Chapter 2

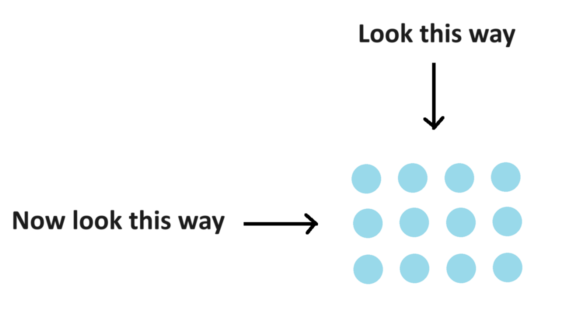
Playing with the Counting Numbers

Chapter 2

15. The Power of a Picture

Now that we've got the counting numbers and their basic arithmetic sorted out, let's engage in some sophisticated mathematical play and mathematical thinking.

We've already seen how changing one's perspective of a picture can lead to mathematical insight. The goal of this section is to do more of the same.



$$4 \times 3 = 3 \times 4$$

Let's start with this picture of 25 dots.

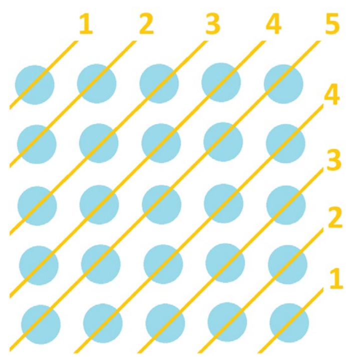
The dots are arranged as five groups of five via rows and simultaneously as five groups of five via columns. It's a picture of 5×5 two ways. That symmetry is kinda cool in-and-of itself. But can you look at this picture another way and see within it the sum of all the counting numbers from 1 up to 5 and back down again?

$$1 + 2 + 3 + 4 + 5 + 4 + 3 + 2 + 1$$

Mull on this before turning the page.

Look at the figure diagonally. That up-and-down sum then comes into view.

As all 25 dots are covered by the diagonals, we can say that the value of that sum just has to be $5 \times 5 = 25$. We don't have to do a lick of arithmetic!



But Check!
Do the arithmetic to verify that $1 + 2 + 3 + 4 + 5 + 4 + 3 + 2 + 1$ does indeed equal 25.

Practice 15.1: Draw a picture of 4 dots that illustrates the sum $1 + 2 + 1$ via diagonals.
Draw a picture of 9 dots that illustrates the sum $1 + 2 + 3 + 2 + 1$ via diagonals.

We're on to a general idea.

A picture of a 10-by-10 square of dots looked at diagonally would show the sum of numbers from 1 up to 10 and back down again. As there are $10 \times 10 = 100$ dots in a ten-by-ten square of dots, this sum must have value 100.

$$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 + 9 + 8 + 7 + 6 + 5 + 4 + 3 + 2 + 1 = 10 \times 10 = 100$$

The sum of all the numbers from 1 up to 1000 and back down again appears as the diagonals of a 1000-by-1000 square of dots. There are $1000 \times 1000 = 1{,}000{,}000$, a million, dots in such a picture. We must have

$$1 + 2 + 3 + \cdots + 1000 + \cdots + 3 + 2 = 1 = 1000 \times 1000 = \text{a million.}$$

And we're doing this without a lick of arithmetic! (Well, we did have to know that a thousand times a thousand is a million.)

Practice 15.2: What is the sum of all the numbers from one up to a million and back down again?

Imagine how long it would take to conduct these ludicrously large sums on a calculator. The mind's eye is mightier than the machine here!

If it is fun to note, here's a way to express what we just discovered as a general result. (Don't memorize this as important. This is all just for mathematical amusement.)

> The sum of the numbers from 1 up to a counting number N and back down again equals $N^2 = N \times N$.
>
> $$1 + 2 + 3 + \cdots + N + \cdots + 3 + 2 + 1 = N^2$$

Let's keep going!

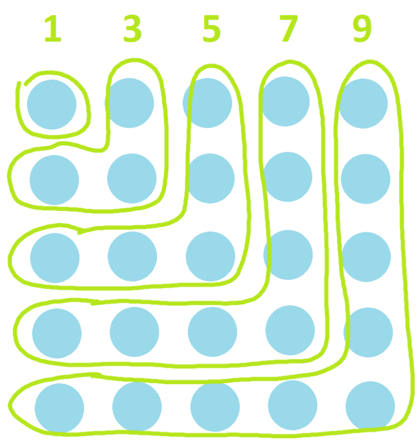
Going back to our five-by-five square of dots, can you look at the picture yet a different way and see the sum of the first five odd numbers $1 + 3 + 5 + 7 + 9$? (We haven't officially discussed even and odd numbers yet, but we shall in the next section. We're just a smidge ahead of ourselves here. I hope that is okay.)

$$1 + 3 + 5 + 7 + 9$$

Again, mull for a while before turning the page.

One can certainly randomly circle groups of 1, 3, 5, 7, and 9 dots in the square array of dots. But we want an approach that generalizes beyond just playing with this one picture.

The key this time is to look at boomerangs. (I am Australian, by the way.)



As the boomerangs I've drawn cover each and every dot, it must be that the sum $1 + 3 + 5 + 7 + 9$ has value $5 \times 5 = 25$, the number of dots in the picture.

Check It! Confirm that $1 + 3 + 5 + 7 + 9$ does indeed equal 25.

Practice 15.3: Draw a picture of 4 dots that illustrates the sum $1 + 3$ via boomerangs. Draw a picture of 9 dots that illustrates the sum $1 + 3 + 5$ via boomerangs.

In the same way, the sum of the first eight odd numbers comes from boomerangs covering an 8-by-8 square of dots.

$$1 + 3 + 5 + 7 + 9 + 11 + 13 + 15 = 8 \times 8 = 64$$

And the sum of the first one-hundred odd numbers must be $100 \times 100 = 10{,}000$. Wow!

$$1 + 3 + 5 + 7 + \cdots + 197 + 199 = 100 \times 100 = 10000$$

In general:

The sum of the first N odd numbers is $N^2 = N \times N$.

Now, having summed odd numbers, it is natural to ask:

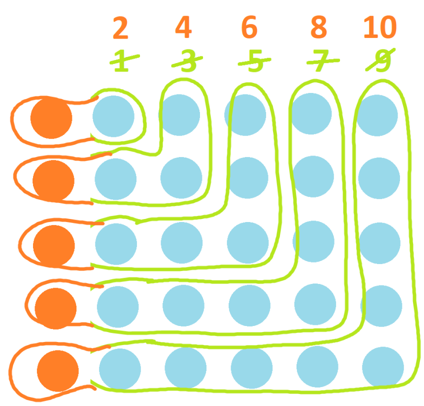
What is the sum of the first N even numbers?

Let's play with what we first learned: the sum of the first five odd numbers is twenty-five. What if we bumped each odd number up by one to get to the even numbers?

$$\underset{+1}{1} + \underset{+1}{3} + \underset{+1}{5} + \underset{+1}{7} + \underset{+1}{9} = \underset{+5}{5 \times 5}$$

$$2 + 4 + 6 + 8 + 10 = 5 \times 5 + 5$$

This is the same as adding an extra dot to each boomerang.



$$2 + 4 + 6 + 8 + 10$$

The picture now looks like a five-by-five square of dots plus five more dots. That makes for $5 \times 5 + 5 = 30$ dots. (We can also regard the picture as a 5-by-6 array of dots and, again, 5×6 equals 30).

$$2 + 4 + 6 + 8 + 10 \quad = 5 \times 5 + 5 \quad = 30$$

Practice 15.4: Draw a picture of $2 \times 2 + 2 = 6$ dots that illustrates the sum $2 + 4$ via boomerangs.

Draw a picture of $3 \times 3 + 3 = 12$ dots that illustrates the sum $2 + 4 + 6$ via boomerangs.

Practice 15.5: What sum of even numbers comes from a picture of $7 \times 7 + 7$?

We're seeing structure here.

The sum of the first six even numbers must equal $6 \times 6 + 6 = 42$.

$$2 + 4 + 6 + 8 + 10 + 12 = 6^2 + 6$$

The sum of the first ten even numbers must equal $10 \times 10 + 10 = 110$.

$$2 + 4 + 6 + 8 + 10 + 12 + 14 + 16 + 18 + 20 = 10^2 + 10$$

The sum of the first one-hundred even numbers must equal $100 \times 100 + 100 = 10{,}100$.

$$2 + 4 + 6 + \cdots + 200 = 100^2 + 100$$

In general:

The sum of the first N even numbers is $N^2 + N$.

Again, nothing to memorize or to keep in your head here. We're just playing.
But we are seeing just how powerful playing with simple pictures can be.

To finish things off for right now, I need to point out something that is a little odd. (Ha!)

We found a quick way to add together the first few odd numbers.

For example, the sum of the first five odd numbers is $1 + 3 + 5 + 7 + 9 = 5 \times 5 = 25$.

We found a quick way to add together the first few even numbers.

For example, the sum of the first five even numbers is $2 + 4 + 6 + 8 + 10 = 5 \times 5 + 5 = 30$.

But we don't yet have a quick way to add together just the first few counting numbers directly.

For example, what is $1 + 2 + 3 + 4 + 5$? What is $1 + 2 + 3 + \cdots + 100$?

Practice 15.6: Here is the sum of the first five even numbers.

$$2 + 4 + 6 + 8 + 10 = 30.$$

Can you see a way to use this sum to figure out the value of $1 + 2 + 3 + 4 + 5$?

Actually, let me answer that question right now. Just halve everything we have in $2 + 4 + 6 + 8 + 10 = 30$. That will give us $1 + 2 + 3 + 4 + 5 = 15$, just like that!

Practice 15.7: The sum of the first twelve even numbers is $12 \times 12 + 12$, which equals 156.

$$2 + 4 + 6 + 8 + 10 + 12 + 14 + 16 + 18 + 20 + 22 + 24 = 156.$$

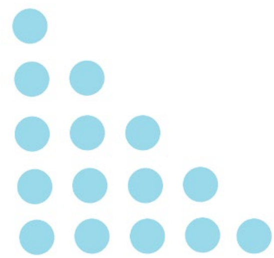
What is the value of the sum $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 + 11 + 12$?

In general, since the sum of the first N even numbers is $N^2 + N$, the sum of the first N numbers is "half of $N^2 + N$."

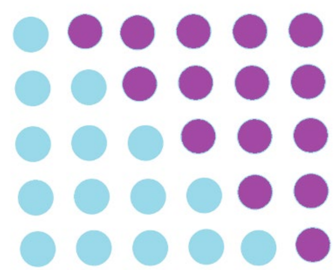
The sum of the fist N even numbers is $N^2 + N$.
The sum of the first N numbers is half of this.

Practice 15.8: Here's another approach to seeing the sum of the first few counting numbers.

I'll start with this picture of 1 dot and 2 dots and 3 dots and 4 dots and 5 dots.
It's a picture of $1 + 2 + 3 + 4 = 5$. (Do you see this via rows?)



Here's two copies of $1 + 2 + 3 + 4 + 5$ together.



a) Can you see that this new picture is telling us that $1 + 2 + 3 + 4 + 5$ is half of $5 \times 6 = 30$?
b) Draw a picture to show that $1 + 2 + 3$ is half of 3×4.
c) Draw a picture to show that $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10$ is half of 10×11. (Or, can you imagine one at least?)

Is what we are seeing here the same as the result on the previous page?

d) Are "half of 5×6" and "half of $5^2 + 5$" the same?
e) Are "half of 3×4" and "half of $3^2 + 3$" the same?
f) Are "half of 10×11" and "half of $10^2 + 10$" the same?
g) Are "half of $N \times (N + 1)$" and "half of $N^2 + N$" the same?

MUSINGS

Musing 15.9

a) On the first day of April, Joi adopts one puppy. On the second day, she adopts two puppies, three on the third day of April, and so on, all the way up to adopting 30 puppies on the 30^{th} day of the month. How many puppies in total did Joi adopt that month?

b) Alberto adopted one kitten on the first day of the year, two on the second day of the year, and so on, up to adopting 365 kittens on the 365^{th} day of the year. How many kittens did he adopt in total that year?

c) If each year on your birthday you were given a birthday cake with as many candles on it as your new age, how many birthday candles would you have blown out in your life?

Musing 15.10

a) What is the sum of the first million odd numbers?
b) What is the sum the first million even numbers?
c) What is the sum of the first million (counting) numbers?
d) **Challenge:** $1{,}000{,}000$ is the millionth counting number. What is the millionth even number?
e) **Challenge:** $1{,}000{,}000$ is the millionth counting number. What is the millionth odd number?

Musing 15.11

a) What is the value of $1 + 3 + 5 + 7 + \cdots + 97 + 99$?
b) What is the value of $2 + 4 + 6 + 8 + \cdots + 498 + 500$?
c) What is the value of $73 + 74 + 75 + 76 + \cdots + 98 + 99 + 100$?
d) What is the value of $512 + 514 + 516 + \cdots + 798 + 800$?

Musing 15.12 Here's another way to compute the sum of the first five counting numbers.

Write the sum twice, in two rows, but with the first sum forwards and the second sum backwards.

$$\begin{array}{rr} 1+2+3+4+5= & \text{answer} \\ 5+4+3+2+1= & \text{answer} \\ \hline 6+6+6+6+6= & 2 \times \text{answer} \end{array}$$

Adding by columns shows that five copies of 6 equals twice the answer. That is, we see 5×6 equals double the answer. This means that $1 + 2 + 3 + 4 + 5$ equals "half of 5×6" (as we learned before).

a) Draw a diagram like the one above that shows that the sum of first one-hundred counting numbers must be half of 100×101.

b) Use this approach to evaluate $4+7+10+13+16+19+22+25+28+31+34+37+40+43+46$.

c) **ABSOLUTELY OPTIONAL:** (But everything in this book is optional!)

Can you make sense of the following statement?

The sum of N evenly-spaced counting numbers equals half of N times the sum of the first and last numbers in the sum.

(For instance, $4+7+10+13+16+19+22+25+28+31+34+37+40+43+46$ is a sum of fifteen evenly-spaced counting numbers.)

MECHANICS PRACTICE

As mentioned, nothing in this section is "important" in, and of, itself.
But we have already played with some of the ideas of this section back in section 6 where we noted that "order does not" matter when computing a long list of additions. That idea is certainly important and helpful.

Practice 15.13 a) Can you see that the sum of all the numbers from 1 up to 8 and back down again,

$$\mathbf{1+2+3+4+5+6+7+8+7+6+5+4+3+2+1}$$

is the same as $8+8+8+8+8+8+8+8$? (That's eight copies of 8 added together and so the answer is $8\times 8 = 64$.)

b) Can you see that the sum of the first eight odd numbers,

$$\mathbf{1+3+5+7+9+11+13+15}$$

is the same as $16+16+16+16$? (Is that answer the same as 8×8, eight copies of 8 added together?)

c) Can you see that the sum of the first eight even numbers,

$$\mathbf{2+4+6+8+10+12+14+16}$$

is the same as $18+18+18+18$? (Is that answer the same as eight copies of 9 added together, which would be 8×9?)

d) Can you see that the sum of the first eight counting numbers,

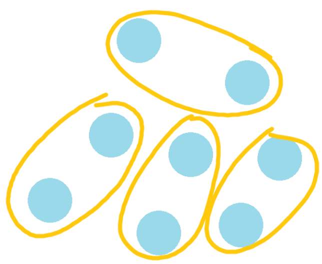
$$\mathbf{1+2+3+4+5+6+7+8}$$

is the same as $9+9+9+9$? (This is half of $9+9+9+9+9+9+9+9=8\times 9$.)

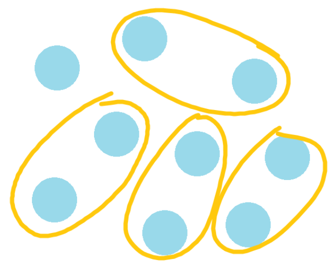
16. Even and Odd Counting Numbers

We played with sums of even and odd numbers in the last section, which was presumptuous of me: I assumed we all knew what even and odd numbers are. Let's talk about them formally now.

> A count of dots is said to be **even** if a set of that many dots can be grouped into pairs without any dots left remaining. (For example, 8 is even as we can group eight dots into pairs of two.)
>
> A count of dots is said to be **odd** if attempting this feat with that many dots always leaves one dot remaining. (For example, 9 is odd.)

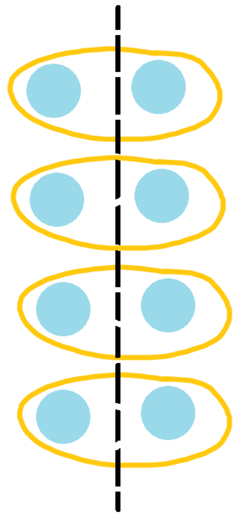


8 is even



9 is odd

Some people prefer to say that a counting number is *even* if we can split a group of that many dots into two equal-sized piles, and *odd* if this cannot be done. (We are assuming here that dots themselves cannot be split.)



8 is still even

Do you see that this is an equivalent definition of "eveneness"?

If we can split a group of dots into two piles of equal size, then move one pile to the left and the other pile to the right. Now we can make pairs by selecting left-and-right dots in turn.

And backwards, if we have a collection of pairs with no dot left over, then we can just split each pair, moving one dot in the pair to the left and the other to the right. This will give you two equal-sized piles in the end.

Practice 16.1: Here's an alternative way to envision matters.

You have two feet.

Imagine you have a huge box of socks. You start putting the socks on, one at a time, first on your left foot, then your right foot, then your left foot, then your right foot, and so on. (You'll have a lot of socks on each of your feet!)

a) Suppose you find at the end that you have the same number of socks on each foot. What does that say about the count of socks that were in the box?

b) If, instead, the count of socks in the box were odd, what would you notice about the count of socks on each of your feet? [Which foot has the greater number of socks?]

Zero

The number zero is always philosophically troublesome!

Is zero even or odd? Neither or both?

If you have zero dots, you can certainly split that collection into two piles of equal size, namely, two empty piles! Zero fits the second definition of evenness.

You could also argue that it readily fits the first definition of eveneness too: From a set of zero dots, one can form zero pairs of dots and, indeed, there are no dots left over.

For this reason(s), zero is considered to be an even number.

Practice 16.2: Is the number 1 even or odd? Convince me that your answer to this is correct.

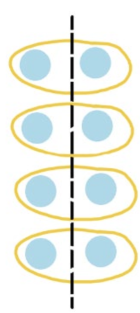
Pratice 16.3:
a) A collection of dots was grouped into seven pairs with no dot left over. How many dots were there? Is that an even or an odd number?

b) Another collection of dots was grouped into 100 pairs with one left over. How many dots were there? Is that an even or an odd number?

c) A third collection of dots was split into two piles of 13 dots each and one dot was left over. Was the original number of dots even or odd?

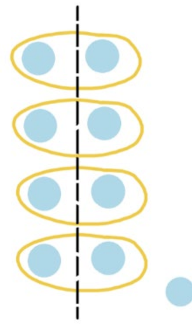
Continuing on …

Every even count of dots can be split into two groups of equal size: just split the pairs in half like we saw before. For example, 8 matches two groups of four: $8 = 2 \times 4$.



Every even number is two times a counting number.

An odd count of dots leaves an extra dot over if we split into two groups of the same size. For example, 9 is one more than two groups of four: $9 = 2 \times 4 + 1$.



Every odd number is one more than two times a counting number.

Some people prefer to use these statements as the definitions for numbers being even or odd. It leads to a more technical-looking approach, but the mathematics is really saying the same thing.

*A counting number N is **even** if we can write $N = 2a$ for some counting number a.*
*A counting number N is **odd** if we can write $N = 2a + 1$ for some counting number a.*

We can be definitive too now about the evenness and oddness of numbers.

Is 0 even or odd?

It is even because 0 equals two times a counting number, namely, it equals 2×0.

Is 1 even or odd?

The number 1 is odd because it equals $2 \times 0 + 1$. (It's one more than double a counting number.)

Is 14 even or odd?

It is even because $14 = 2 \times 7$.

Is 29 even or odd?

It is odd because $29 = 2 \times 14 + 1$.

And so on.

Practice 16.4: Write each of the following numbers either as “two times a counting number” or as “two times a counting number plus one.”

50	51	100	121	A MILLION

Practice 16.5:
a) If a counting number is even, explain why the next counting number sure to be odd.
b) If a counting number is odd, explain why the next counting number sure to be even.

Adding and Subtracting Even and Odd Numbers

The sum of two even numbers is even. We can see why.

If one set of dots grouped into pairs is combined with a second set of dots grouped into pairs, then the resulting conglomeration of dots is a set of pairs with no dot left over.

The sum of an even number and an odd number is odd.

If one set of dots grouped into pairs is combined with a second set of dots grouped into pairs but with one extra dot, the resulting conglomeration of dots is a set of pairs with one extra dot.

The sum of an odd number and an even number is odd.

We can argue just as for the previous example. (But if you want to be particularly “math sophisto,” cite the fact that $a+b=b+a$ *for all counting numbers. Then you can say that this case is really no different from the previous case.)*

Your turn …
Practice 16.6: Explain why the sum of an odd number and an odd number is even.

To summarize, we have this chart.

EVEN + EVEN = EVEN
EVEN + ODD = ODD
ODD + EVEN = ODD
ODD + ODD = EVEN

Practice 16.7: We haven't yet talked about "taking away" (aka, subtraction), but can you convince yourself of these claims too? (We'll examine this properly in Chapter 3.)

EVEN - EVEN = EVEN
EVEN - ODD = ODD
ODD - EVEN = ODD
ODD - ODD = EVEN

Practice 16.8: Will the value of a sum of the following form be even or odd?

EVEN + EVEN + ODD + EVEN + ODD + ODD + ODD + EVEN + ODD

Practice 16.9: Optional Challenge Exercise not for Everybody's Taste
Here's another explanation as to why the sum of two even numbers is even.

Suppose we have two even numbers N and M.

Then we can write $N = 2a$ and $M = 2b$for some counting numbers a and b.

The sum of these two numbers is then

$$N + M = 2a + 2b.$$

By "factoring out a 2" this can be rewritten as

$$2a + 2b = 2(a + b).$$

Thus $N + M = 2(a + b)$. This is also twice a counting number, and so $N + M$fits the definition of also being even.

a) Can you follow the explanation presented? Do you like it? (I personally prefer imagining pairs of dots being combined. But some people like this approach as it is "more mathematical," whatever that means to them.)

b) Are you game for writing a similar explanation as to why the sum of an even number and an odd number is odd?

c) How about writing out an explanation like this for the sum of two odd numbers being even?

Going back to however you like to think of matters, can you convince yourself of the following general result? (Warning: It takes a moment or two to figure out first what exactly each sentence is saying!)

Any sum of counting numbers that involves an even number of odd numbers will be even. Any sum of counting numbers that involves an odd number of odd numbers will be odd.

A sharing moment:

Let me reveal my brain here. It might help.

On the previous page I asked whether a sum of the form

EVEN + EVEN + ODD + EVEN + ODD + ODD + ODD + EVEN + ODD

will give an even or odd answer. I personally look at this and think:

> pairs + pairs + (pairs and **1 extra dot**) + pairs + (pairs and **1 extra dot**) + (pairs and **1 extra dot**) + (pairs and **1 extra dot**) + pairs + (pairs and **1 extra dot**)

That's a whole lot of pairs and 5 extra dots, which can make two extra pairs with 1 dot remaining.

The result is loads of pairs of dots and 1 extra dot. That is, the result is odd.

> **Question:** Can you indeed "see" that any sum involving an odd number of odd numbers will give an odd-numbered answer? And that a sum with an even number of odd numbers will be even?

I think the results on the previous page are astonishingly powerful and astounding!

Let me give some examples to show why I think this.

Example: If you dare to secretly tear twenty random pages out of this book and destroy them, I will then react by telling you that the sum of the missing page numbers is even—and I know I will be right about that without even looking at my mutilated tome!

Why? Each page of a book has an odd page number on one side and an even page number on the other. The sum of the forty missing page numbers is sure to be a sum of twenty even numbers and twenty odd numbers. There is an even number of odd numbers in this sum, and so the sum of the missing page numbers is sure to be even!

Example: Using only pennies (1 cent coins), nickels (5 cent coins), and quarters (25 cent coins), it is impossible to make change for a dollar using precisely fifteen coins.

Why? Fifteen odd values of cents must add to odd value. (An odd number of odd numbers added together.) One dollar--100 cents--on the other hand, is an even number of cents. This task cannot be done because of this mismatch.

Example: The number 699 is the sum of six consecutive counting numbers.
It's $114 + 115 + 116 + 117 + 118 + 119$.

Now write 25,608 as a sum of six consecutive counting numbers too.

Don't bother trying!
In any string of six consecutive counting numbers there will be 3 odd numbers and 3 even numbers. A sum with an odd number of odd numbers in it is sure to be odd. As, 25,608 is an even number, the task ask for is impossible!

Alright. These three examples might now be that exciting. But consider these next four hands-on activities.

ACTIVITY

HANDSHAKE CHALLENGE

It is impossible for an odd number of people in a room to each take part in an odd number of handshakes.

In a room with friends, count how many people are there. If the count is odd, including yourself, great! If it is even, then explain in a moment that you are just going to be a facilitator for the activity and won't take part. (Just have your odd count of friends play.)

Either way, explain the activity:

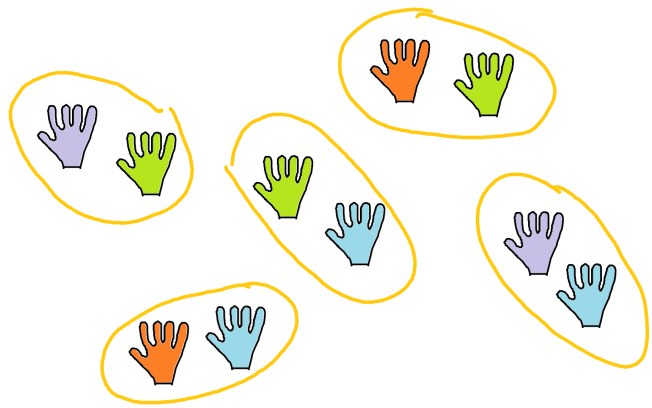
> *Let's all get to know each other by greeting each other and shaking hands. You must greet at least one person, and you may greet the same person more than once. But everyone is to take part in an odd number of handshakes in total. (So, if you shake the same person's hand more than once, you must keep track of those individual counts too.)*
>
> *Once you have taken part in a count of handshakes, any odd number of them you like, feel free to fold your arms to say: "I'm done!". (But know, it is impolite to refuse shaking a hand if someone offers a hand to you.)*
>
> *Okay, let's try it!*

It is impossible for you and your friends to complete this task. Lots of frivolity (usually) happens when people actually attempt this doomed task.

Here's the reason why the task is doomed.

Imagine doing the following:

Each time a handshake occurs, draw a picture of the two right hands that took part in the shake.



At the end of the activity—assuming the task is possible—you will have an **even number of hands drawn on your page**. (The hands are coming in pairs, after all.)

Now, show the picture to all the participants. Each participant will recognize their own hand an odd number of times if each participant really did take part in an odd number of shakes.

As there are an odd number of participants, this means that your picture is composed of an odd number of odd counts of hands. But that makes for an **odd number of hands drawn on your page** in total, not an even total.

We have a mismatch. Someone must have miscounted as an odd number of odd counts cannot add to an even result.

ACTIVITY

CUP-TURNING CHALLENGE

Thirteen cups are placed upside-down on a tabletop.

A "move" consists of choosing two cups and turning them each over. (Cups that are upside down will be turned upright and cups that happen to be upright will be turned back upside down.) Individual cups can be turned over multiple times during the play of this game.

Your job: Turning two cups over at a time, reach a state in which every cup is upright.

[Rather than using cups, try this puzzle using playing cards.]

Here's the reason why this task too is doomed.

In order for a cup to end up upright, it must be turned over an odd number of times in total. (Think about this.) So, with 13 cups, we need to perform an odd numbers of turns thirteen times. Thirteen odd counts add to an odd number of turns in total.

But by being forced to conduct two cup turns at a time, we will never be able to reach an odd number of turns: 0 turns, 2 turns, 4 turns, 6 turns, and so on.

This task described is impossible to complete.

Question: Let's keep playing!

a) Is it possible to solve the cup-turning challenge starting with 12 cups instead of 13? Starting with an even number of cups, can the challenge always be solved?

b) Is it possible to invert 14 cups turning FOUR over simultaneously at a time?

c) Is it possible to invert 15 cups turning FOUR over simultaneously at a time?

Extra Challenge:

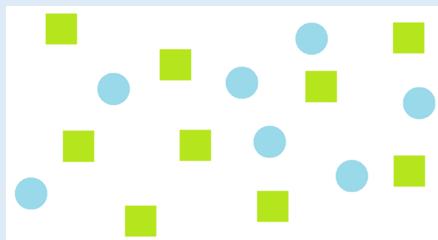
You have some upside-down cups in a row. Now a "move" consists of putting your finger on one cup, keeping it fixed in place, and turning over each of the remaining cups. Your goal is to, after some number of moves, end up with all cups upright.

For which starting counts of cups can this version of the cup-turning puzzle be solved?

ACTIVITY

CIRCLES AND SQUARES GAME

Here is a diagram of some circles and squares.



Edith and Amit will play the following game. They will each take turns choosing two symbols, erasing them, and replacing them with a single symbol determined by the following rule.

ERASE TWO IDENTICAL SHAPES → DRAW A SQUARE
ERASE TWO DIFFERENT SHAPES → DRAW A CIRCLE

The count of symbols on the page decreases by one at each turn and so eventually just one symbol will remain of the page.

If that final symbol is a circle, Edith wins.
If that final symbol is a square, Amit wins.

Play this game a few times. Who wins? Play it several more times. Who keeps winning?

Can you explain what is going on?

Here's why Edith is sure to win the game, even if Amit takes all the turns.

Focus on the count of circles on the page. We are starting with 7 of them.

A move either erases two circles completely or leaves the count of circles the same. (Again, just focus on the circles.) Thus, the count of circles on the page will eventually decrease to 5 and then to 3 and then to 1.

At the end of the game, there is one symbol left. It must be that one circle and Edith wins.

Question: Is the count of squares on the page at all relevant?

Question: Explain why Edith is always sure to win if game starts with any odd number of circles on the page.

Question: If the initial count of circles on the page is instead even, is Amit sure to win?

ACTIVITY

GNOMES, HATS, AND AN EVIL VILLAIN.

Ten gnomes are about to play a game of life and death with an evil villain. They are told that they will be asked to stand in a line, each facing the back of the next, and that hats will be placed on their heads. Each hat will be either black or red, but no indication of how many hats there will be of each color is given. No gnome will be able to see the color of his own hat, but each gnome will be able to see the colors of all the hats in front of him.

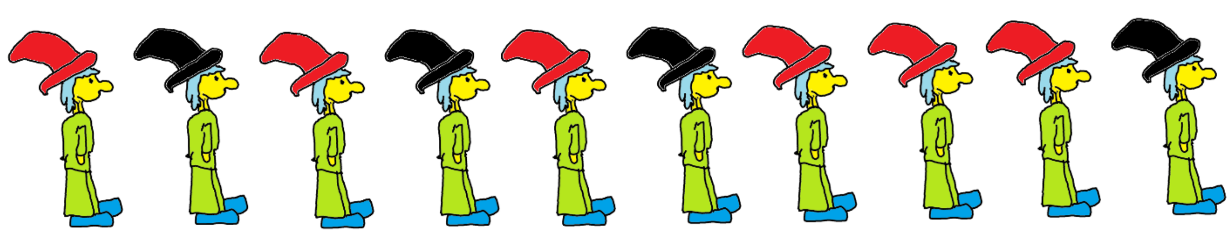
Starting with the gnome at the back of the line (the one who can see nine hats in front of him), and working along the line, the evil villain will ask each gnome in turn to guess the color of his own hat. If a gnome is correct, he will live. If not, he will meet his demise.

Gnomes will not be allowed to speak during the play of this game except for a single word, either red or black when asked, but they will be able to hear the answers – and the subsequent sighs of relief or the screams of horror – of the gnomes behind them. No other information in any shape or form will be transmitted from gnome to gnome.

What scheme could the gnomes agree on before the play of this game that would ensure the survival of a maximum of their number?

Comment 1: The very first gnome is an unfortunate position: No scheme could ever assure him of his survival. But is there a scheme that would absolutely allow for the survival of, say, the next gnome in line? How about every second gnome? Is there a way to ensure the survival of *more* than half the gnomes?

Comment 2: When you do find this special strategy, it is fun to practice it with a group of nine friends. Use playing cards to hold up high, facing backwards along the line, instead of hats.



Surprisingly, it is possible for the team to **ensure the survival of nine out of ten gnomes** (along with a 50% chance of the survival of the first gnome).

Here's how:

> *The gnomes agree on the following strategy:*
>
> *The brave gnome at the front will say "black" if he sees an even number of black hats among the nine hats in front of him, "red" otherwise.*
>
> *Each gnome among the nine will thus know whether there is an even or odd number of black hats among them. Given the number of black hats they see in front of them, and the number of times they hear the word "black" behind them, they can each correctly deduce the color of their own hat.*

In the picture on the previous page, the leftmost gnome will start and will say BLACK as he sees 4 black hats ahead of him, and even count of them. Unfortunately, he himself won't survive (his hat is red), but he has ensured the survival of the remaining nine gnomes.

The next gnome in the line, second from left, sees 3 black hats in front of him. He was told, however, that there are an even number black hats among the nine gnomes, so he deduces that his hat must be black. He says BLACK and survives.

The next gnome in line, third from the left, sees 3 black hats in front of him. He just heard that the hat behind him was black (that accounts for 4 black hats now) and he heard right at the start that there are an even number of black hats among the nine gnomes. The number 4 is even, so he deduces that his hat is red. He says RED and survives.

> **Question**: The next gnome in line, fourth from the left, sees just 2 back hats in front of him. How will he reason to deduce that his hat is black?
>
> Keep going! Explain the reasoning of each gnome as you work down the line.

This strategy works for any number of gnomes in a line.
If 1,000 gnomes play this game, for example, they can be sure of the survival of 999 of them.

That's astounding! And it's just the power of playing with even and odd numbers.

MUSINGS

Musing 16.10

a) Is the sum of seventeen even numbers and thirty-one odd numbers even or odd?
b) Is the sum of the first seven hundred odd numbers even or odd?

Musing 16.11
Is it possible to arrange one-hundred-and-fifty marbles, one of weight 1 gram, one of weight 2 grams, one of weight 3 grams, and so on up to one of weight 150 grams, into two piles of equal total weight?

Musing 16.12
Is 94315 the answer to a sum of one hundred consecutive counting numbers? If so, which sum of counting numbers?

Musing 16.13 In the world of counting numbers, multiplication is repeated addition. Explain then each of the following four claims.

EVEN x EVEN = EVEN
EVEN x ODD = EVEN
ODD x EVEN = EVEN
ODD x ODD = ODD

Musing 16.14 Explain why, if n^2 is even, then n cannot be odd (and so, must be even as well).

Musing 16.15 Explain why the following astounding claim must be true.

> **Right at this very instant, the count of people—living or deceased—who have taken part in an odd number of handshakes is even.**

Musing 16.16
The country of Oddonia only uses 3-cent and 5-cent coins.

a) Is it possible to make change for a dollar (one-hundred cents) using precisely 31 coins in Oddonia?

It is possible to buy something for 1 cent in Oddonia: give the shop clerk two 5-cent coins and receive three 3-cent coins in change, for example.

b) Is it possible to buy an item priced at 2 cents in Oddonia? 4 cents? 7 cents?
Is there a price that cannot be paid in Oddonia?

Musing 16.17 Three counting numbers are chosen at random.

a) Must there be two numbers among them whose sum is even? Explain.
b) Must there be two numbers among them whose sum is odd? Explain.

Musing 16.18 There are 25 people at a meeting. When they greeted each other, could it be that ten of them took part in exactly 4 handshakes, eleven of them in exactly 5 handshakes, and four of them in exactly 6 handshakes? Explain.

Musing 16.19

A **magic square** is a square array of counting numbers with the property that all numbers in the same row, or the same column, or along either of the two diagonals sum to the same value. For example, the picture shows the classic example of a three-by-three magic square with "magic sum" 15. (This magic square happens to use the whole numbers 1 through 9, but a magic square is allowed to repeat values.)

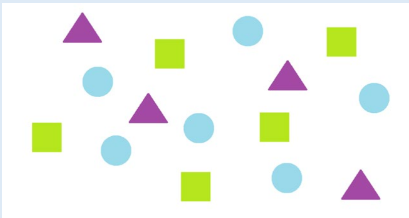
2	9	4
7	5	3
6	1	8

In the classic example, **four** of the numbers are even.

It is possible to create a three-by-three magic square with all **nine** entries even: just put the number 2 in all nine cells, for example. (The magic sum will be 6.) And putting the number 1 in each and every cell creates a magic square with **zero** even entries. (The magic sum here is 3.)

a) Explain why it is impossible to construct a three-by-three magic square with just **one** even entry.
b) Explain why it is impossible to construct a three-by-three magic square with **two** even entries.
c) Can you create a three-by-three magic square with exactly **three** even entries?

Musing 16.20 Here's another game of solitaire. Start with the following diagram of squares, circles, and triangles.



A "move" consists of erasing any two figures of different shape and drawing in their stead a single figure of the third shape. (For example, after erasing a square and a triangle, one is to draw a circle.) Keep going until no more valid moves are possible.

a) Explain why this game is sure to end with nothing but squares left on the page.
b) Why is a game that starts with 107 squares, 225 circles, and 306 triangles on the page sure to end with nothing but triangles?

Hint: The count of each type of shape changes by one with each move. The game will end when the count is zero (an even number) for two shapes.

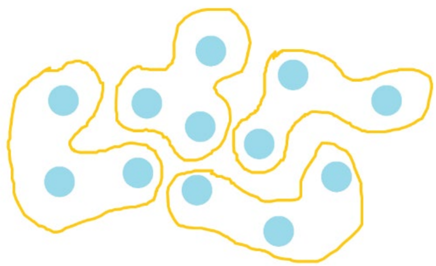
17: Division

We started the last section by saying that a number is **even** if a set of that many dots can be grouped into pairs—sets of 2—without any dots remaining. (And it is **odd** if this feat cannot be done.)

We can extend this idea to beyond focusing on pairs of dots.

> A number is **divisible by 3** (or "can be divided by 3") if a count of that many dots can be grouped into sets of 3 with no dots remaining.

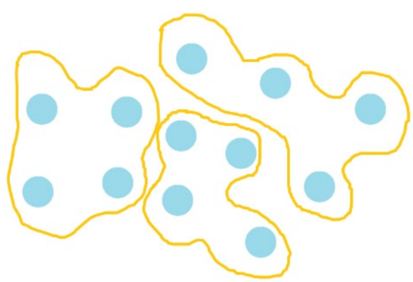
For example, a count of 12 dots can be divided into four sets of 3, and so 12 is divisible by 3.



12 is divisible by 3

A number is **divisible by 4** (or "can be divided by 4") if a count of that many dots can be grouped into sets of 4 with no dots remaining.

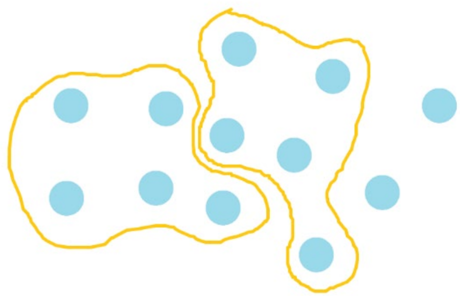
For example, a count of 12 dots can also be divided into three sets of 4, and so 12 is divisible by 4.



12 is also divisible by 4

And, in general, for any counting number N we say:

A number is **divisible by N** ("can be divided by N") if a count of that many dots can be grouped into sets of N dots with no dots remaining.



12 is not divisible by 5

In 1659, Swiss mathematician Johann Rahn used the symbol ÷ (called an **obelus**) in this context of divisibility. He would write

$$12 \div 3 = 4$$

to indicate that 12 objects, when divided into sets of 3, yields 4 such sets. (Though, saying this backwards feels a little more natural: "There are 4 groups 3 among 12 objects.")

Similarly, writing

$$12 \div 4 = 3$$

indicates that dividing set of 12 objects into sets of 4 yields 3 such sets.
("There are 3 groups 4 in a picture of 12 objects.")

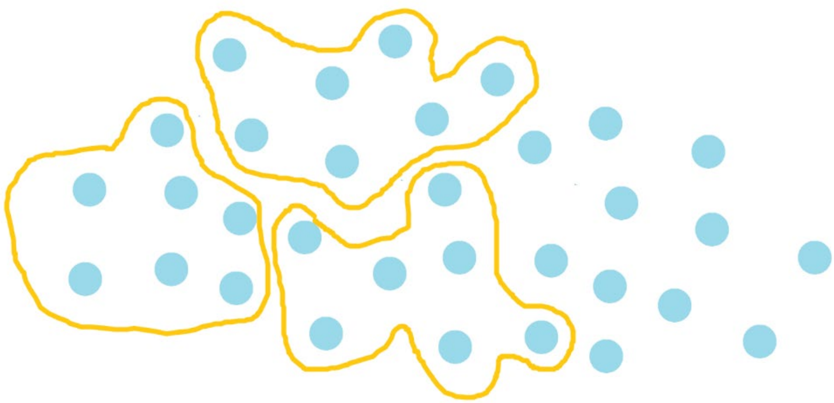
In general, a computation of the form $a \div b$ seeks to know how many groups of b objects one can find within a set of a objects, hopefully without any objects left over. If a set of objects remains unaccounted for, then we say we have a **remainder**.

For example, the picture at the top of this page shows that:

12 leaves a remainder of 2 upon division by 5.

People usually prefer to make the remainder in any division problem is as small as possible. For instance, this picture shows that

32 leaves a remainder of 11 upon division by 7.



This statement is absolutely correct, but it might not be seen as efficient as one can identify yet another group of 7 among the remaining dots. Stating

32 leaves a remainder of 4 upon division by 7

is another correct statement and is probably the preferred one.

Practice 17.1: Quentin was feeling cheeky and wrote:

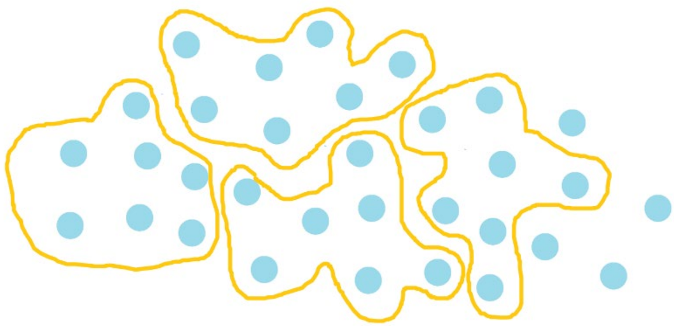
32 *leaves a remainder of* 32 *upon division by* 7.

Is his statement correct? Is it likely helpful?

Practice 17.2: Explain why each number can be said to leave a remainder of either 0, 1, 2, 3, 4, 5, or 6 upon division by 7.

What does a remainder of 0 indicate about the number?

Practice 17.3: This picture shows that $32 = 4 \times 7 + 4$ (four groups of 7 and four remaining dots).



How would you translate the picture at the top of the previous page into a mathematics statement like this?

Three Different Problems with the Same Answer

Consider these three questions. They each have the answer 4, but the real question is why, philosophically, should they have the same answer?

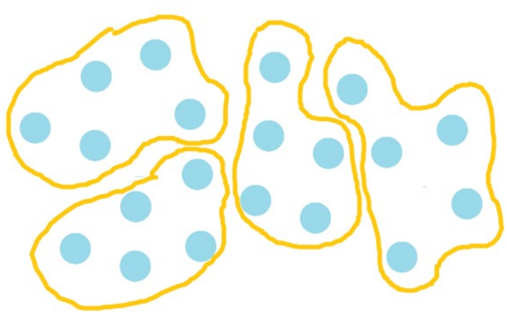
Problem 1: How many groups of 5 can be found within a collection of 20 objects? That is, what is the answer to the division problem $20 \div 5$?

Problem 2: Which number fills in this blank to this multiplication statement?

$$\blacksquare \times 5 = 20$$

Problem 3: I have 20 pies to share equally among 5 students. How many pies will each student receive?

This may seem somewhat cryptic, but do you see that this one picture illustrates the answer to each of the three problems?



With regard to Problem 1: The picture shows 20 dots divided into groups of 5. We count four such groups of 5 and thus conclude

$$20 \div 5 = 4$$

With regard to Problem 2: The picture shows four groups of 5 dots making 20 dots. It's a picture of the multiplication statement

$$4 \times 5 = 20$$

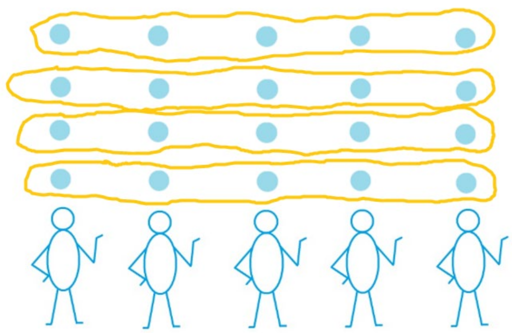
With regard to Problem 3: The picture seems unrelated to sharing pie.

But imagine being asked to distribute 20 pies equally among 5 students.

One could be methodical and hand out one pie to the first student, one pie to the second student, one to the third student, one to the fourth student, one to the fifth student, a second pie to the first student, a second pie to the second student, and so on. But it is hard to tell what final result such a (tedious) approach will yield.

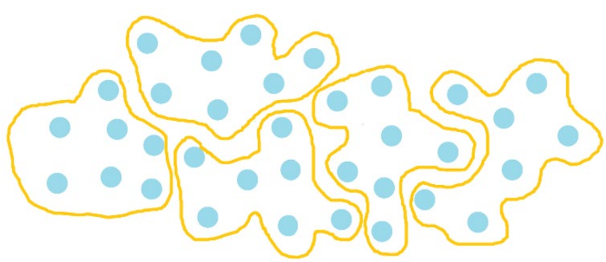
Instead, we can start by organizing pies. Group the pies into sets of 5 and then hand each student one pie from each group of 5.

We can illustrate this organized approach by stretching each group of 5 in our original picture across the five students.



As each students gets one pie per group and there are four groups of 5, each student receives 4 pies.

Example: Here's a picture.



a) A la problem 1, what division statement is it showing?
b) A la problem 2, what multiplication statement is it showing?
c) A la problem 3, what pie-sharing statement is it showing?

Answer:

a) We're seeing 35 dots divided into groups of 7 and there are 5 such groups.
It's a picture of the division statement $35 \div 7 = 5$.

b) We're seeing 5 groups of 7 dots making 35 dots.
It's a picture of the multiplication statement $5 \times 7 = 35$.

c) We're seeing how to share 35 pies equally among 7 students.
We're organizing the pies into groups of seven first and will hand each student a pie from each group. As there are five groups, each student will receive 5 pies.

Practice 17.4: For each of these three statements, find the number that fills in the blank.

1) $91 \div 7 = \blacksquare$

2) $\blacksquare \times 7 = 91$

3) *Sharing 91 avocados equally among 7 bonobos yields* ■ *avocados per bonobo.*

Problem 1 asked as to complete a division problem: how many groups of a given size can we find among a given collection of objects? Let's be specific and call this thinking **division by groups**.

Problem 2 asks about multiplication. But it gave us the answer and asked us about part of the question. Let's call such a challenge a task in **reverse multiplication**.

Problem 3 asks us about equally distributing a collection of objects. Let's call such a problem a **sharing** challenge.

And we have seen that all three problems, the three modes of thinking they represent, are equivalent!

We have three ways to interpret the answer to a division task $N \div a$.

1. It's the number of groups of size a we will find among N objects.

2. It the number that completes this multiplication statement: $\blacksquare \times \mathrm{a} = \mathrm{N}$.

3. If N objects are shared equally among a participants, then it is the number of objects each participant receives.

Schoolbooks call all three of interpretations "division."

So, $20 \div 5$, for instance could regarded as a task of counting groups of size five in a picture of twenty objects (**division by groups**), or as a task of seeking the number that multiplies by 5 to give the answer 20 (**division by reverse multiplication**), or as a task of determining what results when twenty objects are shared equally among five participants (**division by sharing**).

We've seen that all three modes of thinking are equivalent and so there is no harm done in bouncing back and forth between these three modes and in calling all of them **division**.

Practice 17.5: Make up three word problems that could appear in an elementary school textbook that has students think about $42 \div 6$ via each of these frameworks.

a) division by groups.
b) division by reverse multiplication.
c) division by sharing.

ABSOLUTELY OPTIONAL COMMENT

Folk who work in the field of mathematics education use tricky terms for the ways we can interpret a division statement.

They call "division by groups" **quotative division**.
They call "division by sharing" **partitive division**.
(There doesn't seem to be a special term for thinking of division as reverse multiplication.)

Can you research the etymology of these terms?

Practice 17.6 Knowing that $1035 \div 45 = 23$, what then is $1035 \div 23$?
What's the rationale for your answer?

The "reverse multiplication" approach has us compute $48 \div 6$, say, by asking: *What times* 6 *gives* 48*?* The answer is 8. (This is the way most people go about thinking through a basic division question.)

This thinking also provides a way to check answers to division problems.

We see:

$30 \div 6 = 5$ is correct because 5×6 is 30.

$45 \div 5 = 9$ is correct because 9×5 is 45.

$80 \div 6 = 10$ is **not** correct because 10×6 is not 80.

Why Can't You Divide by Zero?

Let's end off this section with an age-old question.

What exactly goes wrong if you try to divide by zero?

Let's explore $5 \div 0$. I think it equals 3.

Following our reverse multiplication check …

$5 \div 0 = 3$ is not correct because 3×0 is not 5.

Okay, I've changed my mind. I think $5 \div 0 = 7$.

$5 \div 0$ is not 7 because 7×0 is not 5.

I can keep guessing and keep seeing I am wrong.

$5 \div 0$ is not 43 because 43×0 is not 5.

$5 \div 0$ is not 679 because 679×0 is not 5.

$5 \div 0$ is not $1{,}000{,}002$ because $1{,}000{,}002 \times 0$ is not 5.

We can see that there is no number N we can dream of to make the statement $5 \div 0 = N$ correct. This is because $N \times 0$ is sure to be zero—not 5—no matter the value of N. (This was a rule of arithmetic we noted back in section 14.)

Math is telling us that there is no answer to be had in dividing 5 by 0.
(Mathematicians say that the expression $5 \div 0$ is **undefined**. There is no possible value for it.)

Practice 17.7: Hang on! Might $5 \div 0$ be equal to 0?

Practice 17.8: Convince me that $123 \div 0$ is also undefined.

It's fun to contemplate the "real-world" interpretations of division by zero.

Practice 17.9: **$5 \div 0$ via Division by Groups**
Here is a picture of five dots. How many groups of zero you see in the picture? Is that a meaningful question?

Practice 17.10: **$5 \div 0$ via Division by Sharing**
I have 5 pies to share equally among zero students. If I conduct this sharing task, how many pies will each student receive?

Does that sound like a meaningful question to you?

Many schoolbooks say that dividing by zero is impossible because it has no meaning when you think of division by groups or division as sharing.

But it is the use of multiplication that gives the true mathematical explanation as to why we must reject division by zero.

Except there is a hiccup!

What's the value of $0 \div 0$?

Playing with reverse multiplication, we see that $0 \div 0$ seems to have value 36.

$0 \div 0 = 36$ could be correct because $36 \times 0 = 0$.

Question: If you look at a picture of zero dots, do you see 36 groups of zero?
If you share no pies equally among no students, will each student get 36 pies?

Both these questions (and answers) feel like nonsense to me!

But multiplication also suggest that $0 \div 0$ could have value 7 or value 777 or value 7056452089!

$0 \div 0 = 7$ could be correct because $7 \times 0 = 0$.

$0 \div 0 = 777$ could be correct because $777 \times 0 = 0$.

$0 \div 0 = 7056452089$ could be correct because $7056452089 \times 0 = 0$.

The trouble with $0 \div 0$ is that **every** value passes our multiplication check.
Too many numbers are possible values for it!

(If you want the fancy language, mathematicians sometimes say that value of $0 \div 0$ is **indeterminate**.)

For $5 \div 0$, there is no number that is a possible value for it.
And $0 \div 0$ has the opposite problem: there are too many numbers that are possible values for it!

We're seeing that dividing by zero is deeply problematic both intuitively—when we are thinking of division as groups and when thinking of division as sharing—and mathematically via multiplication in reverse.

So, people say: **Just don't do it! Don't divide by zero!**

The mathematics and our intuition are aligned on this matter.

Practice 17.11: Can the number 0 be divided by 5? If so, what is the value of $0 \div 5$?

(Try exploring this matter by thinking about division by groups and division by sharing, and then see if your answer is supported by making use of reverse multiplication.)

MUSINGS

Musing 17.12 Every few months a math challenge makes the rounds on social media asking for the value of this expression:

$$8 \div 2(2+2)$$

a) Some people argue vehemently that it has value 1. Can you see how those folk might come to that value?

b) Some people argue vehemently that is has value 16. Can you see how these folk might come to that value?

The conventions of arithmetic dictate that if an expression includes parentheses, we are to compute the value of the expression inside the parentheses first. So, we are certainly being asked to evaluate

$$8 \div 2 \times 4$$

The trouble is that we have no clear convention on how to handle a division sign and a multiplication together in an expression—hence the debate and the confusion!

But we can avoid the confusion by use of more parentheses.

For example, writing $8 \div (2(2+2))$ ensures that the expression will be evaluated to give 1 as the answer.

c) Insert parentheses into the original expression to ensure all who look at it will obtain the value 16.

The original expression is an example of intentionally ambiguous writing. No respectable math author would write this. (Just saying!)

d) Here are some English sentences that are deliberately ambiguous.

"I painted the room with the lights off."
"I saw a man with binoculars."
"She spoke to her friend with an accent."

For fun, try coming up with some more deliberately ambiguous sentences.

Musing 17.13 Give three different division problems all with the answer "6 with a remainder of 2."

Musing 17.14 A number leaves a remainder of 5 upon division by 7. What remainder will double the number leave when divided by 7?

MECHANICS PRACTICE

Practice 17.15 Use multiplication to determine which of these division statements are correct.

a) $103 \div 103 = 1$ b) $1000 \div 125 = 10$ c) $999 \div 1 = 999$

d) $0 \div 5 = 0$ e) $1 \div 3 = 3$ f) $0 \div 1 = 1$

Practice 17.16 In this question a is a counting number different from zero.

a) What is the value of $a \div a$?
b) What is the value of $a \div 1$?

How do you know each of your answers are correct?

Practice 17.17 Fill in the appropriate value for each box.

$7 \times \square = 56$ $56 \div \square = 8$

$7 \times 8 = \square$ $56 \div 7 = \square$

$\square \times 8 = 56$ $\square \div 7 = 8$

$56 \div \square = 7$

(This is not a very exciting problem, I know. But is illustrating that the single math fact "$7 \times 8 = 56$," for instance, can be enquired about in many different forms.)

18. Factors, Prime Numbers, and Composite Numbers

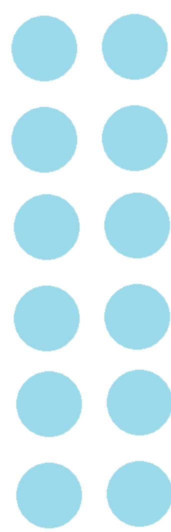
Let's go back to playing with dots directly and ask:

How many different rectangles can one make with 12 dots?

Here's one way, a 2-by-6 rectangle. (Remember, it has become the convention to mention the count of rows first and the count of columns second.)

2×6

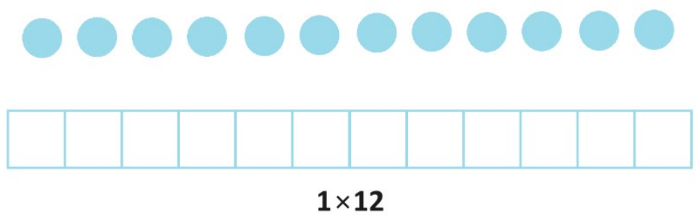
If you turn the picture 90-degrees you get a second rectangle for free.



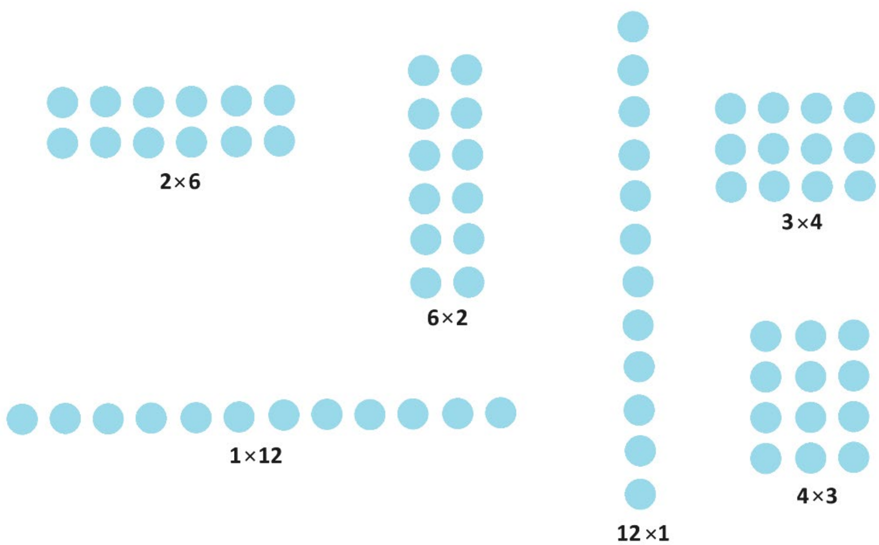
6×2

Practice 18.1: How many different rectangles in total can you make with 12 dots?

There is some debate as to whether or not a single row of 12 dots or a single column of 12 dots really counts as a “rectangle.” But if we were playing this game with 12 unit squares instead of dots, then most everyone agrees a picture representing 1×12 is a rectangle.



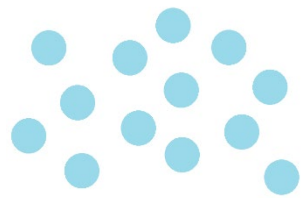
So, let’s include these degenerate cases too. In which case, there are a total of six rectangles you can make with 12 dots.



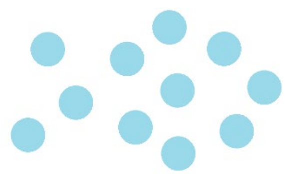
The numbers that arise as the widths and lengths of rectangles – 1, 2, 3, 4, 6, and 12 – are called the **factors** of the number 12. The factors 1 and the number 12 itself are sometimes called the **improper factors** of 12 as the rectangles associated with them might be deemed that way. The remaining factors, 2 3, 4, and 6, are the **proper factors** of 12.

Practice 18.2: List all eight factors of 150.

Practice 18.3: How many different rectangles can one make with 13 dots?



Practice 18.4: How many different rectangles can one make with 11 dots?



Some counts of dots are just resistant to making rectangles. The numbers 13 and 11 each only have two factors, the two improper ones, and you only can make two (improper) rectangles with them.

Practice 18.5: How many different rectangles can you make with just 3 dots?

Practice 18.6: How many different rectangles can you make with just 2 dots?

Practice 18.7: How many different rectangles can you make with just 1 dot?

The numbers 3 and 2 are also resistant to making rectangles—they each have only two factors—and the number 1 is especially resistant: it has only one (improper) factor.

Practice 18.8: Would you regard the picture of a single dot a rectangle? Maybe it's a picture of a one-by-one square?

It is common to regard squares to be examples of rectangles too.

Practice 18.9: How many different rectangles can you make with 9 dots?

Here's the proper language and formal mathematics behind these ideas.

A counting number a is a **factor** of the counting number N if $N = a \times b$ for some counting number b.
(That is, a is the width of some rectangle composed of N dots. b is the rectangle length.)

A factor a of N is a **proper factor** if it is different from 1 and different from the number N itself.
(That is, a is a side-length of a "meaningful" rectangle composed of N dots.)

A number N is called **composite** if has more than two factors.
(That is, you can make at least one meaningful rectangle with N dots. That is, N has at least one proper factor beyond the two improper factors.)

A number N is called **prime** if it has exactly two factors.
(That is, N is a "resistant" number that allows you to make two (not just one) improper rectangles with N dots. So, N is a number different from 1 that has only its two improper factors 1 and N as factors.)

Question 18.10: Explain why the number 0 is composite.

Question 18.11: Give an example of a composite number with exactly one proper factor.

Question 18.12: Does the number 1 fit the definition of being composite? Does it fit the definition of being prime?

The Number 1

The number 1 is neither prime nor composite: it doesn't have more than two factors and it doesn't have exactly two factors. It has one factor, namely, itself.

And yep, there is only one "rectangle" you can make with one dot.

One dot trying to be a rectangle.

During the 1800s, mathematicians decided to keep 1 out of the list of prime numbers. They were studying how prime numbers work in arithmetic, in particular, how they multiply together to create other numbers, and they found it irritating to have to keep track of any 1s that might be part of the product.

For example, 6 is the product of two prime numbers—exactly two!

$$6 = 2 \times 3$$

But if we were to consider 1 to be a prime number, then there infinitely many (irritating) ways to write 6 as a product of prime numbers.

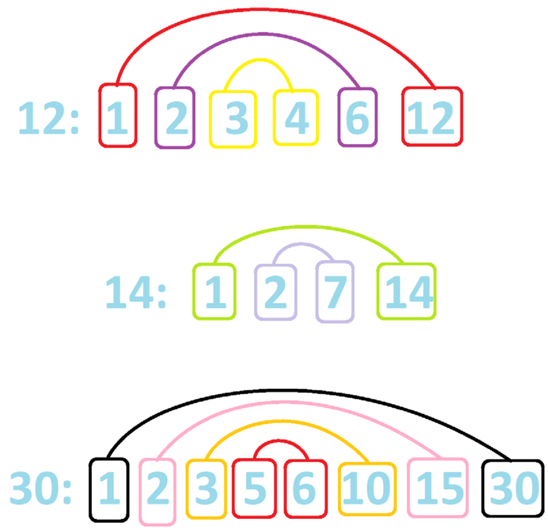
$$6 = 2 \times 3 = 1 \times 2 \times 3 = 1 \times 1 \times 2 \times 3 = \cdots = 1 \times 1 \times 1 \times 1 \times 1 \times 1 \times 1 \times 1 \times 1 \times 2 \times 3 = \cdots$$

For ease, mathematicians decided to deny the number 1 the status of being prime.
It is the only counting number that is neither composite nor prime.

Practice 18.13: If $32 = a \times b$ for two counting numbers and a is a proper factor of 32, explain why b must be a proper factor of 32 as well.

Factor Pairs

It looks like the factors of numbers naturally come in pairs: rectangle lengths and widths. Each pair consists of two numbers that multiply together to the given number.



Practice 18.14: Draw a diagram of the factor pairs for 19.

Since the factors of a number seem to come in pairs, we might conclude that each counting number has an even number of factors. Alas, this is not so.

Practice 18.15:

a) List all the factors of the number of 36 and attempt to identify its factor pairs. What's up?

b) Draw all the rectangles one can make with 16 dots.
(Remember squares are considered to be rectangles too.)

What are the factors of 16? What are the factor pairs? Why does this number have an odd number of factors?

c) Does the number 100 have an even or odd number of factors? How do you know?

d) Why does every square number have an odd number of factors?

e) If a number has an odd number of factors, why must it be a square number?

We have the following observation.

Every square number has an odd number of factors. All other non-zero counting numbers have an even number of factors.

Practice 18.16: Does 0 have an even or odd number of factors? (Is it even possible to answer this question?)

We can have some fun with this.

ACTIVITY

THE CLASSIC LOCKER PROBLEM

A corridor in a school has a line of 25 lockers on its left side. Initially all the locker doors are closed.

Twenty-five students decide to conduct the following experiment:

Student number 1 will walk down the corridor and open every locker door.

Student number 2 will walk down the corridor and close every second locker door. (Locker numbers 2, 4, 6, and so on.)

Student number 3 will walk down the corridor and change the state of every third locker door (lockers 3, 6, 9, …). She will close if it is open and open it if it is closed.

Student number 4 will change the state of every fourth locker door.

Student number 5 the state of every fifth locker door, and so on, all the way to student number 25 who changes the state of every 25^{th} locker door (namely, just the last one).

At the end of this process, which locker doors are left open?

a) Try this experiment using a line of 25 playing cards, all initially face down. What do you notice?
b) Which students touched locker number 12 during this experiment?
c) Which students touched locker number 16 during this experiment?
d) Can you explain the results of this experiment before reading on?

Explaining the Locker Experiment

To get a feel for what is going on, ask: *Which students touched locker 12?*

Answer: Students 1, 2, 3, 4, 6, and 12 each touched locker 12, and only these students. (Do you see this?) This locker door was touched six times—once for each of these factors of 12—and so was returned to a closed state by the end of the experiment.

Which students touched locker 16?

Answer: Students 1, 2, 4, 8, and 16 each touched locker 16, and only these students. Door 16 was thus touched an odd number of times and so was left open at the end of the experiment.

We're now seeing that locker number N is touched only by students whose number is a factor of N. And if N has an even number of factors, its door will be left closed. If N has an odd number of factors, its door will be left open.

Only the square numbers have an odd number of factors. Thus doors 1, 4, 9, 16, and 25 are left open by the end of this experiment.

Practice 18.17: Going Further Here are some more issues to think about if you like.

a) Does it matter in which order the students walk down the corridor? Will the final outcome be the same if students decided to walk down the corridor in a random order?

b) Suppose I want locker number 1 open and all the rest closed. I don't want to send all the students down the corridor (this will leave lockers 4, 9, 16, and 25 open as well), so let's send down only a subset of students. Which students should we send down?

Primes: The Atoms of Arithmetic

People sometimes refer to the prime numbers as the "atoms" of arithmetic since every counting number is built from the prime numbers in the following sense.

Every counting number different from 1 is either a prime number or can be written as a product of prime numbers.

For example:

23 is a prime number. There is nothing more to say about it here.

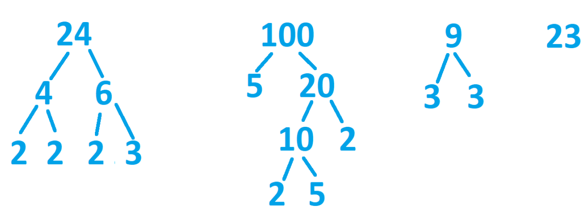
26 is a composite number. It factors as 2×13 and 2 and 13 are each prime.

30 is a composite number. It factors as 3×10, for example, and 10 factors as 2×5. Put these observations together and we see we can write 30 as $3 \times 2 \times 5$. That's a product of three prime numbers.

32 can be written as 4×8, which is equal to $2 \times 2 \times 8$, which equals $2 \times 2 \times 2 \times 4$, which equals $2 \times 2 \times 2 \times 2 \times 2$. Thus 32 is a product of five prime numbers, which each happen to be 2.

In short, if a number is not already a prime number, write it as a product of two proper factors. If each of these factors is prime, then we're done. If not, factor the factors into proper factors and keep doing this until you cannot factor further. At this point, you'll have a nothing but prime factors of the original number, and they all multiply together to make that number.

Question: Did you draw "factor trees" when you were in school? (The factor tree of a prime number is quite short!)



$$24 = 2 \times 2 \times 2 \times 3$$

$$100 = 2 \times 5 \times 2 \times 5$$

$$9 = 3 \times 3$$

$$23 = 23$$

Practice 18.18: Write the number 1000 as a product of primes.

Again, we see why we don't want to regard 1 to be a prime number.

We have

$$24 = 2 \times 2 \times 2 \times 3$$

Allowing "1" into our considerations means we could also write

$$24 = 1 \times 2 \times 2 \times 2 \times 3$$

$$24 = 1 \times 1 \times 2 \times 2 \times 2 \times 3$$

$$24 = 1 \times 1 \times 1 \times 2 \times 2 \times 2 \times 3$$

$$24 = 1 \times 1 \times 1 \times 1 \times 2 \times 2 \times 2 \times 3$$

and so on.

We wouldn't know when to stop!

Practice 18.19: Which prime numbers are even? List them all.
Convince me that your list is complete.

The Prime Numbers are Tricksters

The list of prime numbers begins

$$\mathbf{1, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, \ldots}$$

and it was proved 2300 years ago by Greek scholar Euclid that this list goes on forever. (See Musing 18.26 if you are curious.)

Practice 18.20: Continue the list until you get to the first prime number bigger than 100.

In this list, we see some pairs of consecutive odd numbers that are both prime.

3 and 5	5 and 7	11 and 13	17 and 19
29 and 31	41 and 43	59 and 61	

Such pairs of prime numbers are called **twin primes** and, surprisingly, no one on the planet currently knows if the list of examples of twin primes goes on forever, or if the list of twin primes eventually halts. (This is a famous unsolved problem in mathematics.)

Practice 18.21: According to the internet, what is the largest example of a pair of twin primes currently known? (Can you make sense of the answer offered?)

It is known that the list of **triplet primes**—three consecutive odd numbers, each prime—stops. The list starts with

3 and 5 and 7

and, well, stops there! There is no other example of a triplet of primes.

Hard Challenge 18.22: Can you explain why this is the case?
(A good place to start is to see if you can explain why, for three consecutive odd numbers, one of those numbers must be divisible by 3. Then ask: Which prime number(s) are divisible by 3?)

There are many examples of false patterns that occur with the prime numbers.

Here's a famous one.

Example: In the mid-1700s, Swiss mathematician Leonard Euler noticed that

$0 \times 1 + 41 = 41$ is prime
$1 \times 2 + 41 = 43$ is prime
$2 \times 3 + 41 = 47$ is prime
$3 \times 4 + 41 = 53$ is prime
$4 \times 5 + 41 = 61$ is prime
…
$10 \times 11 + 41 = 151$ is prime
…
$27 \times 28 + 41 = 497$ is prime
…
(Use Google or Siri or some such to check these.)

Does this pattern persist? That is, if we kept going with the pattern, do we obtain a prime number value each-and-every time?

As you can guess, the answer is NO!

Amazingly, this pattern consistently gives prime number values for $0 \times 1 + 41$ and $1 \times 2 + 41$ all the way through to $39 \times 40 + 41$, but not for $40 \times 41 + 41 = 1681$. There luck runs out.

Here's why:

"Factor out a 41*" from* $40 \times 41 + 41$ *to write the expression as*

$(40 + 1) \times 41.$

This is $41 \times 41 = 1681$*, a square number, not a prime number.*

After this incredible run of prime number values, this pattern becomes quite hit and miss (more miss, than hit) as to when it gives prime numbers again.

Practice 18.23: Is $41 \times 42 + 41$ a prime number?

MUSINGS

Musing 18.24 What is the largest known prime number today?
(Can you make sense of the answer presented on the internet?)

Musing 18.25 Take the number 11, which is prime, and keep adding multiples of 30 to it. This produces a list of numbers that begins

$$11, 41, 71, 101, 131, \ldots.$$

Each number mentioned so far is a prime number. (Check this with Alexa if you like.)

If you keep going, does this list produce only prime numbers?

Musing 18.26 Greek scholars some 2300 years ago called a number **perfect** if the sum of all its factors, excluding the number itself, sum to the number. For example, the number 6 is perfect. Its factors (excluding the number 6 itself) are 1, 2, and 3 and we have

$$1 + 2 + 3 = 6.$$

The next perfect number is 28. We have

$$1 + 2 + 4 + 7 + 14 = 28.$$

a) Show that the number 496 is also perfect.

b) Use the internet to find the next few perfect numbers.

c) No one on this planet knows how many examples of perfect numbers should exist—finitely many? Infinitely many? Who knows? (And if they know, why aren't they sharing?)

   According to the internet, how many examples of perfect numbers are currently known?

   No one knows if there is an example of a perfect number that is odd.
   (This is another famous unsolved problem in mathematics!)

Musing 18.27 Some 2300 years ago, Greek scholar Euclid established that the list of primes cannot stop. (That is, the list must go on forever and so there are infinitely many prime numbers.)

Your job in this question—if you choose to keep reading it—is to answer this question at the end of it:

> *On a scale of 1 to 5—where 1 means "not at all, not even one tiny-eensy bit" and 5 means "utterly and completely" –to what degree did you understand what you just read?*

Okay, here's the thing to read.

To prove that the list of primes cannot stop, Euclid argued this way.

If you think you know all the primes there are to know, then I suggest you multiply them all together, and then add one to that product.

My Example:
I am going to pretend I know the prime numbers 2, 5, and 11, but no more!

Euclid wants me to consider $2 \times 5 \times 11 + 1$*, which equals* 111.

The number you now have is one more than a multiple of each of your prime numbers. It is not divisible by any of your prime numbers, and so has none as your prime numbers as a factor.

My Example: True!
$2 \times 5 \times 11 + 1$ *is one more than a multiple of* 2*, and* 2 *is not a factor of* 111.
$2 \times 5 \times 11 + 1$ *is one more than a multiple of* 5*, and* 5 *is not a factor of* 111.
$2 \times 5 \times 11 + 1$ *is one more than a multiple of* 11*, and* 11 *is not a factor of* 111.

When we write this number as a product of primes (draw a factor tree for it), the product will use only primes that are factors. None of your primes are factors, so you must be using new primes that weren't on your first list!

My Example:
Heavens! All right.

111 *factors as* 3×37*.And yes! I've just discovered two new primes to add to my list.*

Okay, I now know the prime numbers 2, 3, 5, 11*, and* 37*, but no more!*

Keep repeating this process to discover more and more new primes to add to your list. Multiply together all the primes you now know, add 1 to the answer, and then draw a factor tree for that final result to discover new primes.

My Example:
Okay, now I am going to consider $2 \times 3 \times 5 \times 11 \times 37 + 1 = 12211$.

The factor tree for this number is quick: 12211 *is already prime according to my computer. Ah! It's new prime number to add to my list.*

Okay … I am aways going to be able to find new primes to add to any list I have.

This process will keep producing more and more and more primes.

No finite list of primes will thus ever be complete!

Here's the question:

On a scale of 1 to 5—where 1 means "not at all, not even one tiny-eensy bit" and 5 means "utterly and completely" –to what degree did you understand what you just read?

MECHANICS PRACTICE

Practice 18.28
The number of degrees in a circle is 360. This number is highly divisible.

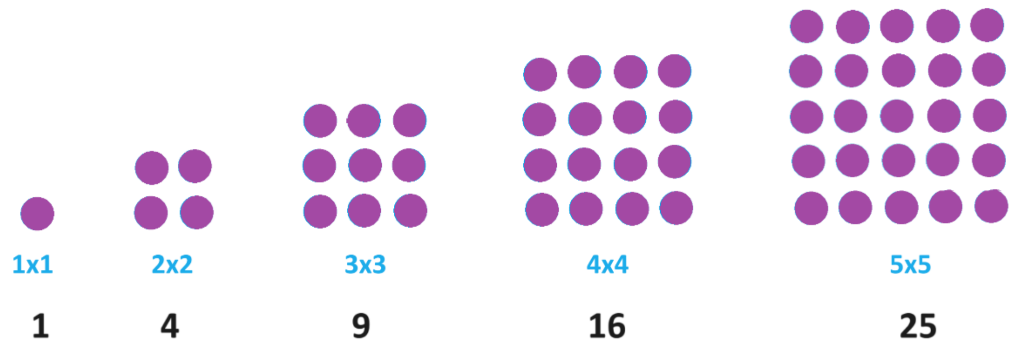
a) List all its factors of the number 360. (It has a total of twenty-four of them.)
b) Is the number 361 prime? If not, list all its factors.
c) Is the number 359 prime? If not, list all its factors.

[It's okay to use google or Siri or some 21st-century tool to ask if a number is prime.]

Musing 18.29 Write the number 360 as a product of prime numbers.

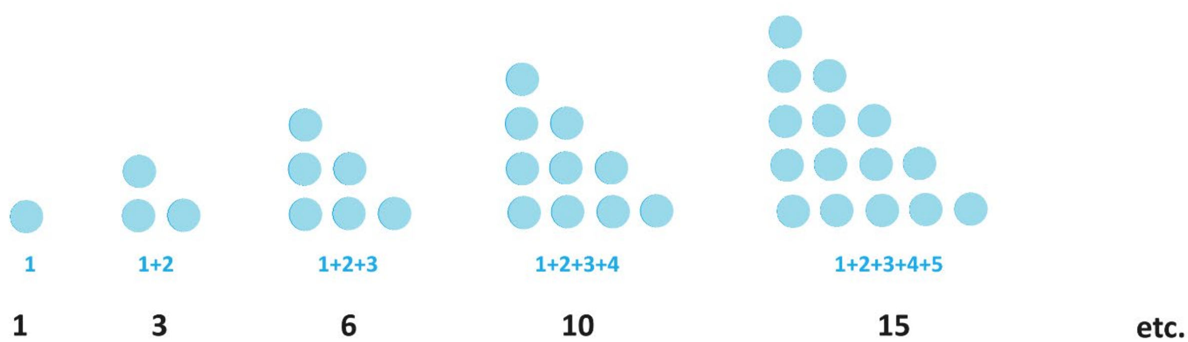
19. Figurate Numbers

We started this chapter with a picture of a square array of dots. The counts of dots that arise in square arrangements of dots are called the **square numbers**.



The fifth square number is $5^2 = 5 \times 5 = 25$.
The seventh square number is $7^2 = 7 \times 7 = 49$.
The thousandth square number is $1000^2 = 1{,}000{,}000$, a million.
In general, the Nth square number in the list is $N^2 = N \times N$.

We also saw in Practice 15.8 an example of a **triangular number** made from a row of one dot, then a row of two dots, then a row of three dots, and so on. Here are the first few triangular numbers.



The fifth triangular number is $1 + 2 + 3 + 4 + 5 = 15$.
The seventh triangular number is $1 + 2 + 3 + 4 + 5 + 6 + 7 = 28$.
And in section 15 we kept going and found a general formula for the Nth triangular number. (Feel free to look up the formula for the sum of the first N counting numbers.)

Greek scholars of some 2,500 years ago enjoyed thinking about square and triangular numbers, and other “figurate numbers” that had a natural geometry associated with them.

They discovered stunning patterns and interactions within and between these numbers. I thought it would be fun to end this chapter with some of that delightful play.

Triangular Numbers: 1 3 6 10 15 21 28 36 45 55 ...
Square Numbers: 1 4 9 16 25 36 49 64 81 100 ...

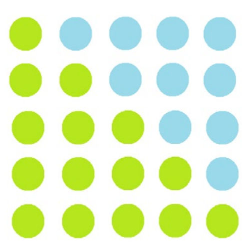
INTERACTION 1

The sum of any two consecutive triangular numbers is sure to be a square number.

$$1 + 3 = 4$$
$$3 + 6 = 9$$
$$6 + 10 = 16$$
$$10 + 15 = 25$$
$$15 + 21 = 36$$

etc.

A single picture reveals why this is so. Here we see that the 4th and 5th triangular arrays lock together to make a square array, and the same will be case for any two consecutive triangular arrays.



Question: Does this picture indeed explain matters for you? Do you see two consecutive triangular arrays in the picture?

Do we need infinitely many pictures like this, or does this one picture show how, in general, any two consecutive triangular arrays of dots will combine to make a square array of dots?

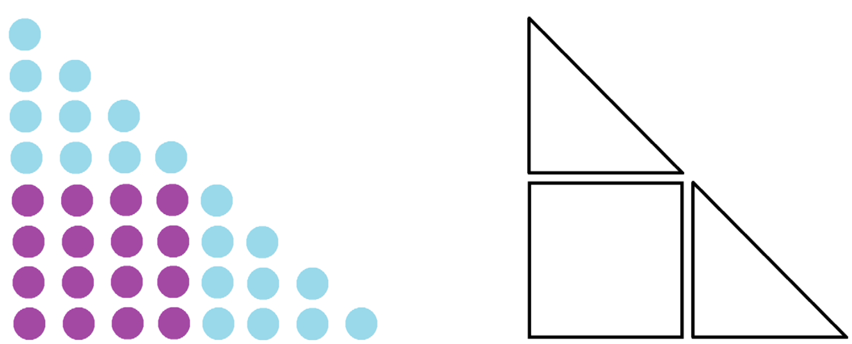
Triangular Numbers: 1 3 6 10 15 21 28 36 45 55 ...
Square Numbers: 1 4 9 16 25 36 49 64 81 100 ...

INTERACTION 2

Take any triangular number and double it. Then add the matching square number to the result. The final answer is sure to be another triangular number.

2 x 1 + 1 = 3
2 x 3 + 4 = 10
2 x 6 + 9 = 21
2 x 10 + 16 = 36
2 x 15 + 25 = 55
etc.

Here's an explanation (maybe).



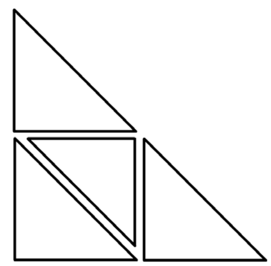
Question: Does this figure provide an explanation for you?

Question: What do you think of the following statement?

> *Twice the Nth triangular number plus the Nth square number gives the $2N$th triangular number.*

(The answer can be: "I don't get it and I don't like it. Don't ask me a question like this again.")

Practice 19.1 What does the following picture suggest about the triangular numbers? (Here we have three triangles the same size and one slightly smaller triangle.)



Triangular Numbers: 1 3 6 10 15 21 28 36 45 55 ...

Square Numbers: 1 4 9 16 25 36 49 64 81 100 ...

INTERACTION 3

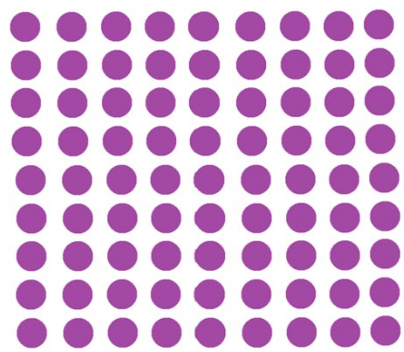
Take any triangular number. Multiply it by eight and then add one to the result.
This is sure to produce an odd square number.

$$8 \times 1 + 1 = 9$$
$$8 \times 3 + 1 = 25$$
$$8 \times 6 + 1 = 49$$
$$8 \times 10 + 1 = 81$$
$$8 \times 15 + 1 = 121$$

etc.

Let's look at one specific example: $8 \times 10 + 1 = 81$.

Here's the nine-by-nine square array of dots that represents the square number 81.



Our job is to see this picture as eight copies of the fourth triangular array of ten dots, plus one extra dot.

Practice 19.2: There is meant to be one special dot in that nine-by-nine array. If you had to make a guess, which dot in that array might be singled out as special?

Can you now draw in eight copies of the small triangular array around that special dot? Can you come up with a design that clearly generalizes beyond this specific example and would work for any odd square array of dots?

Question: What do you think of the following statement?

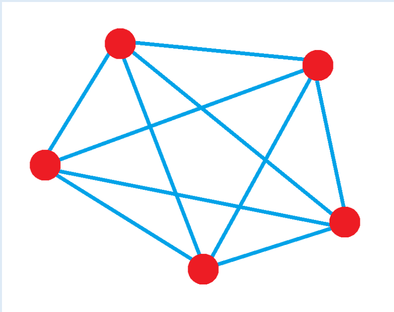
Eight times the Nth triangular number plus one equals the $(2N+1)$th square number.

MUSINGS

Musing 19.3

a) What is the one-hundredth square number?
b) What is the one-hundredth triangular number?

Musing 19.4 In the picture, five dots were drawn in a circle and each dot was connected to each and every other dot with a line segment. A total of 10 line segments were drawn.



a) Draw six dots on a circle and the line segments that connect each and every pair of dots. How many line segments in total did you draw?
b) How many line segments does one draw starting with three dots? With four dots? With two dots? With seven dots?
c) Make a prediction as to how many line segments you would draw if you started with one hundred dots on a circle.
d) Can you explain what you are noticing?

Musing 19.5

The number 1 is both a square number and a triangular number, but that is not very exciting.

The number 36 is also both a square number and a triangular number. (We have $36 = 6 \times 6$ and $36 = 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8$.) It's a **squangular number**!

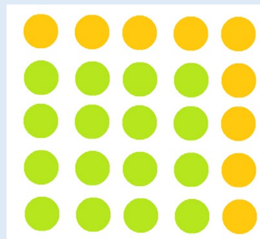
The next squangular number is 1225.

a) Which square number is 1225?
b) Can you figure out which triangular number 1225 is? (This, technically, is a YES/NO question.)

(If you are curious, the list of squangular numbers is infinitely long and it begins $1, 36, 1225, 41616, 1413721, 48024900, 1631423881, \ldots$.)

Musing 19.6

This picture shows that the fourth and fifth square numbers differ by 9. (Do you see that?)



a) What do the 10th and 11th square numbers differ by?
b) Explain why two consecutive square numbers are sure to differ by an odd number.
c) Which two consecutive square numbers differ by 101?
d) Which two consecutive triangular numbers differ by 101?

Musing 19.7

Triangular Numbers:	1	3	6	10	15	21	28	36	45	55	...
Square Numbers:	1	4	9	16	25	36	49	64	81	100	...

The following table suggests that any three consecutive triangular numbers have the following property:

Six copies of a triangular number plus one copy of the triangular number before it and one copy of the triangular number after it gives a square number.

1	+	6 x	3	+	6	=	25
3	+	6 x	6	+	10	=	49
6	+	6 x	10	+	15	=	81
10	+	6 x	15	+	21	=	121

etc.

a) Draw a nine-by-nine square array of 81 dots. Can you identify within it one triangular array of 6 dots, six triangular arrays of 10 dots, and one triangular array of 15 dots?
b) Do you think the pattern suggested in the table is always true?

Musing 19.8 The Prime Numbers are Tricksters!

Consider the list of triangular numbers 1, 3, 6, 10, 15, 21, 28, 36, 45, 55, … .

Starting with 6, do the following:

Add 1 to the triangular number if it is even.
Subtract 2 from the triangular number if it is odd.

Is the result always a prime number? It seems so at first!

$6 + 1 = 7$ is prime
$10 + 1 = 11$ is prime
$15 - 2 = 13$ is prime
$21 - 2 = 19$ is prime
$28 + 1 = 29$ is prime
$36 + 1 = 37$ is prime
$45 - 2 = 43$ is prime
$55 - 2 = 53$ is prime

a) What's the fifteenth triangular number? Does this observation hold for it?
a) What's the eighteenth triangular number? Does this observation hold for it?

Musing 19.9
In Musing 18.25 we were introduced to the perfect numbers: 6, 28, and 496. (Over fifty examples of perfect numbers are currently known and, curiously, each example is an even number.)

6 is also a triangular number. It is the third one, and 3 is a prime number.

28 is also a triangular number. It is the seventh one, and 7 is a prime number.

496 is also a triangular number. It is also in a prime-numbered place on the list of triangular numbers.

Can you figure out which triangular number 496 is?

Swiss mathematician Leonard Euler proved in the mid-1700s that every even perfect number is sure to be a triangular number and in a prime-numbered position in the list of triangular numbers.

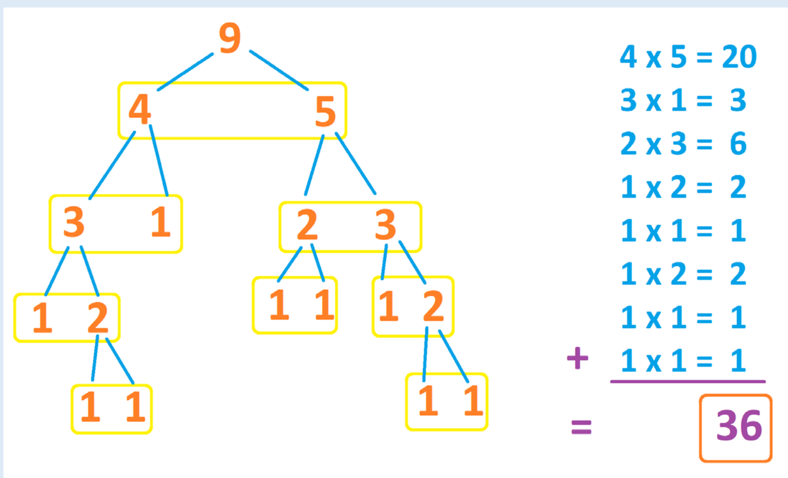
ACTIVITY

Musing 19.10 **PILE SPLITTING**

Sichun splits a pile of 9 buttons into two piles: a pile of 4 and a pile of 5.
He writes on a piece of paper: $4 \times 5 = 20$.

He then splits the pile of 4 buttons into two piles: a pile of 3 and a "pile" of 1.
He writes on a piece of paper: $3 \times 1 = 3$.

He keeps doing this with all the piles he sees—splitting each pile into two and recording the product of the pile sizes on his paper. He does this until he has nothing but single buttons, that is, until he has nine "piles" of 1.



Sichun then adds the results of all the products he wrote down. In the example shown, he gets the "magic sum" of 36.

a) Play the game for yourself. Start with nine objects and keep splitting piles into two, recording products along the way. Make different choices than Sichun did for splitting piles.
What magic sum do you get?

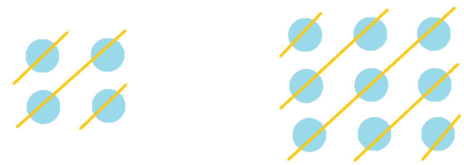
Play the game a second time–but predict first what magic sum you think you will get. Do you?

b) Play the game again, but this time start with 8 buttons. What magic sum do you get this time?
c) Play the game yet again, starting with 2 buttons, with 3 buttons, with 4 buttons, and so on, up to 7 buttons. Any comments?

OPTIONAL HARD CHALLENGE: Can you explain what is going on?

SOLUTIONS

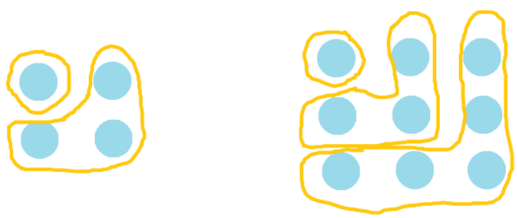
15.1



$1 + 2 + 1 = 4$ $\quad$ $1 + 2 + 3 + 2 + 1 = 9$

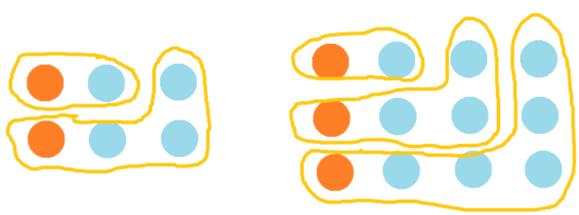
15.2 It's a $1{,}000{,}000 \times 1{,}000{,}000$. That's a million millions, which is a trillion.

15.3



$1 + 3 = 4$ $\quad$ $1 + 3 + 5 = 9$

15.4



$2 + 4$ $\quad$ $2 + 4 + 6$

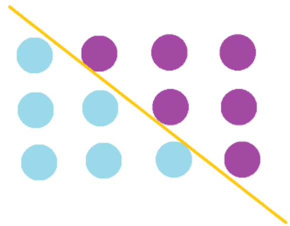
15.5 It is the answer to the sum of the first seven even numbers: $2 + 4 + 6 + 8 + 10 + 12 + 14$.

15.6 Read on!

15.7 Half of 156, which is 78.

15.8

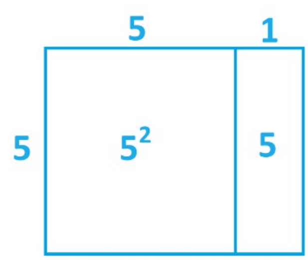
a) I hope you can see this.
b)



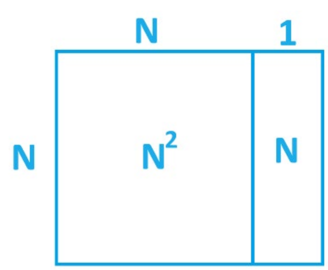
Two copies of 1 + 2 + 3

c) Do imagine a ten-by-eleven array of dots and a diagonal line, like the one above, that cuts the array into two triangles.

d) The picture shows that 5×6 and $5^2 + 5$ are the same. So, half of each of these are the same too.



e) f) g) The same sort of picture works in general to show that $N \times (N + 1)$ and $N^2 + N$ are sure to be the same, for any number N.



15.9

a) $1 + 2 + 3 + \ldots + 30$ equals half of $30^2 + 30$. That's half of 930, making for 465 puppies.
b) $1 + 2 + 3 + \ldots + 365$ equals half of $365^2 + 365$. This turns out to give 66,795 kittens.
c) If you are currently N years old, you have blown our half of $N^2 + N$candles.

15.10

a) $1{,}000{,}000^2 = 1{,}000{,}000{,}000{,}000$, a trillion.

b) $1{,}000{,}000^2 + 1{,}000{,}000 = 1{,}000{,}001{,}000{,}000$

c) Half of $1{,}000{,}001{,}000{,}000$, which is $500{,}000{,}500{,}500$

d) $2{,}000{,}000$

e) $1{,}999{,}999$

15.11

a) $1 + 3 + 5 + 7 + \cdots + 97 + 99$ is the sum of the first fifty odd numbers.
The sum is $50^2 = 2500$.

b) $2 + 4 + 6 + 8 + \cdots + 498 + 500$ is the sum of the first two-hundred-and-fifty even numbers.
The sum is $250^2 + 250 = 62500 + 250 = 62750$.

c) $73 + 74 + 75 + 76 + \cdots + 98 + 99 + 100$ is the sum of the first one-hundred numbers (which is half of $100^2 + 100$, which is 5050) but we're missing the sum of the first seventy-two numbers (which is half of $72^2 + 72$, which is 2628).

So, $73 + 74 + 75 + 76 + \cdots + 98 + 99 + 100$is $5050 - 2628 = 2422$.

d) Now $2 + 4 + \cdots + 510 + 512 + 514 + 516 + \cdots + 798 + 800$ is $400^2 + 400 = 160400$.
And $2 + 4 + \cdots + 510$ is $255^2 + 255 = 65280$.

So, $512 + 514 + 516 + \cdots + 798 + 800$ is $160400 - 65280 = 95120$.

15.12

a)

$$\begin{array}{l} 1 + 2 + 3 + \dots + 99 + 100 = \text{answer} \\ 100 + 99 + 98 + \dots + 2 + 1 = \text{answer} \\ \hline 101 + 101 + 101 + \dots + 101 + 101 = 2 \times \text{answer} \end{array}$$

100 of these

100 x 101 = twice the answer

b) Fifteen 50s is twice the answer, that is, $15 \times 50 = 750$ is twice the answer.
Thus, the answer must be 375.

$$\begin{array}{l} 4 + 7 + 10 + 13 + 16 + 19 + 22 + 25 + 28 + 31 + 34 + 37 + 40 + 43 + 46 = \text{answer} \\ 46 + 43 + 40 + 37 + 34 + 31 + 28 + 25 + 22 + 19 + 16 + 13 + 10 + 7 + 4 = \text{answer} \\ \hline 50 + 50 + 50 + 50 + 50 + 50 + 50 + 50 + 50 + 50 + 50 + 50 + 50 + 50 + 50 = 2 \times \text{answer} \end{array}$$

15 copies of 50

c) The previous example illustrates what is going on.

Write two copies of the sum, one under the other, one forward and one backward, and add the columns. If there N numbers in the sum, you are adding N columns and each column has two numbers summing to the same value. Why? The first column consists of the first and last number in the original sum. The second column essentially has the same two numbers, one adjusted up by a certain amount and the other down by that same amount, thereby giving the same sum value again. And so on.

So, we have N copies of the of the first and last numbers. And this equals twice the answer we seek.

Thus, the answer we seek is indeed as described in the question.

15.13

a) Look at the sum like this.

$$\begin{array}{l} \mathbf{1 + 2 + 3 + 4 + 5 + 6 + 7 + 8} \\ \mathbf{+ 7 + 6 + 5 + 4 + 3 + 2 + 1} \end{array}$$

b)

$$1 + 3 + 5 + 7 + 9 + 11 + 13 + 15$$

c) d) Do the same “rainbow” technique on each of these.

16.1
a) You have an even number of socks. (You’ve split the supply into two sets of equal size: the socks on your left foot and the socks on your right roof.)
b) You have an odd number of socks. Your left foot has one more sock on it than the right foot.

16.2 The number 1 is odd.

If given one dot and try to make pairs, you’ll have zero pairs and that one dot will be left over.
Or, if you try to split your supply of one dot into two piles of equal size, you’ll have two piles of zero size and that one dot left over.

16.3
a) 14, and even number.
b) 201, an odd number.
c) Odd

16.4 $50 = 2 \times 25$; $51 = 2 \times 25 + 1$; $100 = 2 \times 50$; $121 = 2 \times 60 + 1$; million $= 2 \times 500{,}000$

16.5
a) If a count of dots is even, then the dots can split into pairs. Adding 1 dot to this count gives pairs and an extra dot, making the new count odd.

[OR … If N is even, then N equals $2a$ for some counting number a.
Then $N + 1$ equals $2a + 1$, showing it is odd.]

b) If a count of dots is odd, then the dots can split into pairs with one extra dot. Adding 1 dot to this count allows us to make make an additional pair. The new count even.

[OR … If N is odd, then N equals $2a + 1$ for some counting number a.
Then $N + 1$ equals $2a + 2 = 2(a + 1)$, by factoring, showing that $N + 1$ is even.]

16.6 If one set of dots grouped into pairs and an extra dot is combined with a second set of dots grouped into pairs with one extra dot, the resulting conglomeration of dots is a set of nothing but pairs—the two extra dots combine to make an additional pair.

16.7 This is technically a yes/no question!

But you can argue like we did for adding even and odd numbers, imaging quantities as pairs of dots or pairs of dots with an extra dot.

16.8 Odd.

Imagine quantities as pairs of dots or pairs of dots with an extra dot, we see that this sum has five extra dots on the scene. We can make two more pairs with them, but there will still be one extra dot outside of the pairs.

16.9 I'll leave this question to you.

16.10 a) Odd b) Even

16.11 The total weight of the marbles is $1 + 2 + 3 + \cdots + 149 + 150$ grams, a sum with an 75 odd numbers and 75 even numbers. It therefor adds to an odd number of grams. This cannot be split into two piles of equal weight (without splitting a marble).

16.12 Any set of one-hundred consecutive numbers contains 50 even numbers and 50 odd numbers and will thus sum to an even value. The number 94315 cannot be the answer to such a sum.

16.13

"Even x Even" is a sum of an even number of even numbers, and so will be even.

"Even x Odd" is a sum of an even number of odd numbers, and so will be even.

"Odd x Even" is a sum of an odd number of even numbers, and so will be even.

"Odd x Odd" is a sum of an odd number of odd numbers, and so will be odd.

16.14 If n is an odd number, then, by the previous question, $n \times n$ would be odd.

But we are told that $n^2 = n \times n$ is not odd. So, it cannot be the case then that n is odd. It must be even.

16.15 This is just a larger version of this Handshake Game described in this section. Exactly the same reasoning applies.

16.16

a) A sum of thirty-one 3s and 5s must be odd. One cannot make an even number 100 from such a sum.

b) You can buy an item of any price in Oddonia: just pay 1 cent at time if you like until the full price is reached. (Of course, you can probably be much more efficient than this approach, but it will work!)

16.17

a) Yes! You have either three, two, one, or no even numbers among your three numbers.

If you have three or two even numbers, then you can add two even numbers together to get an even sum.
If you have just one or no even numbers, then you have two odd numbers, which you can add together to create an even sum.

b) No! You might have three even numbers.

16.18

If we asked each person to submit the number of handshakes they took part in and tallied the list, then the sum would be $10 \times 4 + 11 \times 5 + 4 \times 6 = 119$. But as we saw in this Section, this total tally should be an even number (two hands at a time take part in a handshake). There is a problem with the data.

16.19

a) The row that contains the one even entry must sum to an even value (it has one even and two odd entries). Any other row consists of three odd entries and so sum to an odd value. This means the rows are not adding to the same sum and so the square fails to be magic.

b) If the two even entries are in different rows, then there is a third row with nothing but odd entries. We now have two rows that sum to an even value and one to an odd value. The square fails to be magic.

If the two even entries are in the same row, then we have two columns each containing one even entry and one column with three odd entries. By adding the entries in columns now, we see that the square again fails to be magic.

c) It is possible. Here's an example.

6	**1**	**5**
3	**4**	**5**
3	**7**	**2**

16.20 For the start of the game we have:

squares = 5 = odd
circles = 6 = even
triangles = 4 = even

The count of squares is odd, and the count of each of the other shapes is even.

A move changes the count of each shape by one—up 1 or down 1. So, after a first move we'll have:

squares = even
circles = odd
triangles = odd

After a second move we'll be back to

squares = odd
circles = even
triangles = even

And so on. The count of squares, in terms of evenness and oddness, is always different from the other two counts. Thus, if we do reach the state of 0 (even) of two shapes and some number of a remaining shape (which must be an odd count), that remaining shape must be squares.

b) Now the count of triangles is distinguished from the other two. We must have triangles remaining.

17.1 The statement is absolutely correct, but I doubt it is helpful.

17.2 In looking for groups of 7 among a collection of objects, if one has seven or more objects "left over," then we can identify at least one more group of 7. So, if we are being as efficient as possible, only if we have 0, 1, 2, 3, 4, 5,or 6 objects remaining will we no longer be able to identify additional groups of 7.

If the count of remaining objects is 0, then all objects are part of groups of 7. The original count is divisible by 7.

17.3 $32 = 3 \times 7 + 11$

17.4 These three statements are equivalent, and the number 13 does the trick in each case.

17.5 Something like …

a) We need to put crackers in lunch boxes, 6 per box. But we only have 42 crackers. Hmm. How many lunchboxes can we fill?
b) How many lily pads are there if each lily pad has 6 frogs sitting on it and there are 42 frogs in total?
c) I need to evenly distribute 42 rotten apples into 6 rubbish bins. How many apples per bin will that be?

17.6 From $1035 \div 45 = 23$ we deduce that $23 \times 45 = 1035$. We can switch the order of the multiplication and write $45 \times 23 = 1035$ to deduce that $1035 \div 23 = 45$.

17.7 $5 \div 0$ can't be zero because 0×0 is not 5.

17.8 There is no value N to fit this statement: $123 \div 0 = N$. This is because $N \times 0$ is sure to equal 0, not 123, no matter what value N is.

17.9 I personally don't know what to make of this question.

17.10 Nor do I personally know what to make of this question.

17.11 $0 \div 5 = 0$ passes our multiplication check because 0×5 does equal 0.
This answer also passes our intuitive checks: there are certainly 0 groups of five among no dots, and sharing 0 pies among 5 students gives 0 pies per student.

17.12 a) They are seeing $8 \div 8 = 1$.
b) Since $8 \div 2 = 4$ and $2 + 2 = 4$, they are seeing $4 \times 4 = 16$.
c) $(8 \div 2)(2(2 + 2))$
d) Go for it!

17.13 Three examples are $20 \div 3$ and $26 \div 4$ and $32 \div 5$.

17.14

a) $103 \div 103 = 1$ is correct because $1 \times 103 = 103$.

b) $1000 \div 125 = 10$ is not correct because $10 \times 125 \neq 1000$. (I am using the **not equal to** sign here.)

c) $999 \div 1 = 999$ is correct because $999 \times 1 = 999$

d) $0 \div 5 = 0$ is correct because $0 \times 5 = 0$

e) $1 \div 3 = 3$ is not correct because $3 \times 3 \neq 1$

f) $0 \div 1 = 1$ is not correct because $1 \times 1 \neq 0$

17.15
a) $a \div a = 1$ because $1 \times a = a$. (intuitively, there is one group of size a among a collection of a objects.)
b) $a \div 1 = a$ because $a \times 1 = a$. (intuitively, there are a groups of size 1 among a collection of a objects.)

17.16 Just keep filling in the number 7, 8, and 56, as appropriate.

18.1 Read on!

18.2 $1, 2, 3, 5, 30, 50, 75$, and 150.

18.3 Just two rectangles: 1 by 13, and 13 by 1.

18.4 Just two rectangles: 1 by 11, and 11 by 1.

18.5 Just two rectangles: 1 by 3, and 3 by 1.

18.6 Just two rectangles: 1 by 2, and 2 by 1.

18.7 There is not much you can do with one dot. Just make a one-by-one "square"?

18.8 Probably.

18.9 Three. A 1-by-9 rectangle, a 9-by-1 rectangle, and a 3-by-3 square.

18.10 Every number N is a factor of 0 because $N \times 0 = 0$. Thus 0 has infinitely many factors, which is at least two. Thus 0 fits the definition of being composite.

18.11 9 only has 3 as a proper factor.

18.12 The number 1 only has one factor, namely 1 itself since $1 \times 1 = 1$. Thus 1 fails to fit the definition of being prime nor composite.

18.13
If b is 1, then a is 32 and not a proper factor.

If b is 32, then a is 1 and not a proper factor.

So b better be a proper factor if we are told a is.

18.14 It's not very exciting.

18.15
a) 1 and 36; 2 and 18; 3 and 12; 4 and 9; and 6. It has an odd number of factors. (We have that 6 "wants

to" pair with itself!)

b) You should have 1-by-16, 16-by1, 2-by-8, 8-by-2 rectangles (as per factor pairs) and a 4-by-4 square. The "factor pair 4 and 4" produces only one picture: rotating the square 90-degrees gives the same picture.

c) Because $100 = 10 \times 10$ we have a factor that "pairs with itself." The number 100 will have an odd number of factors.

d) By definition, a square number is of the form $a \times a$ for some number a, and so will have one factor that "pairs with itself." This will cause it to have an odd number of factors.

e) All factors come in pairs unless one factor a pairs with itself as $a \times a$ which must be the case if there are an odd number of factors. So our number equals $a \times a$, making it a square number.

18.16 It is not possible to answer this question as zero has an infinite number of factors.

18.17 a) It does not matter. Each student will touch the lockers they touch and it does not matter when they walk down the line to do so.
b) Students 1, 2, 3, 5, 6, 7, 10, 11, 13, 14, 15, 17, 19, 21, 22, and 23.
(Send down student 1 and then decide if you need to send student 2 down to close door 2, then decide if you need to send student 3 down to close door 3, and so on.)

18.18 $1000 = 2 \times 2 \times 2 \times 5 \times 5 \times 5$

18.19 The only even prime number is 2. (All other even numbers have 2 as a proper factor and so are not prime.)

18.20 101 is the first three-digit prime number.

18.21 Check it out.

18.22 Suppose n, $n + 2$, and $n + 4$ are three consecutive odd numbers.
If n is a multiple of three, then we have a number that is a multiple of three.
If n is 1 more than a multiple of three, then $n + 2$ is a multiple of three. (Do you see why?)
If n is 2 more than a multiple of three, then $n + 4$ is a multiple of three. (Again, do you see why?)

Either way, one of the numbers n, $n + 2$, and $n + 4$ is a multiple of three. The only prime number that is a multiple of three is 3. So the number 3 is one of the three numbers.

We must have the triple 3, 5, 7.

18.23 No. It factors as $41 \times 42 + 41 = (42 + 1) \times 41 = 43 \times 41$.

18.24 Check it out on the internet.

18.25 Add the multiple 11×30 to 11 and get $11 + 11 \times 30 = 11 \times (1 + 30) = 11 \times 31$, which is not prime!

(And, more simply, the next umber in the list, 161, is not prime! It's 7×23.)

18.26 Check the internet.

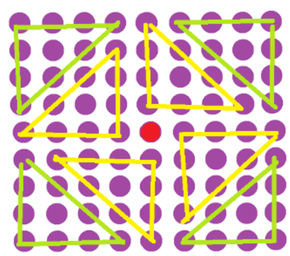
18.27 What number on the scale did you choose?

18.28
a) 1, 2, 3, 4, 6, 8, 9, 10, 12, 15, 18, 20, 24, 30, 36, 40, 45, 60, 90, 120, 180, and 360.
b) No. $361 = 19 \times 19$.
c) Yes. 359 is prime.

18.29 $360 = 2 \times 2 \times 2 \times 3 \times 3 \times 5$

19.1 Three copies of one triangular number and one copy of the next smaller triangular together make another triangular number. (e.g. $3 \times \mathbf{10} + \mathbf{6} = \mathbf{36}$.

19.2



19.3 a) $100 \times 100 = \mathbf{10,000}$ b) Half of $10{,}000 + 100$ is $\mathbf{5050}$.

19.4 a) 15 b) One gets the triangular numbers c) The ninety-ninth triangular number, $\mathbf{4950}$.

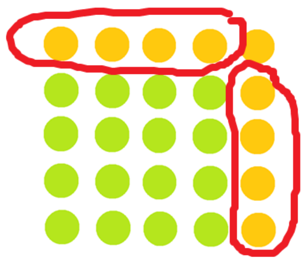
d) Imagine six dots on a circle, for example.

Pick a dot and draw all the lines from it. Now go to the next dot over and draw all the lines from it not already drawn. And so on, marching around the circle. Really do this (!) and you will see that you are drawing $5 + 4 + 3 + 2 + 1$ lines. That's a triangular number.

In general, with N dots on a circle, you will draw $(N - 1) + \cdots + 3 + 2 + 1$ lines, a triangular number.

19.5 It's the 35th square number and 49th triangular number.

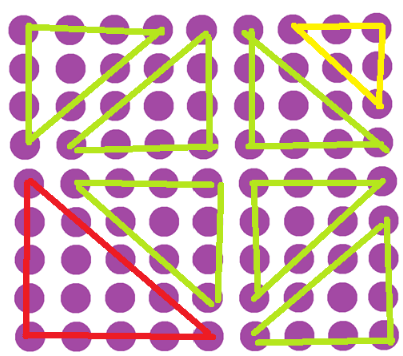
19.6 a) 21
b) The L-shape we add on consists of two copies of a set of dots and one more dot. That makes it an odd number of dots.



c) $101 = 2 \times 50 + 1$ and so the 50th and 51st square numbers differ by 101.

d) The 100th and 101st square numbers differ by 101. (One is the sum of all the numbers from 1 up to 100, and the second is the sum of the all the numbers from 1 up to 101.)

19.7 The pattern does persist.



19.8 a) The 15th triangular number is 120 and $121 = 11 \times 11$ is not prime!
b) The 18th triangular number is 171 and $169 = 13 \times 13$ is not prime!
This is a false pattern.

19.9 496 is the 31st triangular number.

19.10 The magic number is always a triangular number!
If you are game, here's a piece I wrote explaining what is going on.