So, each of your friends has $\frac{1}{9}$ of the original sheet.

But consider how they each received that paper. They each received it as

$$\frac{1}{10}+\frac{1}{100}+\frac{1}{1000}+\cdots$$

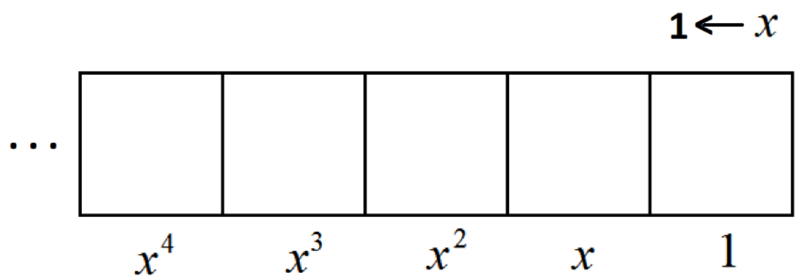
It must be that this infinite sum corresponds to $\frac{1}{9}$ of the paper!

This is, of course, a mind game (as is the nature of infinite decimals!). But it feels like we can imagine an infinite sharing process like this going on forever and having a sense of what "final" result all is heading to.

Practice 70.2 OPTIONAL Imagine repeatedly sharing in this way a sheet of paper equally among you and three friends. Can you see that $\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\frac{1}{80}+\cdots$ wants to be $\frac{1}{4}$?

Here's another way to play with the infinite.

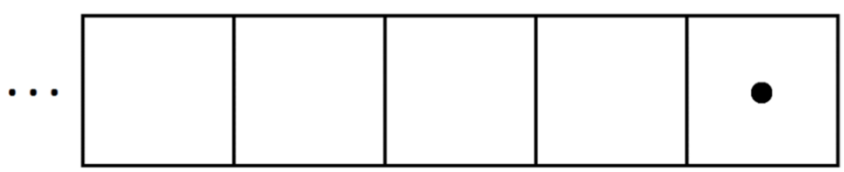
Consider the $1 \leftarrow x$ machine again with boxes heading infinitely far to the left.

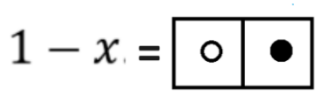


Let's use this machine to compute the strange division problem

$$\frac{1}{1-x}$$

It is the number 1, just a single dot in the machine, divided by $1-x$.
If we think of $1-x$ as $-x+1$ we can see it as an antidot next to a dot.



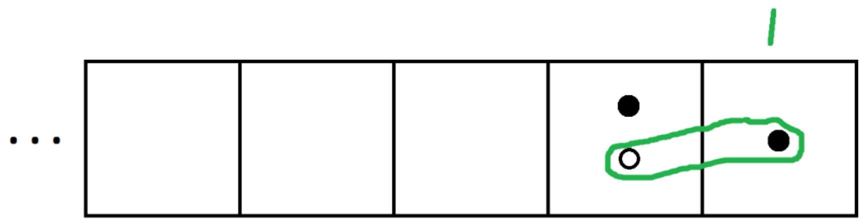


Do you see any antidot-dot pairs in this picture of just one dot?
I don't.

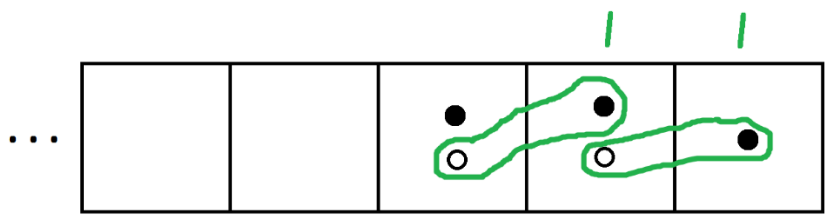
But remember

> If there is something in life you want, make it happen!
> (And deal with the consequences.)

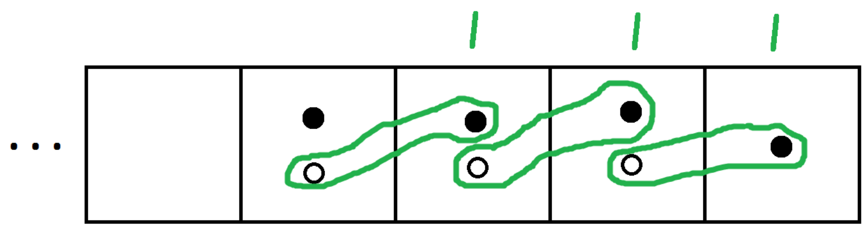
Let's create we want by adding a dot-antidot pair to the picture.



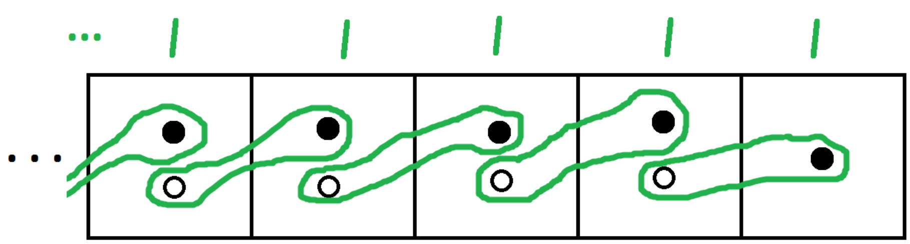
And let's do it again.



And again.



And we can see that we'll be doing this forever.



Whoa!

How do we read this answer?

Well, we have one antidot-dot pair at the 1 level, one at the x level, one at the x^2 level, one at the x^3 level, and so on and so on.

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots$$

The answer is an infinite sum.

Practice 70.3 Let's imagine that x represents the number $\frac{1}{10}$.

a) Do you see that $1 + x + x^2 + x^3 + x^4 + \cdots$ is then $1 + \frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + \frac{1}{10000} + \cdots$?

b) Make $\frac{1}{1-\frac{1}{10}}$ look friendlier.

c) Do a) and b) together show (again) that $\frac{1}{9} = \frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + \frac{1}{10000} + \cdots$?

The equation we obtained is a famous formula in mathematics. It is called the **geometric series formula** and it is often given in many upper-level high school text books for students to use. But textbooks often write the formula the other way round, and with the letter r rather than the letter x.

$$1 + r + r^2 + r^3 + \cdots = \frac{1}{1-r}$$

Practice 71.4 Suppose x represents the number 2.

Do you believe the equation $\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots$ in this case?

The previous two practice problems show that playing with the infinite is dangerous.
The formula $\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots$ seems believable for some values of x (like $\frac{1}{10}$) but not for others (like 2).

All through the 1600s and beyond scholars tried to understand what formulas involving the infinite could mean and when, exactly, they are to believed. This helped spur on the famous subject called **calculus**.

Just so you have it, scholars have come to understand that he geometric series formula

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots$$

is consistent with arithmetic if x represents a number between -1 and 1 on the number line.

MUSINGS

Continuing the dangerous play with the infinite …

Musing 70.5 Show, according to an $1 \leftarrow x$ machine, that $\frac{1}{1+x}$ equals

$$1 - x + x^2 - x^3 + x^4 - \cdots$$

Musing 70.6: Compute $\frac{x}{1-x^2}$. Do you get a sum of odd powers of x?

Musing 70.7: Compute $\frac{1}{1-x-x^2}$ and discover a very famous sequence of numbers. (Draw very big boxes for this one. The picture gets messy quickly!)

71. COMPLETELY OPTIONAL AND COMPLETELY WILD: Base One-and-a-Half

In all our base machines, a set number of dots “explode,” to disappear to be replaced **one** dot, one place to their left.

In a $1 \leftarrow 10$ machine, groups of **ten** dots are replaced by **one** dot.
In a $1 \leftarrow 2$ machine, groups of **two** dots are replaced by **one** dot.
In a $1 \leftarrow 3$ machine, groups of **three** dots are replaced by **one** dot.

And so forth.

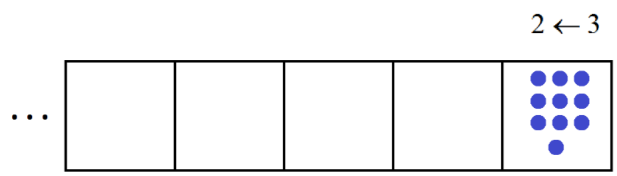
What if we mix things up a bit?

Consider a $2 \leftarrow 3$ machine.

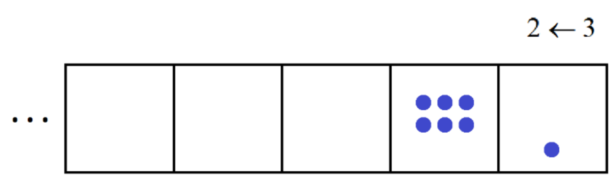
This machine replaces **three** dots in any one box with **two** dots one place to their left.

Curious!

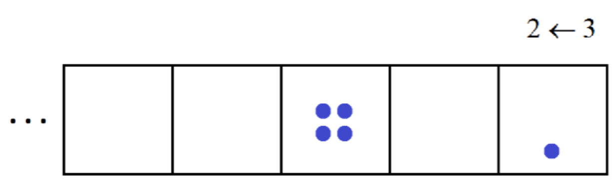
To get a feel for the machine, let’s try putting ten dots into the machine.



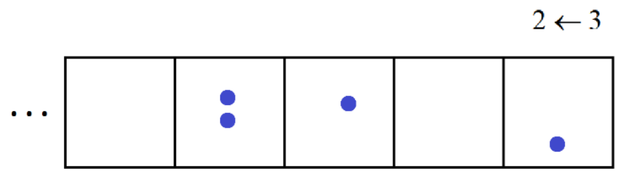
We immediately get three explosions creating two and two and two dots in the next box over.



Now there are two more explosions creating two and two dots.



One more explosion.



We see the code 2101 appear for the number ten in this 2 ← 3 machine.

Practice 71.1 Show that the number thirteen has code 2121.

Here are the 2 ← 3 codes for the first fifteen numbers. (Check these!)

1: 1	**6: 210**	**11: 2102**
2: 2	**7: 211**	**12: 2120**
3: 20	**8: 212**	**13: 2121**
4: 21	**9: 2100**	**14: 2122**
5: 22	**10: 2101**	**15: 21010**

Practice 71.2 Does it make sense that only the digits 0, 1, and 2 appear in these codes? Explain why you won't see a digit of 3 or higher in any 2 ← 3 machine code.

Practice 71.3 Does it make sense to you that the final digits of these codes cycle 1, 2, 0, 1, 2, 0, 1, 2, 0, …?

Even if we don't know what these codes mean, we can still do arithmetic in this weird system!

For example, ordinary arithmetic says that $6 + 7 = 13$, and the codes in this machine say the same thing too.

Six has code 210.
Seven as code 211.
Adding gives 2121, which is the code for thirteen.

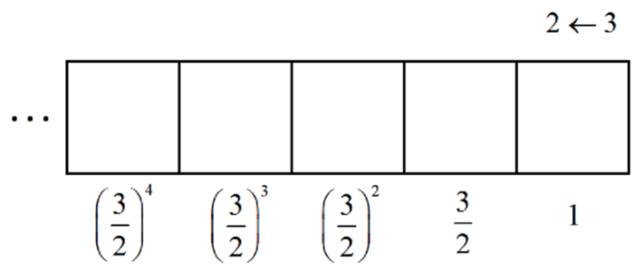
$$\begin{array}{r} 210 \\ +\ 211 \\ \hline =\ 421 \end{array} \quad = 2121$$

Practice 71.4 Compute $10 + 5$ purely by 2 ← 3 machine codes. Do you get the code for fifteen?

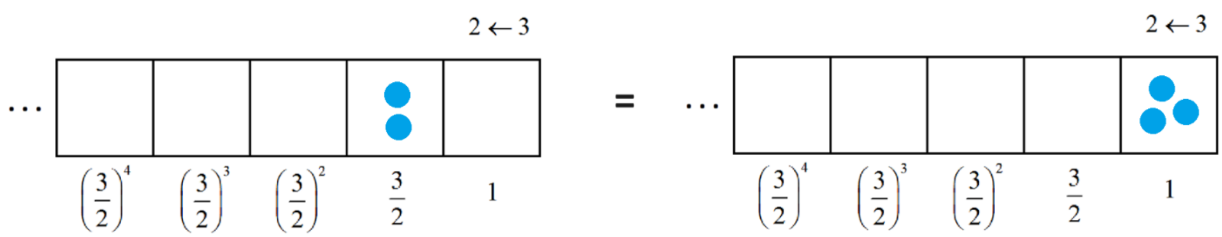
But the real question is:

What are these codes? Are these codes for place-value in some base?
If so, which base?

Of course, the title of this Section gives the answer away. So, let's check if the powers of one-and-half, $\frac{3}{2}$, really are the correct place values here.



For starters, three dots in the 1s place should be equivalent to two dots in the $\frac{3}{2}$s place. Are they?

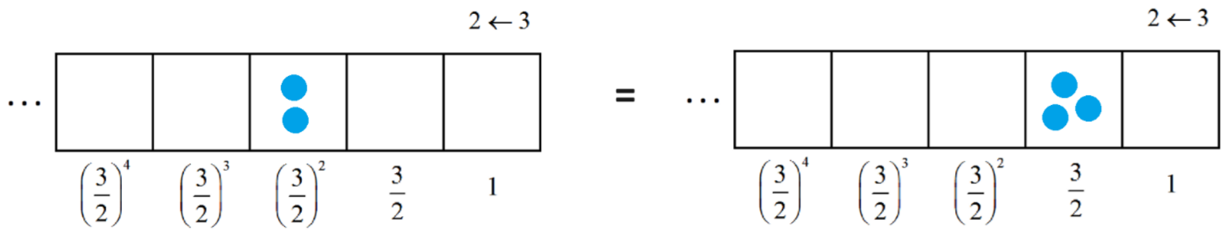


Now

three 1s are: $3 \times 1 = 3$
two $\frac{3}{2}$s are: $2 \times \frac{3}{2} = 3$

Yes! These are the same.

We also need three dots in the $\frac{3}{2}$s place to be equivalent to two dots in the $\left(\frac{3}{2}\right)^2$s place. Let's check.

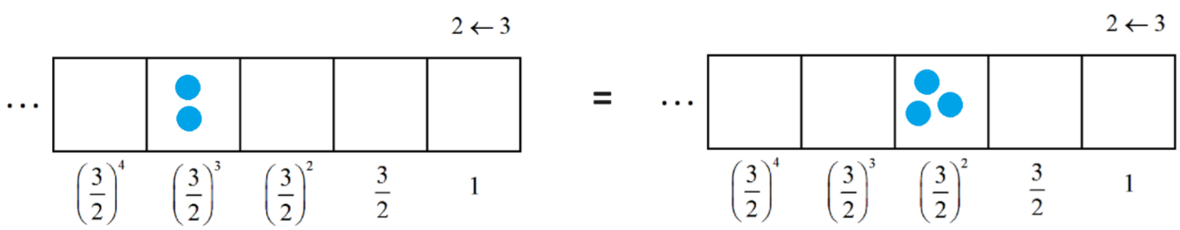


Three $\frac{3}{2}$s are: $3 \times \frac{3}{2} = 3 \times 3 \times \frac{1}{2}$

Two $\left(\frac{3}{2}\right)^2$s are: $2 \times \left(\frac{3}{2}\right)^2 = 2 \times \frac{3}{2} \times \frac{3}{2} = 2 \times 3 \times \frac{1}{2} \times 3 \times \frac{1}{2} = 3 \times 3 \times \frac{1}{2}$

These are the same!

Let's check one more.



Three $\left(\frac{3}{2}\right)^2$s are: $3 \times \left(\frac{3}{2}\right)^2 = 3 \times 3 \times \frac{1}{2} \times 3 \times \frac{1}{2}$

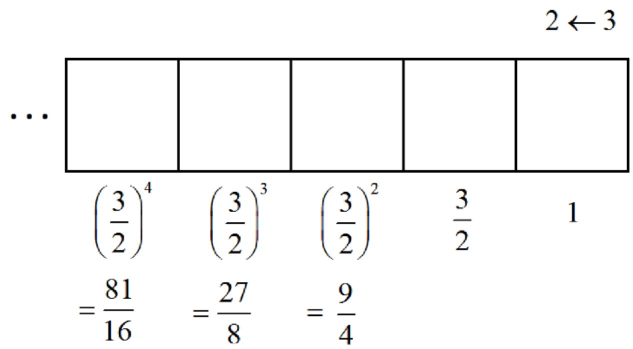
Two $\left(\frac{3}{2}\right)^3$s are: $2 \times \left(\frac{3}{2}\right)^3 = 2 \times 3 \times \frac{1}{2} \times 3 \times \frac{1}{2} \times 3 \times \frac{1}{2} = 3 \times 3 \times 3 \times \frac{1}{2} \times \frac{1}{2}$

These are the same!

Practice 71.5 OPTIONAL If you can, show in general, that three copies of one power of $\frac{3}{2}$ will always match two copies of the next power of $\frac{3}{2}$.

We are indeed working in something that looks like base one-and-a-half!

And if you like we can evaluate the powers of $\frac{3}{2}$ if we want (but they are awkward fractions!)



$\left(\frac{3}{2}\right)^2 = \frac{3}{2} \times \frac{3}{2} = \frac{9}{4}$ $\left(\frac{3}{2}\right)^3 = \frac{3}{2} \times \frac{3}{2} \times \frac{3}{2} = \frac{27}{8}$ $\left(\frac{3}{2}\right)^4 = \frac{3}{2} \times \frac{3}{2} \times \frac{3}{2} \times \frac{3}{2} = \frac{81}{16}$ $\left(\frac{3}{2}\right)^5 = \frac{3}{2} \times \frac{3}{2} \times \frac{3}{2} \times \frac{3}{2} \times \frac{3}{2} = \frac{243}{32}$

This codes from our $2 \leftarrow 3$ machine use the digits 0, 1, and 2, which is a little weird.

There is no digit ten or bigger in base ten.
There is no digit two or bigger in base two.
There is no digit three or bigger in base three.
And so on.

Yet there is a digit bigger than the base number for these $2 \leftarrow 3$ codes.

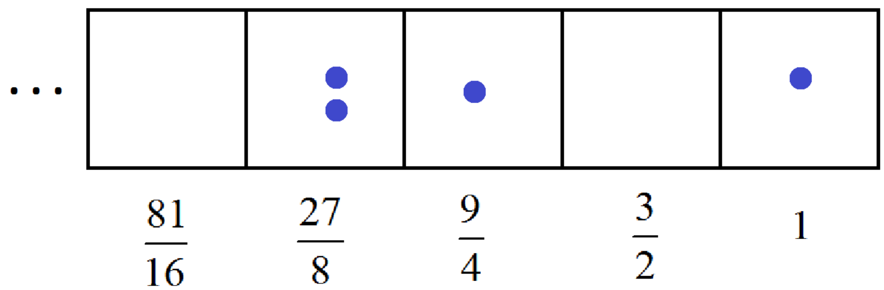
Comment: Mathematicians are trying to develop a notion of base one-and-a-half, and other fractional bases, that don't use digits that exceed the base value. (If you are game, look up *beta expansions* and *non-integer representations* on the internet.)

This alternative version of base one-and-a-half was first conceived by mathematician Dr. James Propp. It is called **Propp Base One-and-a-Half**.

I personally find $2 \leftarrow 3$ machine codes intuitively alarming!

We are saying that each and every counting number can be represented as a combination of the ghastly fractions $1, \frac{3}{2}, \frac{9}{4}, \frac{27}{8}, \frac{81}{16}, \ldots$.

For example, we saw that the number ten has code 2101.



Is it true that this combination of fractions

$$2 \times \frac{27}{8} + 1 \times \frac{9}{4} + 0 \times \frac{3}{2} + 1 \times 1$$

turns out to be the perfect whole number ten?

Check that it does.

(And to that, I say WHOA!)

There are plenty of questions to be asked about the $2 \leftarrow 3$ machine codes of numbers, and many are unsolved to this day.

For reference, here are the first forty numbers in Propp base one-and-a-half (along with zero at the beginning).

0			
1	**2102**	**21220**	**212021**
2	**2120**	**21221**	**212022**
20	**2121**	**21222**	**212210**
21	**2122**	**210110**	**212211**
22	**21010**	**210111**	**212212**
210	**21011**	**210112**	**2101100**
211	**21012**	**212000**	**2101101**
212	**21200**	**212001**	**2101102**
2100	**21201**	**212002**	**2101120**
2101	**21202**	**212020**	**2101121**

Practice 71.6:

Explain why, after a small initial "hiccup," all the codes begin with the digit 2.

Explain why, after a slightly bigger initial "hiccup," all the codes begin with 21.

(The first three digits of the codes, alas, don't stabilize, even after a big "hiccip.")

Practice 71.7: With the exceptions of 0, 1, and 2, explain why if you delete the final digit of any code in this list, what remains is another code in the list. (For example, deleting the final digit of 2101121, the code for forty, leaves 210112, the code for twenty six.)

The Musings to end of this volume (assuming you are reading this optional section) represent a lifetime of work! Musing on them and making progress with them and possibly solving them is likely opening up new original mathematics currently unknown to the world.

Enjoy musing on them. Just play and have fun. And if you happen to make significant headway on a problem, let me know! I'll help the world learn what you've accomplished.

Have fun!

MUSINGS

Musing 72.8 **Which combinations of 0s, 1s, and 2s are Propp codes for numbers?**

Look at our list of codes for the first forty numbers. We don't see "201" or 21102."

a) The code 201 represents the mixed number $5\frac{1}{2}$. (Why?)
What mixed number does 21102 represent?

One way to work out whether or not a combination of digits represents a whole number is to simply work out the sums of powers of $\frac{3}{2}$ it represents. But that doesn't seem fun! For instance, how long would it take to determine the number represented by this long code?

2102212020212002102022002101011220002212212220210201220110021020102022 12

b) **Unsolved Challenge:** Develop a quick and efficient means to look at a sequence of 0s, 1s, and 2s to determine whether or not it corresponds to the code of a whole number. (Of course, how one defines "quick" and "efficient" is up for debate.)

Musing 72.9 **Divisibility Checks**

Look at the list of the first forty 2 ← 3 machine codes of numbers.

Starting with number zero, every third code ends with a zero, and only every third code (Why?)

0			
1	2102	21220	212021
2	2120	21221	212022
20	2121	21222	212210
21	2122	210110	212211
22	21010	210111	212212
210	21011	210112	2101100
211	21012	212000	2101101
212	21200	212001	2101102
2100	21201	212002	2101120
2101	21202	212020	2101121

This leads to the following check.

Divisibility by Three
A number written in 2 ← 3 code is divisible by three precisely when its final digit is zero.

This makes it easy to tell if a number expressed as a 2 ← 3 machine code is a multiple of three.

a) Find, and explain, a similar divisibility check for 9.

b) Find, and explain, a similar divisibility check for 27 and for 81. (Could you keep going with higher and higher powers of three?)

Every fifth code in our list of 2 ← 3 machine codes of numbers has a curious property too: the alternating sum of its digits is a multiple of five.

0			
1	2102	21220	212021
2	2120	21221	212022
20	2121	21222	212210
21	2122	210110	212211
22	21010	210111	212212
210	21011	210112	2101100
211	21012	212000	2101101
212	21200	212001	2101102
2100	21201	212002	2101120
2101	21202	212020	2101121

2-2 = 0

2-1+0-1+1-1 = 0

2-1+2-0+2-0 = 5

c) Can you explain why this is so for every fifth entry, and only the fifth entries?

This then gives a check for divisibility by five.

Divisibility by Five

A number is divisible by 5 only if the alternating sum of the digits of its $2 \leftarrow 3$machine code is a multiple of five.

c) **Unsolved Challenge: How do you tell if a number is even?**

Is there a divisibility check for the number two? Is there an interesting property to the codes of every second number?

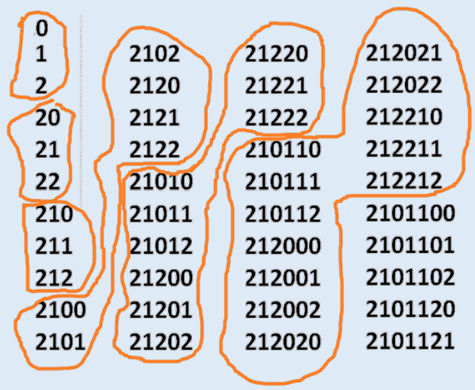
As far as I am aware, no one knows how to look at the $2 \leftarrow 3$ machine code a number and quickly determine if that number is even!

Do you see—and can you prove always hold—something interesting about every second code?

0			
1	2102	21220	212021
2	2120	21221	212022
20	2121	21222	212210
21	2122	210110	212211
22	21010	210111	212212
210	21011	210112	2101100
211	21012	212000	2101101
212	21200	212001	2101102
2100	21201	212002	2101120
2101	21202	212020	2101121

Musing 72.9 Understanding Code Lengths

Look at the lengths of the 2 ← 3 machine codes for numbers.



There are 3 codes one digit long, 3 codes two digits long, 3 codes three digits long, 6 codes four digits long, 9 digits five digits long, 12 codes six digits long, and so on.

If you kept going you will get the following sequence of numbers for how many codes have a given length.

$$3, 3, 3, 6, 9, 12, 18, 27, 42, 63, 93, 141, 210, 315, 474, \ldots$$

Is there a pattern of some kind to these numbers?

Challenge: As far as I am aware, no one knows a formula for the numbers in this sequence that would quickly tell you the 500th or the 1200th or millionth number is in this sequence.

Your challenge: Find a direct way to find the nth number in this list of numbers for any counting number n.

Musing 72.10 **Palindromes**

In mathematics, a **palindrome** is a number that reads the same way forwards as it does backwards (in whatever base we happen to be discussing).

Among the first forty $2 \leftarrow 3$ machine codes, there are six palindromes: the codes for 0, 1, 2, 5, 8,17, and 35.

0			
1	2102	21220	212021
2	2120	21221	212022
20	2121	21222	212210
21	2122	210110	212211
22	21010	210111	212212
210	21011	210112	2101100
211	21012	212000	2101101
212	21200	212001	2101102
2100	21201	212002	2101120
2101	21202	212020	2101121

It turns out that the codes for 170, 278, 422, and 494 are also palindromes in Propp's base one-and-half.

Challenge: Is there another number with a palindromic Propp base one-and-a-half code?

(Dr. Propp argues in his essay here that there might not be any more examples. Or, if he is wrong, at most finitely many more examples.)

Final Comment:

In these musings I have brought you to an edge of mathematical knowledge.

The school world gives the impression that all in mathematics is "solved" and that mathematicians—most likely—are spending their days just doing calculations on bigger and bigger numbers.

The truth of matters is much more fun and much more interesting. (Calculations, in and of themselves, are boring!) There are so many deep questions and wonderings for mathematicians—and everyone in the world—to explore.

We've asked here just a few curious and unsolved questions about the behavior of the number $\frac{3}{2}$. It is such a simple fraction, yet there is clearly so much we don't yet understand about it!

One of the world's current foremost mathematicians, Dr. Terrence Tao, has written about the mysteries of this basic fraction here. And other scholars too have looked at the mysteries of Propp's base one-and-a-half. (For instance, see this work by ben Chen et al.)

It is astounding to me that we don't understand the mathematical properties of such a simple fraction. That's just wonderful! And that's typical of the source of the adrenalin that pushes mathematicians forward and makes them want to want to strive for more. It's the pursuit of deeper understanding and deeper clarity on how our intellectual universe works.

I hope this volume has given you a glimpse of how mathematics can be so compelling to so many humans.

Solutions

65.1 $15 = 3 \times 5$ and $63 = 7 \times 9$ and $255 = 5 \times 51$.

65.2 Read on!

65. 3 $2^9 - 1 = 511 = 7 \times 73$ and $2^{10} - 1 = 1023 = 3 \times 341$ and $2^{12} - 1 = 4095 = 5 \times 819$.

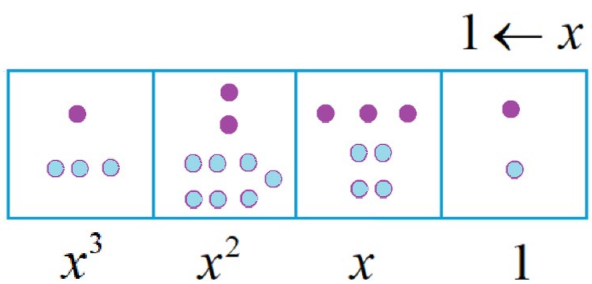
65.4 a) One more b) 2^{300} being one more than a multiple of three would mean that $2^{300} - 1$ is a multiple of three. c) Actually, there are ways to justify this pattern. Can you think of one?

66.1 We can. We have that $x \times x^2$ is $x \times x \times x$, which is x^3, and so forth.

66. 2 $45 \div 5 = 9$

66.3 a) Did you get it? b) $2556 \div 12 = 213$.

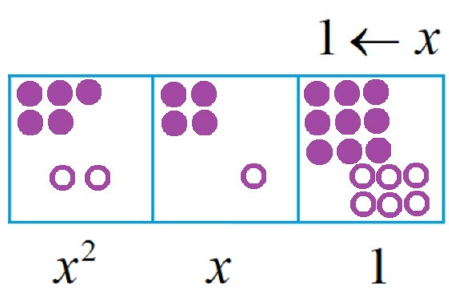
66.4



Adding these together gives $4x^3 + 9x^2 + 7x + 2$.

For $x = 10$ we are saying: $1231 + 3741 = 4971$.

66.5 Don't forget tods!



We get $3x^2 + 3x + 3$.

For $x = 10$ we are saying that $549 - 216 = 333$.

66.6 a) It's $2541 \div 21 = 121$ b) It's $(2x^3 + 5x^2 + 4x + 1) \div (2x + 1) = x^2 + 2x + 1$

66.7 a) $x^3 + x^2 + 2x + 1$ b) $x^2 + 2x + 3$
If $x = 10$, we computed $23541 \div 21 = 1121$ and $13653 \div 111 = 123$.

66.8 You are looking for one-dot-blank-three-dots. (That blank is annoying!)
You'll find one at the x^2 level, two at the x level, and one at the 1 level. The answer is $x^2 + 2x + 1$.

66.9 You can do it!

For $x = 10$ it says 14641 ÷ 11 = 1331

For $x = 2$ it says 81 ÷ 3 = 27

For $x = 3$ it says 256 ÷ 4 = 64

For $x = 4$ it says 625 ÷ 5 = 125

For $x = 5$ it says 1296 ÷ 6 = 216

For $x = 6$ it says 2401 ÷ 7 = 343

For $x = 7$ it says 4096 ÷ 8 = 512

For $x = 8$ it says 6561 ÷ 9 = 729

For $x = 9$ it says 10000 ÷ 10 = 1000

For $x = 11$ it says 20736 ÷ 12 =1728

For $x = 0$ (is this allowed? What's a $1 \leftarrow 0$ machine?) it is saying that $1 \div 1 = 1$

67.1 Read on!

68.1 Try it!

68.2 Try it!

68.3 Try it!

68.4 $972 \div 12 = 81$

68.5 $\frac{x^3-3x^2+3x-1}{x-1} = x^2 - 2x + 1$

68.6 $\frac{4x^3-14x^2+14x-3}{2x-3} = 2x^2 - 4x + 1$

68.7 $\frac{4x^5-2x^4+7x^3-4x^2+6x-1}{x^2-x+1} = 4x^3 + 2x^2 + 5x - 1$

68.8 $2x + 3 + \frac{1}{x+2}$

68.9 $\frac{x^4}{x^2-3} = x^2 + 3 + \frac{9}{x^2-3}$

68.10 $5x^2 - 2x + 21 + \frac{-14x^2+86x-14}{x^3-4x+1}$

68.11 a) $x^3 - 2x^2 + x$
b) $x^3 - 4x^2 + 5x - 2$
c) $x^5 - 5x^4 + 10x^3 - 10x^2 + 5x - 1$
d) $x - 1 + \frac{2}{x^2-2x+1}$

68.12 a) It is. (But this was lucky!)
b) We have $\frac{3x^2+x+2}{x+3} = 3x - 8 + \frac{26}{x+3}$, which looks surprising!
But have x be 10 and you see that it does read $\frac{312}{12} = 22 + \frac{26}{13}$, which is correct!

Each division problem done in base x represents an infinitude of arithmetic problems, including one in base ten. But one problem in base ten does not represent an infinitude of problems in base x.

The trouble is that in base ten we can explode and unexplode and have a whole slew of equivalent division statements:

$$3|1|2 \div 1|2 = 2|4$$

$$2|11|2 \div 1|2 = 2|4$$

$$2|10|12 \div 1|2 = 1|14$$
$$1|19|22 \div 1|2 = 2|2 \text{ with remainder } 2|6$$

We don't know which, if any, of these match a base x version.

69.1 We have $\frac{x^4-1}{x-1} = x^3 + +x^2 + x + 1$, so $x^4 - 1 = (x-1) \times (x^3 + +x^2 + x + 1)$.
The remaining problems are similar.

69.2 Write 637 for x in $x^{52} - 1 = (x-1) \times (something)$.

69.3 Write 8 for x in $x^{100} - 1 = (x-1) \times (something)$.

69.4 a) Since $8 = 2 \times 2 \times 2$ a product of three-hundred 2s is the same as a product of one-hundred 8s.
b) $2^{300} - 1 = 8^{100} - 1 = (8-1) \times (something)$.

69.5 a) Since $16 = 2 \times 2 \times 2 \times 2$ a product of three-hundred 2s is the same as a product of seventy-five 16s.
b) $2^{300} - 1 = 16^{75} - 1 = (16-1) \times (something)$.

69.6 a) $2^{300} - 1 = 32^{60} - 1 = (32-1) \times (something)$.

b) $2^{300} - 1 = 64^{50} - 1 = (64-1) \times (something)$.

69.7 a) $2^{44} - 1 = 16^{11} - 1 = (16 - 1) \times (something)$
b) $2^{55} - 1 = 32^{11} - 1 = (32 - 1) \times (something)$
c) $2^{2222} - 1 = 4^{1111} - 1 = (4 - 1) \times (something)$
d) If n is even, then $n = 2a$ for some counting number a.

$$2^n - 1 = 4^a - 1 = (4 - 1) \times (something)$$

e) If n is is a multiple of three, then $n = 3a$ for some counting number a.

$$2^n - 1 = 8^a - 1 = (8 - 1) \times (something)$$

69.8

d) $\frac{x^3+1}{x+1} = x^2 - x + 1$; $\frac{x^5+1}{x+1} = x^4 - x^3 + x^2 - x + 1$ and $\frac{x^7+1}{x+1} = x^6 - x^5 + x^4 - x^3 + x^2 - x + 1$

e) $\frac{x^{107}+1}{x+1} = x^{106} - x^{105} + x^{104} - x^{103} + \cdots - x + 1$

f) $78^{107} + 1 = (708 + 1) \times (something)$

69.9 a) The product ab is the same as $b \times a$, which means "b groups of a." So, a product of ab 2s can be seen as a product of b groups of $\overbrace{2 \times 2 \times \cdots \times 2}^{a\ times}$. Thus 2^{ab} can be seen as M^b where M is 2^a.

b) $2^n - 1 = 2^{ab} - 1 = M^b - 1 = (M - 1) \times (something)$

70.1 Do it!

70.2 Think about it.

70.3 a) It does.
b) Multiply top and bottom by ten to get: $\frac{1}{1-\frac{1}{10}} = \frac{10}{10-1} = \frac{10}{9} = 1 + \frac{1}{9}$.
c) Yes! We have $1 + \frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + \frac{1}{10000} + \cdots = \frac{1}{1-\frac{1}{10}}$, which equals $1 + \frac{1}{9}$.
So,

$$1 + (\mathbf{\frac{1}{10}} + \mathbf{\frac{1}{100}} + \mathbf{\frac{1}{1000}} + \mathbf{\frac{1}{10000}} + \cdots) = 1 + (\mathbf{\frac{1}{9}})$$

It must be that $\frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + \frac{1}{10000} + \cdots = \frac{1}{9}$.

70.4 I don't. We get that $1 + 2 + 4 + 8 + 16 + \cdots$ is meant to add to -1. That sounds like nonsense!

70.5 Try it.

70.6 You get $\frac{x}{1-x^2} = x + x^3 + x^5 + \cdots$.

70.7 $\frac{1}{1-x-x^2} = 1 + x + 2x^2 + 3x^3 + 5x^4 + 8x^5 + 13x^6 + 21x^7 + \cdots$

71.1 Try it. Put in thirteen dots and conduct all the explosions.

71.2 If there are three or more dot in a box, three will explode away. Consequently, you will only every see 0, 1, or 2 dots in a box.

71.3 Yes. If you keep putting dots into the rightmost box one at a time, and perform explosions as you go, the count of dots in that box will start with 0, then 1, then 2, and then head back to 0 because of an explosion. And so on.

71.4 Try it. It works!

71.5

$3 \times \overbrace{\frac{3}{2} \times \frac{3}{2} \times \cdots \times \frac{3}{2}}^{n\ times}$ equals $\overbrace{3 \times 3 \times \cdots \times 3}^{n+1\ times} \times \overbrace{\frac{1}{2} \times \frac{1}{2} \times \cdots \times \frac{1}{2}}^{n\ times}$

and

$2 \times \overbrace{\frac{3}{2} \times \frac{3}{2} \times \cdots \times \frac{3}{2}}^{n+1\ times}$ equals $2 \times \frac{3}{2} \times \overbrace{3 \times 3 \times \cdots \times 3}^{n\ times} \times \overbrace{\frac{1}{2} \times \frac{1}{2} \times \cdots \times \frac{1}{2}}^{n\ times}$.

And this equals

$3 \times \overbrace{3 \times 3 \times \cdots \times 3}^{n\ times} \times \overbrace{\frac{1}{2} \times \frac{1}{2} \times \cdots \times \frac{1}{2}}^{n\ times}$, which equals the first line!

71.6 Once explosions start happening, two dots will always sit in the leftmost position.

Since two dots will start sitting in the leftmost position, soon explosions will start spilling into those two dots to make 4 dots there. But this induces another explosion to create "21".

71.7 The dots that enter the second box at a time do so two at a time. So, only even counts of dots accumulate there, which then explode in the ways they must. So, if we ignore the rightmost box, we must be seeing the codes of all the even numbers.

71.8 a) 21102 is the number nineteen and a quarter.

71.9 a) A number is divisible by nine precisely when its code ends with two zeros.

Since deleting the final digit gives us a code of another integer, this new integer is divisible by a second factor of three if it also ends with a zero.

b) A number is divisible by twenty-seven precisely when its code ends with three zeros.
A number is divisible by eighty-one precisely when its code ends with four zeros.

And so on.

c) Adding 1 or 2 dots to the rightmost box increases the alternating sum by 1 or 2. So, if this sum was divisible by five, it is not so any more. Adding 3 dots induces an explosion, which changes the alternating by 2. If it was divisible by five before, it is not so any more. Adding 4 dots also induces an explosion that changes the alternating by 1. If it was divisible by five, it is not so any more. But adding 5 dots induces an explosion, which keeps the alternating sum the same. So, if it was divisible by five before, it still is.

Every fifth code has an alternative sum that is divisible by five.