COLLEGE ALGEBRA FOR HUMANS

A Refreshingly Joyous, Human, and Accessible approach to Algebra for all those who may have experienced it otherwise

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PART 2

Algebra: All Arithmetic, All at Once

Photo: Erick Mathew, Tanzania

Algebra is the practice of avoiding the tedium of doing arithmetic problems one instance at a time, to take a step back and see a general structure to what makes arithmetic work the way it does, and so open one's mind to more than the one view of what arithmetic can be.

CONTENT FOR PART 2

Contents

Introduction: Why Part 2? Where's Part 1?

I remember as a youngster back in Australia in the late 70s eagerly awaiting the release of a new movie that I heard was creating a sensation in America. It was called *Star Wars.* Us Aussies had to wait several months before it was released on our continent, and that waiting was torture for this eleven-year-old lad.

I was, of course, immediately enthralled by the production when I finally got to see it. But I was also perplexed. It was called "STAR WARS: EPISODE IV." Where were episodes I, II, and III? What had I missed?

Well, it turns out much of the world was perplexed too by this in the opening titles and nothing was missed.

I dare not equate these math notes with the magic George Lucas's work (nor try to explain why I think Lucas decided to first share his epic galactic saga to the world midway through its story), but I will settle one similar issue with regard to these notes:

Where are chapters 1 through 8 that supposedly comprise Part 1 of this math story?

The answer is that they are here, in this link, freely available to one and all, including you. Because I put them on a website, you may have missed them.

These first eight chapters are a gift to the world as part of a program I co-found called the *Global Math Project* whose goal is to prove that mathematics—school mathematics even—truly can serve as a portal to intense human connection, meaning, and joy. (This video outlines the history of the project and the impact it has had.)

Part 1 of this series covers the story of grade-school arithmetic. You likely already know the mechanics of what is covered there.

For instance, you no doubt know about the **counting numbers** $0, 1, 2, 3, \ldots$ and how to do arithmetic with them: add them, subtract them, multiply them, and divide them. You likely know the "opposite" numbers $-1, -2, -3, \ldots$, that then bring us to the world of **integers**. And you have probably been trained to do arithmetic with these numbers too. (To recognize that $(-4) \times 5 = -20$ and $(-2) \times (-3) = +6$, for instance.)

And you have no doubt gone further and worked with **fractions** and **decimals** as well.

But notice that I chose to say "mechanics." Doing math versus truly owning and deeply understanding the math you are doing are very different things.

I remember back in grade 5 of my schooling receiving gold star after gold star on my long division worksheets. I was even called the star of the class for apparently truly "getting" long division. But I didn't get it. I had just memorized the algorithm (quite well, apparently) and was performing simply to please my teacher. I was fully cognizant of the fact that I didn't understand what I was doing one whit.

Part 1 invites us to reexamine all we think we know about school arithmetic and see it in new, clear, and sensical light.

However, starting on this second volume without working though Part 1 will likely be fine. Just be willing to refer to sections of Part 1 every now and then as you go along. To get a sense of what I mean by this, have a look at the six Musings at the end of this introduction. They are not prerequisites, but they do illustrate the depth of understanding developed in that first volume.

Each of our educational journeys has bits and bobs that are hazy or are outright missing. That's okay! Try this volume and see how it goes. I'll do my best to direct you to relevant sections of Part 1 as we move along to help out.

Be kind to yourself.

Despite what you might have been trained to believe from school math, there is actually no rush to make sense of mathematics. Don't hurry! Just let a profound beautiful sense of mathematics unfold over whatever amount of time it takes.

So, take your time. Linger. Enjoy!

Mathematics is a gift for you to truly savor.

MUSINGS

Here are some thorny issues covered in detail in Part 1. How do personally feel about attending to these questions?

Musing 1: *People say you can't divide by zero. But why?*
Can you personally explain what goes wrong mathematically if you try to divide by zero?
[See Sections 17 and 38.]

Musing 2: *Why, exactly, is negative times negative positive? Have you a convincing explanation?*
[See Section 25.]

Musing 3: *Every few months the following problem makes the rounds on the internet.*

What is the value of $\mathbf{8 \div 2(2+2)}$?

a) Some people say this expression has value 1*. Do you see why they might say that?*
Other people say that it has value 16*. Do you see how such folk must be thinking?*
b) Can you insert parentheses into the expression to ensure everyone evaluates it as 1*?*
c) Can you insert parentheses into the expression a different way to make sure everyone evaluates it as 16*?*
[See Sections 8 and 9.]

Musing 4: *Are you comfortable evaluating* 17×18 *each of these two ways?*
(The second picture looks like it is a rectangle with negative side lengths and negative areas. Is that allowed?)

	10	8
10	100	80
7	70	56

100 + 70 + 80 + 56 = <u>306</u>

	20	-2
20	400	-60
-3	-40	+6

400 - 60 - 40 + 6 = <u>306</u>

[See Sections 11 and 24.]

Musing 5: *Does "*$3n+1$*" make sense to you as a shorthand way to write "triple an unspecified number and add one to the result"?*

Musing 6: *Here's a piece of math that looks scary:* $6x^2 - 3ax = 3x(2x - a)$.
After a moment and a deep breath, can you start to make some sense of it?
Could you perhaps draw a picture to demonstrate what it is saying?
[See Section 13 and 24.]

Chapter 9

Solving, Graphing, Seeing

72. Math is a Language

People say that "mathematics is a language." I personally am not sure I know what that means.

Presumably this remark is a curt summary of a statement made by Italian physicist Galileo Galilei (1564-1642) in his work *Opere II Saggiatore*:

> *[The universe] cannot be read until we have learnt the language and become familiar with the characters in which it is written. It is written in mathematical language, and the letters are triangles, circles and other geometrical figures, without which means it is humanly impossible to comprehend a single word.*

Galileo is making a deep philosophical statement that no doubt has been probed and debated by folk cleverer than me.

Rather than comment on Galileo's musing myself, I will offer instead a non-erudite interpretation of the curt summary statement.

Mathematics *is* a language, in a very literal way.

Since we are currently communicating in English, the language of math is … English!
(And if I was writing in Hindi or in Korean, then the language of mathematics would be Hindi or Korean.)

The fact is that every mathematical statement is a sentence.

For example, the statement

$$5\ = 2 + 3$$

has a subject (the quantity "5"), a verb ("equals"), and an object (the quantity "$2 + 3$").

As such, the sentence when written as a stand-alone statement should come with proper English punctuation: it needs a period at its end.

$$5\ = 2 + 3.$$

Look up any published mathematics paper and you will see it littered with words and symbols and formulas replete with all the Englich punctuation to go with them—periods, commas, semi-colons, you name it. You'll see that punctuation even within and throughout the symbols and lines of formulas. (There are occasional instances where the mathematics community has agreed to let go of some punctation for the sake of visual clarity, but it is implied that it is there.)

One can literally read a math paper out loud as though it were an essay from English class.

Let's run with this idea.

Consider, for instance, the statement:

$$7 > 4 + 9.$$

The subject is the number 7, the verb is "is greater than," (well, that's a bit more than just a verb in this case), and the object is the quantity $4 + 9$. (We talked about the *greater than* symbol in section 50.)

> A sentence with *equals* as its verb is called an **equation**.
>
> A sentence that compares quantities or states that two quantities are not equal is called an **inequality**.

Our first sentence, $5 = 2 + 3$, happens to be a true sentence about numbers and our second sentence, $7 > 4 + 9$, a false one. (Seven is not larger than thirteen!)

And although we know in life that sentences made need not be true, mathematics tends to focus on truth and wants to present sentences that are true statements about numbers.

But there is another issue: sentences can also be ambiguous. They may need further context or information before being deemed true or false.

For instance, the sentence

Harold is over six feet tall

cannot be determined as true or false until we are told which particular Harold of the world the speaker of this sentence is referring to.

And any sentence in math class that uses a name for a number without specifically stating what that number is in the author's mind cannot be determined as true or false either. For example,

$$N + 3 = 10$$

is a sentence about a number being called "N." The sentence is currently neither true nor false. Of course, if we are later told that N represents the number 7, then we can deem it a true sentence. If, instead, we are later told that N represents 13, then we can say we have a false sentence.

The sentence

$$x = 2$$

is currently neither true nor false. But we do realize that if x represents the number 2, then it would be a true sentence.

Practice 72.1: The symbol for "does not equal" is $\neq$. (Have you seen this before?)

a) Give a value for a number being called p that makes the sentence

$$p \neq 2$$

a true sentence.

b) Now give value for p that would make it a false sentence.

MUSINGS

Musing 72.2 (We talk about numbers being squared in Section 11.)

Here's a sentence about an unspecified number being called n.

$$n^2 = 25$$

Without knowing anything about n, the sentence is currently neither true nor false.

a) Let me tell you that the unspecified number n is actually 7 in my mind. Knowing that, is the given sentence true or false?

b) Actually, that's not the case: the number n is really 5. Given that, is the sentence true or false?

c) There are two numbers n could be that would make the sentence true. I believe you've just found one of those values. What's the other one?

Musing 72.3 Here is another math sentence about a number being called r. Knowing nothing about r, the sentence is currently neither true nor false.

$$r + 3 > 100$$

Describe all the possible numbers r could represent that would make this a true sentence.

Musing 72.4 Here's a math sentence about two numbers being called a and b.

$$ab = 0$$

Give some examples of numbers that a and b could represent to make this a true sentence.

[The given sentence could also be written $a \times b = 0$ using the multiplication symbol.
See Section 9 for the various ways people indicate two numbers being multiplied together.]

Musing 72.5 Have you noticed spots in my writing where I have not put in the proper punctuation around a math sentence? There is the convention in math writing not to put in punctuation if a math sentence is being "displayed," that is written on its own line, usually centered in the line.

73. Collecting Math Data

When presented with a math sentence that includes the names of numbers not explicitly specified, one feels compelled to think of values for those undeclared numbers that make the sentence true.

For example, in seeing the sentence

$$N + 3 = 10$$

with N the name of some unspecified value, we can't help but think: "N should be 7!"

Practice 73.1 Here is a sentence about an unspecified number M.

$$(M-2) \times (M-8) \times (M-100) \times (M-42\frac{1}{2}) = 0$$

Think of some values for M that would make the sentence true.

People call a symbol, letter, or name that represents a number a **variable**. It's a scary-sounding word. But it comes from the idea that the value the symbol represents could vary if the author of the sentence changes their mind about the number they are actually thinking of. (Or, maybe the math sentence comes from some physical experiment the author is conducting and each run of the experiment produces slightly different values for the symbols representing numbers.)

Those just reading a math sentence without any context might use the word **unknown** instead for the symbol representing an unspecified number.

And if a math sentence contains unknowns, it feels compelling to think of values for the unknowns that make the sentence true.

Collecting Data from Math Sentences

It is natural to collect from a math sentences the values for the unknown(s) in the sentence that bring truth.

For example, for the sentence

$$N + 3 = 10$$

there is just one data value to collect: Only the value 7 for N makes the sentence true.

There is a lot of data we can collect from this equation:

$$a \times b = 12$$

For example, we could have

a is 3 and b is 4 $\quad$ a is 4 and b is 3 $\quad$ a is 2 and b is 6

a is 12 and b is 1 $\quad$ a is $\frac{1}{2}$ and b is 24 (We don't have to stick with whole numbers!)

People usually organize the data they collect in tables.

a	b
3	4
4	3
2	6
12	1
$\frac{1}{2}$	24

This data table is certainly incomplete: there are an infinitude of values to be had that involve fractions and decimals making the statement $ab = 12$ true. And there are also negative data values we can add to the table such as a is -3 and b is -4.

Example: Collect all relevant data for the equation $x = 2$.

Answer: There is only one data value to collect. We need x to be the number 2 for this to be a true sentence.

Example: Describe all relevant data for the inequality $x \neq 2$.

Answer: There is an infinite amount of data to collect for this sentence: having x be any number but 2 will make the sentence true!

Practice 73.2 Back to this sentence about the unknown M.

$$(M-2)\times(M-8)\times(M-100)\times(M-42\frac{1}{2})=0$$

a) Do you have four data values for this sentence?
b) Why isn't 98 one of your data values?
c) Why isn't -2 one of your data values?
d) Convince me that your four data values are the only data values that make this sentence true.

Practice 73.3: Here is a compound sentence:

$$3<q<7$$

It reads "the value 3 is smaller than the value q, which, itself, happens to be smaller than the value 7." (So many words are condensed into those math symbols!)

a) Give six data values that make the sentence true.
b) How many data values are there that make the sentence true?

Here is another math sentence.

$$w^2 = b$$

The subject is an unspecified number w that is squared, the verb is "equals," and the object of the sentence is an unspecified number b.

Again, it is natural to collect values for w and b that make the sentence true. Using trial and error to do so is fine. (But if you have some judicious reasoning to use, go for it!)
Here's my start to a data table.

w	b
2	4
1	1
3	9
-3	9
0	0
$\frac{1}{2}$	$\frac{1}{4}$

We have that w is 3 and b is 9 is a line in the table because $3^2 = 9$ is a true statement.
I didn't put w is 16 and b is 4 in the table because $16^2 = 4$ is a false statement.

Practice 73.4: Start a data table for the inequality

$$s \cdot t < 0$$

Have at least six lines within your table.

MUSINGS

Musing 73.5 Which of these math sentences about two unspecified numbers a and b have an infinitude of data values that make the sentence true?

i) $a + b = 0$ ii) $a - b = 0$

iii) $a^2 + b^2 = 0$ iv) $a^2 - b^2 = 0$

Musing 73.6 Write down a math sentence about an unspecified number x that has no data values that make the sentence true.

Musing 73.7 Consider the math sentence

$$\frac{b}{b} = 1$$

Describe all the data values that make this sentence meaningful and true.

MECHANICS PRACTICE

Practice 73.8 Give all the data values for this math sentence:

$$36 = w^2$$

Practice 73.9 Draw a data table at least six lines long for this math sentence:

$$a \times b = 1$$

Practice 73.10 Find at least one data value for the inequality

$$2x + 3y \neq 5$$

Practice 73.11 Find at least one data value for the equality

$$2x + 3y = 5$$

Practice 73.12 Find all the data values for the math sentence

$$-x = -7$$

74. Visualizing Data: Graphs

Here's a story that is not true.

One time, some students and I conducted an experiment.

We were wondering if eating carrots has any effect on sleeping patterns. So, we decided to each eat some carrots before going to bed one night and note how many hours of sleep we each got for that night.

Here's the (fake) data we collected.

Number carrots eaten	Number hours of sleep
6	9
3	5
4	6
3	6
10	13
1	3
0	0
5	8

You can see that one student forgot to conduct the experiment and then could not sleep a wink out of guilt.

> **Question:** Are there other data values that stand out to you?

It's hard to see—literally see—if this data is indicating anything of note. Is there a connection between carrot-eating and sleep?

Can we make this data visual?

In our early grades, we represented numbers visually on a number line. It has become the convention to draw this line horizontally, with the positive numbers heading off to the right, negative numbers off to the left. We can put dots on the line to highlight certain numbers.

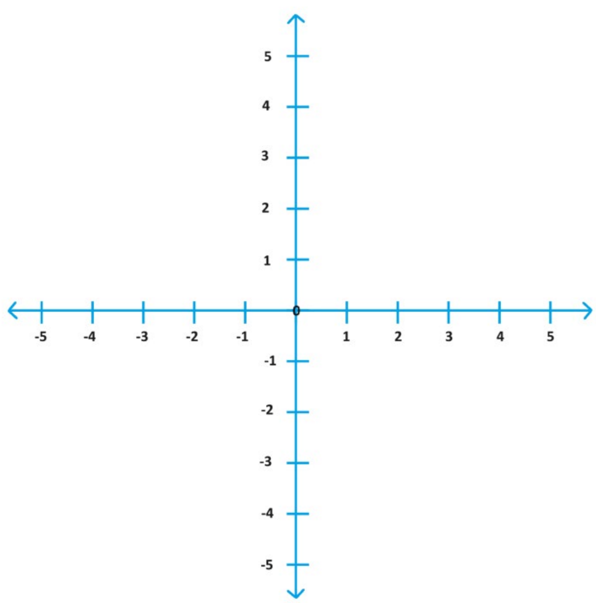
So, we could represent the various counts of carrots eaten with dots on one number line, and the counts of hours slept with dots on a second number line. That's one way to display our data.

But the two number lines should be linked in some way since the data is linked: 6 carrots eaten matches 9 hours of sleep, and so forth.

Hmm.

It took mathematicians a very long time to figure out a way to put two number lines together in a way that would help visualize linked data. In fact, the number line itself wasn't "invented," or seen as useful at least, until the mid-1600s when English mathematician John Wallis suggested using it as way to visualize basic arithmetic. French mathematician René Descartes at that time thought to put dots above and below a number line to start visualizing data from physical and geometric problems.

But it wasn't until a century later in the 1700s that scholars started drawing two numbers explicitly together. They kept one horizontal but made the second line vertical (with positive numbers going upwards). And they had the two lines cross at each of their zeros.



So, let's do this too.

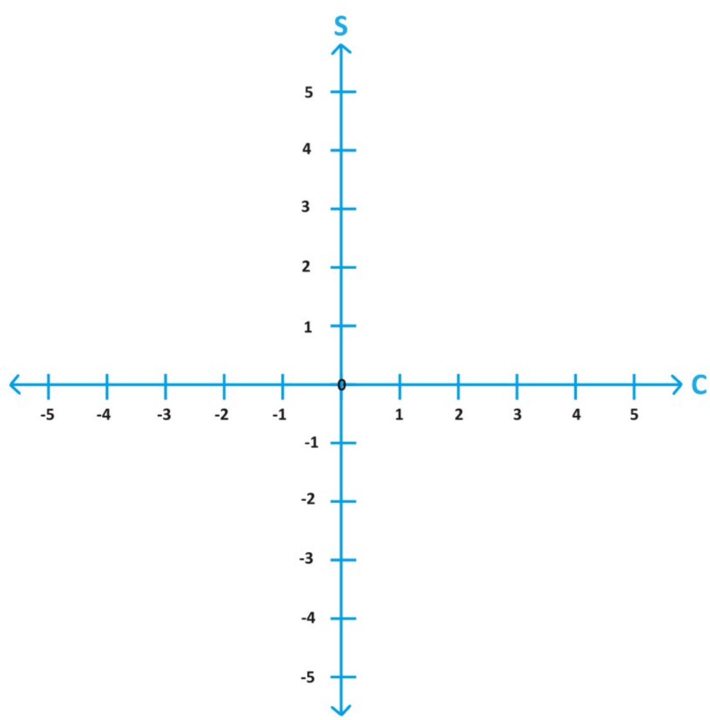
But now there's a question: Which number line should we use for counts of carrots, the horizontal one or the vertical one? And which number line should we use then for the counts of hours of sleep?

Mathematics won't care which one we choose for which, but scientists have come up with a convention:

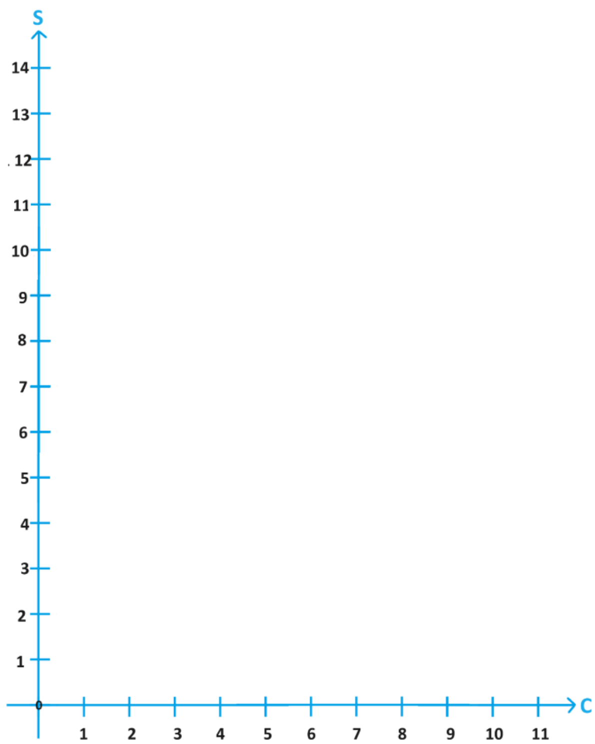
> In an experiment, you are usually in control of one quantity and are looking to see what the response shall be to various control values.
>
> **Convention**: Use the horizontal number line for the control data and the vertical number line for the response data.

In our experiment, students were in control of the number of carrots they ate. So, counts of carrots are our "control data" and we'll consequently use the horizontal number line to represent that data. The number of hours of sleep students got as a result is our "response data" and we'll use the vertical number line for that data.

Let's label our number lines C and S, for "carrots" and "sleep" to show this.



Actually, since our data never gives us negative number results, let's focus on just the positive numbers for each of our number lines.

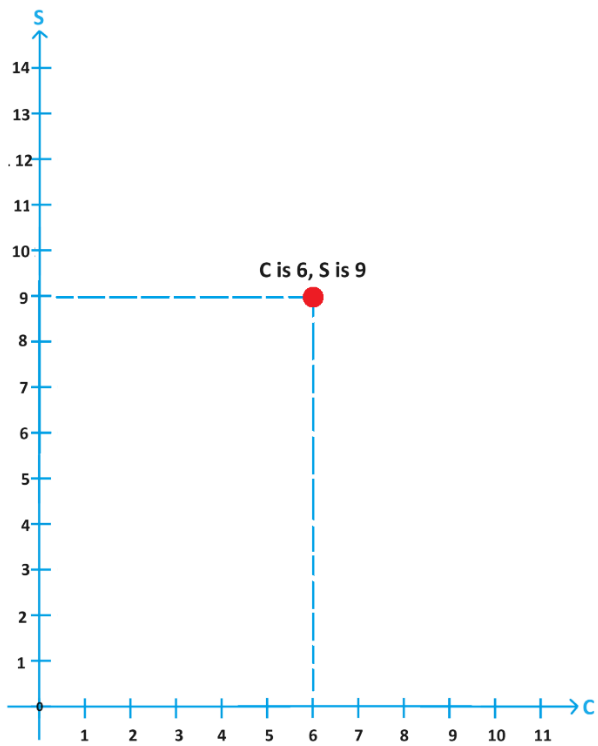


Number carrots eaten	Number hours of sleep
6	9
3	5
4	6
3	6
10	13
1	3
0	0
5	8

Here’s how folk thought to make linked data visual using these two lines some 300 years ago:

The first line of the data table has C as 6 and S as 9 (one student ate six carrots and got nine hours of sleep).

To show this piece of data, look for the number 6 on the carrot number line. Then move 9 places vertically up from it and draw a dot at that point. (And notice that you can use the vertical number line for S to help identify that height of 9.)

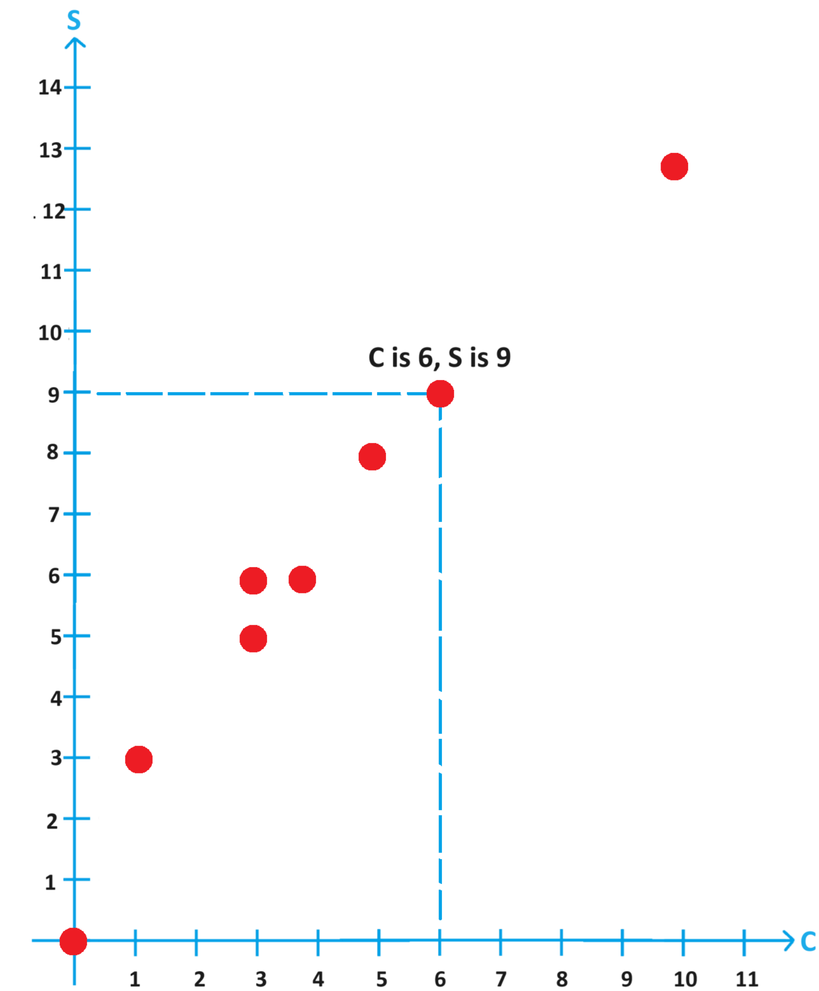


Number carrots eaten	Number hours of sleep
6	9
3	5
4	6
3	6
10	13
1	3
0	0
5	8

Then do the same for the remaining piece of data.

Plotting the data piece C is 0 and S is 0 is interesting: you have to go “up” a height of zero from the number 0 on the carrot number line.

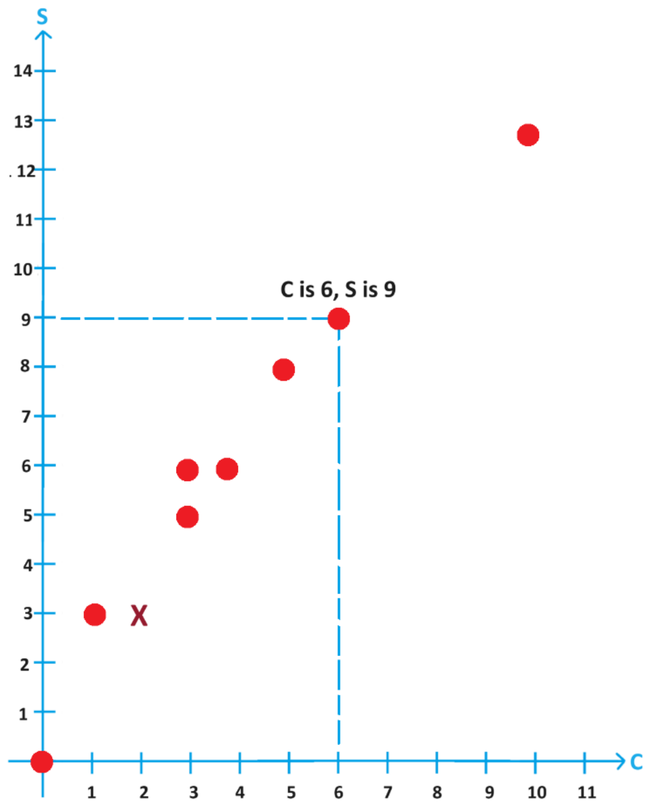
Here's the result of making the data table visual.



Number carrots eaten	Number hours of sleep
6	9
3	5
4	6
3	6
10	13
1	3
0	0
5	8

There does seem to be a trend: the more carrots you eat, the more hours of sleep you get!
The data is showing us that there is something worth investigating about the effect of evening carrot eating for sleep effects (assuming you believe this nonsense story about my students and me eating carrots).

Practice 74.1: A student says we omitted her data point. She marked it on the picture with an X. How many carrots did she eat and how many hours of sleep did she get?



Number carrots eaten	Number hours of sleep
6	9
3	5
4	6
3	6
10	13
1	3
0	0
5	8

We have just made a **graph** of our data. Some people might also call this a **scatter plot**.

Each piece of data is represented as a point, and you may have heard people describe a piece of data as a **data point**.

We had two number lines in our graph and data appears as points in a two-dimensional page.

Some More Jargon and Notation

Each number line in a graph is usually called an **axis**.

And rather label points as "C is 6, S is 9," which is cumbersome, people will write $(6, 9)$ with the first number mentioned in the set of parentheses the value of the control variable, the second the value of the response variable. We call $(6, 9)$ the **coordinates** of that particular data point.

It is unfortunate that mathematicians have settled on using parentheses in this context as they are usually used to represent "groupings" and order of operations as per Section 9. (But maybe folk were thinking that this is appropriate as data values are being grouped together?)

Practice 74.2: Which of the following points are not data points in our graph?

$(3, 5)$ $(5, 3)$ $(5, 6)$ $(5, 8)$ $(0, 0)$ $(4, 5)$ $(6, 4)$ $(4, 8)$

The point where the two axes (number lines) cross has coordinates $(0,\ 0)$.
People call this point the **origin**.

Now to something that sounds like it comes from a high-school algebra textbook.

Example: Graph the equation

$$a^2 = b^2$$

Let's take it slowly to unravel what exactly is being asked of us here.

To start, we see we have a math sentence. The subject of the sentence is a^2, an unspecified number that is squared, the verb is *equals*, and the object b^2, another unspecified number that is squared.

And no doubt we will want to obtain data values, values for a and b, that make this sentence true.

The command of the example is to "graph." The word is being used as a verb, not a noun, but the author of the question must surely mean: "make a graph."

Okay, So, we are to collect data and display it visually with a graph.

Next: What sorts of numbers for a and b make the sentence $a^2 = b^2$ true?

By trial and error, I got these examples.

a	b
3	3
2	2
-2	-2
-2	2
2	-2
0	0
1	1
-1.5	-1.5
-0.7	0.7

Check: Do verify that each of these data points do indeed make the math sentence true.

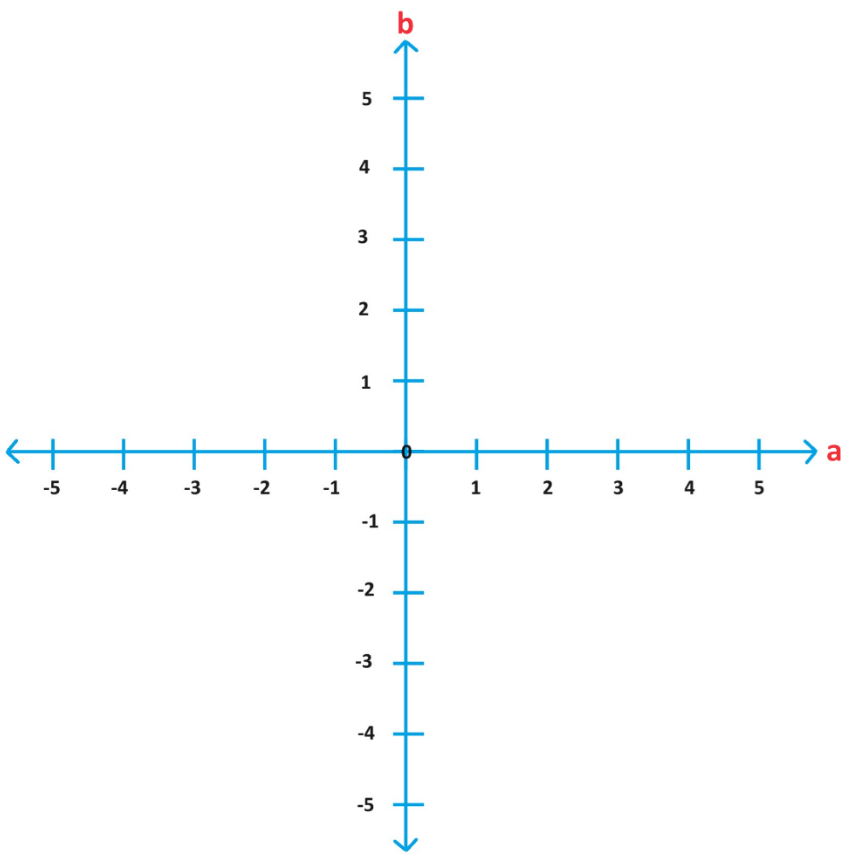
Let's start creating a graph.

We'll need one axis (number line) for the data representing a and a second axis for the data representing b. Let's label the axes "a" and "b."

But there's a question: Which axis should be which?
Should we label horizontal axis "a" and the vertical one "b"? Or the other way round?

There is no information in the question to indicate if one of these variables is in "control" and the other is in "response." So, we can choose to do whatever we wish.

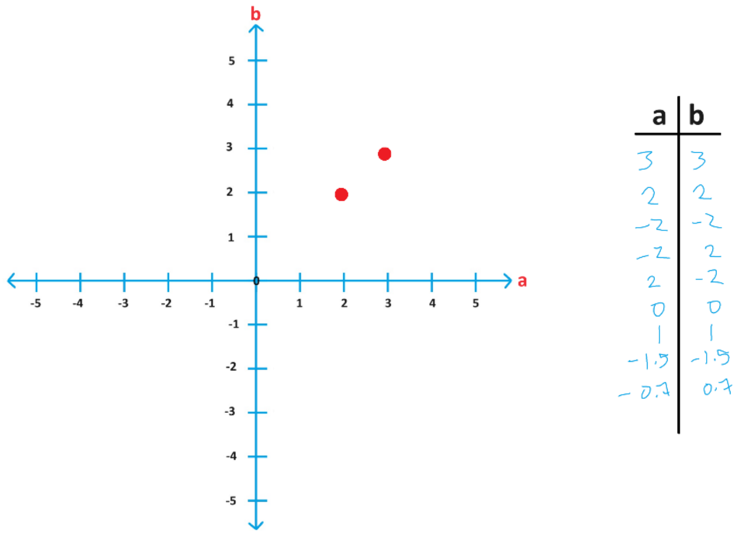
So, let's label the horizontal axis "a" and the vertical one "b." (Mathematics does not care.)



a	b
3	3
2	2
-2	-2
-2	2
2	-2
0	0
1	1
-1.5	-1.5
-0.7	0.7

Let's now plot the data points.

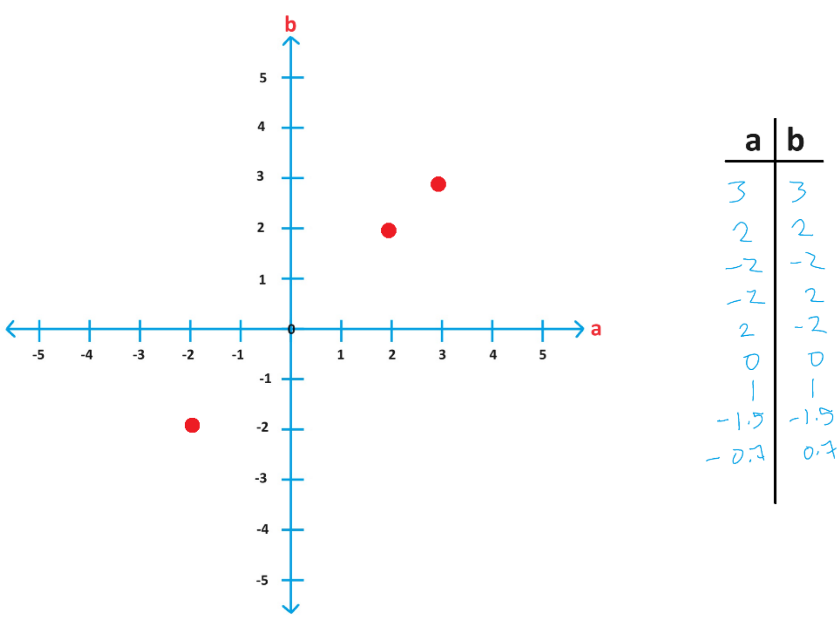
The first two data points $(3,3)$ and $(2,2)$ are manageable, but I want to pause on the third one.



Plotting $(-2,-2)$ requires finding -2 on the number line for a—not a problem—and then "going up a height of -2" from it. That sounds a bit strange.

But a negative height must be the opposite of a positive height and we go <u>downwards</u> rather than upwards. A point -2 units high, must be 2 units below.

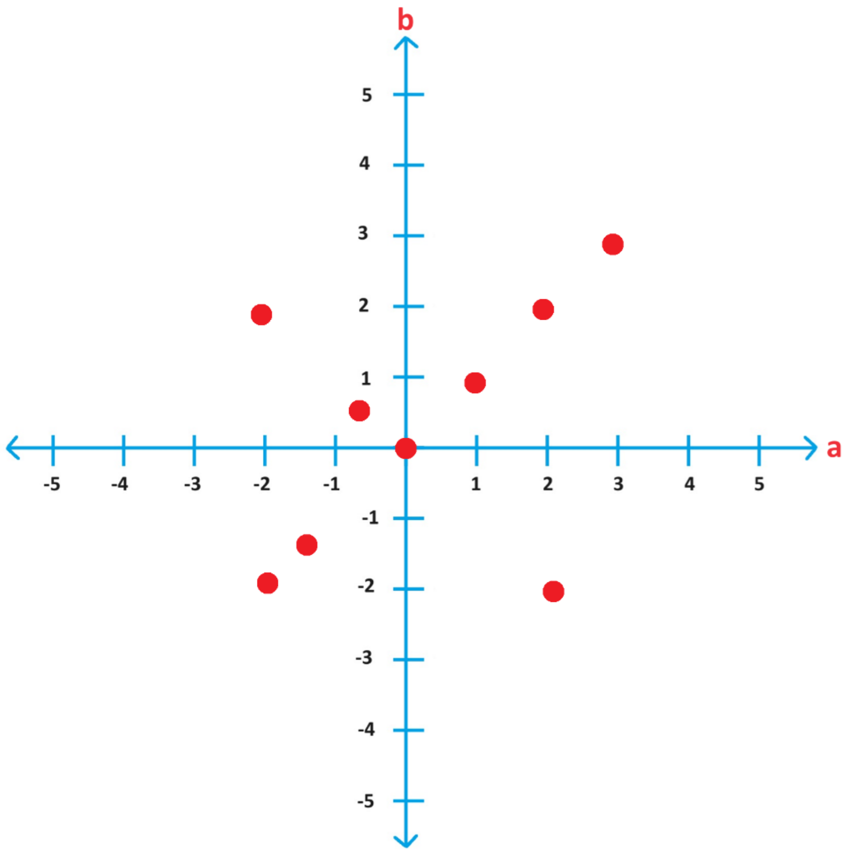
We can plot $(-2,-2)$.



Practice 74.3: Try plotting the remaining six data points before turning the page.

Here's what I got. Did you get the same picture?

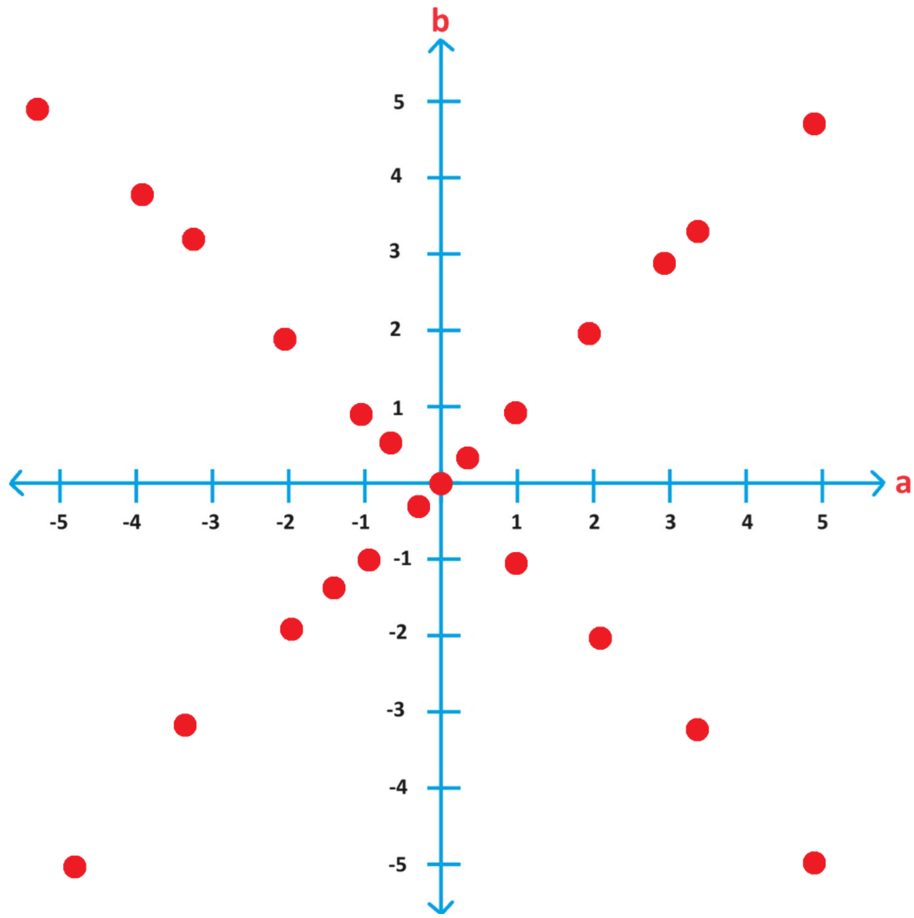
(One should really conduct this work on graph paper. I am currently eye-balling the proper locations of the data points!)



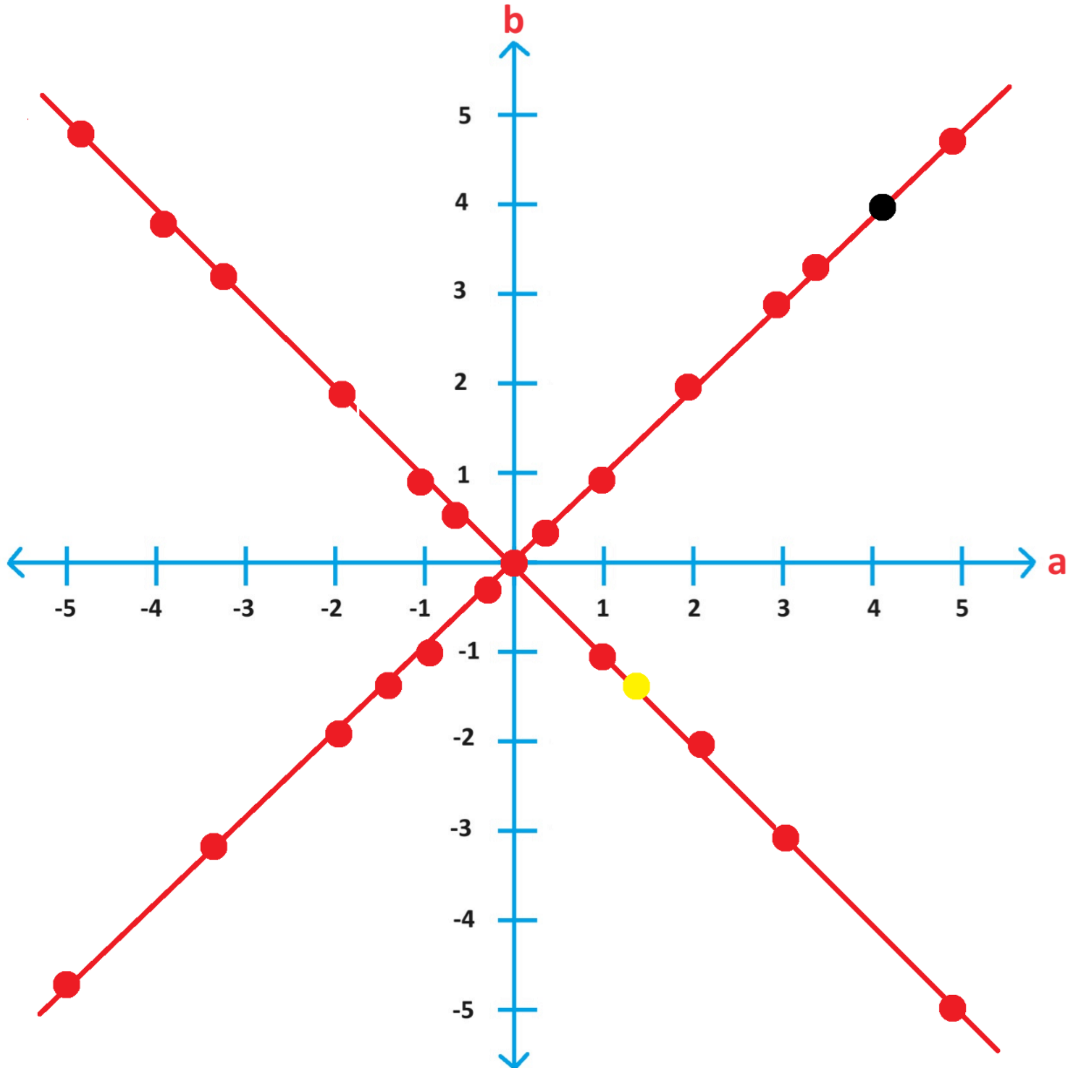
a	b
3	3
2	2
-2	-2
-2	2
2	-2
0	0
1	1
-1.5	-1.5
-0.7	0.7

Practice 74.4: Collect at least ten more data points that make the equation $a^2 = b^2$ a true sentence and plot those points as well on the picture above. Be sure to choose some data points with fractional or decimal values.

As I plot more and more data points, a compelling picture seems to be falling into place.



Actually, if we kept going and going and going with this, I can imagine a whole continuum of dots appearing to make a picture of big X for my graph.



And I can argue that this would be legitimate thing to conclude.

For example, I chose the black dot at random on one part of the X and it looks like it has coordinates $(4.1, 4.1)$. And this is a data value that makes $a^2 = b^2$ a true statement.

In fact, whenever we set a and b to have the same value, the statement $a^2 = b^2$ will be true, and it looks like every point on the north-east diagonal line provides values for a and b that are the same.

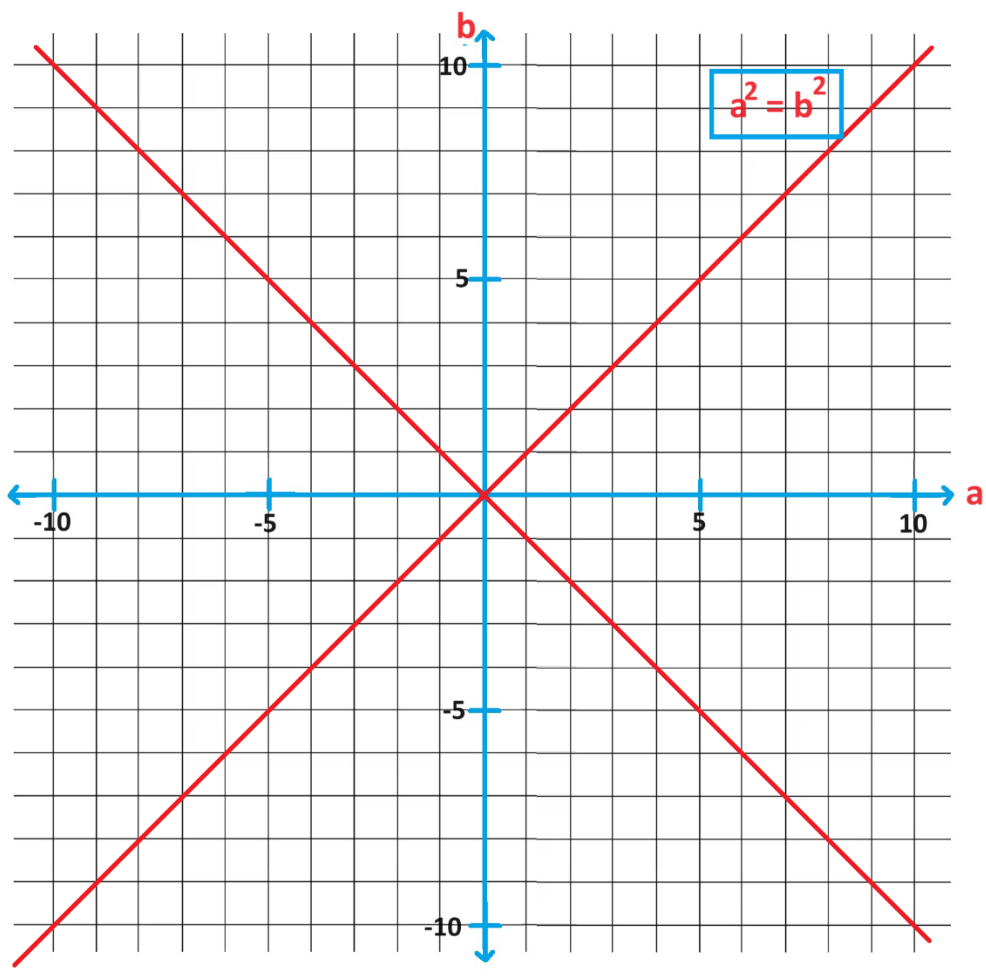
I also chose the yellow dot on the south-east diagonal line at random, it is looks like it has coordinates $(1.3, -1.3)$ and this too is a data value that makes $a^2 = b^2$ a true statement.

In fact, whenever we set a and b to have the same value but make one of the values negative and the other positive, the statement $a^2 = b^2$ will be true. Every point on the south-east diagonal line gives such data values.

The graph of the equation $a^2 = b^2$ really is a pair of lines making an X shape: every point on one of the lines corresponds to a data point that makes the sentence $a^2 = b^2$ true. And every data point that makes the sentence true lies on one of these lines.

The origin happens to lie on both lines.

Here's a picture of the graph properly drawn on graph paper.



Practice 74.5: What would the graph of the equation look like if we had instead labeled the horizontal number line as "b" and vertical one as "a"?

Practice 74.6: Consider this equation:

$$a = b + 6$$

Find some data values for this equation, plot the data points you find, and create the graph of this equation on the same picture as the one above.

What is special about the data point that are on both graphs simultaneously?

Practice 74.7: In each of the following, label the horizontal axis "x" and the vertical axis "y."

a) Sketch a graph of the equality $xy = 0$.

b) Sketch a graph of the inequality $xy \neq 0$.

c) Sketch a graph of the inequality $xy > 0$.

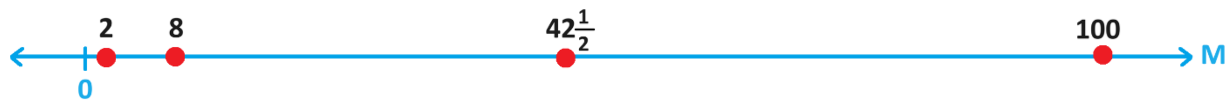
One-Dimensional Graphs

Consider this math sentence from the previous section.

$$(M-2)(M-8)(M-100)(M-42\frac{1}{2})=0$$

There are only four data values that make the sentence true: M must be 2, 8, 100, or $42\frac{1}{2}$.

To display this data visually we need only one number line. We can label it "M"and show these four values as points on it.



This picture is a graph of the equation.

Practice 74.8 Graph the equation $x = 2$.

A graph of the inequality

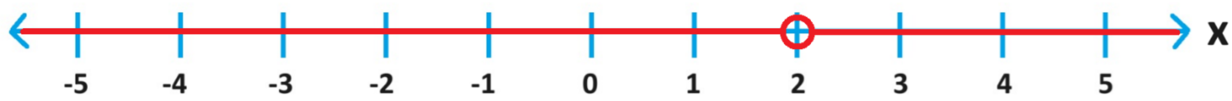
$$x \neq 2$$

would also be one dimensional and would show a number line labeled "x" that has every point but 2 shaded on it. This is tricky to draw.

Mathematicians have settled on the following drawing conventions:

- To indicate that a particular point is <u>not</u> part of the graph, draw an open dot at that point.
- If it helps to clarify that a particular point <u>is</u> part of the graph, use a solid dot at that point.
- To indicate that a region of points is part of the graph, shade the region. (The idea is to have it look like an infinitude of solid dots drawn throughout.)

Here’s the graph of $x \neq 2$.



Practice 75.9 Graph the inequality $2 < x$.

The symbol $\leq$ means “less than, or possibly equal to.”

Example: Graph the statement $-4 < p \leq 3$.

Answer:

Here’s a tricky example.

Example: Graph the inequality $w^2 \geq 4$.

Answer: What sorts of values for w make this statement true? Certainly w could be 3 or 10 or 10.067, for instance. It could also be 2.

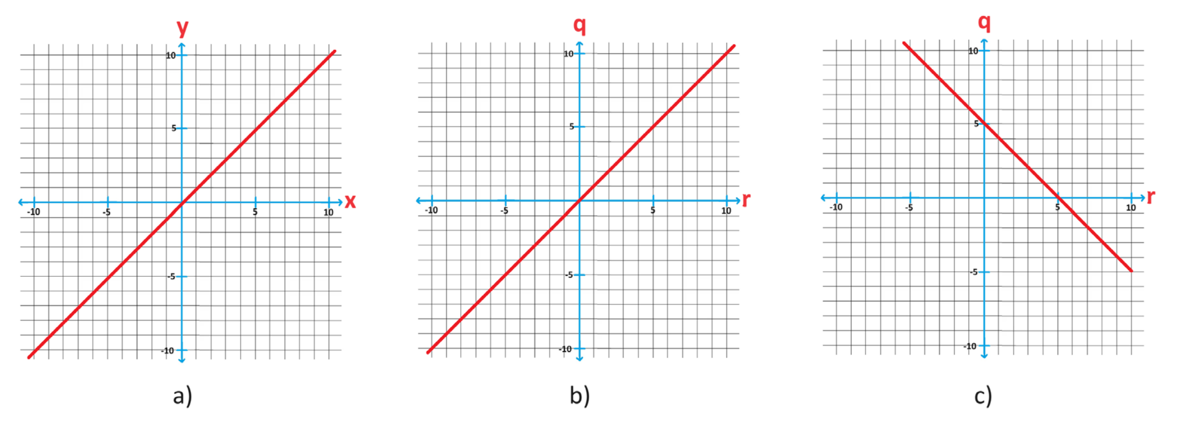
Actually, it could also be -2 or -3 or -10 or -10.067. (Do you see why?)

We have the following graph (with filled-in dots).

Practice 74.10 Graph $r^2 < 1$.

MUSINGS

Musing 74.11 For each of the following, give a mathematics statement whose graph could be as shown.



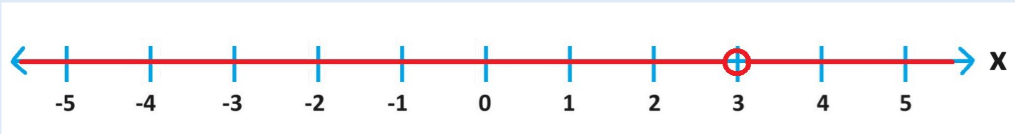
Musing 74.12 Let's look at some one-dimensional graphs.
a) Graph the equation $a = a$.
b) Graph the inequality $a \neq a$.
c) Graph the equation $\frac{a}{a} = 1$.

Musing 74.13 Sketch a graph of

$$x + 0 \times y = 3$$

(Most people would just write this sentence as "$x = 3$," but I want to point out that it is really a statement about two unknowns x and y to thus obtain a two-dimensional graph.)

Musing 74.14 Challenge
Give TWO math statements whose graph would match this picture.

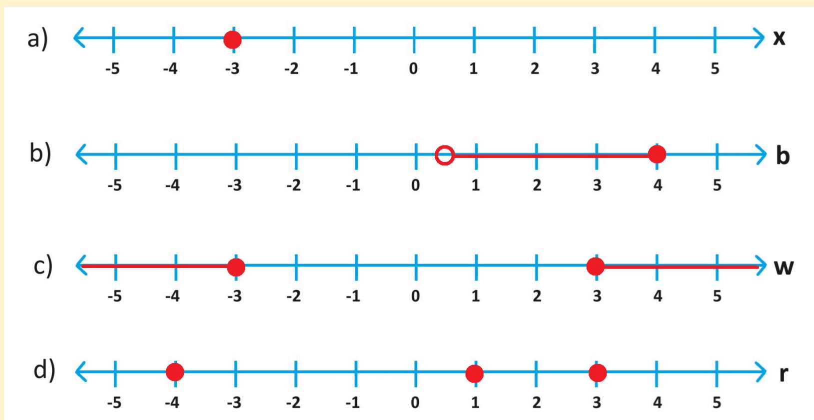


MECHANICS PRACTICE

Practice 74.15 Sketch (one-dimensional) graphs for each of these statements.

a) $x(x-3)(x+4)=0$
b) $5>p\geq 3$
c) $w^2=0$

Practice 74.16 Give a math statement whose graph would match each of these pictures.



Practice 74.17 Graph

$$2x+y=9$$

with the horizontal axis labeled "x."

Start by collecting data points in a table and plotting those. (Is there a systematic way to do this? Perhaps ask "If x is 1, then y would have to be …?" for instance. Don't forget fractional values too.)

Then imagine collecting more and more and more data points.
Would you have a continuum of data points in the graph?

75. What does it mean to Solve and Equation or Inequality?

One hears all the time in algebra class, “Please solve …” (but often without the “Please”!)

Simply put, to **solve** a sentence about numbers is to identify all the data points that make the sentence true. We’ve been doing this already.

For example, the equation

$$w + 4 = 10$$

has only one value for w that would make this a true sentence, namely 6. We say that this equation has one only **solution**, namely that w must be 6.

The inequality

$$2 < x$$

has a whole range of solutions: all values larger than 2.

The equation

$$c^2 = 16$$

has two solutions, namely, c can be 4 or -4 to make this a true sentence.

We can express the solutions to a sentence about numbers in any way that successfully communicates to the reader what they are. For example, the inequality

$$ab > 0$$

has solutions:

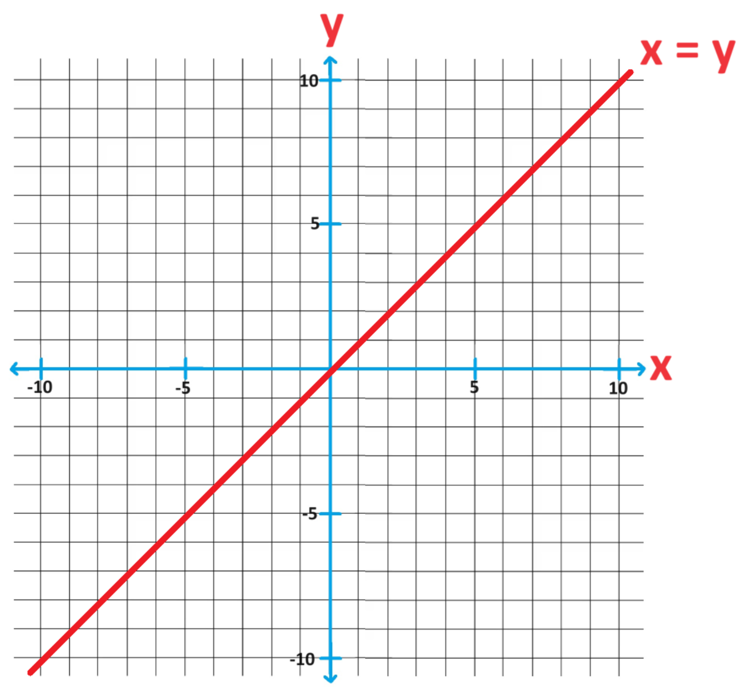
Any two values for a and b that are either both positive or both negative.

There! We have just solved the inequality $ab > 0$.

Practice 75.1: Solve the equation $x = y$.

A graph is a visual representation of all the solutions to a sentence about numbers. So, one could communicate the solutions of an equation or an inequality simply by presenting a graph of the equation or inequality.

This graph answers Practice 75.1. It's a picture of all the data points that have x and y the same value.



Here's an example that confuses many people, including math teachers.

Example: Solve $x = 3$.

Answer: There is only one possible value for x that makes this a true sentence, namely, x must represent the number 3.

This is confusing to math folk because they have forgotten that a statement like $x = 3$ is a sentence that is technically neither true nor false as it stands. It is only when someone tells you the value for x they have in mind do we know whether or not the sentence is true.

And why have people forgotten this?

Because people do something that is a tad lazy and use an equal sign to express the solutions to a number sentence.

As an example of that I mean, consider the very first equation I presented in this section

$$w + 4 = 10$$

People might write this response to this

If $w + 4 = 10$, then $w = 6$

and stop there.

Actually, people will usually write even less

$$w + 4 = 10$$
$$w = 6$$

and skip all the in-between words.

And what they mean in these curt writings is the following:

The solutions to $w + 4 = 10$ are the same as the solutions to $w = 6$, and everyone knows what the solutions to this second sentence are.

As another example, people might write

$$\textit{If } c^2 = 16, \textit{ then } c = 4 \textit{ or } -4$$

or just

$$c^2 = 16$$

$$c = 4 \text{ or } -4$$

These both stands for:

The solutions to $c^2 = 16$, are the same as the solutions to the sentence " $c = 4$ or -4," and it is clear what the solutions to this second sentence are.

People have gotten into the habit of not bothering to write "and the solutions to this second sentence are clear," assuming it is understood. As a result, many have forgotten that sentences like "$w = 6$" and "$c = 4 \text{ or } -4$" are sentences that have solutions that need to be described too.

But you can see a process here.

The **art of solving a sentence about numbers** is to

Take the given math sentence and turn into a new sentence which you are confident has exactly the same solutions as the original sentence, and whose solutions are blatantly clear and obvious to anyone who reads the new sentence.

And how do we do conduct such a transformation?

The process we use to do so was first developed by the Persian scholar al-Khwarizmi (ca. 780 – ca. 850) in his book "The Science of Restoring and Balancing." He described methods for converting one sentence about numbers into a new sentence without changing the solutions it has. He called his method *al-jabr* from the Arabic term for a "reunion of broken parts." From that term came the word **algebra** we use today.

(In addition to this, the word **algorithm**, which we use for any set of instructions for carrying out a task, is derived from his name!)

MECHANICS PRACTICE

Practice 75.2 Here's a (curt) piece of written mathematics.

$$(x-2)(x-3)(x+5)=0$$

$$x=2\,,3\,,\text{or}-5$$

Write out in full what is being said (and understood to be said) here.

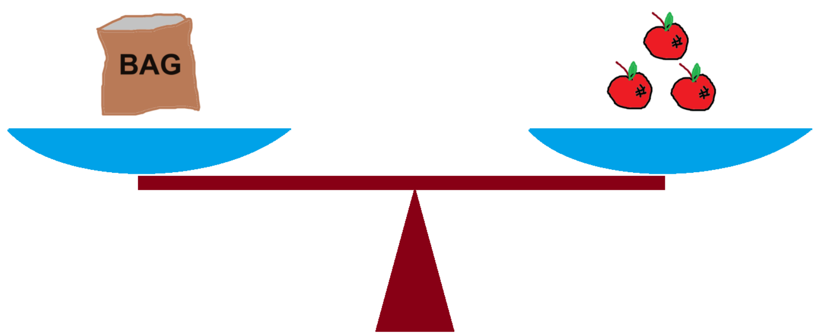
76. The Art of Balancing: What We Say is True about Equality

al-Khwarizmi evoked the idea of two quantities in a math sentence being in "balance" if the two quantities are deemed equal.

This provides lovely imagery that has been adopted by many math curriculums: two quantities are shown as "equal" if they balance on a simple two-pan balance.

For example, here's a picture of a bag of apples balancing perfectly with three apples. What can we conclude?

[Let's assume henceforth that all apples we discuss are of identical weight, that bags only contain apples, and that the weight of the material of any bag is immaterial compared to the weight of apples.]



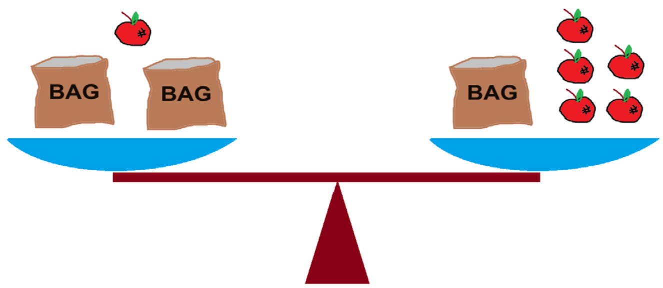
The picture represents the equation

$$BAG = 3$$

and the solution must be that "BAG" represents the number 3. That is, there must be three apples in the bag.

Let’s use the imagery of “balance” to identify what we believe is true about equality.
And let’s do that by playing with this more complicated balance picture.

(Assume all bags in this picture contain the same count of apples.)

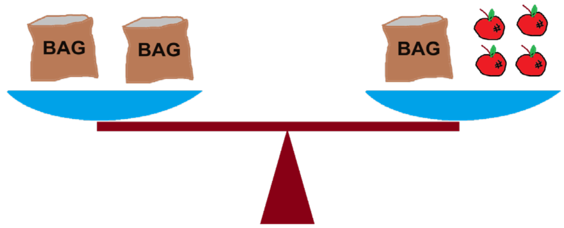


You can no doubt see that each bag must contain four apples for this picture to be true.
But that solution is less easy to see if we just look at the math sentence that describes the picture.

$$2\ BAG + 1 = BAG + 5$$

Let’s make seeing that easy via al-Khwarizmi’s method of *al-jabr*.

Our everyday experience suggests removing one apple from each side of the scale won't affect the balance of the scales and thus won't affect the truth of the situation. The number of apples in each bag that make our original picture true is precisely the same as the number of apples in each bag to make this slightly less complicated picture true.



In terms of math sentences, we've just converted the statement

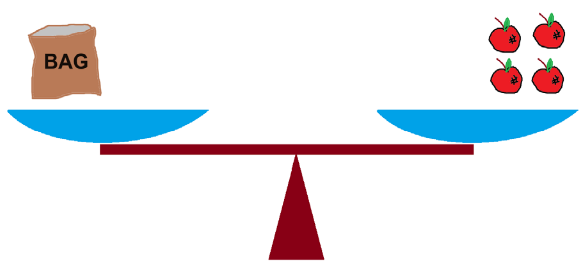
$$2\,BAG + 1 = BAG + 5$$

to the statement

$$2\,BAG = BAG + 4$$

believing that we haven't affected the truth of the statement and hence its solutions.

Since all bags are identical, we can also take a bag off of each side of the scale and not affect the truth of the picture.



This means we have just converted the statement

$$2\,BAG = BAG + 4$$

to

$$BAG = 4$$

without affecting the truth and the solutions.

From

$$BAG = 4$$

it is clear that BAG represents the number 4. There are four apples in the bag.

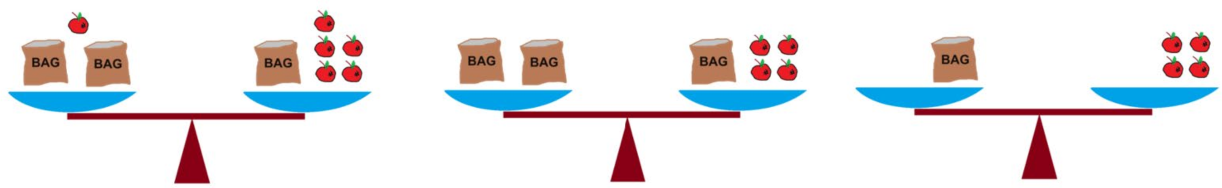
In summary: We turned the statement

$$2\,BAG + 1 = BAG + 5$$

into much simpler statement

$$BAG = 4$$

without affecting the truth along the way. They are equivalent statements. Any value for BAG that makes the first statement true, makes the second statement true too, and vice versa.



Jargon: Two math sentences about the same unknowns are **equivalent** if they have exactly the same solutions (that is, the same set of values that make each sentence true).

Algebra is the art of turning one math sentence into an equivalent sentence whose solutions are more readily seen.

As we worked through this example, we employed a feature we like to believe is true of equality:

> Suppose $A = B$ is a statement about equality of quantities.
>
> For any number k, the statements
>
> $$A = B$$
>
> and
>
> $$A + k = B + k$$
>
> are equivalent.

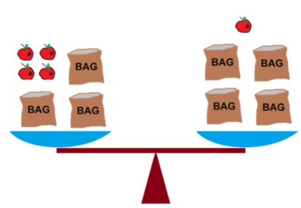
(If k is a negative number, then we're really making a statement about subtraction here.)

Adding an apple to each side of balance pan ($k = 1$) does not affect truth.
Subtracting an apple to each side of a balance pan ($k = -1$) does not affect truth.
Adding a bag to each side of a balance pan ($k = BAG$) does not affect truth.
And so on.

Example: Solve

$$4 + 3BAG = 4BAG + 1$$

Answer: The left side of the math sentence is $4 + 3 \times BAG$ and the right side is $4 \times BAG + 1$. We have this picture:



4 + 3BAG = 4BAG +1

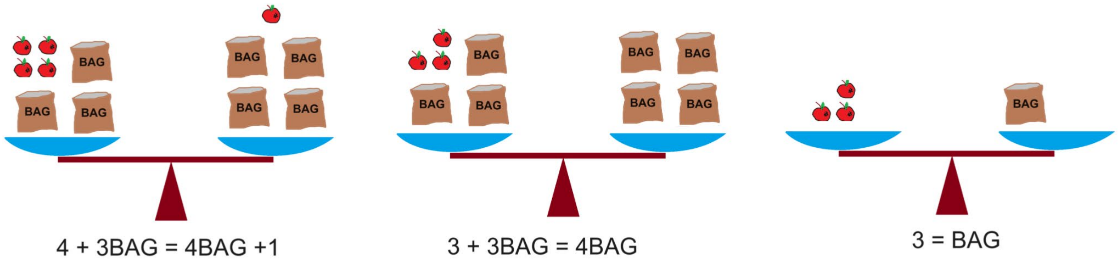
Subtracting 1 from each side of the equation gives

$$3 + 3BAG = 4BAG$$

Subtracting $3BAG$ from each side of the equation gives

$$3 = BAG$$

(Are you comfortable with "$4BAG - 3BAG$" giving "BAG"? See Sections 13 and 25.)



We have converted the sentence $4 + 3BAG = 4BAG + 1$ into the equivalent sentence $3 = BAG$.

It must be that BAG represents the number 3.

Here's the curt way to present this reasoning.

$$4 + 3BAG = 4BAG + 1$$

$$3 + 3BAG = 4BAG$$

$$3 = BAG$$

thus

$$BAG = 3$$

Students are usually required to annotate their work

$4 + 3BAG = 4BAG + 1$ **(subtract 1 from both sides)**

$3 + 3BAG = 4BAG$ **(subtract 3BAG from both sides)**

$3 = BAG$

thus

$BAG = 3$

I need to point out two more things.

1. In the previous solution, I converted the sentence $3 = BAG$ into the sentence $BAG = 3$ without comment. This presumed another belief about equality.

If $A = B$, then $B = A$, and vice versa.

Not a big deal, but I thought I should be explicit about this.

2. If a math expression, like $4 + 3BAG$, involves expressions sitting between $+$ and – signs and some of these pieces are just numbers and others involve unknowns, then it has become the convention to write the pieces that involve unknowns first and the pieces that are just numbers second.

Consequently, people prefer to $4 + 3BAG$ as

$$3BAG + 4$$

There is a seemingly contradictory convention that if a number and an unknown are multiplied together, then one should write the number first and the unknown second. (So, write $3BAG$ and not $BAG3$.)

Got that?

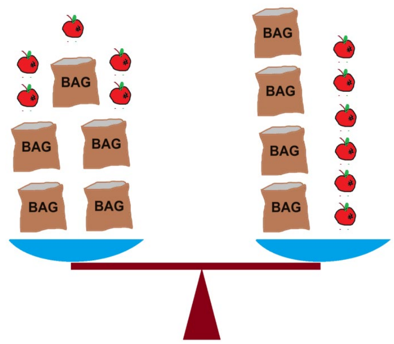
Mathematics doesn't care about any of this. It is just societal style thing.

Practice 76.1: For each picture below, write a math sentence that matches the picture.

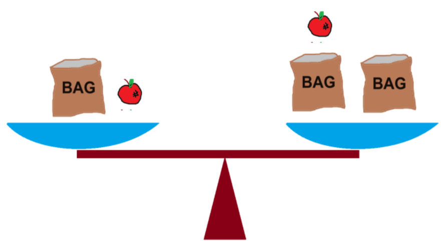
Then perform the steps of algebra to convert the sentence to an equivalent one that makes it blatantly clear how many apples must be in each bag. (Giving a curt-style presentation is fine.)

Imagine how the balance picture is changing as you perform each of your steps.

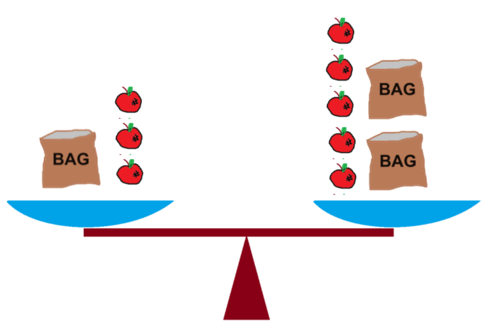
a)



b)

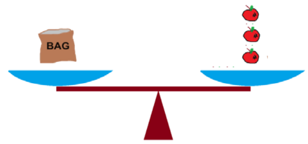


c)



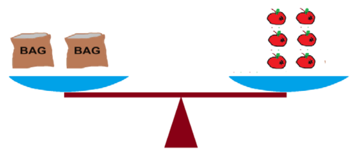
(Anti-apples?)

Here’s a picture and its matching sentence.



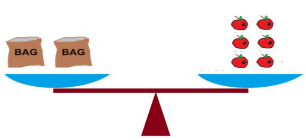
BAG = 3

Let’s double the quantities on each side of the pan-balance scale. Commonsense tells us that this won’t change the truth of the picture.

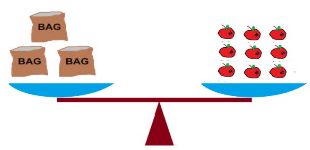


2BAG = 6

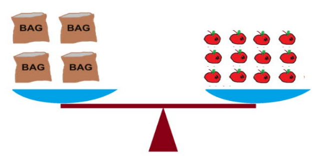
We could triple or quadruple the quantities on each side and still have no effect on the truth of matters.



2BAG = 6



3BAG= 9



4BAG = 12

We can even multiply quantities on each side of the scale by fractions and preserve truth. For example, multiplying each side of “$4BAG = 12$” (the right picture) by $\frac{1}{2}$ gives “$2BAG = 6$” (the left picture.)

We have a second belief about sentences that are equations.

> Suppose $A = B$ is a statement about equality of quantities.
>
> For any non-zero number k, the statements
>
> $$A = B$$
>
> and
>
> $$k \cdot A = k \cdot B$$
>
> are equivalent.

(See chapters 4 and 5 to see that multiplying a quantity by a fraction of the form $\frac{1}{n}$ is the same as dividing that quantity by n. There really is no such thing as division. It’s just multiplication by a fraction.)

Practice 76.2: Why shouldn't k be zero in this belief?
Could two statements of the form $A = B$ and $0 \cdot A = 0 \cdot B$ have different solution sets?

Actually, let me answer this practice question.

Consider, for instance, the math sentence

$$x = 7$$

This sentence has just one solution: x must be 7 for the sentence to be true.

Now consider the sentence

$$0 \cdot x = 0 \cdot 7$$

This sentence is true no matter what number x represents: multiplying any number by zero gives zero in all cases.

The sentence $x = 7$ has solution set: the single number 7.
The sentence $0 \cdot x = 0 \cdot 7$ has solution set: the set of all numbers.

Multiplying each side of math sentence by zero gives a sentence that is suddenly true for all values for the unknowns.

Another issue: Do you believe in anti-apples?

I snuck some in for practice problem 76.1, but I am not sure if you liked that.

The solution to 76.1 c) requires each bag to contain two anti-apples. If each (proper) apple exerts a downward force due to gravity, each anti-apple does the opposite and wants to float upward by that same amount of force (so that an apple and anti-apple together have zero combined effect).

But, of course, making such "factual" claims is pointing to the absurdity of trying to make mathematics apply, in full, to any one physical model. As we saw in Part 1, physical models can inspire mathematics, motivate mathematics, and provide intuition for some aspects of mathematics—but not all of it. Each model starts to become "absurd" when pushed beyond its natural parameters.

Nonetheless, given that we do have negative numbers in our mathematical universe, it is natural to test our stated belief and decide if it should extend to negative numbers as well.

For example, taking $k = -1$, does it seem right that $A = B$ and $-A = -B$ are equivalent statements?

Practice 76.3
a) What are the solutions to $r = 3$? What are the solutions to $-r = -3$?
b) What are the solutions to $x = y$? What are the solutions to $-x = -y$?

The solution sets for each pair of equations in this problem simply must be the same. And here's why.

Example: Explain, in general, why $A = B$ and $-A = -B$ are equivalent statements. (Here A and B each represent expressions that involve numbers and unknowns.)

Answer: Start with $A = B$ and add the number $-A$ to each side.

$$A = B$$

$$A + -A = B + -A$$

This is

$$0 = B + -A$$

Now add $-B$ to each side.

$$0 + -B = B + -A + -B$$

$$-B = -A$$

We can rewrite this as $-A = -B$.

So, believing that adding the same number to each side of an equation does not alter truth forces us to conclude that $A = B$ and $-A = -B$ are two equivalent equations.

So, it does not matter whether or not you believe in and want to play with apples and anti-apples. Mathematics offers a way to move forward and find solutions to equations that involve negative quantities nonetheless.

Okay, let's practice some *al-jabr*.

Example: Solve

$$5b + 2 = 2b + 14$$

(I am getting tired of writing BAG. I am just writing b now.)

Answer: Let's add -2 to both sides of the equation. (That is, if I am picturing bags and apples on a balance scale, let's remove two apples from each side.)

$$5b + 2 = 2b + 14$$

$$5b + 2 + -2 = 2b + 14 + -2$$

$$5b = 2b + 12$$

Let's now add $-2b$ to each side of the equation. (In my mind and removing two bags from each side of the balance scale.)

$$5b + -2b = 2b + 12 + -2b$$

$$3b = 12$$

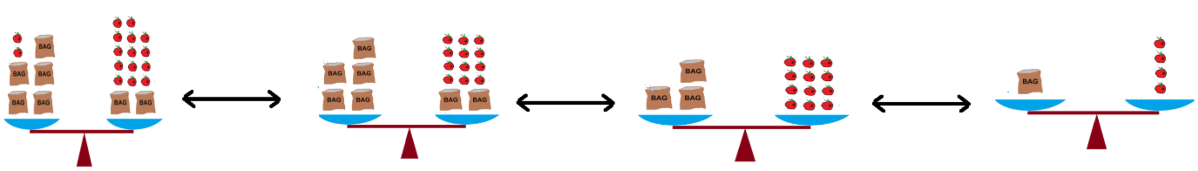
Let's now multiply each side of the equation by $\frac{1}{3}$. (That is, let's scale down each side of the balance scale by a factor of three.)

$$\frac{1}{3} \times 3b = \frac{1}{3} \times 12$$

$$\frac{1}{3} \times 3 \times b = \frac{1}{3} \times 3 \times 4$$

$$b = 4$$

Thus, the equation $5b + 2 = 2b + 14$ is equivalent to the equation $b = 4$.



For truth, b must be the number 4.

Curt, Wordless, Makes-Math-Seem-Inhuman-and-Scary Presentation:

$5b + 2 = 2b + 14$

$5b + 2 - 2 = 2b + 14 - 2$

$5b = 2b + 12$

$5b - 2b = 2b + 12 - 2b$

$3b = 12$

$\frac{1}{3} \times 3b = \frac{1}{3} \times 12$

$b = 4$

Remember that technically the final line here is not a solution – it's just another equation with the same solution set as the original equation. One should write a final sentence along the lines "And so b having the value 4 is the solution to the original equation," but the practice is to omit such a final sentence and assume it is understood.

Example: Solve

$$19z + 2 = 17z + 3$$

(If I am still thinking about bags, I guess I am labeling them "z" now.)

Answer: You've probably sensed a general strategy:

1. Add or subtract a number to each side of the equation to reduce the number of pieces in the math sentence that are numbers.

2. Add or subtract some number of the unknown to each side of the equation to reduce the number of pieces in the math sentence that involve the unknown.

3. Follow your nose from there.

Here goes:

$$19z + 2 + -2 = 17z + 3 + -2$$

$$19z = 17z + 1$$

Now let's work the pieces involving the unknown:

$$19z + -17z = 17z + 1 + -17z$$

$$2z = 1$$

I can see that for this equation to be true, z better be $\frac{1}{2}$, and I can stop here having said that.

But … if you are giving a curt, wordless presentation, then you might want to go a little further and add these two lines

$$\frac{1}{2} \times 2z = \frac{1}{2} \times 1$$

$$z = \frac{1}{2}$$

Example: Solve

$$2 - 2w = w + 6$$

Answer: Here's how my brain went with this:

$$2 - 2w + -2 = w + 6 + -2$$
$$-2w = w + 4$$
$$-2w + 2w = w + 4 + 2w$$
$$0 = 3w + 4$$
$$0 + -4 = 3w + 4 + -4$$
$$-4 = 3w$$
$$\frac{1}{3} \times (-4) = \frac{1}{3} \times 3 \times w$$
$$-\frac{4}{3} = w$$

so

$$w = -\frac{4}{3}$$

(See Chapters 5 and 6 to be clear on this fractions work.)

MUSINGS

Musing 76.4 What do you believe about "not equal to"?

What would you say is/are the solution(s) to the following inequality?

$$5x + 7 \neq x + 5$$

As you work through this, imagine a pan balance with apples and bags of apples (each labeled "x") on each side but not in balance. As you add and remove apples and bags, does it feel right to say that the pan-balance is still out of kilter? Is this still the case if you scale the contents of each pan by a non-zero number k?

MECHANICS PRACTICE

Practice 76.5 Solve each of the following equations. (Giving curt presentations of your work is fine.)

a) $2w = -4$

b) $19z + 2 = 17z - 3$

c) $8x + 7 = 5x + 31$

d) $2p + 1 = 12p$

e) $3R + 5 + 2R + 9 = 4R + 22$

77. Another Algebraic Move

In Part 1 we established ten basic rules of arithmetic—and their logical consequences. (See the **Appendix**.)

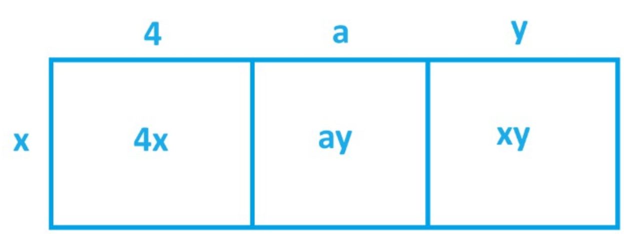
We understand these rules to speak truth in, and of, themselves.

For example, we believe we can change the order of the sum of two numbers without contradiction (**Rule 1**):

> $a + b = b + a$ *is a true sentence no matter which two numbers* a *and* b *represent.*

and we believe the distributive rule (a.k.a. "chopping up rectangles," **Rule 8**). For instance:

> $x(4 + a + y) = 4x + ax + xy$
>
> *is a true sentence no matter which numbers* x, y *and* a *represent.*



It seems right to believe that applying a fundamental rule of arithmetic to any part of a math sentence does not change the truth, and hence the solutions, of that sentence.

For example, the sentences

$$3 + 2w + 5 = w - 7$$

and

$$2w + 8 = w - 7$$

are equivalent math sentences.

Why? Because Rule 3 shows that we can add a string of addition in any order we like, so $3 + 2w + 8$ can be deemed no different than $2w + 3 + 5$, which in turn, is no different than $2w + 8$.

Example: Kindly solve

$$3d + 2 - d = 4(d - 1)$$

Answer: Let's try applying some basic rules of arithmetic first to portions of this sentence.

For starters, the expression $3d + 2 - d$ is no different than $d + d + d + 2 + -d$, which is just $2d + 2$.

Also, $4(d - 1)$ is the same as $4(d + -1)$, which is $4d - 4$.

So, our given math sentence is equivalent to the sentence

$$2d + 2 = 4d - 4$$

This looks like the type of example we solved in the last section.
Adding 4 to each side of the equal sign gives the equivalent sentence

$$2d + 6 = 4d$$

Adding $-2d$ to each side of the equal sign then gives

$$6 = 2d$$

Multiplying each side of the sentence by $\frac{1}{2}$ gives

$$3 = d$$

The solution to the original math sentence is that d must be the number 3.

Curt Presentation:

$3d + 2 - d = 4(d - 1)$

$2d + 2 = 4d - 4$ (arithmetic)

$2d + 2 + 4 = 4d - 4 + 4$

$2d + 6 = 4d$ (arithmetic)

$2d + 6 + -2d = 4d + -2d$

$6 = 2d$ (arithmetic)

$\frac{1}{2} \times 6 = \frac{1}{2} \times 2d$
$3 = d$ (arithmetic)

So,

$d = 3$

Example: Please solve

$$2(x-3)+2=2(x+3)$$

Answer:

$2(x-3)+2=2(x+3)$
$2x-6+2=2x+6$ (arithmetic)
$2x-4=2x+6$ (arithmetic)
$2x-4+\ -2x=2x+6+\ -2x$
$-4=6$ (arithmetic)

The original math sentence is equivalent to a math sentence that is never true.
There are no solutions to the given equation. That is, there are no values for x that could make the sentence true.

Some people will say "The solution set is empty."

Example: Please solve

$$2(x-3)+12=2(x+3)$$

Answer:

$2(x-3)+12=2(x+3)$
$2x-6+12=2x+6$ (arithmetic)
$2x+6=2x+6$ (arithmetic)
$2x-4+\ -2x=2x+6+\ -2x$
$6=6$ (arithmetic)

The original math sentence is equivalent to a math sentence that is always true, irrelevant to whatever value x may be.
Every number is a solution to the given equation.

Some people will phrase this as "The solution set is the set of all numbers."

MUSINGS

Musing 77.1 Are you a logic purist? Did you realize that we have already been using the idea of this section in the previous section? (This means I should have presented this section of text first!)

For example, in Section 76, I presented this example:

Example: Solve $5b + 2 = 2b + 14$
Answer:

$5b + 2 = 2b + 14$
$5b + 2 - 2 = 2b + 14 - 2$
$5b = 2b + 12$ (arithmetic)
$5b - 2b = 2b + 12 - 2b$
$3b = 12$ (arithmetic)
$\frac{1}{3} \times 3b = \frac{1}{3} \times 12$
$b = 4$ (arithmetic)

Do you see that in going from the second line to the third we used the idea that $5b + 2 - 2$ is no different than $5b$ (by the rules of arithmetic) and that $2b + 14 - 2$ is no different than $2b + 12$. And so on throughout this answer, and throughout the entire Section.

Musing 77.2 Make up a complicated, scary-looking math sentence using the unknown y, whose solution is that y must be 1.

MECHANICS PRACTICE

Practice 77.3 Kindly solve each of these equations.

a) $6m + 5 - 3m + 4 + 7m = 4 + 3m + 2 + 8m + 2$

b) $2(t + 2) - 3t = 2(t + 1)$

c) $4(w - 5) - 5(w - 4) = w - 6$

d) $p(p + 5) = 5p + 36$

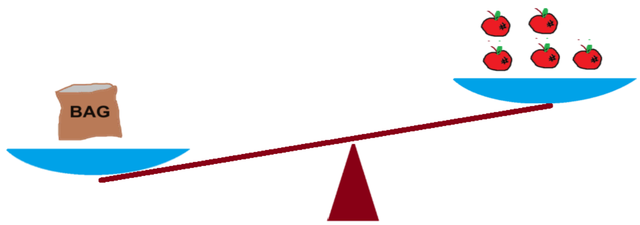
e) $3(x - 3) + 9 = 3x + 1$

f) $z = z$

78. What We Say is True about Inequality

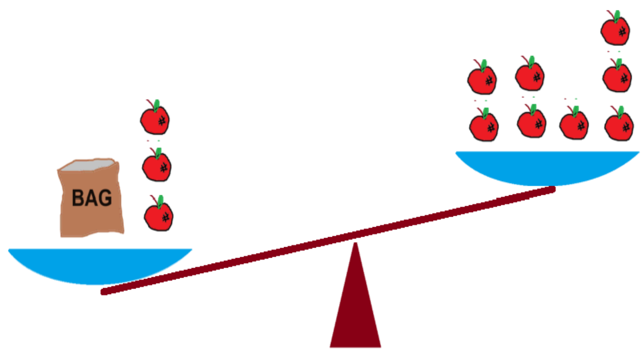
Here is a picture of a balance-scale not in balance.

$$BAG > 5$$



For the picture to be true, the bag must contain more than five apples. That is, the solution set the inequality $BAG > 5$ is the set of all numbers greater than 5.

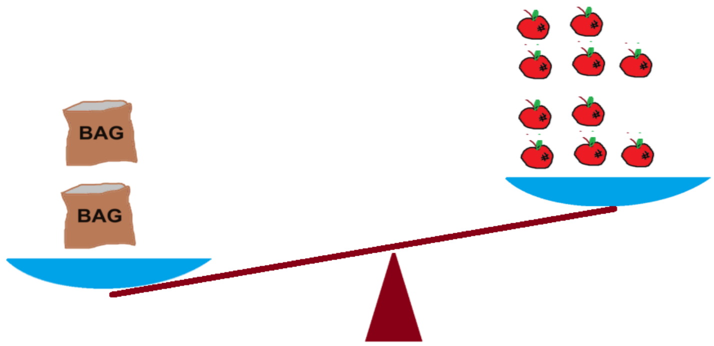
What happens if we add, say, three apples to each pan on each side of the balance scale? Common sense tells us that the scale will again be unbalanced and tilted in the same direction as before.



It seems that the sentences $BAG > 5$ and $BAG + 3 > 5 + 3$ are equivalent statements. They have the same solutions, namely, that the unknown in each case must be a number greater than 5.

What if, instead, we doubled the quantities on each side of the balance scale?

Common sense tells us that the balance pan will again be titled and in the same direction as before.



It seems that the sentences $BAG > 5$ and $2 \times BAG > 2 \times 5$ are equivalent statements too, each having the same solutions of requiring the unknown to represent a number larger than five.

The same would be the case if we tripled, quadrupled, or centupled the quantities on each side of the balance scale: the scale will remain tipped in the same direction. We can even scale each quantity by a fractional amount, the tilt of the scale will not change.

It is unclear, however, if we change the quantities on each side of the balance scale by a negative factor. This will require thinking through the meaning of "anti-apples" and "anti-bags." and it is not clear if they are even meaningful!

At the very least, we have:

Suppose $A < B$ is a statement about the inequality of two quantities.

For any number k, the statements

$$\boldsymbol{A < B}$$

and

$$\boldsymbol{A + k < B + k}$$

are equivalent.

If k is positive, then we have a third equivalent statement.

$$\boldsymbol{k \cdot A < k \cdot B}$$

This observation holds too for the inequalities $\leq$, $>$, and $\geq$.

Question: Does it feel right to you to say that $A \geq B$ and $A + k \geq B + k$ are equivalent statements for a number k?

If k is a positive number, does it feel right to you that we actually have three equivalent statements?

$$A \geq B$$

$$A + k \geq B + k$$

$$k \cdot A \geq k \cdot B$$

(Remember, $A \geq B$ means that the quantity A is larger, or possibly equal to, the quantity B.)

Practice 78.1 Draw lines in this picture to connect pairs of statements that are equivalent.

$B < A$ $\quad$ $B > A$ $\quad$ $A > B$ $\quad$ $A < B$

$A \geq B$ $\quad$ $A \leq B$ $\quad$ $B \leq A$ $\quad$ $B \geq A$

Example: Please solve

$$3w - 2 \leq 16$$

Answer:

$$3w - 2 \leq 16$$

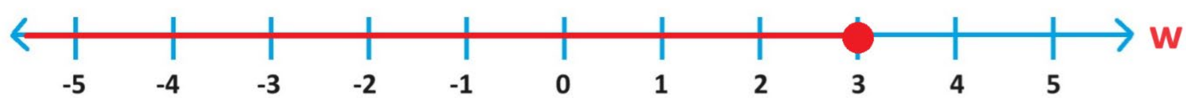
$$3w - 2 + 2 \leq 16 + 2$$

$$3w \leq 18$$

$$\frac{1}{3} \times 3w \leq \frac{1}{3} \times 18$$

$$w \leq 3$$

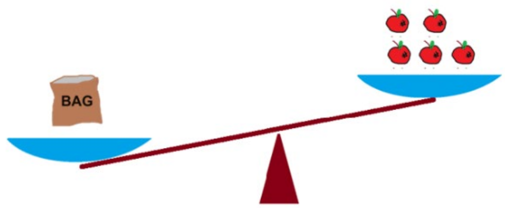
Solution set: All numbers less than, or possibly equal to, 3



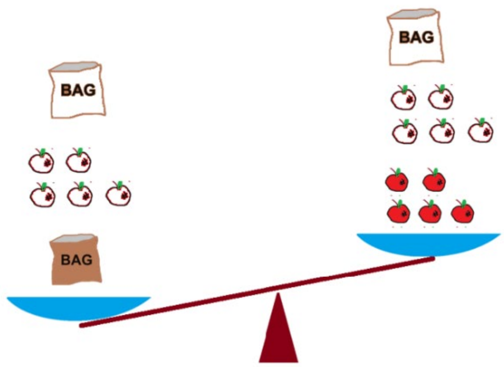
Practice 78.2: Solve $4s + 7 > 2s + 5$ and present the solution set as a graph.

Practice 78.3: Let's try playing with anti-apples and anti-bags.

Here's a picture of $BAG > 5$.



Let's now add five anti-apples and one anti-bag to each side of the pan balance.



Is there something to deduce from this picture?

The approach of Practice problem 78.3 suggests a means to understand inequalities that involve negative quantities (without resorting to "anti" shenanigans).

We know that

$$\boldsymbol{A < B} \qquad \text{and} \qquad \boldsymbol{A + k < B + k}$$

are equivalent statements for any number k. Well, the expressions A and B themselves, even if they contain unknowns, do represent numbers—we just might not be privy to what numbers they actually are.

Let's choose k to be the number $-A + -B$.

Consequently, the statement

$$A < B$$

is equivalent to

$$A + -A + -B < B + -A + -B$$

Tidying this up, it reads

$$-B < -A$$

We could rewrite this as

$$-A > -B$$

if we like.

Suppose $A < B$ is a statement about the inequality of two quantities.

Then

$$\boldsymbol{A < B}$$

and

$$\boldsymbol{-A > -B}$$

are equivalent statements.

Similar observations hold too the inequalities $\leq$, $>$, and $\geq$.

Practice 78.4: Does this feel right to you?
For instance, we know that

$$3 < 5$$

From what we've just established, it must be that

$$-5 < -3$$

Do you agree that -5 is "less than" -3?

a) Draw a number line and show the location of the points 3 and 5, and the location of the points -3 and -5 on it.

If a number to the left is always considered "less than" a number its right, is $3 < 5$ and is $-5 < -3$?

b) Recall from Section 50 that we say $a < b$ (read as "less than") if there is a positive number n so that

$$a + n = b$$

(that is, we need to "adding something to a to get up to the number b").

For example, $3 < 5$ because $3 + 2 = 5$.

i) Establish that $-5 < -3$ according to this definition.

ii) Show that $-13 < 100$ according to this definition.

iii) Establish that every negative number is "less than" zero.

iv) Suppose $a < b$. Then there is a positive number n so that $a + n = b$.

What is the value of $-b + n$?
Explain why we have $-b < -a$.

Practice 78.5: Establish that $A \geq B$ and $-2A \leq -2B$ are equivalent sentences.

Did you learn the following rule in school?

If you multiply an inequality through by a negative number, then you must flip the sign of the inequality.

Hopefully now you can see that no flipping is actually involved.

For example, to answer the practice problem, we have

$$A \geq B$$

$$A + -A + -B \geq B + -A + -B$$

$$-B \geq -A$$

Now multiply through by the positive number 2

$$-2B \geq -2A$$

Notice that the inequality sign has **not** "flipped."

It is only when we choose to rewrite $-2B \geq -2A$ as

$$-2A \leq -2B$$

does "flipping" seem to occur.

I often find it much clearer when working with an inequality to not multiply through by negative number: I just add quantities to each side of the inequality. That way I can keep the direction of the inequality straight.

Example: Kindly solve

$$3 - 2r \geq 3 - 2s$$

and graph the solutions (with "r" as the horizontal axis in the graph).

Answer: Let's start by subtracting 3 from each side of the inequality.

$$3 - 2r + -3 \geq 3 - 2s + -3$$

$$-2r \geq -2s$$

Let's now add $2r$ throughout

$$-2r + 2r \geq -2s + 2r$$

$$0 \geq -2s + 2r$$

and add $2s$ throughout.

$$0 + 2s \geq -2s + 2r + 2s$$

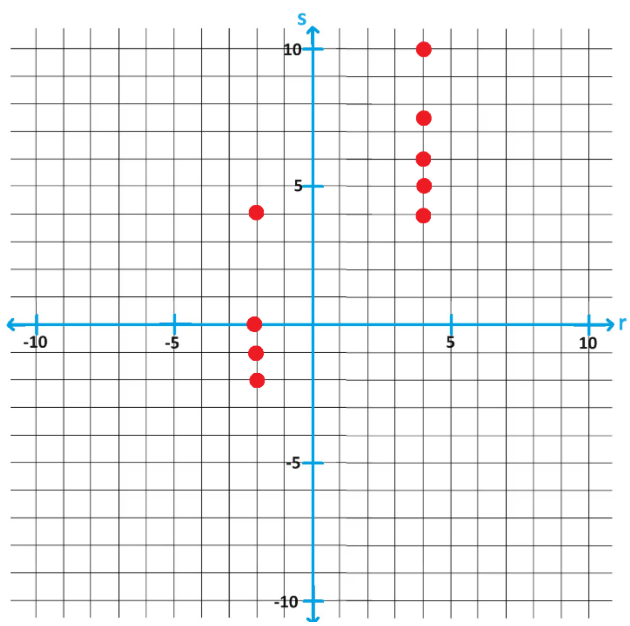
$$2s \geq 2r$$

Now multiply though by $\frac{1}{2}$ to obtain

$$s \geq r$$

The solution is the set of all values for s and r with s having a value larger than or equal to r.

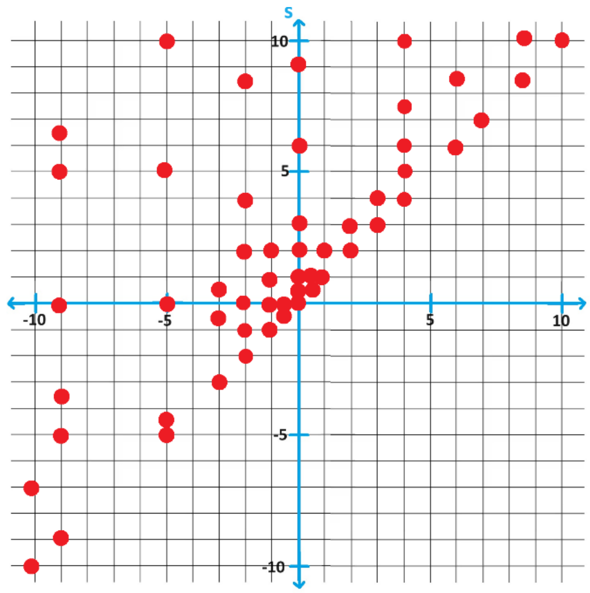
To graph this, let's collect some data values.



r	s
4	4
4	5
4	6
4	$7\frac{1}{2}$
4	10
-2	-2
-2	-1
-2	0
-2	4

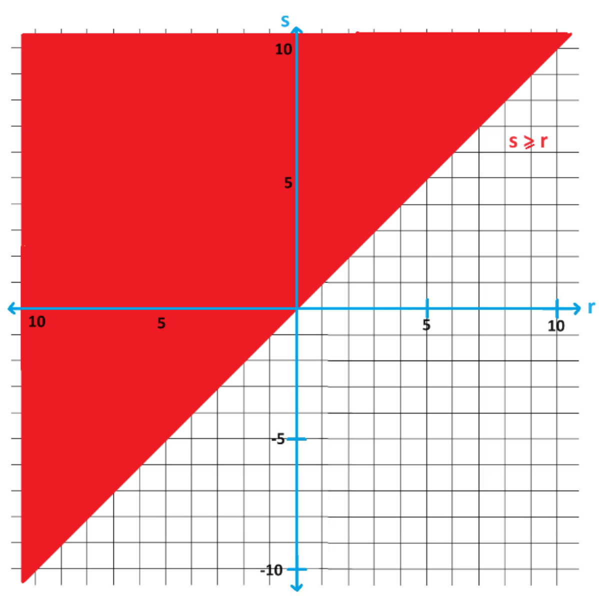
This is not enough data to see a meaningful picture.

Here's some more data points.



Every point along the northeast diagonal line has r and s equal in value, which is a valid data point. And every point vertically above a point on this line represents a point with s larger than r.

The graph of the solutions is a diagonal half of the entire plane.



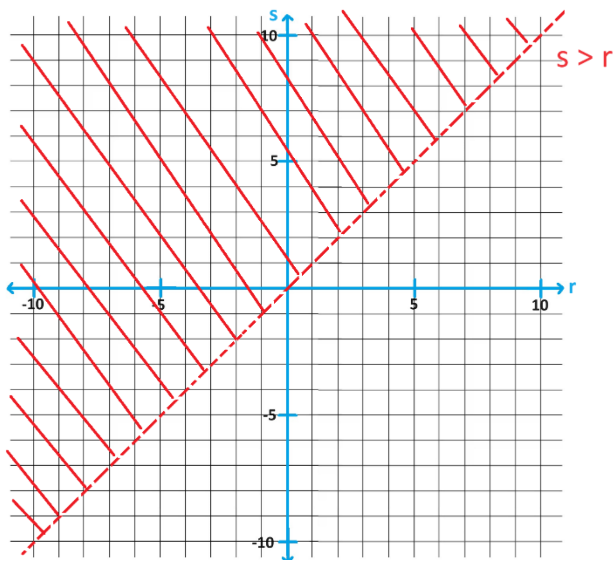
Practice 78.6

a) Would the graph of $3 - 2r \geq 3 - 3s$ be the same as the graph of $s \geq r$?

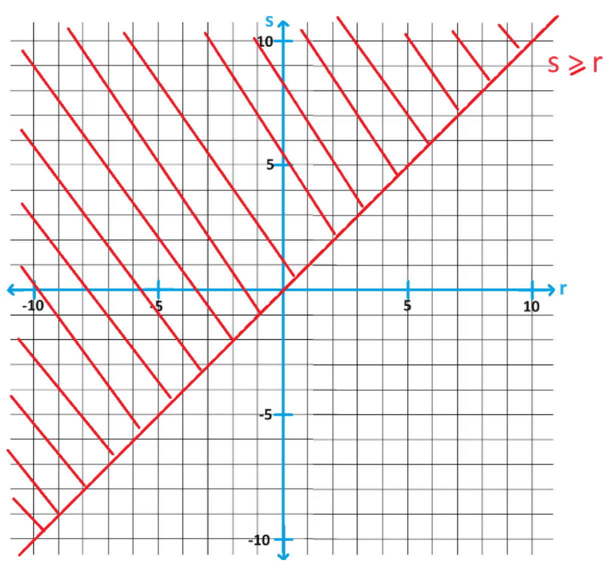
b) In general, would two equivalent statements have the same graph?

The graph of $s > r$ would be similar to the graph of $s \geq r$. It just has the points along the northeast diagonal omitted.

The way people indicate that is to draw a dashed line for the line of omitted points and not use a solid block of color for a region of points, using instead "line strokes" to indicate that a region is filled in.



We could draw our previous graph similarly using a solid line for the included points along the boundary.



Practice 78.7 Graph

$$2(b - a) + 2 < 6 - 2a$$

with the horizontal axis of your graph labeled "a."

Practice 78.8

a) Graph $x + y = 0$.

b) Graph $x + y \geq 0$

c) Graph $x + y > 0$

d) Graph $x + y \neq 0$

Label the horizontal axis of your graph "x" in each case.

MUSINGS

Musing 78.9
a) Describe the graph of the solutions to $(a-a)\cdot b=0$.
b) Describe the graph of the solutions to $(a-a)\cdot b\geq 0$.
c) Describe the graph of the solutions to $(a-a)\cdot b>0$.

Musing 78.10
a) Create an inequality in one unknown using the symbol $>$ whose solution set is all numbers different from zero.
b) Create an inequality in one unknown using the symbol $\geq$ whose solution set is just the number zero.

MECHANICS PRACTICE

Practice 78.11 Solve each of these inequalities and graph its solutions. (Always choose to label the horizontal axis "x.")

a) $-3x+9\leq 3x+9$

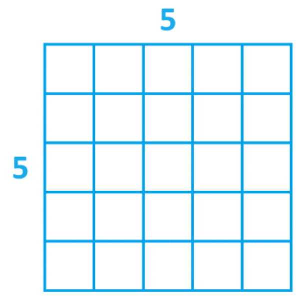
b) $x(x-2)+2x<4$

c) $3(y-2)-(x-2)\neq 2(y-x+2)+2$

d) $2x+y+3\leq 3(y+1)$

79. Squares and Square Roots

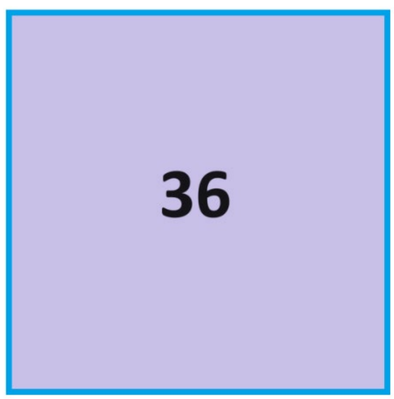
A square with side length 5 units has area $5 \times 5 = 25$ unit squares.



As we saw in Sections 11 and 60, we write 5^2 for 5×5 and call it five **squared**. The connection to geometry in this language is deliberate.

And we can go in reverse.

Suppose I gave you the area of a square first. Let's say we have a square with area 36 squared units. It is natural to wonder what the base or "root" feature of this square must be.



Of course, we are all thinking that the side length of the square must be 6 units and that I am using strange language to ask this.

But this is the language al-Khwarizmi used to when thinking of an equation of the form $s^2 = 36$. He used the Arabic word for *root*, which when transcribed into Latin by western Scholar became **radix**.

The symbol for "the root feature of a square," that is, for **square root** is √, which we call a **radix**. But the symbol usually comes attached with a vinculum (as was discussed in Section 11).

We write

$$\sqrt{36} = 6$$

In the same way, we have $\sqrt{25} = 5$.

Practice 79.1 Compute the following square root values.
(Remember, we are talking about quantities related to geometric square figures.)

$\sqrt{100}$ $\quad$ $\sqrt{49}$ $\quad$ $\sqrt{196}$ $\quad$ $\sqrt{1024}$ $\quad$ $\sqrt{1.21}$ $\quad$ $\sqrt{\frac{81}{4}}$

It is understood that the radix (and its vinculum) $\sqrt{\blacksquare}$ is a symbol from geometry, in which case all quantities being discussed when using the symbol are assumed to be from geometry. As such, they must be positive numbers. (All measurements of lengths and areas are positive numbers.)

Writing

$$\sqrt{-9}$$

is meaningless, as there is no square of area -9 in geometry.

And writing

$$\sqrt{9} = 3 \quad \text{or} \quad -3$$

is also meaningless as a square cannot have a side length of -3.

This latter point is important. The equation

$$x^2 = 9$$

has solutions

$$x = 3 \quad \text{or} \quad -3$$

and writing this is good and correct. No radix was used here and so there is no implicit command keep this piece of work in the context of geometry.

Practice 79.2

a) Is writing $\sqrt{11 + -2}$ technically meaningful? If so, what is its value?

b) Is writing $\sqrt{(-3)^2}$ technically meaningful? If so, what is its value?

Practice 79.3 a) Describe the solution set to $s^2 = -3$.

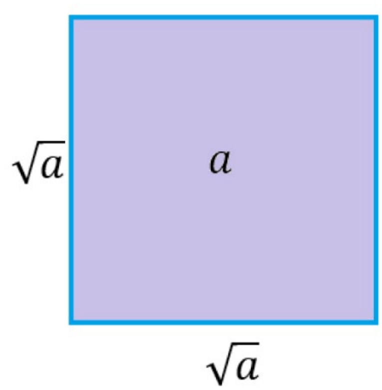
b) Describe the solution set to $s^2 = 0$.

There is one exception to this "geometry rule." People will consider squares of zero area. Such squares have a side length of zero units. It is accepted to observe and write:

$$\sqrt{0} = 0$$

Here's the formal definition of the square root of an allowed number.

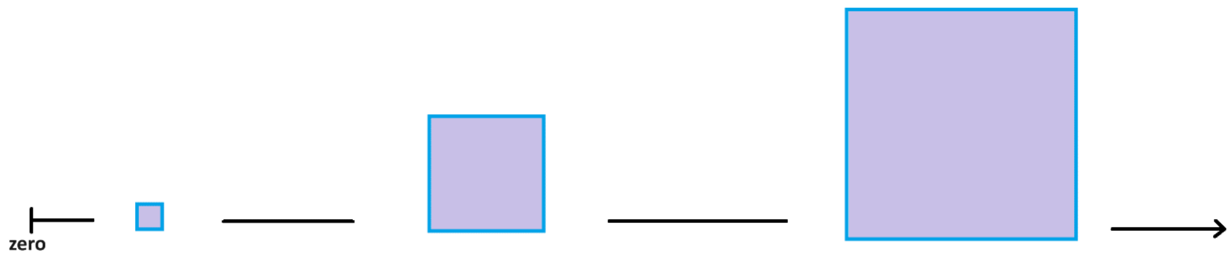
> If a is a positive number, or possibly zero, then the **square root** of a is a number $\sqrt{a}$ with the property that $\sqrt{a} \times \sqrt{a} = a$.



We can check that $\sqrt{16} = 4$, for instance, by noting that $4 \times 4 = 16$.
(Picturing an actual square is always a good move.)

Also, $\sqrt{\frac{81}{4}} = \frac{9}{4}$ is not correct because $\frac{9}{4} \times \frac{9}{4} = \frac{81}{16}$, which is not $\frac{81}{4}$.

As squares can shrink and grow to any size we want, every positive number (and zero) has a square root.

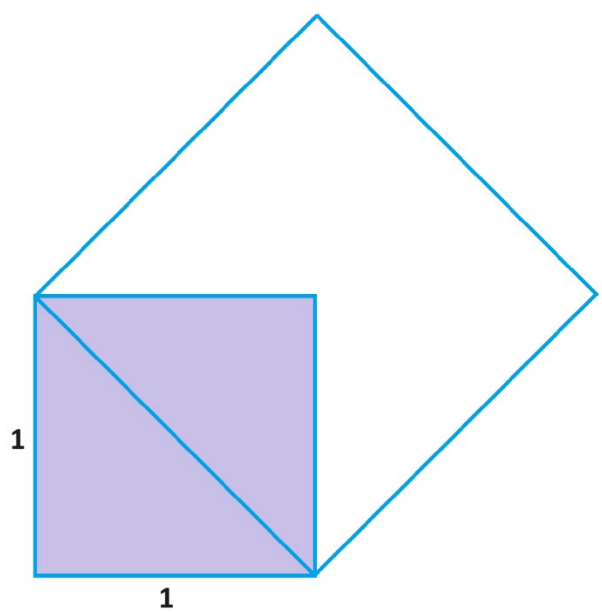


Practice 79.4 *Does the square root of two exist?*

One way to answer this question is to exhibit a square of area 2. The side length of that square is then, by definition, the square root of 2.

Draw a square with a side length 1 and draw a tilted square with side the diagonal of that unit square.

Can you see that the titled square has area 2.

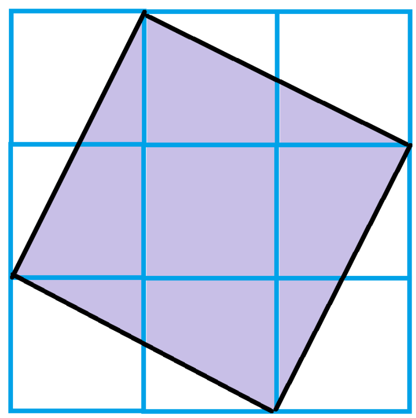


Comment: We mentioned that the diagonal of the unit square has length $\sqrt{2}$ back in Section 59, but this picture gives a more natural way to see that this must be so.

We proved that $\sqrt{2}$ is an irrational number in Section 59, and we tried to write $\sqrt{2}$ as a decimal in Practice Problem 56.17.

Practice 79.5 On a calculator what is $\sqrt{2}$ as a decimal rounded to some large number of decimal places?

Practice 79.6 Use this picture to demonstrate that $\sqrt{5}$ exists.



Example: Kindly solve $2\sqrt{w} + 4 = 0$.

Answer: The use of the radix indicates that we must be thinking of positive (or zero) values with regard to squares and square roots.

We have

$$2\sqrt{w} + 4 = 0$$

$$2\sqrt{w} = -4$$

$$\sqrt{w} = -2$$

The original equation is equivalent to an equation which can never be true. (A square root value cannot be negative.)

The original equation has no solutions.

MUSINGS

Musing 79.7 Collect some data for the equation $b = \sqrt{a}$. Graph the data with the horizontal axis labeled "a."

MECHANICS PRACTICE

Practice 79.8 Make the expression $\sqrt{\frac{\sqrt{3}}{2.1}} \times \sqrt{\frac{\sqrt{3}}{2.1}}$ a tinier bit friendlier.

Practice 79.9 What is the value of $\sqrt{\sqrt{\sqrt{256}}}$?

Practice 79.10 Show that $\frac{3\sqrt{20}}{2}$ is the same number as $\frac{30}{\sqrt{20}}$.

Practice 79.11 Please solve each of these math sentences.

a) $4\left(\sqrt{r} + 3\right) = 3\sqrt{r} + 5$

b) $x(x + 3) - 2x + 2 \leq x - 5$

c) $\sqrt{a} - 3 \neq 3 - \sqrt{a}$

80. Going Rogue

Two mathematical sentences (mentioning the same unknowns) are equivalent if they have exactly the same solutions. And the process of solving an equation or an inequality is to carefully transform the given sentence into an equivalent one whose solutions we readily recognize.

Thanks to the al-Khwarizmi, we have a number of algebraic steps we can take that won't alter the truth, and hence solutions, of a given math sentence.

- **Applying a fundamental rule of arithmetic to one piece of a math sentence won't affect the truth of the sentence.**

For example,

$$3(x+6) = 2x \quad \text{and} \quad 3x + 18 = 2x$$

are equivalent sentences because a rule of arithmetic allows us to expand brackets and rewrite the piece of the sentence $3(x+6)$ as $3x + 18$.

- **Adding (or subtracting) identical quantities to each side of a math sentence does not affect the truth of the sentence.**

For example, the adding $-a$ to each side of this inequality

$$3a + 2 \geq a - 1$$

gives the equivalent sentence

$$3a + 2 + \ -a \geq a - 1 + \ -a$$

which, by the first point, is equivalent to

$$2a + 2 \geq -1$$

- **Multiplying each side of a math sentence by a quantity known to be a <u>positive</u> number won't affect the truth of the sentence.**

For example, scaling each side of the sentence $2a + 2 \geq -1$ by a factor of $\frac{1}{2}$ gives the equivalent sentence

$$a + 1 \geq -\frac{1}{2}$$

And that's it! These are the three algebraic operations that preserve the truth of math sentences.

Practice 80.1 Are

$$b > 2$$

and

$$(b^2 + 1)b > 2(b^2 + 1)$$

equivalent sentences?

One need not fuss about what happens when we multiply through by a negative value: the addition and subtraction of quantities will handle that.

Practice 80.2
a) Show that

$$-A = -B \quad \text{and} \quad A = B$$

are equivalent sentences by adding $A + B$ to each side of the first equation.

b) Show that

$$-3 \leq x \quad \text{and} \quad -x \leq 3$$

are equivalent sentences.

But we did see "rules" we can follow if you prefer the shortcut they offer.

- Multiplying each side of an equality ($=$) by a negative number does not affect the truth of the equality.
- In multiplying each side of an inequality ($>$ or $\geq$ or $<$ or $\leq$) by a negative number, one most also "flip" the direction of the inequality sign in order to preserve truth.

Practice 80.3 Is there a special "rule" to be deduced for multiplying an inequality of the from $A \neq B$ through by a negative number?

One is, of course, welcome to transform a math sentence in any way one desires.

But if you deviate from the bulleted items listed on the previous two pages, then all bets are off as to what remains true about the new math sentence you create. It is up to you to decide if the solutions to your new sentence bear any relevance to the solutions of the original sentence.

For example, here's an equation.

$$x = 2$$

It has just one solution, namely that x must have the value 2 to make it a true sentence.

Now let's square each side of the equation. (This is a rogue move: it's not one of the allowed moves of algebra.)

We get the sentence

$$x^2 = 4$$

This new sentence has two solutions: x must be 2 or -2 to make the sentence true.

The sentences $x = 2$ and $x^2 = 4$ are not equivalent sentences.

Solution sets will likely change if you deviate from the standard steps!

However, there is some potential value here.

We can observe that if two numbers are equal, then the squares of those two numbers will also be equal. So, if we have some values that make a math sentence

$$A = B$$

true, then those values will also make the sentence

$$A^2 = B^2$$

true.

The solutions to $A = B$ will appear among the solutions to $A^2 = B^2$.

And we saw that: the solution to $x = 2$ is among the solutions to $x^2 = 4$.

So, if you feel like squaring each side of an equation, go for it. But you now know that your equation will likely have more solutions than the original equation, but the solutions to the original equation you seek will be among them.

You can then just check each potential solution in turn to see which ones actually work.

Comment: School curricula call the appearance of additional potential solutions a phenomenon of **extraneous solutions**. They require students to "check all your solutions" to weed out which ones don't apply to the original equation. And that's appropriate if a student has taken a step that's broken away from the standard steps of algebra (our previous bullet points).

But if a student has not deviated from the standard steps of algebra, then they can be assured that all the solutions obtained are all the solution of the original equation. There is no need to "check all your solutions" (except to catch arithmetic errors, perhaps).

Practice 80.4

a) Describe the set of solutions to

$$\sqrt{w} = -3$$

b) Squaring each side of this equation gives

$$w = 9$$

What are the solutions to this equation? Are any of the solutions "extraneous"?

Going rogue is dangerous.

For example, consider this equation

$$(b-5)^2 = 36$$

Students are often encouraged to take the square root of each side of the equation and to even draw in the radix.

$$\sqrt{(b-5)^2} = \sqrt{36}$$

This leads students to then write

$$b - 5 = 6$$

$$b = 11$$

I have no idea what taking the square root of each side of math sentence typically does to the set of solutions of the original sentence. I am not sure what to say about concluding "b must have value 11."

Practice 80.5

a) Does b having the value 11 make the sentence $(b-5)^2 = 36$ true?

b) Have we fully solved $(b-5)^2 = 36$? Are we missing solutions?

Like I said, if you choose to go rogue, you are completely on your own!

Example: Solve

$$\frac{x}{x} = 1$$

Answer Attempt 1: Multiply each side of the equation by the number x to obtain

$$x \times \frac{x}{x} = x \times 1$$

$$x = x$$

Every possible value for x makes this final sentence true.

Solution set: all numbers.

The trouble with this answer is that we've subtly gone rogue.

We are welcome to multiply an equation through by a number, as long as that number is positive or negative—but not zero!

So, we have to be careful with our thinking.

Answer Attempt 2: If x represents a number that is not zero, then we can multiply each side of our equation x to obtain

$$x \times \frac{x}{x} = x \times 1$$

$$x = x$$

Every possible value for x makes this final sentence true—but remember, we are only considering non-zero numbers at present.

Solutions so far: The set of all non-zero numbers.

So, what if we do consider x to be the number zero?

Well, in that case our math sentence $\frac{x}{x} = 1$ is not meaningful, yet alone true. (We can't have a denominator of zero in a fraction.) So, zero is not a solution after all.

Final answer: The solutions are all non-zero numbers.

Practice 80.6 Godspeed was asked to solve this equation:

$$a \times a = a$$

He presented this solution:

$$a \times a = a$$

$$\frac{1}{a} \times a \times a = \frac{1}{a} \times a$$

$$a = 1$$

I conclude that there is one solution: a is 1

Any commentary?

Example: Solve

$$\frac{1}{x-3} = 2$$

Answer: The equation does not make sense, yet alone be true, if x is 3. We have to keep the value 3 out of our considerations.

But if x is not 3, then $x - 3$ is not zero and we are welcome to multiply both sides of the equation by $x - 3$ to get an equivalent equation (but still under the proviso that x is not 3).

$$(x-3) \times \frac{1}{x-3} = (x-3) \times 2$$

$$1 = 2x - 6$$

We can keep going

$$7 = 2x$$

$$\frac{1}{2} \times 7 = \frac{1}{2} \times 2x$$

$$3\frac{1}{2} = x$$

We have the solution that x is $3\frac{1}{2}$ (and the fits the proviso of not being 3!

MUSINGS

Musing 80.7 Consider the equation

$$x < \frac{1}{x}$$

a) Is 2 a solution to this equation? Is -2 a solution?

b) Is $\frac{1}{2}$ a solution to this equation? Is $-\frac{1}{2}$ a solution?

c) Why can't zero be considered as a potential solution to this equation?

d) If we restrict our minds to consider only positive values for x, show that the equation is then equivalent to $x^2 < 1$. What then are the solutions to the equation (within this restricted mindset)?

e) If we restrict our minds to consider only negative values for x, show that the equation is then equivalent to $x^2 > 1$. What then are the solutions to the equation (within this restricted mindset)?

f) Describe the full set of solutions to the original equation.

MECHANICS PRACTICE

Practice 80.8 Fully solve

$$\frac{x+3}{x-3} = 0$$

explaining your reasoning with care as you go along.

Practice 80.9 Give two solutions to the equation

$$(a+3)^2 = 100$$

Practice 80.10 Kindly solve

$$\sqrt{w+6} = 9$$

Solutions

72.1 a) Any value that is not 2 makes the sentence true. So, let p represent the number 3, for example.
b) If p represents the number 2, then it a false sentence.

72.2 a) False. b) True c) If n is the number -5, then we have another true statement.

72.3 r could be any value larger than 97.

72.4 a is 0 and b is 45 or a is 2 and b is 0 or a is 0 and b is 0, for example.
(We just need to make sure at least one of the symbols a or b represents the number zero.)

72.5 Yep. That is the case!

73.1 and **73.2**
a) If Mrepresents the number 2 or 8 or 100 or $42\frac{1}{2}$, then we have a true sentence.
b) If M is the number 98, then we have the sentence

$$96 \times 90 \times (-2) \times 45\frac{1}{2} = 0$$

which is not true.
c) If M is the number -2, then we have the sentence

$$(-4) \times (-10) \times (-102) \times (-44\frac{1}{2}) = 0$$

which is not true.
d) If M represents a value different number 2 or 8 or 100 or $42\frac{1}{2}$, then the sentence says that a product of four non-zero numbers is zero. That would be a false sentence.

73.3 a) 4 and 5 and 6 and 6.002, for example. b) Infinitely many.

73.4 For example,

s	t
−2	3
−2	6
5	−1
102	−6
−3	3
−9	96

73.5 i), ii), and iv) [Only a is 0 and b is 0 make sentence iii) true. Can you see why?]

73.6 Here's an example:

$$x = x + 1$$

73.7 We cannot have b be 0. For all non-zero values for b, the sentence is true.

73.8 w can be 6 or -6.

73.9 For example:

a	b
1	1
2	$\frac{1}{2}$
$\frac{1}{2}$	2
3	$\frac{1}{3}$
$\frac{1}{3}$	3
17	$\frac{1}{17}$

73.10 x is ten thousand and y is one, for example.

73.11 x is 1 and y is 1, for example.

73.12 For this to true, x must be the number 7.

74.1 She ate 2 carrots for 3 hours of sleep.

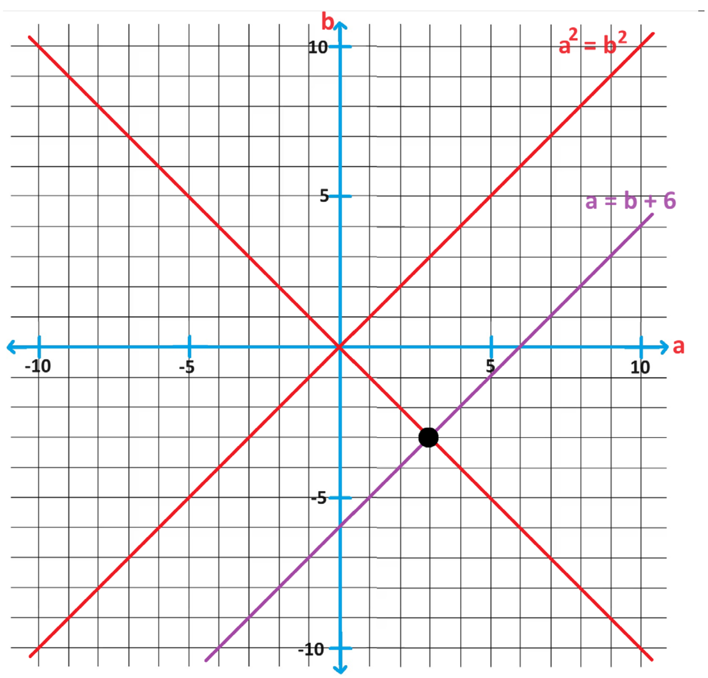
74.2 These points do not appear in the graph: $(5,3)$ $(5,6)$ $(4,5)$ $(6,4)$ $(4,8)$

74.3 Do try.

74.4 Do try.

74.5 It would actually look the same, except the axes will have their labels switched.
It is not normally the case that the graphs would look identical with such a change. This picture just happened to be highly symmetrical.

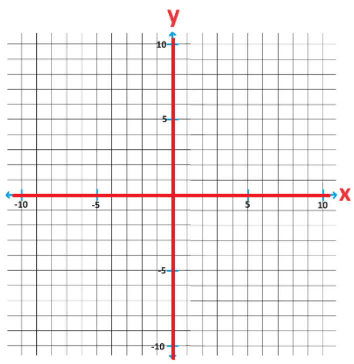
74.6



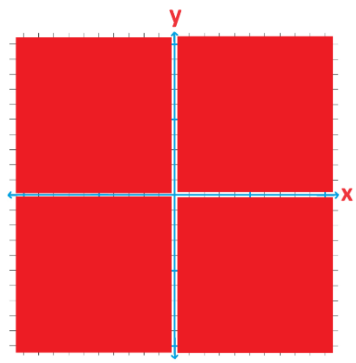
The point of intersection $(3,-3)$ represents a data point with a as 3 and b as -3 that represents truth for both statements $a^2 = b^2$ and $a = b + 6$ simultaneously. (And indeed, we do have $3^2 = (-3)^2$ and $3 = (-3) + 6$.)

74.7 a) All the points on the axes.
b) All the points not on the axes.

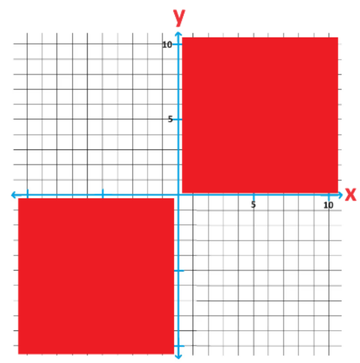
c) All the points (not on axes) in the top right and bottom left quadrants.



xy = 0



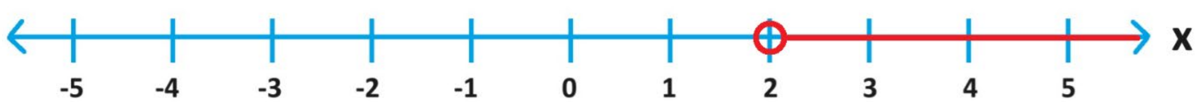
xy ≠ 0



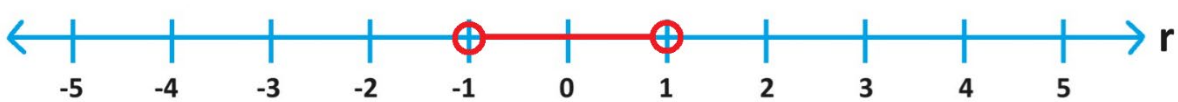
xy > 0

74.8

74.9

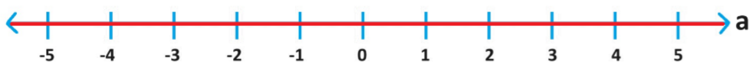


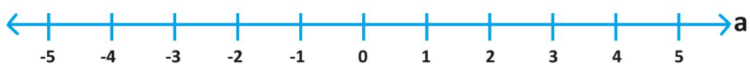
74.10



74.11 a) $x = y$ works. b) $r = q$ works. c) $r + q = 5$ works.

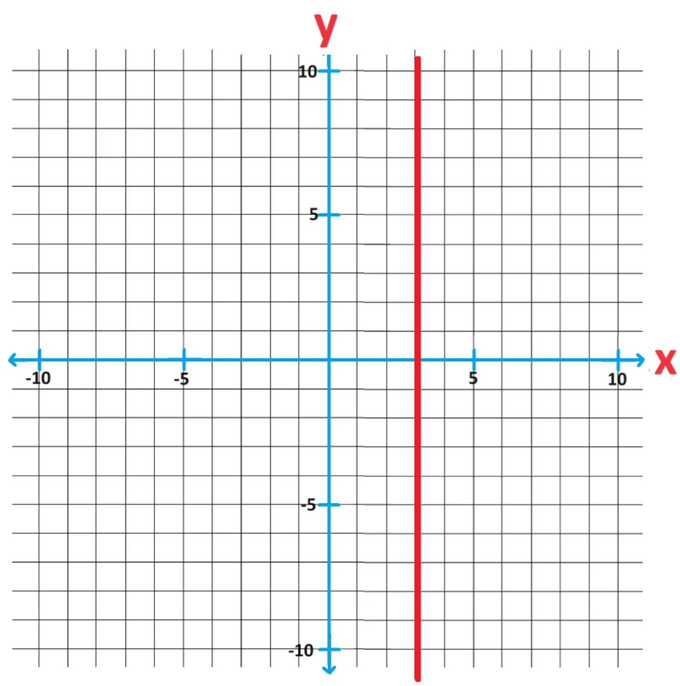
74.12 a) Every possible value for a gives truth.
b) No possible value for a gives truth.
c) Every value, except zero, gives a (meaningful and) true statement.

a) 

b) 

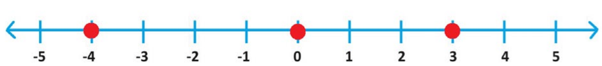
c) 

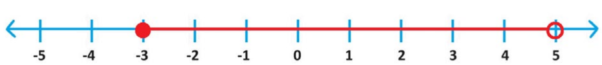
74.13

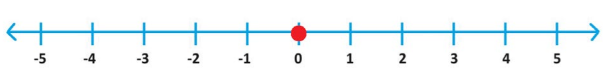


74.14 The statements $\frac{x-3}{x-3} = 1$ and $x \neq 3$ both do the trick.

74.15

a) 

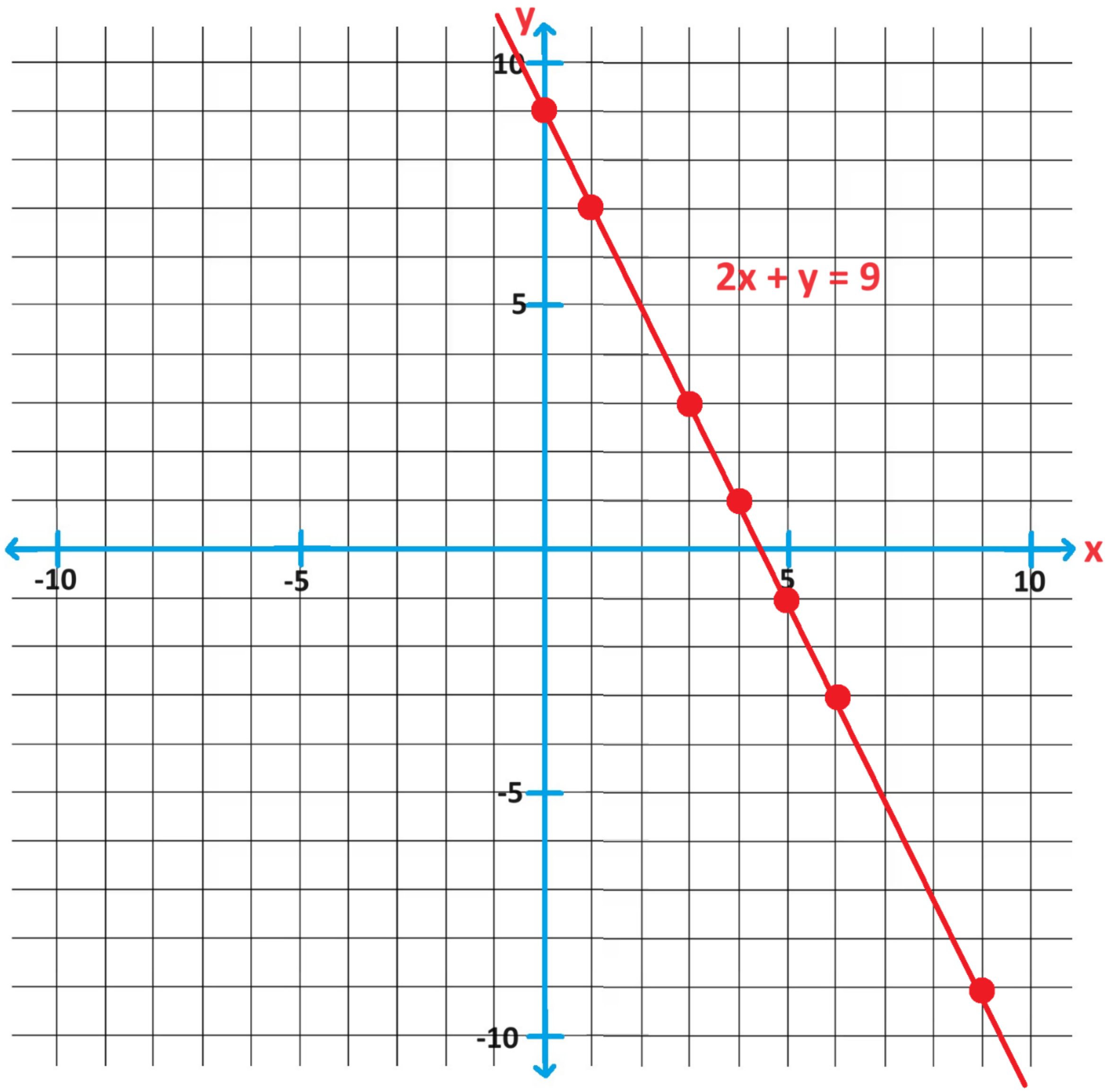
b) 

c) 

74.16 These statements, for instance, do the trick.

a) $x = -3$ b) $\frac{1}{2} < b \le 4$ c) $w^2 \ge 9$ d) $(r+4)(r-1)(r-3) = 0$

74.17



75.1 Read on!

75.2 The solutions to the equation

$$(x-2)(x-3)(x+5)=0$$

are the same as the values of x that make this next sentence true:

$$x = 2, 3, \text{or} -5$$

This second sentence is clearly true if x represents the number 2, 3, or -5.

76.1 I am writing B for BAG.

a)

$$\begin{aligned} 5B+5 &= 4B+6 \\ 5B &= 4B+1 \\ B &= 1 \end{aligned}$$

There is one apple in each bag.

b)

$$\begin{aligned} &B+1 = 2B+1 \\ &B = 2B \\ &0 = B \\ &B = 0 \end{aligned}$$

Each bag is empty.

c)

$$\begin{aligned} &B+3 = 2B+5 \\ &B = 2B+2 \\ &0 = B+2 \\ &0 + -2 = B + -2 \\ &-2 = B \\ &B = -2 \end{aligned}$$

Each bag contains negative two apples! (Anti-apples?)

The picture of $0 = B + 2$ has two apples and a bag of apples on one side of the balance scale balancing with nothing. So, whatever is in the bag must be floating upwards to counteract the weight of two apples! (This is presuming you want to keep believing this physical model of scales, apples, and bags.)

76.2 Read on!

76.3

a) They both have the single solution of r being 3.
b) They both have the solution set of all numbers x and y that have the same value.

76.4 Thinking about a pan-balance out of kilter, adding or removing apples or bags, and scaling the quantities on each side of the balance will keep the system akilter.

It seems we have that $A \neq B$ and $A + k \neq B + k$ and $kA \neq kB$, for a nonzero number k, are equivalent statements.

We have following series of equivalent statements.

$$5x + 7 \neq x + 5$$
$$5x + 7 - 5 \neq x + 5 - 5$$
$$5x + 2 \neq x$$
$$5x + 2 - x \neq x - x$$
$$4x + 2 \neq 0$$
$$4x + 2 - 2 \neq 0 - 2$$
$$4x \neq -2$$
$$\frac{1}{4} \times 4x \neq \frac{1}{4} \times (-2)$$
$$x \neq -\frac{1}{2}$$

Any value for x different from $-\frac{1}{2}$ is a solution to the original inequality.

76.5

a)

$$2w = -4$$
$$\frac{1}{2} \times 2w = \frac{1}{2} \times (-4)$$
$$w = -2$$

b)

$$19z + 2 = 17z - 3$$
$$19z + 2 - 2 = 17z - 3 - 2$$

$$19z = 17z - 5$$

$$19z - 17z = 17z - 5 - 17z$$

$$2z = -5$$

$$\frac{1}{2} \times 2z = \frac{1}{2} \times (-5)$$

$$z = -\frac{5}{2}$$

c) Getting a bit swifter …

$$8x + 7 = 5x + 31$$

$$8x + 7 - 5x - 7 = 5x + 31 - 5x - 7$$

$$3x = 24$$

$$x = 8$$

d)

$$2p + 1 = 12p$$

$$1 = 10p$$

$$p = \frac{1}{10}$$

e)

$$3R + 5 + 2R + 9 = 4R + 22$$

$$5R + 14 = 4R + 22$$

$$R + 14 = 22$$

$$R = 8$$

77.1 Did you indeed notice this?

77.2 Solve your scary-looking equation to check that you are right!

77.3

a)

$6m + 5 - 3m + 4 + 7m = 4 + 3m + 2 + 8m + 2$

$10m + 9 = 11m + 6$ (arithmetic)

$9 = m + 6$

$3 = m$

So, $m = 3$.

b)

$2(t + 2) - 3t = 2(t + 1)$
$2t + 4 - 3t = 2t + 2$ (arithmetic)
$-t + 4 = 2t + 2$ (arithmetic)

$4 = 3t + 2$

$2 = 3t$

$t = \frac{3}{2}$

c)

$4(w + \ -5) + (-5)(w + \ -4) = w + \ -6$ (making the subtractions explicit)
$4w + \ -20 + (-5)w + 20 = w \ + \ -6$ (arithmetic)
$-w = w - 6$ (arithmetic)
$-w + 6 = w$
$6 = 2w$
$3 = w$
So, $w = 3$.

d)

$p(p + 5) = 5p + 36$
$p^2 + 5p = 5p + 36$ (arithmetic)
$p^2 = 36$
So, $p = 6 \ \ or \ -6$.

e)

$3(x - 3) + 9 = 3x + 1$
$3x - 9 + 9 = 3x + 1$ (arithmetic)
$3x = 3x + 1$ (arithmetic)
$0 = 1$

Our original sentence is equivalent to a patently false statement. There are no values of x that make the given equation true.

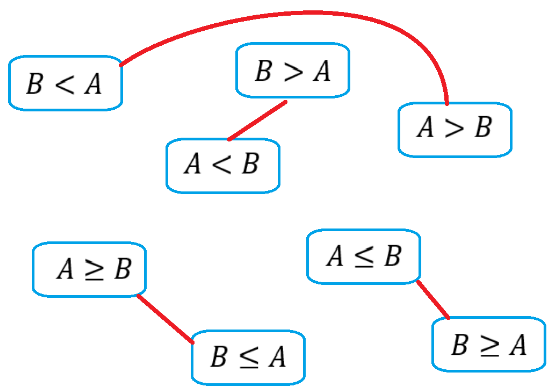
There are no solutions.

f)
$z = z$

This sentence is patently true no matter the value of z.

Solutions: The set of all numbers.

78.1



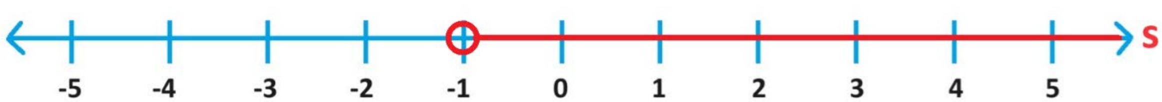
78.2

$$4s + 7 > 2s + 5$$

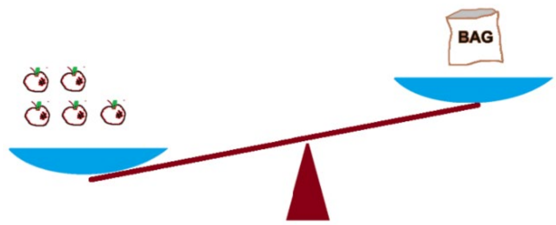
$$2s + 7 > 5$$

$$2s > -2$$

$$s > -1$$



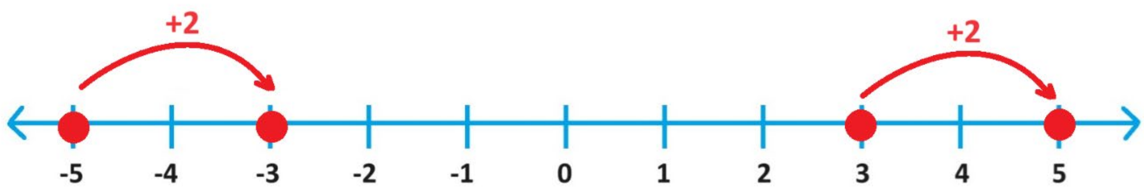
78.3 Presumably apples and anti-apples "cancel," as do bags and anti-bags. This then leaves us with this picture.



Are we seeing that $BAG > 5$ and $-5 > -BAG$ are equivalent statements?
Read on!

78.4

a)



b) i) $-5 < -3$ because $-5 + 2 = -3$.
ii) $-13 < 100$ because $-13 + 113 = 100$.
iii) If a is a positive number, then $-a < 0$ because $-a + a = 0$.
iv) From $a + n = b$ we get

$$a + n + -a + -b = b + -a + -b$$

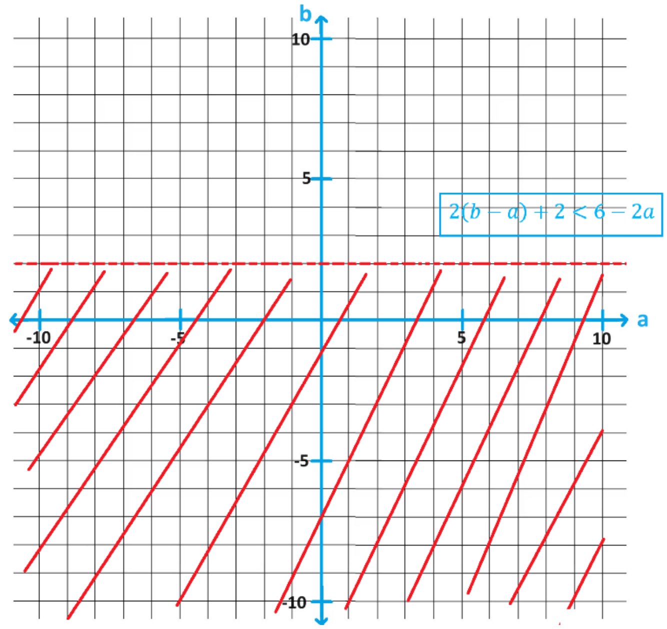
$$-b + n = -a$$

which shows that $-b < -a$.

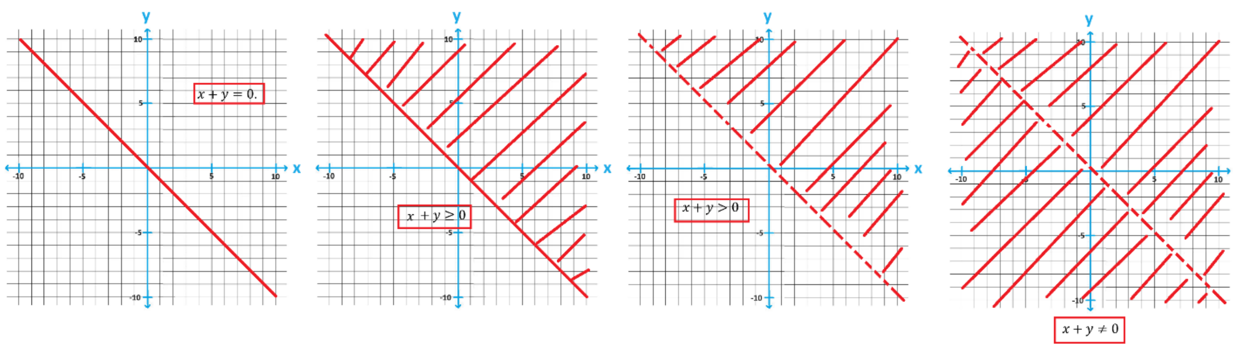
78.5 Read on.

78.6 Yes and Yes. The graph of a math sentence is a visual representation of all the data values that make the sentence true. If two statements are equivalent, that means they have the same solutions and so have the same graphs.

78.7 The inequality is equivalent to $b < 2$.



78.8



78.9

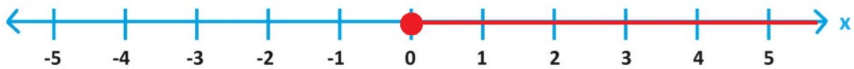
a) All pairs of values a and b make the equation true. The graph is the entire two-dimensional plane shaded in.

b) Again, all pairs of values a and b make the equation true. The graph is the entire two-dimensional plane shaded in.

c) No pairs of values a and b make the equation true. The graph is the entire two-dimensional plane with nothing marked on it.

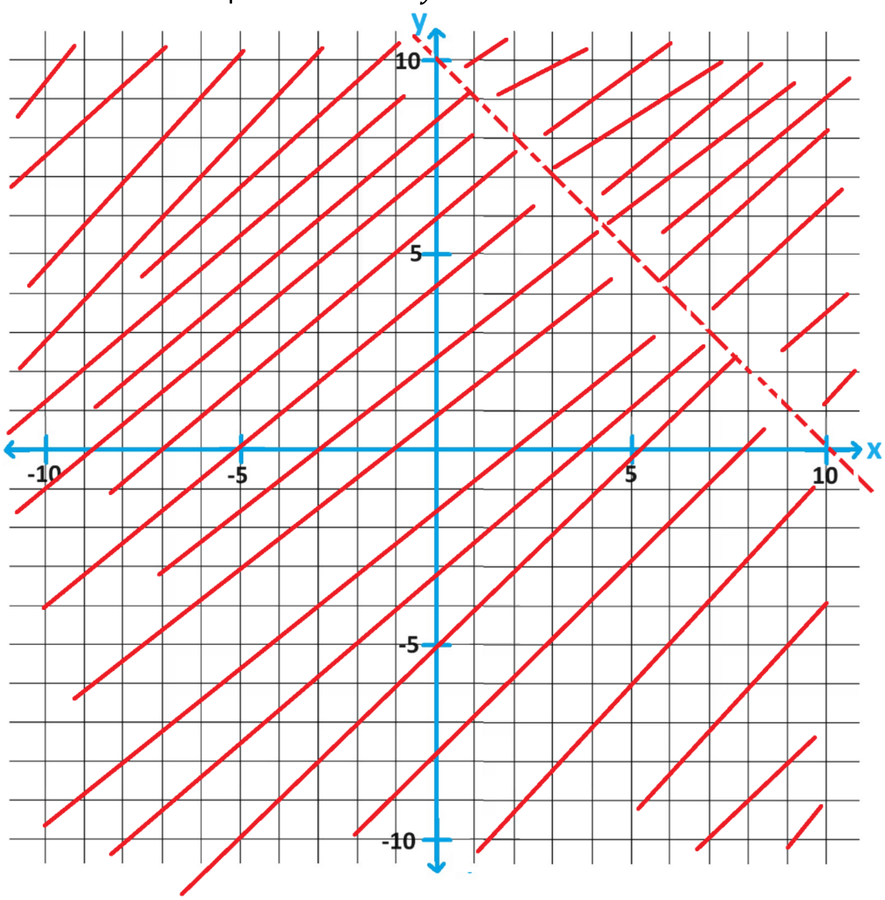
78.10 a) $x^2 > 0$ works. b) $x^2 \leq 0$ works.

78.11 a) The solution set is all values for x that are greater than or equal to zero.

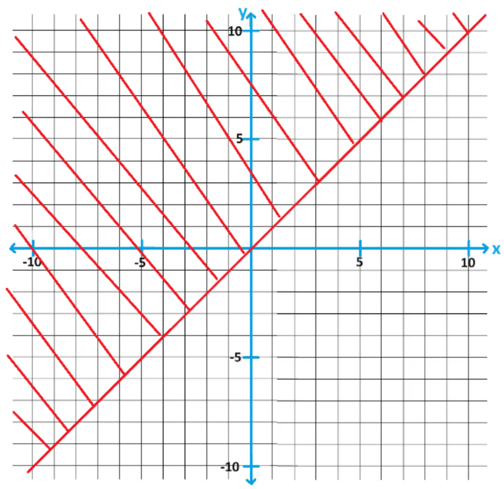


b) The statement is equivalent to $x^2 < 4$. The solutions are as shown.

c) The statement is equivalent to $x + y \neq 10$. The solutions are as shown.



d) The statement is equivalent to $y \geq x$ which solutions are as shown.



79.1

$\sqrt{100} = 10$ $\sqrt{49} = 7$ $\sqrt{196} = 14$ $\sqrt{1024} = 32$ $\sqrt{1.21} = 1.1$ $\sqrt{\frac{81}{4}} = \frac{9}{2}$

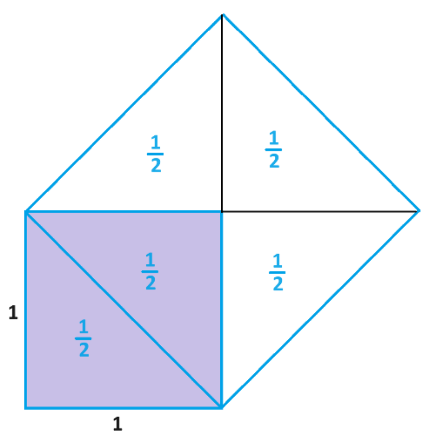
79.2

a) For $\sqrt{11 + -2}$ the vinculum tells us to compute $11 + -2$ first. So, the given quantity is $\sqrt{9}$ which is meaningful and has value 3.

b) For $\sqrt{(-3)^2}$ the vinculum tells us to compute $(-3)^2$ first. So, the given quantity is $\sqrt{9}$ which is meaningful and has value 3.

79.3 a) The solution set is empty. There are no solutions to this equation.
b) There is just one solution, namely, s is 0.

79.4 The tilted square is composed of four triangles each of area $\frac{1}{2}$.



79.5 Here are the first one hundred decimal places of $\sqrt{2}$.

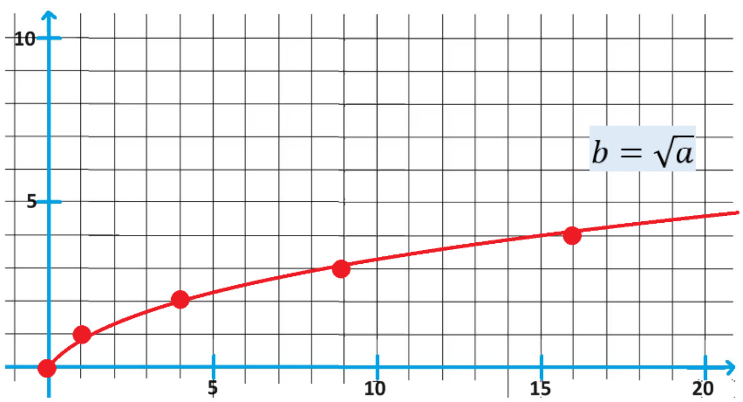
1.4142135623730950488016887242096980785696718753769480731766797379907324784621070388503875343276415727…

79.6 Let's assume each blue square has area 1.

Do you see that the titled square is surrounding by four white triangles and that each triangle is half of a 1-by-2 rectangle? The area of each triangle is thus "half of 1×2" which is 1.

So the area of the titled square is $9 - 4 = 5$, and so its side length is $\sqrt{5}$.

79.7



79.8 $\frac{\sqrt{3}}{2.1}$

79.9 We have a vinculum within a vinculum within a vinculum.

$$\sqrt{\sqrt{\sqrt{256}}} = \sqrt{\sqrt{16}} = \sqrt{4} = 2$$

79.10 Multiply top and bottom by $\sqrt{20}$

$$\frac{3\sqrt{20}}{2} = \frac{3 \times \sqrt{20} \times \sqrt{20}}{2 \times \sqrt{20}} = \frac{60}{2\sqrt{20}} = \frac{30}{\sqrt{20}}$$

79.11

a)

$$4(\sqrt{r}+3)=3\sqrt{r}+5$$

$$4\sqrt{r}+12=3\sqrt{r}+5$$

$$\sqrt{r}+12=5$$

$$\sqrt{r}=-7$$

There are no solutions.

b)

$$x(x+3)-2x+2\leq x-5$$

$$x^2+3x-2x+2\leq x-5$$

$$x^2+x+2\leq x-5$$

$$x^2+2\leq -5$$

$$x^2\leq -7$$

No number squared is negative.
This inequality has no solutions.

c)

$$\sqrt{a}-3\neq 3-\sqrt{a}$$

$$2\sqrt{a}-3\neq 3$$

$$2\sqrt{a}\neq 6$$

$$\sqrt{a}\neq 3$$

This final statement is true only if a is a positive number different from 9.

80.1 They are. We know that b^2, no matter what value b may have, is sure to be a number greater than or equal to zero, and so b^2+1 is a value more than this and is thus certain to be a positive number. Multiplying an inequality by a positive number gives an equivalent statement.

80.2

a)

$$-A = -B$$

$$-A + A + B = -B + A + B$$

$$B = A$$

b)

$$-3 \leq x$$

$$-3 + 3 + \ -x \leq x + 3 + \ -x$$

$$-x \leq 3$$

80.3 The sentence $A \neq B$ is equivalent to $A + \ -A + \ -B \neq B + \ -A + \ -B$ which is equivalent to $-B \neq -A$. So, multiplying through by -1 maintains equivalency.

We can multiply each of $A \neq B$ and $-A \neq -B$ each by a positive number k to get two more equivalent statements: $kA \neq kB$ and $-kA \neq -kB$.

In short, we can multiply an inequality of the form $A \neq B$ by a positive or negative number and maintain truth. No special rule is needed.

80.4

a) No solutions.

b) Just one solution, namely that w must be 9, and that is indeed extraneous: It is not a solution to the original equation.

80.5 a) It does. b) We are missing that b could be -1.

80.6 Godspeed subtly made a rogue assumption. By working with the quantity $\frac{1}{a}$ he assumed that a does not have the value 0.

And he is right, among all the non-zero numbers, having a be 1 is indeed all the solutions.

But what about the about that value 0 itself?

Since $0 \times 0 = 0$ we see that it is also a solution!

So, the original equation has two solutions: a could be 0 or 1.

80.7 a) 2 is not a solution, but -2 is.

b) $\frac{1}{2}$ is a solution, but $-\frac{1}{2}$ is not.

c) As a fraction cannot have a denominator of zero, x being zero gives a meaningless, let alone true, sentence. We cannot have x being 0 in our considerations.

d) We can multiply each side of the inequality by the positive number x in this case and obtain the equivalent statement $x^2 < 1$. Normally this has solutions all numbers strictly between -1 and 1, but remember, we are considering only positive numbers. So, in this setting, we have the solutions: all numbers between 0 and 1.
e) We can multiply each side of the inequality by the negative number x in this case and obtain the equivalent statement $x^2 > 1$. (Remember: We have a “flip”” of the inequality in this case.) Normally $x^2 > 1$ this has solutions all numbers larger than 1 and all numbers less than -1, but remember, we are considering only negative numbers. So, in this setting, we have the solutions: all numbers less than -1.
f) Putting it all together, $x < \frac{1}{x}$ has solutions: All numbers less than -1 along with all numbers between 0 and 1. (And this matches what we saw in parts a) and b).)

80.8 For starters, x cannot be 3 (for then we would have a meaningless, let along true, math sentence).

So, in our next considerations, let’s assume x is not 3.
Then we can be sure that $x - 3$ is a number different than zero. We can multiply our equality through by this number.

$$\frac{x+3}{x-3} = 0$$

$$(x-3) \times \frac{x+3}{x-3} = (x-3) \times 0$$

$$x + 3 = 0$$

$$x = -3$$

The original equation has solution x is -3.

80.9 a can be 7 or -13

80.10 Let’s square both sides. We know that will expand the size of the solution set.

$$\sqrt{w+6} = 9$$

$$w + 6 = 81$$

Working with this we get

$$w = 75$$

But is this actually a solution or is it extraneous?

Checking: $\sqrt{75+6}=\sqrt{81}=9$ is good!

The solution is that w must be 75.

APPENDIX: A Summary of All the Rules of Arithmetic

For reference, here's a summary of the rules of arithmetic we established in Part 1.

We have a mathematical universe of numbers, which includes the counting numbers

$$0, 1, 2, 3, \ldots$$

There are two operations—**addition** and **multiplication**—on these numbers which behave just as we expect them to when applied to the counting numbers. But they also apply to all numbers in our number universe and behave as follows:

Rule 1: For any two numbers a and b we have $a + b = b + a$.

Rule 2: For any number a we have $a + 0 = a$ and $0 + a = a$.

Rule 3: In a string of additions, it does not matter in which order one conducts individual additions.

Rule 4: For any two numbers a and b we have $a \times b = b \times a$

Rule 5: For any number a we have $a \times 1 = a$ and $1 \times a = a$.

Rule 6: In a string of multiplications, it does not matter in which order one conducts individual multiplications.

Rule 7: For any number a we have $a \times 0 = 0$ and $0 \times a = 0$.

Rule 8: "We can chop up rectangles from multiplication and add up the pieces."

Rule 9: For each number a, there is one other number "$-a$" such that $a + -a = 0$.

Some Logical Consequences of Rule 9: For any two numbers a and b

i) $-0 = 0$
("The opposite of zero is zero")

ii) $- - a = a$
("The opposite of the opposite is back to the original")

iii) $-(a+b) = -a + -b$
(We can "distribute a negative sign")

iv) $(-a) \times b$ and $a \times (-b)$ and $-(a \times b)$ all have the same value
(We can "pull out a negative sign")

v) $(-1) \times a = -a$
("Multiplying by -1 gives you the opposite number")

Rule 10: For each non-zero number a there is one number, $\frac{1}{a}$ such that $a \times \frac{1}{a} = 1$.
(That is, we can fill in the blank to $a \times \blacksquare = 1$.)

Convention: For a number a and a non-zero number b, the notation $\frac{a}{b}$ is shorthand for $a \times \frac{1}{b}$.

Some Logical Consequences of Rule 10:

i) $\boldsymbol{\frac{a}{1} = a}$ ("We can put numbers over 1.")

ii) $\boldsymbol{\frac{a}{a} = 1}$ for a not zero ("We can write 1 in many forms.")

iii) $k \times \frac{a}{b} = \frac{ka}{b}$ for b not zero

iv) $\boldsymbol{b \times \frac{a}{b} = a}$ for b not zero ("We can cancel a denominator.")

v) $\boldsymbol{\frac{ka}{kb} = \frac{a}{b}}$ for k and b each not zero ("We can simplify fractions.")

vi) **ADDING/SUBTRACTING FRACTIONS**: $\frac{a}{N} \pm \frac{b}{N} = \frac{a \pm b}{N}$ for N not zero

vii) **MULTIPLYING FRACTIONS**: $\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d}$ for b and d each not zero

viii) **PULLING OUT NEGATIVE SIGNS:** $-\frac{a}{b} = \frac{-a}{b} = \frac{a}{-b}$ for b non-zero

ADDING/SUBTRACTING FRACTION WITH DIFFERENT DENOMINATORS
Either rewrite each fraction using v) to create a common denominator, or just put the quantity over 1 and use v).

DIVIDING FRACTIONS Rewrite $\frac{a}{b} \div \frac{c}{d}$ as $\frac{\frac{a}{b}}{\frac{d}{c}}$ and use v).

And finally …

ix) $\boldsymbol{a \div b}$ **and** $\boldsymbol{\frac{a}{b}}$ **are the same number**, assuming b is not zero, or course.

Point ix) is just a reinterpretation of iv).