Chapter 5

Fractions

32. Too Many Models for Intuition

Many people fear fractions. And the reasons for this fear are natural and absolutely appropriate!

I think that the school curriculum, where one learns about fractions, is caught between too contradictory demands: the demand to be age appropriate as it develops the story of what a fraction is, and the demand of the truth of what mathematics says a fraction is. The latter has to be sophisticated and abstract conversation along the lines we had in section 12:

> "Real world examples lead us to believe that there are these things called the *counting numbers*, along with two operations, *addition* and *multiplication*, that behave according to the following rules."

An honest discussion of fractions requires a similar discussion.

The typical curriculum is good at offering lots of "real world examples" that illustrate how fractions should behave in different contexts. But it is not typical for a curriculum to pause at some point later on and take stock of matters, that is, to look back at all the various intuitive models discussed and try to develop the mathematics that encapsulates and summarizes all the observed behavior. In short, the curriculum never answers: *Here's what mathematics says a fraction is.*

You, as reader of this text, I think are ready for this discussion.

So, let's start by looking back at the ways fractions are introduced in the typical curriculum. I'll offer here my personal recollection of matters. But I wonder if what I describe in the pages that follow resonates for you with regard to your early experiences with fractions?

WARNING: **Read the following at your own peril!**

What I am about to share—although likely familiar—is going to come across as a brain-hurty jumble of ideas. It is! Our job for later in this chapter is to find a consistent and clear way to mathematically organize this jumble of ideas.

Okay … on to the jumble.

VERY EARLY GRADES: Fractions are "parts of a whole." Nothing more.

I remember as a very young student being given problems like the following.

Example: *Circle a third of these six kittens.*

Example: *Circle half the stars.*

Clearly, the goal of the curriculum at this point was simply to help me become comfortable with the words, *half*, *third*, *quarter*, *fifth*, and so on, that is, to become familiar with the idea of dividing sets into equal parts and using the names that go with those equal parts.

Here, fractions don't seem to be numbers at all. They are more "calls to action," that is, things to do. If someone asked me at this age, "What is a half plus a third?" I might have answered with this picture. Do you see why?

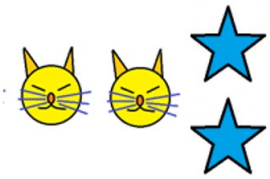


Figure $\frac{1}{3}+\frac{1}{2}$?

I think I might have been introduced to the idea that *two thirds*, for example, is as the wording suggests: two copies of one third.

Figure Two thirds

In the same way, "three fifths" would be "three copies of one fifth." And so on.

LESS EARLY GRADES: Fractions should come from the same whole.

The previous page shows that fractions coming from different sets—sets of kittens and sets of stars, for instance—are hard to compare and combine. At some point in the curriculum, it becomes important to insist that fractions come from the same whole.

In my schooling, the "same whole" was always a pie.

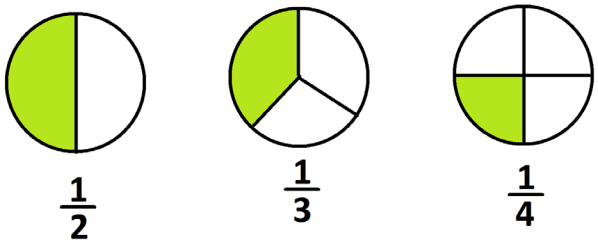


Figure Fractions from a round pie.

We started using the notation $\frac{1}{2}$ for "one part of a one pie divided into two equal parts" and $\frac{1}{3}$ for "one part of one pie divided into three equals parts," and so on.

I don't know why it took me so long, but I eventually realized it is easier to draw square (or rectangular) pies. (Why does everyone draw circular pies?)

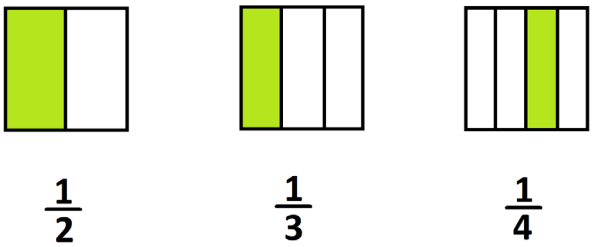
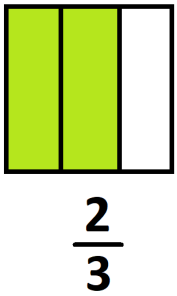
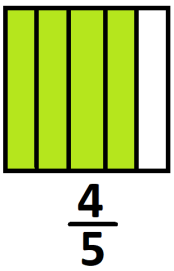


Figure Square pies are easier to draw

In this model, I learned that $\frac{2}{3}$ is indeed interpreted as "two copies of $\frac{1}{3}$" (just as saying "two thirds" out loud suggests) and $\frac{4}{5}$ is interpreted as "four copies of $\frac{1}{5}$" (just as saying "four fifths" out loud suggests), and so on. This meshed well with the idea that "multiplication is repeated addition," as I was learning around the time.



$$\textit{Figure } \frac{2}{3} = \text{two copies of a third} = \frac{1}{3} + \frac{1}{3} = 2 \times \frac{1}{3}$$



$$\textit{Figure } \frac{4}{5} = \text{four copies of a fifth} = \frac{1}{5} + \frac{1}{5} + \frac{1}{5} + \frac{1}{5} = 4 \times \frac{1}{5}$$

In more adult language, I was being told to interpret a fraction $\frac{a}{b}$ as "a copies of $\frac{1}{b}$," where $\frac{1}{b}$ is one part of a whole (a pie) being divided into b equally-sized parts. (Got that?)

These things being called "fractions"—whatever they actually are—are more number-like now. But I still don't think I was required to think of fractions actually as numbers at this point.

Question: Suppose I was required to think of $\frac{1}{2}$ and $\frac{1}{3}$ as numbers at this point in my education. Here $\frac{1}{2}$ represents half a pie and $\frac{1}{3}$ represents a third of a pie.

If they really are numbers, we should be able to add them and multiply them.

Addition: I think $\frac{1}{2}+\frac{1}{3}$ means: "Push half a pie and a third of a pie together." That makes sense.

Multiplication: What on Earth could $\frac{1}{2}\times\frac{1}{3}$ mean? What is "half a pie times a third of a pie"?

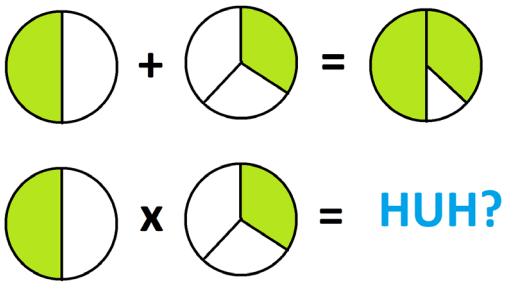


Figure $\frac{1}{2}+\frac{1}{3}$ makes sense, but $\frac{1}{2}\times\frac{1}{3}$ makes no sense.

I can't make sense of multiplication here. Can you?

SLIGHTLY LATER GRADES: Fractions are numbers on the Number Line

At some point in my schooling, we constructed a geometric model of the number system, namely, the **number line**. This line is composed of unit lengths stacked together. The number 4 on the line, for instance, indicates that four unit lengths stacked together from the zero reference point reach that position on the line.

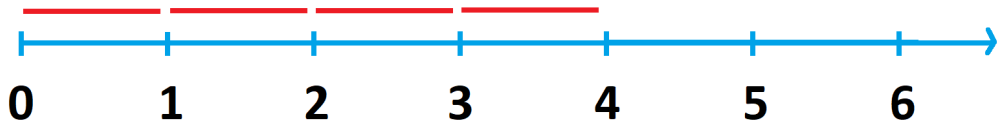


Figure 4 on the number line

I was then instructed to consider this unit length as my new "pie," that is, my new whole, that could be divided into fractional parts. For example, $\frac{1}{2}$ represents half a unit length and we can mark on the number line the reach of this part of the whole.

Figure $\frac{1}{2}$ on the number line

And since I was taught that $\frac{7}{2}$, for example, means $7 \times \frac{1}{2}$, that is , seven copies of a half, I could mark this point on the number line too. I could see this is the same as three plus and an extra half.

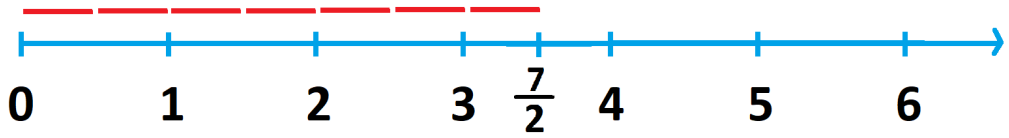


Figure $\frac{7}{2}$ on the number line

Question: Were you also taught to recognize, on the number line at least, that $\frac{6}{2}$ is the same as 3, and that $\frac{7}{3}$ is the same as $2+\frac{1}{3}$, and so on?

I see now, looking back, that the curriculum I went through was really trying to say to me:

> "James. Fractions are numbers! We're now putting them on the number line. They just must be numbers!"

If I thought through this (I didn't) I might have had the same questions as before.

> Okay then. If fractions are numbers, I should be able to add them and multiply them.
> I see how to make sense of something like $\frac{3}{4}+\frac{1}{2}$.
> But $\frac{3}{4}\times\frac{1}{2}$ is still meaningless, even on a number line!

Figure $\frac{3}{4}+\frac{1}{2}$. But what is $\frac{3}{4}\times\frac{1}{2}$?

LATER GRADES: Fractions really are Numbers, plus a jumble of previous ideas

At some point I was just told that fractions really are numbers. In particular, they are the answers to division problems. For example, $\frac{7}{3}$ is the answer to the division problem $7 \div 3$, seven divided by three, and $\frac{1}{2}$ is the answer to the division problem $1 \div 2$, one divided by two. I was given the impression that this is "obvious" (and thus, as a dutiful student, I didn't question it).

Question: We are adults. Let's now question this!

The number-line model makes me think that $1 \div 2$ and $\frac{1}{2}$ being the ssame is plausible.

Figure $1 \div 2$ matches $\frac{1}{2}$

But is it obvious that $7 \div 3$ matches $\frac{7}{3}$ (seven copies of $\frac{1}{3}$)?

Figure $7 \div 3$

Even trying to follow what I am asking here is making my own brain hurt!

But it gets worse!

I remember, at the same time, learning about "equivalent fractions." I was being shown pictures that weren't about number lines. (Are they pies? I thought we were now focusing on number lines.)

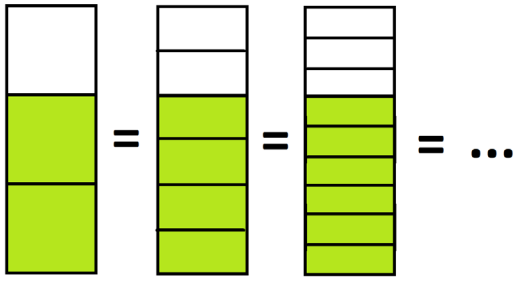


Figure $\frac{2}{3} = \frac{4}{6} = \frac{6}{9} = \cdots$

The idea of "equivalent fractions" allowed me to add fractions. For example, think of $\frac{1}{3} + \frac{1}{2}$ as $\frac{2}{6} + \frac{3}{6}$. Clearly "two-sixths and three-sixths" makes "five-sixths:" $\frac{1}{3} + \frac{1}{2} = \frac{5}{6}$. This is just as "easy" as two apples plus three apples makes five apples. (Is it really this easy?)

Then something really strange occurs: "OF MEANS MULTIPLY"

I remember being give tasks like the following to do.

Example: Draw a picture of two thirds of four fifths.

To do this, I was instructed to go back to draw a (square) pie. Here's $\frac{4}{5}$ of a pie.

Figure $\frac{4}{5}$ of a pie

And here is two-thirds of $\frac{4}{5}$ of a pie.

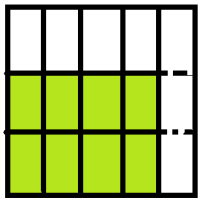


Figure Two thirds of $\frac{4}{5}$ of a pie

This look like a pie divided into 15 equals parts with 8 of those parts highlighted. I was instructed to say then that

$$\frac{2}{3} \text{ of } \frac{4}{5} \text{ equals } \frac{8}{15}.$$

What's going on exactly?

But the strangeness continued and got even stranger! The following conversation occurred.

Do you remember our area model for multiplication?
It looks like we have $\frac{2}{3}$ *on one side and* $\frac{4}{5}$ *on the other.*
And the picture looks like it is showing that $\frac{2}{3} \times \frac{4}{5} = \frac{8}{15}$.

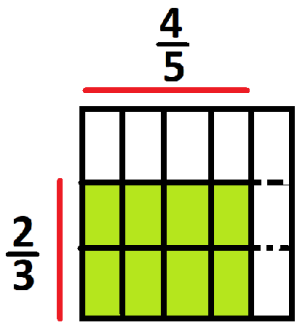


Figure Pretending that the area model makes sense in this context

Now look closely at what we wrote: $\frac{2}{3} \times \frac{4}{5} = \frac{8}{15}$.

I was to deduce:

To multiply fractions, just multiply the numerators (the top numbers) and multiply the denominators (the bottom numbers).

Question: Going back to image just above … Ummm. What's the "whole" here?
Is it the left side of the pie? The top side of the pie? The whole pie?
All three different things all at once?

(Don't try to answer this question: There are three different "wholes" going on here all at once and it is not at all clear if this is really allowed.)

When it came to dividing fractions, I was simply taught to “flip and multiply.” No explanation was given. For example, to compute $\frac{2}{3} \div \frac{4}{5}$ just compute $\frac{2}{3} \times \frac{5}{4}$ instead. This equals $\frac{10}{12}$.

Question: Why on Earth is “flip and multiply” the right thing to do to divide fractions? (This is completely and utterly mysterious!)

HIGH SCHOOL

By ninth grade my training on fractions was “complete.” I was not allowed to have any questions about them from that point on.

> **Question:** After reading the past number of pages (and recalling your own school experience in learning fractions), do you personally feel you “get” fractions?

I didn’t “get” fractions. And the way I coped from this point on was to just memorize the rules for adding, subtracting, multiplying, and dividing fractions, and simply declaring that any picture I drew that could be described with the word “of” was a picture of fractions being multiplied.

I really had very little to hold on to or to make coherent sense of.

MUSINGS

Musing 32.1 We think of $\frac{5}{7}$ of "five copies of a seventh."

$$\frac{5}{7} = \frac{1}{7}+\frac{1}{7}+\frac{1}{7}+\frac{1}{7}+\frac{1}{7} = 5\times\frac{1}{7}$$

We counted out five copies of a "type of thing," namely, a seventh.

The late Latin word for "a count of things" is *numerator* and the word for "the type of a thing" is *denominator*. These are the words we use today for the two numbers in fraction notation. (In the fraction $\frac{5}{7}$, we have that 5 is the numerator and 7 is the denominator.)

These two words are used today almost solely in the context of fractions, but we have the related words being used in everyday like. For example *enumerate* means to count out something and *denomination* means the name of a certain type.

a) What is the official name of the horizontal line used in fraction notation? (Most people call it a "fraction bar," but what is its Latin name? Why that name?)

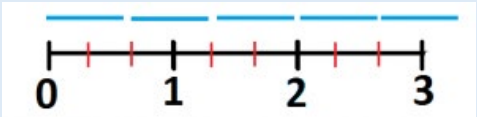
b) Care to look up the general history of fraction notation? Why do we write fractions the way we do today?

Musing 32.2 Some curricula have students play with the division of fractions using a number line. They have students draw a picture and ask: "How many groups of what you are looking for do I see?"

Here are two examples.

Example: *Draw a picture to evaluate* $3\div\frac{2}{3}$.

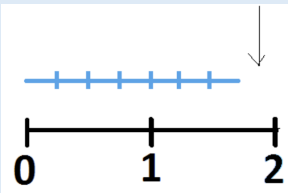
Answer:



We see four-and-a-half copies of $\frac{2}{3}$ in 3. We have $3\div\frac{2}{3}=4\frac{1}{2}$.

Example: *Draw a picture to evaluate* $2 \div 1\frac{3}{4}$.

Answer: We see that if we had one more seventh of the blue segment shown, we'd have a complete "2." Thus, we see one full $1\frac{3}{4}$ and one seventh of $1\frac{3}{4}$ in 2.



This means $2 \div 1\frac{3}{4} = 1\frac{1}{7}$.

a) Do you like what's going on in these two examples?
(The answer can be: "No I don't! I don't get it. I don't like it. And I am turning the page.)

b) Would you like to complete each of these with number-line pictures?

i) $2\frac{1}{2} \div \frac{3}{4}$ *ii)* $\frac{2}{3} \div \frac{1}{2}$ *iii)* $\frac{1}{2} \div 3$.

(The answer can be no.)

33. One (still not perfect) Model

There is one model of fractions that, although not perfect nor complete, I think would have helped me tremendously in high school. It would have given me a good blend of the concrete and "real world," with the necessary abstract thinking to push beyond the concrete and literal.

I call it the **Pies-per-Student model**.

It is based on the idea that "a fraction is an answer to a division problem." And the type of division I am going to think about is sharing. So, here:

> *A fraction is the answer to a sharing problem.*

What shall we share?

You guessed it: pies. We'll share pies equally among students. (And using pies allows us to see links to the early-grade thinking we all went through.)

Okay … here's the pie-per-student model.

33.1 Pies and Students

Suppose I have 6 pies to share equally among 3 students. That makes for 2 pies per student.

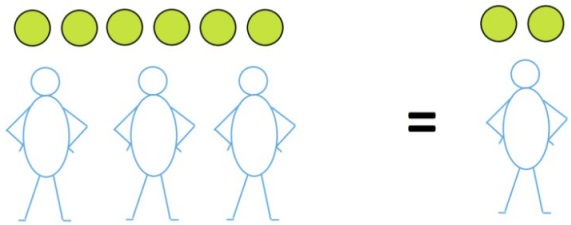


Figure 6 pies for 3 students is 2 pies per student

Now most people would express this as $6 \div 3 = 2$, as this is a division problem. But since I am saying that a fraction *is* the answer to a division problem, I am going to use fraction notation for this and write $\frac{6}{3} = 2$.

$$\frac{6}{3} = 2$$

Figure 6 pies for 3 students is 2 pies per student

In the same way

Sharing 20 pies equally among 5 students yields $\frac{20}{5} = 4$ pies per student,

and

Sharing 100 pies among 2 students yields $\frac{100}{2} = 50$ pies per student.

Figure The "pies per student" model

Consider the sharing problem $\frac{1}{2}$, the act of sharing just 1 pie equally among 2 students.

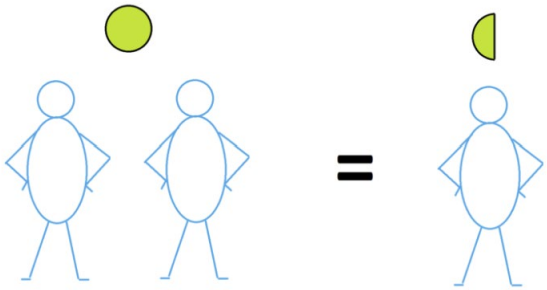


Figure $\frac{1}{2}$

This gives, what we called in our early days a "half." Each student receives half a pie.

$$\frac{1}{2} = \text{half}$$

In the same way,

$$\frac{1}{3} = \text{a third}$$

and $\frac{1}{4}$ is what we called a *quarter*, and so on. This model matches some of early work.

We even have that $\frac{2}{3}$ is "two copies of one third," just as before. To see this, imagine how you would physically share two pies equally among three students. You would probably divide each pie into three equal portions (thirds) and give a portion from each pie to each student. In this way, each student literally received two copies of one third of a pie!

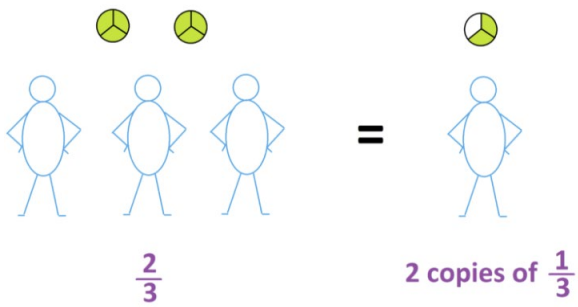


Figure $\frac{2}{3}$

Exercise: Draw a picture to show the process and result of the sharing problem $\frac{4}{3}$.

Are you seeing four copies of one third?

33.2 Going Backwards

Let's have some fun going backwards.

Question: Here's the result of a sharing problem.

What was the question? That is, how many pies were shared among how many students to produce this result?

Well, since this pie is divided into 5 equal parts, one might guess that 5 student were involved. And another moment's thought has suggests that just 1 pie was shared among them.

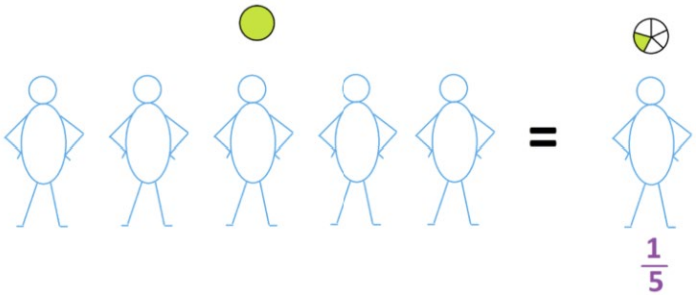


Figure $\frac{1}{5}$

Question: Consider now this answer to a sharing problem. How many students? How many pies?

We see "fifths," again, as the basis of this picture, suggesting 5 students were involved. But we have 3 copies of one fifth in this picture. The previous problem of one pie shared among five students gave one copy of a fifth per student, so it must be the case that now 3 pies were shared equally among those five students—each student receives a copy of a fifth from each of the three.

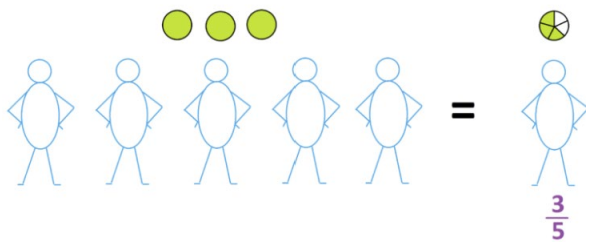


Figure $\frac{3}{5}$

Let's now go up a notch in difficulty.

Question: Here's the answer to a sharing problem. How many pies? How many students?

Again, "fifths" seem to be at the base of matters. A good start is to guess that five students, again, are involved.

Each student receives a fifth of a pie (and more!) and sharing 1 pie among them accounts for that fifth each. But each student also receives an additional two pies each. This suggests that 11 pies were shared among them.

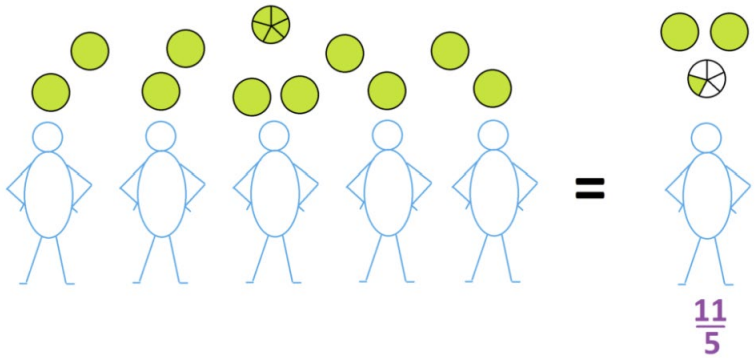


Figure $\frac{11}{5}$

One can also see 11 copies of a fifth directly in the amount of pie given to each student.

Figure 11 copies of $\frac{1}{5}$

Also, we have not made the claim that the solutions we're offering in these challenges are unique. In fact, other solutions are possible.

For example, one can also interpret the amount of pie shown above as 22 copies of a tenth, and so we could also say that we shared 22 pies equally among 10 students instead. And I am sure there are many more, equally valid, alternative solutions.

Figure 22 copies of $\frac{1}{10}$

Practice 33.1 Why am I drawing circular pies? Let's draw rectangular pies.

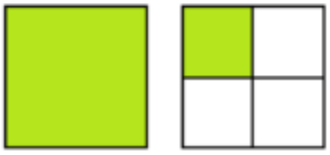
Each diagram below shows the amount of pie each student received when some number of rectangular pies were shared equally among some number of students. Determine a possible count of pies and matching count of students for each picture.

Warning: Look at the firs picture. From our early-grade training we will likely write $\frac{2}{9}$ as a knee-jerk response. Although correct, make sure that you truly can see the picture as the result then of sharing 2 rectangular pies equally among 9 students.

a)

b)

c)



Practice 33.2: Is $\frac{10}{13}$ double $\frac{5}{13}$?
How would you justify this as so using the model of sharing pies among students?

33.3 Some Apparent Properties of Fractions

Suppose we have 5 pies to share among one (lucky) student. How many pies per student is that? It's 5 pies per student!

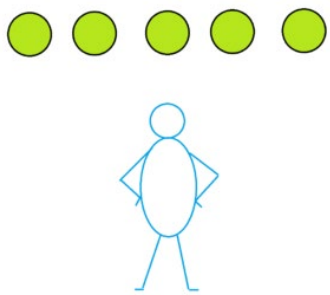


Figure $\frac{5}{1} = 5$

In the same way 20 pies for one (even luckier) student makes for 20 pies per student, and 503 pies for one (extra lucky?) student makes for 503 pies per student.

$$\frac{20}{1} = 20$$
$$\frac{503}{1} = 503$$

In general, it seems we have

$$\boxed{\frac{a}{1} = a \text{ for each number } a.}$$

(This seems true for positive counting numbers, at least. But maybe it is true for other types of numbers too?)

Now suppose we share 5 pies equally among 5 students. How many pies per student does that yield?

It makes for 1 pie per student.

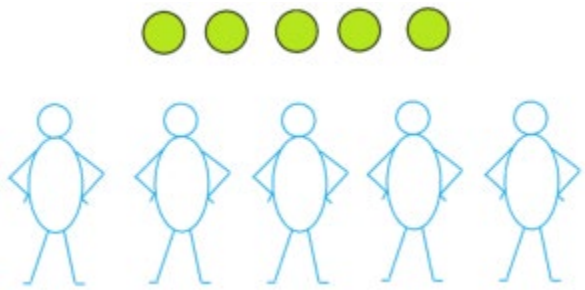


Figure $\frac{5}{5}=1$

In the same way, 20 pies shared equally among 20 students makes for 1 pie per student, as does sharing 503 pies among 503 students.

$$\frac{20}{20}=1$$
$$\frac{503}{503}=1$$

It seems we also have

$$\boxed{\frac{a}{a}=1 \text{ for each number } a.}$$

(This seems true for positive counting numbers, at least. But maybe it is true for other types of numbers too?)

Practice 33.3 "I have no pies to share among 7 students."
Might this scenario lead to another basic property of fractions?

Consider again the scenario of sharing 6 pies equally among 3 students.

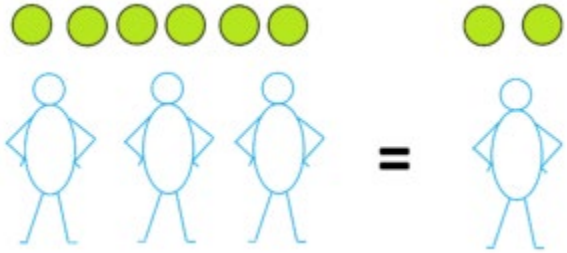


Figure $\frac{6}{3} = 2$

Suppose I am feeling generous and want to double the amount of pie each student receives. How might I do that? Well, I would have to double the count of pies to 12 pies.

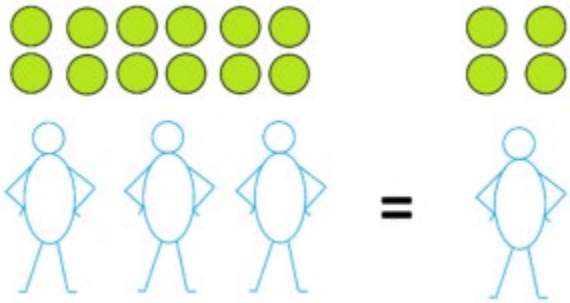


Figure $2 \times \frac{6}{3} = \frac{12}{3}$

Tripling the total count of pies triples the amount of pie each student receives.

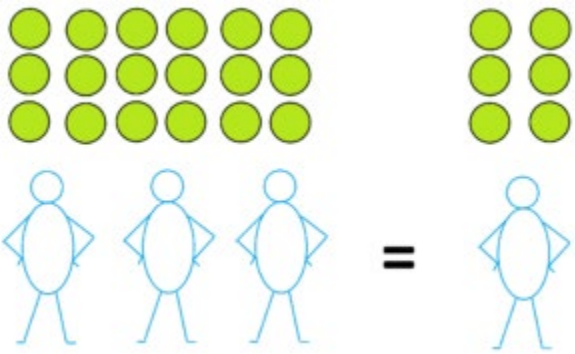


Figure $3 \times \frac{6}{3} = \frac{18}{3}$

In general, $\frac{a}{b}$ is the amount of pie each student receive when sharing a pies equally among b students. To double, triple or quadruple or pentuple the amount of pie each student receives, change the number of pies we give out to $2a$ or $3a$ or $4a$ or $5a$.

We have

$$k \times \frac{a}{b} = \frac{ka}{b} \text{ for a number } k \text{ and a fraction } \frac{a}{b}.$$

Again, this seems right for positive counting numbers, as least. To change the amount of pie each student receives by a factor k , change the number of pies by that factor .

Example: We have that $5 \times \frac{2}{3} = \frac{10}{3}$.

Here's a "consistency check" of exercise.

$\frac{2}{3}$ is two copies of one third: $\frac{1}{3} + \frac{1}{3}$.

$5 \times \frac{2}{3}$ is five copies of this: $\left(\frac{1}{3}+\frac{1}{3}\right)+\left(\frac{1}{3}+\frac{1}{3}\right)+\left(\frac{1}{3}+\frac{1}{3}\right)+\left(\frac{1}{3}+\frac{1}{3}\right)+\left(\frac{1}{3}+\frac{1}{3}\right)$.

This is indeed $\frac{10}{3}$, ten copies of one third.

All is hanging together.

Practice 33.4 In a sharing scenario, some pies were shared equally among 7 students. If each student received 3 pies, how many pies were there in total? (That is, if $\frac{a}{7} = 3$, what is a ?)

More generally, if $\frac{a}{b} = n$, what is a ?

33.4 ASIDE: Something Quirky

Okay! Let's be wild, and let's get ahead of ourselves by putting some non-counting numbers in unexpected places. This aside is just for quirky fun.

Let's try to make sense of $\frac{1}{1/2}$.
If one pie is shared "equally among" half a student, how much pie per student is that?

We have one pie for half a student. But all our work so far has been about finding the total amount of pie each student receives, that is, the amount each *whole* student receives.

It seems we do have to look at the two halves of a student.

Ahh! If each half student is assigned one pie, that makes for 2 pies for the whole student!

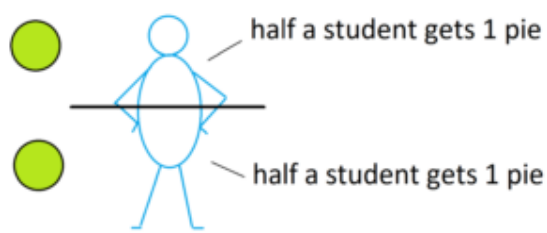


Figure $\frac{1}{1/2} = 2$

Whoa!

In the same way, distributing one pie to each third of a student yields 3 pies for the whole student.

$$\frac{1}{1/3} = 3$$

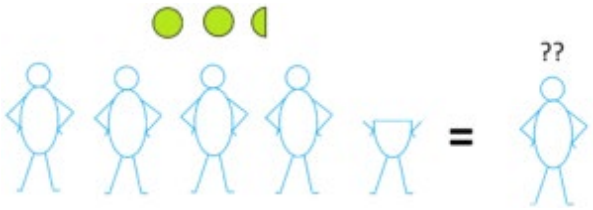
And distributing one pie to each N th of a student gives a total of N pies to the full student.

$$\frac{1}{1/N} = N$$

Question: Can you reason that $\frac{5}{1/7}$ must be 35?

Challenge: Can you make sense of $\frac{8}{2/3}$?

Absurd Challenge: Two-and-a-half pies are to be shared equally among four-and-a-half students! How much pie does an individual (whole) student receive?



Actually, don't do this challenge. We'll find an easy way to handle it soon enough.

33.5 More Apparent Properties of Fractions

A fraction is an answer to a sharing problem.

$\frac{a}{b}$ represents the amount of pie an individual student receives when a pies are distributed equally among b students.

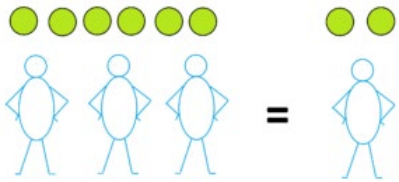


Figure $\frac{6}{3} = 2$

What happens if we double the number of pies and double the number of students? Nothing! The amount of pie per student is still the same.

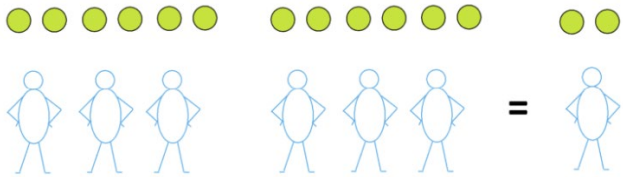


Figure $\frac{2 \times 6}{2 \times 3} = \frac{6}{3} = 2$

Tripling both the number of pies and the number of students also affects no change in the final amount of pie given to each student. Nor does quadrupling the counts of each or one-trillion-billion-tupling the each of numbers!

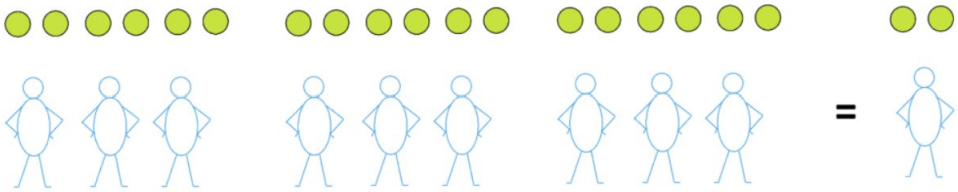


Figure $\frac{3 \times 6}{3 \times 3} = \frac{6}{3} = 2$

In general, we have $\frac{a}{b} = \frac{2a}{2b} = \frac{3a}{3b} = \frac{4a}{4b} = \cdots$.

$$\frac{ka}{kb} = \frac{a}{b} \text{ for each number } k \text{ and each fraction } \frac{a}{b}.$$

This seems true for positive counting numbers k. But maybe it is true for other types of numbers too?

For example, sharing three pies among five students

$$\frac{3}{5}$$

yields the same result as

$$\frac{2 \times 3}{2 \times 5} = \frac{6}{10},$$

that is, as sharing six pies among ten students, and as

$$\frac{100 \times 3}{100 \times 5} = \frac{300}{500},$$

sharing 300 pies among 500 students.

Going backwards, sharing 20 pies among 32 students

$$\frac{20}{32}$$

is the same problem as

$$\frac{4 \times 5}{4 \times 8} = \frac{5}{8},$$

sharing five pies among eight students.

Many people say we have **cancelled** a common factor of 4 from with the fraction and, in doing so, we have **simplified** the expression $\frac{20}{32}$. Indeed, $\frac{5}{8}$ is easier to conceptualize than $\frac{20}{32}$. Some might say that we have **reduced** the fraction. This verb is a misleading, alas. It is true that we have reduced the count of pies and the count of students, but the fraction itself has not changed at all. It is the still the same final amount of pie per student. A "reduced fraction" is just the same size as a "non-reduced" one!

As another example $\frac{280}{350}$ can certainly be made to look more manageable by noticing that there is a common factor of 10 in both the count of pies and the count of students. Thus

$$\frac{280}{350} = \frac{10 \times 28}{10 \times 35} = \frac{28}{35}.$$

We can go further as 28 and 35 are both multiples of 7.

$$\frac{28}{35} = \frac{7 \times 4}{7 \times 5} = \frac{4}{5}$$

Thus, sharing 280 pies among 350 students gives the same result as sharing just 4 pies among 5 students, which is much easier to conceptualize.

$$\frac{280}{350} = \frac{4}{5}$$

As 4 and 5 share no common factors, this is as far as we can go with this example (while staying with whole numbers).

Question: Jennie says that $\frac{4}{5}$ does "simplify" further is you are willing to move away from whole numbers. She writes

$$\frac{4}{5} = \frac{2 \times 2}{2 \times 2\frac{1}{2}} = \frac{2}{2\frac{1}{2}}.$$

Is she right? Does sharing 4 pies among 5 students yield the same result as sharing 2 pies among 2½ students? What do you think? (And is her expression "simpler" than the original?)

33.6 A Logical Consequence of our Four Properties

We've identified four properties of fractions in this pies-per-student model of fractions.

Property 1: $\frac{a}{1} = a$ for each number a

Property 2: $\frac{a}{a} = 1$ for each number a.

Property 3: $k \times \frac{a}{b} = \frac{ka}{b}$ for a number k and a fraction $\frac{a}{b}$.

Property 4: $\frac{ka}{kb} = \frac{a}{b}$ for each number k and each fraction $\frac{a}{b}$.

Here's a question:

What is $7 \times \frac{3}{7}$?

We can work it out!

By Property 3 this is

$$\frac{7 \times 3}{7}.$$

By our ordinary rules of arithmetic, we can write the 7 in the denominator as 7×1. So, our quantity is

$$\frac{7 \times 3}{7 \times 1}.$$

By Property 4, this is $\frac{3}{1}$, which is just 3 by property 1.

Figure $7 \times \frac{3}{7} = 3$

I remember being taught that “multiplying a fraction by its denominator just “cancels” the denominator. I can see now the logic behind this!

Logical Consequence 5: $b \times \frac{a}{b} = a$ for each fraction $\frac{a}{b}$.

We have, for instance

$$3 \times \frac{20}{3} = 20$$

$$5 \times \frac{4}{5} = 4$$

$$17 \times \frac{92}{17} = 92$$

and

$$\frac{92}{17} \times 17 = 92$$

(since, for from the general rules of arithmetic, we believe that $a \times b = b \times a$).

We also have

$$1 \times \frac{20}{1} = 20$$

(though this example might be true for other reasons too!)

33.7 Comparing Fractions

Here's a fun question.

> Which is larger: $\frac{5}{9}$ or $\frac{6}{11}$?
> Does sharing 5 pies equally among 9 students yield more pie per student than sharing 6 pies among 11 students? Or does sharing 6 pies among 11 produce more pie per student than sharing 5 pies equally among 9 students?

It's hard to tell!

One way to compare these fractions is to rewrite each of them, via Property 4, with at least one count—either the count of pies or the count of students—the same. For instance, we have

$$\frac{5}{9}=\frac{5\times 11}{9\times 11}=\frac{55}{99}$$

$$\frac{6}{11}=\frac{6\times 9}{11\times 9}=\frac{54}{99}$$

Now we are comparing the results of sharing 55 versus 54 pies among 99 students. It is clear that the first scenario provides more pie per student. The fraction $\frac{5}{9}$ is the larger of the two.

> **Question:** Consider writing instead
>
> $$\frac{5}{9}=\frac{5\times 6}{9\times 6}=\frac{30}{54}$$
>
> $$\frac{6}{11}=\frac{6\times 5}{11\times 5}=\frac{30}{55}$$
>
> Do you see again that the first scenario yields more pie per student?

Practice 33.5 Arrange the fractions $\frac{5}{9}$ and $\frac{6}{11}$ and $\frac{15}{28}$ in order from smallest to largest.

Practice 33.6 Which is larger: $\frac{1}{11}$ or $\frac{1}{12}$? (Actually think your way through this.)

33.8 Zero as a Denominator

The logical consequence we developed

Logical Consequence 5: $b \times \frac{a}{b} = a$ for each fraction $\frac{a}{b}$.

is powerful. It allows us to compare fractions, for example, and it also warns us about a danger with fractions.

Question: Is $\frac{5}{0}$ meaningful?

Answer: If $\frac{5}{0}$ is a meaning quantity, then our logical consequence says that

$$0 \times \frac{5}{0} = 5 .$$

On the other hand, our rules of arithmetic from section 12 (and then section 19) have that

Multiplying a quantity by zero gives zero.

In which case.

$$0 \times \frac{5}{0} = 0 .$$

We have two contradictory statements.

It must be that $\frac{5}{0}$ is <u>not</u> meaningful!

Question: In the pies-per-students, is the task of sharing 5 pies equally among no students meaningful?

We have then a caveat about fractions:

For a fraction $\frac{a}{b}$ to be mathematically meaningful, we require b not be zero.

33.9 Checking Values of Fractions

People like to use Logical Consequence 5 to check their fraction thinking.

Logical Consequence 5: $b \times \frac{a}{b} = a$ for each fraction $\frac{a}{b}$.

For example, suppose I am having trouble computing $\frac{20}{4}$. I think the answer is 6.

We can check.

We know that $4 \times \frac{20}{4} = 20$.

My guess makes this read: $4 \times 6 = 20$.

This is patently wrong.

In general, one can check whether or not the computation of a fraction is correct by performing multiplication. For example

$\frac{6}{3} = 2$ looks correct because 3 times 2 is indeed 6.

$\frac{20}{4} = 5$ looks correct because 4 times 5 is indeed 20.

$\frac{83}{9} = 11$ is definitely not correct because 9×11 is not 83.

$\frac{\frac{1}{2}}{5} = \frac{1}{12}$ is definitely not correct because $5 \times \frac{1}{12}$ is not $\frac{1}{2}$.

Practice 33.7

a) Cecille says that $\frac{5}{0}$ equals 2. Why is this definitely not correct?

b) Arjun says that $\frac{5}{0}$ equals 74. Why is this definitely not correct?

c) Maria says that there no number n such that $\frac{5}{0} = n$ is correct. Why is she right?

Practice 33.8

a) Cyril says that $\frac{0}{0} = 1$ looks like it could be correct. Do you agree?

b) Ethyl says that $\frac{0}{0} = 2$ looks like it could be correct. Do you agree?

c) Wonhi says that $\frac{0}{0} = 73$ looks like it could be correct. Do you agree?

d) Daune says that there can be no consistent answer to $\frac{0}{0}$. Do you agree?

We saw in the previous section showed that the we have an inconsistency in the rules of arithmetic if we allow a fraction to have a denominator of zero. These questions are re-affirming that!

33.10 Mixed Numbers

Some people are surprised that I start a discussion on fractions with examples with numerator larger than denominator.

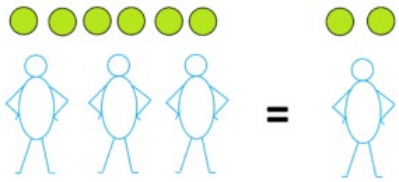


Figure $\frac{6}{3}$ has numerator larger than denominator

After all, the word "fraction" itself is derived from the old French word *fraccion* meaning "a break" causing an object to shatter into pieces. This suggests pieces smaller than the original whole. Yet I started off this section with the example $\frac{6}{3}$ with the big number answer 2.

(Maybe we're "breaking" the original group of six pies into smaller pieces?)

People sometimes call a fraction with numerator larger than the denominator an **improper fraction**. There is nothing "improper" about such a fraction. Nonetheless, people often choose to rewrite a given improper fraction in a way that only exhibits only a proper fraction. (They also like to do this because it helps give a sense of the size of an improper fraction.)

For example, consider the improper fraction $\frac{5}{2}$. As a pies-per-student problem, this corresponds to two-and-a-half pies per student.

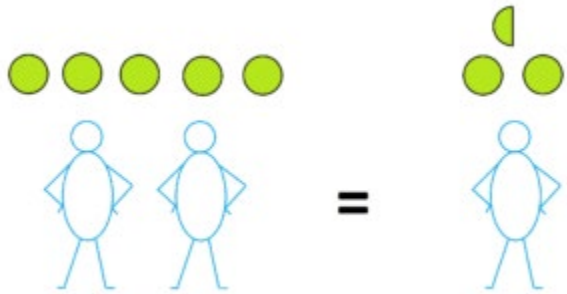


Figure $\frac{5}{2} = 2 + \frac{1}{2}$

In writing this answer. people usually omit the plus sign and express $2+\frac{1}{2}$ as $2\frac{1}{2}$, even they though still read $2\frac{1}{2}$ out loud as "two-<u>and</u>-a-half."

A **mixed number** is an expression of the form $a\frac{b}{c}$ with a a counting number and $\frac{b}{c}$ a fraction. It is really the number $a+\frac{b}{c}$.

Thus

$3\frac{4}{17}$ means $3+\frac{4}{17}$

(each student received 3 whole pies and $\frac{4}{17}$ of a pie in a sharing scenario),

and

$200\frac{1}{200}$ means $200+\frac{1}{200}$

(each student received 200 whole pies and $\frac{1}{200}$ of a pie in a sharing scenario),

and so on.

Practice 33.9*:* Show that the improper fraction $\frac{16}{3}$ is the mixed number $5\frac{1}{3}$.

Some Mixed-Number Arithmetic

Example: Please double $2\frac{1}{2}$.

The best way to handle arithmetic with mixed numbers is to reinsert the plus signs that should be there.

We have, using the distributive rule and our logical consequence 5,

$$\begin{aligned} 2\times 2\frac{1}{2} &= 2\left(2+\frac{1}{2}\right) \\ &= 4+2\times\frac{1}{2} \\ &= 4+1 \\ &= 5 \end{aligned}$$

Example: Kindly triple $5\frac{1}{3}$.

We have

$$\begin{aligned} 3\times 5\frac{1}{3} &= 3\left(5+\frac{1}{3}\right) \\ &= 15+1 \\ &= 16 \end{aligned}$$

Practice 33.10 a) Why is $\frac{10}{7}$ the same as $1\frac{3}{7}$?
(If you had to share 10 pies equally among 7 students, you could start by giving each student one whole pie, and then go from there.)

b) Why is $\frac{21}{8}$ the same as $2\frac{5}{8}$?

Practice 33.11 In a sharing problem each student received $200\frac{1}{200}$ pies.
How many pies (a whole number) might have been shared among how many students (also a whole number)?

Can you give more than one answer this problem?

33.11 Dividing Fractions

We've identified a number of properties of fractions with our pies-per-student model.

Property 1: $\frac{a}{1} = a$ for each number a

Property 2: $\frac{a}{a} = 1$ for each number a.

Property 3: $k \times \frac{a}{b} = \frac{ka}{b}$ for a number k and a fraction $\frac{a}{b}$.

Property 4: $\frac{ka}{kb} = \frac{a}{b}$ for each number k and each fraction $\frac{a}{b}$.

Logical Consequence 5: $b \times \frac{a}{b} = a$ for each fraction $\frac{a}{b}$.

(We're assuming there is no denominator of zero in any of the fractions.)

Let's now attend to doing arithmetic with fractions.

Perhaps surprisingly, division is the easiest operation to first explore. In fact, let's start by dividing mixed numbers.

Compute $2\frac{1}{2} \div 4\frac{1}{2}$.

Recall, we started of this section by interpreting the act of division as sharing. This division problem reads:

Share four-and-a-half pies equally among two-and-a-half students.
How many pies does each whole student receive?

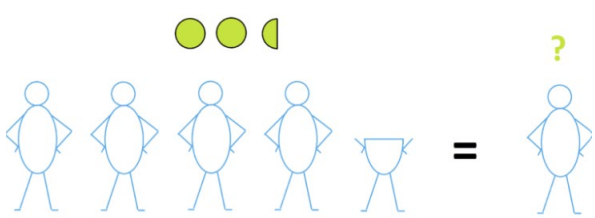


Figure $2\frac{1}{2} \div 4\frac{1}{2}$

We've been using fraction notation to denote such division problems, so our task is to make sense of

$$\frac{2\frac{1}{2}}{4\frac{1}{2}}.$$

The halves appearing in the numerator and the denominator are annoying.

How then might we make matters less complicated?

Let's double the numerator and double the denominator. That is, lets double the count of pies and double the count of students. We know this changes nothing. (Property 4.)

$$\frac{2\frac{1}{2}}{4\frac{1}{2}} = \frac{2+\frac{1}{2}}{4+\frac{1}{2}} = \frac{2\left(2+\frac{1}{2}\right)}{2\left(4+\frac{1}{2}\right)} = \frac{4+1}{8+1} = \frac{5}{9}$$

Sharing two-and-a-half pies equally among four-and-a-half students is equivalent to sharing five pies among nine students. Each student gets five-ninths of a pie. That's much more manageable to envision!

Another example.

Compute $7\frac{2}{3} \div 5\frac{3}{4}$.

In fraction notation this is $\frac{7\frac{2}{3}}{5\frac{3}{4}}$, which is quite ghastly. We have thirds in the numerator and fourths in the denominator.

To contend with the third in the numerator, let's multiply both the top and bottom of this fraction each by 3.

$$\frac{7\frac{2}{3}}{5\frac{3}{4}} = \frac{7+\frac{2}{3}}{5+\frac{3}{4}} = \frac{3\left(7+\frac{2}{3}\right)}{3\left(5+\frac{3}{4}\right)} = \frac{21+3\times\frac{2}{3}}{15+3\times\frac{3}{4}}$$

There is some complicated stuff still in the expression. We have $3\times\frac{2}{3}$ and $3\times\frac{3}{4}$.

But by Logical Consequence 5,

$$3\times\frac{2}{3}=2$$

And by property 3,

$$3\times\frac{3}{4}=\frac{9}{4}.$$

This gets us a bit further.

$$\frac{7\frac{2}{3}}{5\frac{3}{4}}=\frac{21+3\times\frac{2}{3}}{15+3\times\frac{3}{4}}=\frac{21+2}{15+\frac{9}{4}}$$

We still have an annoying fourth. Let's multiply numerator and denominator each by 4 to handle this.

$$\frac{21+2}{15+\frac{9}{4}}=\frac{4(21+2)}{4\left(15+\frac{9}{4}\right)}=\frac{84+8}{60+4\times\frac{9}{4}}$$

And this equals

$$\frac{84+8}{60+9}=\frac{92}{69}.$$

This shows that sharing $7\frac{2}{3}$ pies equally among $5\frac{3}{4}$ students is equivalent to sharing 92 pies among 69 students. (Is that friendlier?)

Well, each of the 69 students will receive one pie plus the result of sharing the remaining $92-69=23$ pies. We see

$$\frac{92}{69}=1\frac{23}{69}$$

But we can keep going!

We have that $\frac{23}{69} = \frac{1 \times 23}{3 \times 23} = \frac{1}{3}$ by property 4. So, $\frac{7\frac{2}{3}}{5\frac{3}{4}}$ actually equals $1\frac{1}{3}$.

Sharing $7\frac{2}{3}$ pies equally among $5\frac{3}{4}$ students gives one-and-a-third pies per student. Whoa!

Another example.

Compute $\frac{3\frac{1}{2}}{1\frac{1}{2}}$.

Multiplying the numerator and denominator each by 2 would be a good move.

$$\frac{3\frac{1}{2}}{1\frac{1}{2}} = \frac{2\left(3+\frac{1}{2}\right)}{2\left(1+\frac{1}{2}\right)} = \frac{6+1}{2+1} = \frac{7}{3}$$

enough to make the expression look friendlier.

Consider these four examples. What could you do with each expression to make it look more manageable?

a) Make $\dfrac{4\frac{2}{3}}{5\frac{1}{3}}$ look friendlier.

b) Make $\dfrac{2\frac{1}{5}}{2\frac{1}{4}}$ look friendlier.

c) Make $\dfrac{1\frac{4}{7}}{2\frac{3}{10}}$ look friendlier.

d) Make $\dfrac{\frac{3}{5}}{\frac{4}{7}}$ look friendlier.

The fourth one is particularly interesting. It is really asking us to divide two fractions. It wants us to compute $\frac{3}{5} \div \frac{4}{7}$. (But let's leave the problem in fraction in notation.)

Let me work on this one

I don't like the fifths in the numerator. Multiply numerator and denominator each by 5. (I'll use properties 3 and 5 along the way.)

$$\frac{\frac{3}{5}}{\frac{4}{7}} = \frac{5 \times \frac{3}{5}}{5 \times \frac{4}{7}} = \frac{3}{\frac{20}{7}}$$

I don't like sevenths in the denominator. Multiply the numerator and denominator each by 7.

$$\frac{3}{\frac{20}{7}} = \frac{7 \times 3}{7 \times \frac{20}{7}} = \frac{21}{20}$$

We have just learned how to divide fractions!

Practice 33.12: Compute $\frac{2}{3} \div \frac{5}{7}$ this way.

Here's a general, abstract example.

Compute $\frac{a}{b} \div \frac{c}{d}$.

Can you follow this line of reasoning? I am just mimicking the example we just did, but with abstract numbers this time. (Make sure you can see what is happening from one equals sign to the next.)

$$\frac{a}{b} \div \frac{c}{d} = \frac{\frac{a}{b}}{\frac{c}{d}} = \frac{b \times \left(\frac{a}{b}\right)}{b \times \left(\frac{c}{d}\right)} = \frac{a}{\frac{bc}{d}} = \frac{d \times a}{d \times \left(\frac{bc}{d}\right)} = \frac{da}{bc} = \frac{ad}{bc}$$

Of course, there is no need to memorize this "rule" for dividing fractions. If given the challenge of dividing two fractions, just rewrite the problem in fraction notation and make all the fractions within the fraction go away!

Here are some practice problems to try if you like.

Practice 33.13 Compute each of the following.

a) $\frac{1}{2} \div \frac{1}{3}$ b) $\frac{4}{5} \div \frac{3}{7}$ c) $\frac{2}{3} \div \frac{1}{5}$

Practice 33.14 Compute $\frac{45}{45} \div \frac{902}{902}$. (Or can you see what the answer must simply be?)

Practice 33.15 Compute $\frac{10}{13} \div \frac{2}{13}$. Any general comments about this one?

Practice 33.16 Compute $\frac{3/4}{2/\left(\frac{3}{5}\right)}$.

Practice 33.17 Compute $\dfrac{1}{a/b}$.

Here are some more challenging thinking questions. (Remember, everything in this book is optional reading and doing!)

Practice 33.18 Some curricula have students solve division problems by rewriting terms to have a common denominator (using property 4). For example, to compute

$$\frac{3}{4} \div \frac{2}{3}$$

rewrite the problem as

$$\frac{9}{12} \div \frac{8}{12}$$

The claim is then made that the answer to the original problem is $\dfrac{9}{8}$, the two numerators you see, the first divided by the second. Whoa!

a) Does $\dfrac{3}{4} \div \dfrac{2}{3}$ indeed equal $\dfrac{9}{8}$?

b) Work out $\dfrac{5}{4} \div \dfrac{7}{9}$ via the method of this lesson, and then again by the method described above. Are the answers indeed the same?

c) Can you figure out why this "common denominator method" works?

Practice 33.19 Work out $\dfrac{12}{15} \div \dfrac{3}{5}$ and show that it equals $\dfrac{4}{3}$.

Now notice that

$$12 \div 3 = 4$$
$$15 \div 5 = 3$$

and

$$\frac{12}{15} \div \frac{3}{5} = \frac{4}{3}.$$

Whoa!

Is this a coincidence or does $\dfrac{a}{b} \div \dfrac{c}{d}$ always equal $\dfrac{a \div c}{b \div d}$?

33.12 Multiplying Fractions

We've been playing the five properties of fractions that naturally arose from the pie-per-student model of fractions.

Property 1: $\frac{a}{1} = a$ for each number a

Property 2: $\frac{a}{a} = 1$ for each non-zero number a.

Property 3: $k \times \frac{a}{b} = \frac{ka}{b}$ for a number k and a fraction $\frac{a}{b}$.

Property 4: $\frac{ka}{kb} = \frac{a}{b}$ for each number k and each fraction $\frac{a}{b}$.

Logical Consequence 5: $b \times \frac{a}{b} = a$ for each fraction $\frac{a}{b}$.

And we've allowed ourselves to be somewhat quirky about the values of a and b we use to make fractions. For example, in doing division, we did have fractions within fractions. That is, we allowed a and b to be numbers different from counting numbers.

In which case, let's now push property 3, knowing we don't have to play solely with counting numbers.

Example*:* Give a mathematical value for $\frac{2}{3} \times \frac{7}{5}$ as a logical consequence of our fraction properties.

Let's use Property 3 with k having the value $\frac{2}{3}$.

$$\frac{2}{3} \times \frac{7}{5} = \frac{\frac{2}{3} \times 7}{5}$$

And we can use property 3 again within the numerator we see.

$$\frac{2}{3} \times \frac{7}{5} = \frac{\frac{2}{3} \times 7}{5} = \frac{\frac{14}{3}}{5}$$

The thirds we see are annoying. Let's multiply the top and bottom of what we have each by 3. (Property 4.)

$$\frac{\frac{14}{3}}{5} = \frac{3\times\left(\frac{14}{3}\right)}{3\times 5} = \frac{14}{15}$$

We see that the mathematics wants $\frac{2}{3}\times\frac{7}{5}$ to equal $\frac{14}{15}$.

Example: More generally, give a meaningful value to $\frac{a}{b}\times\frac{c}{d}$..

We have

$$\begin{aligned}
\frac{a}{b}\times\frac{c}{d} &= \frac{\frac{a}{b}\times c}{d} && \text{(Property 3)}\\
&= \frac{\frac{ac}{b}}{d} && \text{(Property 3)}\\
&= \frac{b\times\frac{ac}{b}}{b\times d} && \text{(Property 4)}\\
&= \frac{ac}{bd} && \text{(Property 3)}
\end{aligned}$$

We mathematics wants the multiplication rule we were all taught in school!.

"To multiply two fractions, multiply their numerators and multiply their denominators."

$$\frac{a}{b}\times\frac{c}{d} = \frac{ac}{bd}$$

Question: In the previous section we saw that $\frac{a}{b}\div\frac{c}{d} = \frac{ad}{bc}$.
Does this correspond to a rule you were taught in school?

One can try to be efficient with the arithmetic.

Practice 33.20 Ibrahim was asked to compute $\frac{39}{7} \times \frac{14}{13}$ and within three seconds he said that the answer was 6. How did he see this so quickly?

Practice 33.21 What is the value of $\frac{39}{35} \times \frac{14}{13}$?

33.13 A Happy Coincidence: The word “of”

Let's go back to our intuitive "parts of a whole thinking" and consider the following question.

What is two-thirds of four-fifths?

One issue with this question is that it feels incomplete: two-thirds of four-fifths of what?

Well, people usually assume the full phrase reads, “two-thirds of four-fifths of one complete whole," and that you get to choose the whole.

Let’s draw a square pie for the whole. Here’s four-fifths of one.

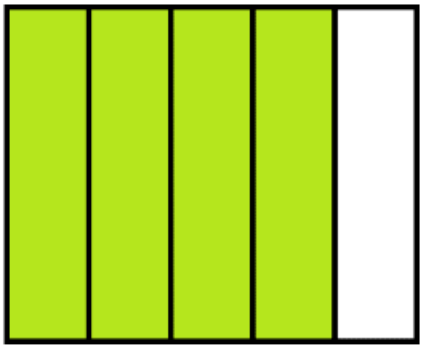


Figure Four-fifths of a pie.

We want two-thirds of this portion of the pie. Here’s it is. (Is it two copies of one third of this portion of the pie.)

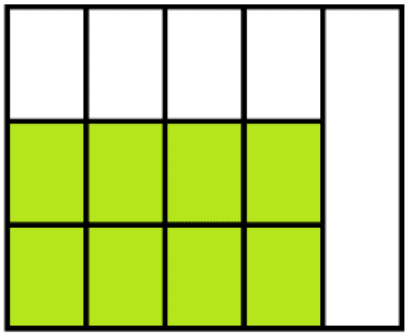


Figure Two-thirds of four-fifths of a pie

If we draw extra lines in the picture, we see the shaded region as 8 parts out of 15 equal parts of the pie. This picture is the result of sharing 8 pies equally among 15 students.

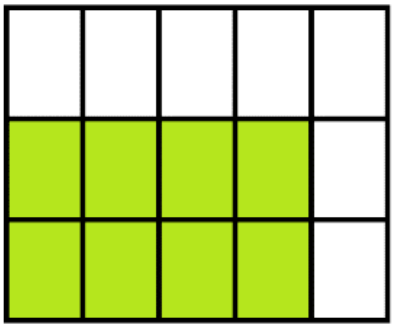


Figure Still two-thirds of four-fifths of a pie

That is, combining the fractions $\frac{2}{3}$ and $\frac{4}{5}$ via the word "of" led us to the fraction $\frac{8}{15}$.

Here's the happy coincidence.

> The number 8 comes from the pieces in the shaded 2-by-4 rectangle: $8 = 2 \times 4$.
> The number 15 comes from the pieces in the big 3-by-5 rectangle: $15 = 3 \times 5$.
> Our answer of $\frac{8}{15}$ is $\frac{2 \times 4}{3 \times 5}$.
> And this happens to be the answer to $\frac{2}{3} \times \frac{4}{5}$, the product of the two fractions in consideration.

WHOA!

Here's the general picture. When considering "$\frac{a}{b}$ of $\frac{c}{d}$ of a pie" we …

1. Draw a pie.
2. Then draw $\frac{c}{d}$ of the pie. (That is, we identify c copies of one d th of the pie.)

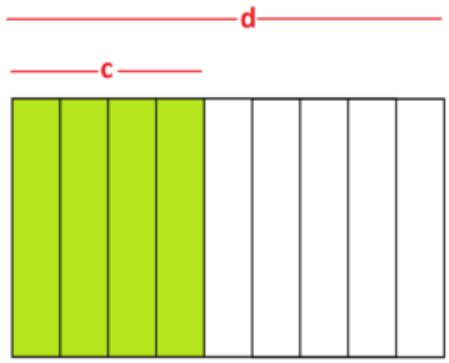


Figure $\frac{c}{d}$ of a pie

3. Next, we identify $\frac{a}{b}$ of this portion of the pie. (That is, we divide this portion of pie into b ths and select a copies of them.)

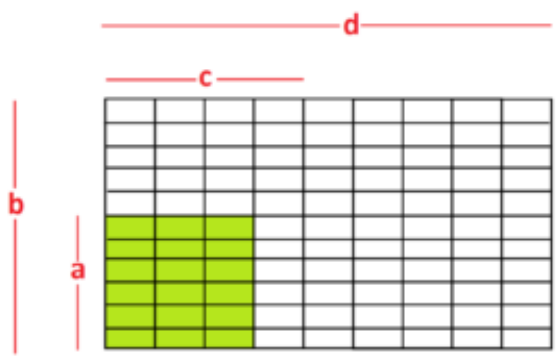


Figure $\frac{a}{b}$ of $\frac{c}{d}$ of a pie

The result is a pie divided into a total of $b \times d$ pieces of which $a \times c$ of them are shaded. We thus associate with this picture the fraction $\frac{ac}{bd}$.

Remarkably, this fraction happens to match the result of computing $\frac{a}{b} \times \frac{c}{d}$.

What a lovely coincidence!

Many curricula just make the assertion that "of means multiply" in the world of fraction arithmetic. But the truth is that had a system of arithmetic first and when we push that system of arithmetic to include fractions, the logic of the mathematics shows that the mechanics of fraction multiplication happen to match the mechanics of “of.”

This is reassuring. It shows that the mathematics we are developing is aligned with the different real-world contexts we like to consider.

Have you noticed a subtle turn-around of perspective here? Many school curricula give real-world scenarios and use those scenarios to say what math is.

We’ve developed a system of mathematics—our 9 arithmetic rules and our 5 properties—and, yes, it was guided by real-world intuition. But then we gave these rules a “life of their own” to see what they force us to say about numbers.

I personally don’t know what multiplication means in the pies-per-student model. (What’s “2 pies for 5 students times 3 pies for 5 students”? Huh?) But the mathematics we created is telling me how it wants multiplication to behave. (It wants $\frac{2}{5}\times\frac{3}{5}=\frac{6}{25}$.)

Whether this this is meaningful in a real-world scenario is now a secondary question, <u>not</u> the starting question as school curricula suggest.

The multiplication of fractions, as far as I can tell, has no meaning in the pie-per-student scenario.

It does have useful meaning in the “of” world of thinking.

It likely has useful meaning in other real-world contexts too.

33.14 Division and "of"

Even though people say "of means multiply," they often use the word "of" when thinking about division.

For example, "half of six" means the result of dividing a group of six items into two equal parts and selecting one of those parts (just as one does in sharing six pies among two students). And yet the multiplicative expression

$$\frac{1}{2}\times 6$$

is read out loud as "half of six."

It seems to be understood by society that multiplying a quantity by a fractional amount corresponds to dividing that quantity.

Property 3 has captured this idea. It shows

$$\frac{1}{2}\times 6=\frac{6\times 1}{2}=\frac{6}{2}.$$

That is, mathematics has that $\frac{1}{2}\times 6$ and $\frac{6}{2}$ are the same quantity—just as our everyday language wants them to be!

Practice 33.22 We have to be careful here. Consider the phrase: "multiplying a quantity by a fractional amount corresponds to dividing that quantity." For which of the following expressions would you agree with this?

$$\frac{1}{3}\times 12 \qquad \frac{1}{5}\times 100 \qquad \frac{3}{2}\times 100 \qquad \frac{2}{3}\times 600$$

33.15 Adding Fractions

Have you noticed that none of our five properties of fractions mention addition? How are we meant to add fractions?

The pie-per-student model doesn't really help here.

> *Two pies being shared equally among three students is to be added to (huh?) the act of sharing seven pies among five students. Does this have any meaning?*
>
> Are we meant to think: A total of $2+7=9$ pies is being shared among $3+5=8$ students?
>
> But there are already 3 students among the 5 students, so maybe in sharing the 7 pies among the 5 we are meant to give a special 3 students among them 2 extra pies?
>
> Or perhaps we ask: A student was part of a group in which 2 pies were shared among 3, and then later she was part of a group in which 7 pies were shared among 5. How much pie did she end up with in total?

This is confusing!

Let's see what the math wants us to do.

For reference, here are our five properties.

Property 1: $\frac{a}{1}=a$ for each number a

Property 2: $\frac{a}{a}=1$ for each number a.

Property 3: $k\times\frac{a}{b}=\frac{ka}{b}$ for a number k and a fraction $\frac{a}{b}$.

Property 4: $\frac{ka}{kb}=\frac{a}{b}$ for each number k and each fraction $\frac{a}{b}$.

Logical Consequence 5: $b\times\frac{a}{b}=a$ for each fraction $\frac{a}{b}$.

Our challenge:

> Give a mathematically meaningful value to $\frac{2}{3}+\frac{7}{5}$.

Here's a sneaky way to do this.

By Property 1 we have

$$\frac{2}{3}+\frac{7}{5} \quad = \frac{\frac{2}{3}+\frac{7}{5}}{1}.$$

I don't like the third and the fifth in the numerator. Let's do what we also do when faced with such an annoyance: use Property 4. (With Properties 3 and 5 along the way.)

$$\frac{\frac{2}{3}+\frac{7}{5}}{1} \quad = \frac{3\left(\frac{2}{3}+\frac{7}{5}\right)}{3\times 1} \quad = \frac{2+\frac{21}{5}}{3}$$

$$= \frac{5\left(2+\frac{21}{5}\right)}{5\times 3} \quad = \frac{10+21}{15} \quad = \frac{31}{15}$$

Practice 33.23 Compute $\frac{1}{2}+\frac{1}{3}$ this way.
(If a student were part of a group in which 1 pie was shared equally among 2 students and then later part of a group in which 1 pie was shared equally among 3 students, draw a picture to show how much pie she was receive, in total.)

Challenge: Follow the same mathematical argument abstractly to show that $\frac{a}{b}+\frac{c}{d}$ equals $\frac{ad+bc}{bd}$.

(Please don't ever memorize such a formula!)

Common Denominators

Early-grade school had as consider a problem such as $\frac{2}{7}+\frac{3}{7}$ just as it reads.

Two copies of a seventh plus three copies of a seventh makes for five copies of a seventh.

Figure $\frac{2}{7}+\frac{3}{7} \quad =\frac{1}{7}+\frac{1}{7} \; + \; \frac{1}{7}+\frac{1}{7}+\frac{1}{7} \quad =\frac{5}{7}$

And this thinking does align with correct mathematics.

$$\frac{a}{N}+\frac{b}{N}=\frac{\frac{a}{N}+\frac{b}{N}}{1}=\frac{N\left(\frac{a}{N}+\frac{b}{N}\right)}{N\times 1}=\frac{a+b}{N}$$

Adding fractions with a common denominator is just as straightforward—it turns out—as adding apples:

a apples plus b apples makes $a+b$ apples.
a N ths plus b N ths makes $a+b$ N ths.

This gives us another way to add fractions that do not initially have a common denominator. This is the method usually taught in schools.

$$\frac{2}{3}+\frac{7}{5}=\frac{5\times 2}{5\times 3}+\frac{3\times 7}{3\times 5}=\frac{10}{15}+\frac{21}{15}=\frac{31}{15}$$

Practice 33.24 Compute each of the following

a) $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}$ b) $\frac{6}{11}+\frac{7}{10}$ c) $\frac{3/4}{5}+\frac{3}{20}$

33.16 Subtracting Fractions

As you know, I do not believe subtraction exists.

Subtraction is the addition of the opposite.

This means we need to be clear on the meaning of the opposite of a fraction, at least mathematically.

(We could try conduct a scenario of sharing anti-pies among ant-students, or some such—which might be fun—but we are pulling away from the "real world" models and letting the mathematics now guide us.)

We have our five properties of fractions to help us, as well as the properties of negative numbers and their logical consequences we identified in section 19. We saw, for example, the following.

$$-(-a) = a$$
$$(-1)\times a = -a$$
$$-0 = 0$$
$$(-a)\times b = a\times(-b) = -ab$$
$$(-a)\times(-b) = ab$$

Let's see what math has to say about the quantity $-\frac{3}{5}$, say.

We have that

$$-\frac{3}{5} = (-1)\times\frac{3}{5}.$$

By Property 3 of fractions, this is

$$(-1)\times\frac{3}{5} = \frac{(-1)\times 3}{5} = \frac{-3}{5}$$

By Property 4 of fractions, this is

$$\frac{-3}{5} = \frac{(-1)\times(-3)}{(-1)\times 5} = \frac{3}{-5}$$

We've just shown that $-\frac{3}{5}$ and $\frac{-3}{5}$ and $\frac{3}{-5}$ are all the same!

$$-\frac{a}{b}=\frac{-a}{b}=\frac{a}{-b} \text{ for a fraction } \frac{a}{b}.$$

People call rewriting $\frac{-a}{b}$ as $-\frac{a}{b}$, or rewriting $\frac{a}{-b}$ as $-\frac{a}{b}$ as "pulling out a negative sign."

Practice 33.25 Explain why $\frac{-a}{-b}$ is the same as $\frac{a}{b}$.

Practice 33.26 What is $\frac{-8}{9}\times\frac{2}{-5}$?

Question: We have that $-\frac{3}{5}=\frac{-3}{5}=\frac{3}{-5}$. If we want to think about pies-per-student, is this saying anything meaningful? "Sharing three anti-pies among five actual students is the same as ..."

We're now ready to explore subtraction.

Example: Compute $\frac{2}{3}-\frac{1}{4}$.

Answer: We have

$$\begin{aligned}\frac{2}{3}-\frac{1}{4} &= \frac{2}{3}+-\frac{1}{4} = \frac{2}{3}+\frac{-1}{4}\\ &= \frac{8}{12}+\frac{-3}{12} = \frac{8+-3}{12} = \frac{5}{12}\end{aligned}$$

Example: Compute $1-\frac{1}{20}$.

Answer: $1-\frac{1}{20}=\frac{20}{20}-\frac{1}{20}=\frac{20}{20}+\frac{-1}{20}=\frac{19}{20}$.

Practice 33.27 Compute

a) $\frac{1}{3}-\frac{1}{4}$ b) $1+\frac{1}{2}+\frac{1}{4}-\frac{7}{2}$ c) $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}$

33.17 Mixed Numbers Again

Recall that a **mixed number** is $a+\frac{b}{c}$, usually written as $a\frac{b}{c}$, with a, b, and c positive integers.

For example, $2\frac{1}{3}$ is read as "two and a third" and it means $2+\frac{1}{3}$.

Idle Question: Do textbooks allow $2\frac{7}{3}$ and $3\frac{-5}{6}$ and $5\frac{0}{9}$ as examples of mixed numbers?

Or must these particular numbers be written as $4\frac{1}{3}$ and $2\frac{1}{6}$ and 5, respectively?
(Did I do the arithmetic correctly?)

Textbooks often give a strict definition of a mixed number: it is one of the form $a+\frac{b}{c}$ with each of a, b, c a counting number and with b smaller in value than c.

This means that textbooks will never allow us to interpret $-7\frac{1}{2}$, say, as $(-7)+\frac{1}{2}$. (Negative numbers cannot be part of mixed numbers.) So, what does "$-7\frac{1}{2}$" represent?

A **negative mixed number** is a number of the form

$$-\left(a+\frac{b}{c}\right)$$

again, with each of a, b, c a counting number.

$$\boxed{-a\frac{b}{c} \text{ is taken to mean } -\left(a+\frac{b}{c}\right).}$$

Thus, $-7\frac{1}{2}$ is $-\left(7+\frac{1}{2}\right)$, a number halfway between -7 and -8 on the number line.

Example: Compute $13\frac{3}{10}-9\frac{7}{10}$.

Answer: We are being asked to compute

$$\left(13+\frac{3}{10}\right)+-\left(9+\frac{7}{10}\right).$$

This is

$$13+\frac{3}{10}+-9+-\frac{7}{10},$$

which equals

$$4+\frac{3-7}{10}=4+\frac{-4}{10}.$$

To handle this, we could do the following.

$$4+\frac{-4}{10}=3+\frac{10}{10}+\frac{-4}{10}=3+\frac{6}{10}.$$

This is the same as $3\frac{3}{5}$.

Practice 33.28 Compute each of the following if you would like the practice.

a) $2\frac{2}{5}+5\frac{4}{5}$ b) $10\frac{1}{7}-9\frac{3}{7}$ c) $6\frac{3}{4}-5\frac{8}{9}$ d) $100\frac{1}{2}-20\frac{1}{3}$

Recall too, that a fraction with a numerator greater than its denominator is called **improper**, and that often people prefer to rewrite improper fractions as mixed number. There is no mathematical reason for this. It is just an issue of style.

Example: Write $\frac{39}{8}$ as a mixed number.

Answer: Now that we have some more arithmetic of fractions at our hands, we can proceed this way.

$$\begin{aligned}\frac{39}{8} &= \frac{32+7}{8} = \frac{32}{8} + \frac{7}{8} \qquad \text{(from how we add fractions)}\\ &= \frac{8\times 4}{8\times 1} + \frac{7}{8}\\ &= \frac{4}{1} + \frac{7}{8}\\ &= 4\frac{7}{8}\end{aligned}$$

Of course, we can convert mixed numbers into (improper) fractions too.

Example: Write $20\frac{1}{20}$ as a single fraction.

Answer:

$$\begin{aligned}20\frac{1}{20} &= 20 + \frac{1}{20}\\ &= \frac{20}{1} + \frac{1}{20}\\ &= \frac{20\times 20}{20\times 1} + \frac{1}{20}\\ &= \frac{400}{20} + \frac{1}{20} \qquad = \frac{401}{20}\end{aligned}$$

Alternatively …

$$20\frac{1}{20} = 20 + \frac{1}{20} = \frac{20+\frac{1}{20}}{1} = \frac{20\left(20+\frac{1}{20}\right)}{20\times 1} = \frac{400+1}{20} = \frac{401}{20}.$$

(Are you able to follow all the details in both approaches?)

Practice 33.29 Write each of the following as a mixed number.

a) $\frac{8}{5}$ b) $\frac{100}{13}$ c) $\frac{200}{199}$ d) $\frac{199\frac{1}{2}}{199}$

Write each of the following a single fraction.

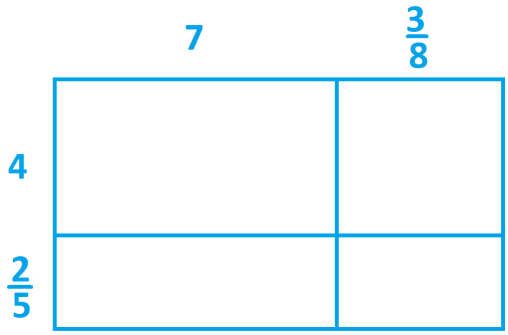
e) $7\frac{2}{9}$ f) $2\frac{3}{4}+5\frac{2}{7}$ g) $300\frac{299}{300}$ h) $2\frac{3}{4}-5\frac{2}{7}$

Multiplying Mixed Numbers

Recall that we have the area model at our disposal for conducting arithmetic. (It's our eighth rule or arithmetic, the "distributive rule.") The area model helps significantly with correctly multiplying mixed numbers.

Example: Compute $4\frac{2}{5}\times 7\frac{3}{8}$.

Answer: We're being asked to compute $\left(4+\frac{2}{5}\right)\left(7+\frac{3}{8}\right)$. We see that there shall be four "pieces."



We have

$$\left(4+\frac{2}{5}\right)\left(7+\frac{3}{8}\right)=4\times 7+\frac{2}{5}\times 7+4\times\frac{3}{8}+\frac{2}{5}\times\frac{3}{8}$$
$$=28+\frac{14}{5}+\frac{12}{8}+\frac{6}{40}$$
$$=28+2+\frac{4}{5}+1+\frac{1}{2}+\frac{3}{20}$$
$$=31+\frac{16}{20}+\frac{10}{20}+\frac{3}{20}$$
$$=31+\frac{29}{20}$$
$$=32\frac{9}{20}$$

(Not fun arithmetic, but it is doable. What does Siri say the answer is?)

Question: Often people will say that $4\frac{2}{5}\times 7\frac{3}{8}$ is $28+\frac{6}{40}$.
Do you understand the impulse for saying that? Do you see that this answer accounts for only two of the four "pieces" that should be in the computation?

Here's an alternative approach to conducting this computation.

$$4\frac{2}{5}\times 7\frac{3}{8} \quad =\frac{22}{5}\times\frac{59}{8} \quad =\frac{1298}{40}$$

Question: Is $\frac{1298}{40}$ the number $32\frac{9}{20}$ in disguise?

Practice 33.30: Compute $1\frac{2}{3}\times 3\frac{1}{2}$ two ways.

33.18 Multiplying and Dividing by Positive Numbers Bigger and Smaller than One

People say that multiplying a quantity by a number bigger than one gives an answer bigger than the quantity. Is this true?

For instance, let's ask:

Is $\frac{5}{4} \times N$ larger than N for a positive number N ?

Well, yes. We have

$$\begin{aligned} \frac{5}{4} \times N &= \left(1 + \frac{1}{4}\right) \times N \\ &= N + \frac{1}{4} \times N \\ &= N + \textit{something more} \end{aligned}$$

In general, any number larger that 1 can be written as $1 + \varepsilon$, with ε positive, and

$$(1 + \varepsilon) N = N + \varepsilon N = N + \textit{more}$$

In the same way, multiplying a positive quantity by a positive number smaller than 1 is sure to give an answer smaller than the original quantity. For example,

$$\frac{4}{5} \times N = \left(1 - \frac{1}{5}\right) N = N - \textit{something}$$

In general, any positive quantity smaller than 1 can be written in the form $1 - \varepsilon$, with ε positive, and we have

$$(1 - \varepsilon) N = N - \varepsilon N = N - \textit{something}$$

(By the way, mathematicians like Greek letters. The symbol ε is the letter epsilon.)

Does dividing a quantity by a positive number smaller than 1 give a bigger or smaller result?

Let's try an example.

Compute $\frac{100}{4/5}$? Is the result bigger or smaller than 100?

We have

$$\frac{100}{\frac{4}{5}} = \frac{5\times 100}{5\times\frac{4}{5}} = \frac{500}{4} = 125 .$$

This is larger.

In general, can we say that $\frac{N}{1-\varepsilon}$, with ε small and positive, is sure to be larger than N?

To compare these quantities, write them each with a common denominator.

$$\frac{N}{1-\varepsilon}$$

$$N = \frac{N}{1} = \frac{(1-\varepsilon)\times N}{(1-\varepsilon)\times 1} = \frac{N - something}{1-\varepsilon}$$

We see that $\frac{N}{1-\varepsilon}$ is larger than N.

Practice 33.31

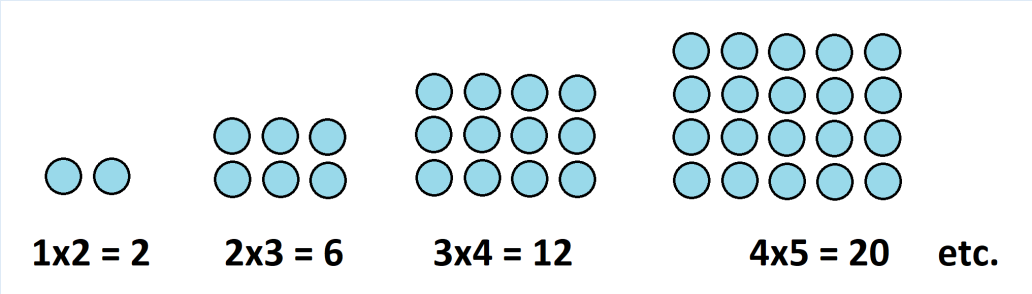
a) Compute $\frac{100}{5/4}$. Is the result larger or smaller than 100?

b) Show, in general, that $\frac{N}{1+\varepsilon}$, with ε small and positive, is sure to be smaller than N.

MUSINGS

Musing 33.32 Did you try the various practice problems sprinkled throughout this (absurdly long) section?

Musing 33.33 Here are the **oblong numbers**. These are the numbers that arise from arranging dots into rectangular arrangements with one side one unit longer than the other.



a) What is the 100th oblong number?

b) Compute each of the following. What do you notice?

$$\frac{1}{1}-\frac{1}{2} \qquad \frac{1}{2}-\frac{1}{3} \qquad \frac{1}{3}-\frac{1}{4} \qquad \frac{1}{4}-\frac{1}{5} \qquad \frac{1}{100}-\frac{1}{101}$$

c) Can you rewrite the following in an illuminative way?

$$\frac{1}{n}-\frac{1}{n+1}$$

34. The Mathematical Truth about Fractions

We've been using concrete, real-world scenarios to guide us in identifying the ways numbers behave. But we have also pulled away from the real-world examples to let the logic of the mathematics guide us beyond the real-world. For example,

> It is hard to give a "real-world" explanation of why negative times negative is positive, yet the logic of the arithmetic we identified makes it clear this has to be so.
>
> It makes no sense of multiply portions of pie (what's half a pie times a third of a pie?), yet the logic of the arithmetic we identified tells us what the product of two fractions must be.

Often the mathematics we develop harks back to real-word contexts.

> The product of two fractions matches how we think about "fractions of fractions of pie."

But the driving force is now the mathematics, not the real-world examples.

WARNING: This section is heavy on the abstract and the theoretical. Read only at your own joy. Or ignore this section! We're showing here that everything we established, and surmised, in the pies-per-students model of the previous section is mathematically on mark.

> To add, subtract, or multiply fractions, follow what you were taught in school.
>
> To divide fractions, follow the approach of 33.11.(It's more sensical than what is usually taught in schools.)
>
> To compare fractions, rewrite them with a common denominator.

And so on.

In section 19 we listed the rules of arithmetic for the integers: the counting numbers, zero, and the negative counting numbers.

Numbers come with two operations, addition and multiplication, which satisfy:

Rule 1: For any two numbers a and b we have $a+b=b+a$.

Rule 2: There is a number 0 such that $a+0=0$ and $0+a=0$ for all numbers a.

Rule 3: In a string of additions, it does not matter in which order one conducts individual additions.

Rule 4: For any two numbers a and b we have $ab=ba$.

Rule 5: There is a number 1 such that $a\times 1=a$ and $1\times a=a$ for all numbers a.

Rule 6: In a string of multiplications, it does not matter in which order one conducts individual multiplications.

Rule 7: For any number a we have $a\times 0=0$ and $0\times a=0$.

Rule 8: "We can chop up rectangles from multiplication and add up the pieces."
(Simple version: $a(b+c)=ab+ac$ for all numbers a, b, and c.)

Rule 9: For each number a, there is one other number "$-a$" such that $a+-a=0$.

Logical Consequences 10:

i) $-0=0$

ii) $--a=a$ for all numbers a.

iii) $-(a+b)=-a+-b$ for all numbers a and b.

iv) $(-a)b=-ab$ and $a(-b)=-ab$ for all numbers a and b.

v) $(-1)\times a=-a$ for all numbers a.

In the previous (very long!) section we identified five properties of fractions that the unlocked all the arithmetic of fractions we were taught in school.

Here's the surprising thing: We need to add only one more rule to the list above, to give us the five properties we need all as logical consequences, to then make everything we saw in the previous section mathematically valid. Just one!

Here's the one additional rule we need.

Rule 11: For each non-zero number a there is one number, "$\frac{1}{a}$" such that $a \times \frac{1}{a} = 1$.

That is, we are introducing new quantities $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, …, and such into our number system .

We were cautioned against a quantity "$\frac{1}{0}$" in section 33.8, where we saw that such a quantity would be logically inconsistent with our other rules.

Rule 11 not only introduced some new types of numbers, it also captures their basic behavior. We like to say that "two halves make a whole," and "three thirds make a whole," and so on. Rule 11 is saying this.

For those who like the fancy language, for a non-zero number a, the quantity $\frac{1}{a}$ is called the **multiplicative inverse** of a.

Each non-zero number has precisely one multiplicative inverse, and I added the word "one" in Rule 11. But like for rule 9, it is not actually needed: logic allows us to deduce that this is so.

(If you are curious, here is the logic.

Consider a non-zero number a.

We know $a \times \frac{1}{a} = 1$.

Suppose b is another number that works this way: $a \times b = 1$.

Let's now work out $\frac{1}{a} \times a \times b$. Using the other rules of arithmetic we see this as

$$\left(\frac{1}{a} \times a\right) \times b = 1 \times b = b$$

or as

$$\frac{1}{a} \times (a \times b) = \frac{1}{a} \times 1 = \frac{1}{a}.$$

The answer must be the same. So, our "other number" b was $\frac{1}{a}$ all along.)

Is Our System of Arithmetic Lacking?

Where are fractions like $\frac{2}{3}$ and $\frac{9}{20}$ in our rules? All we have so far are $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, …, it seems.

Early grades have as read $\frac{2}{3}$ as "two thirds," that is, as two copies of $\frac{1}{3}$, and they have us read $\frac{9}{20}$ as nine copies of $\frac{1}{20}$.

And in section 33.16, we connected $2 \times \frac{1}{3}$ with $\frac{2}{3}$, and $9 \times \frac{1}{20}$ with $\frac{9}{20}$.

(Well, we showed $\frac{1}{N} \times a = \frac{a}{N}$.)

Okay. Let's now really *own* this idea.

Convention 12: For a number a and a non-zero number b, the notation "$\frac{a}{b}$" is simply shorthand for $a \times \frac{1}{b}$.

This now gives Rule 11 the power to "create" all the fractions we expect.

Establishing the Five Properties of Fractions

Let's now play with Rule 11 and see some of its logical consequences.

The quantity $\frac{1}{1}$ is just the number 1.

This is not very exciting. But it would be quite shocking if this is not so! We'd better check.

Reason: The quantity $\frac{1}{1}$ is the multiplicative inverse of 1. That is, it is the number you can multiply 1 by to get the answer 1.

But multiplying 1 by 1 gives 1.

Okay. So, 1 is the value of $\frac{1}{1}$.

A quantity $\frac{a}{1}$ is just the number a.

Reason:

We have that $\frac{a}{1}$ is shorthand for $a \times \frac{1}{1}$.

We just saw that $\frac{1}{1} = 1$, so this is $a \times 1$, which is indeed a.

A quantity $\frac{a}{a}$ is just 1.

Reason:

We have that $\frac{a}{a}$ is shorthand for $a \times \frac{1}{a}$. By Rule 11, this is 1.

We have that $k \times \frac{a}{b}$ and $\frac{ka}{b}$ are the same.

Reason:

We have

$$k \times \frac{a}{b} = k \times a \times \frac{1}{b}.$$

And we have

$$\frac{ka}{b} = ka \times \frac{1}{b}.$$

These are indeed the same!

This covers our first three properties.

To keep going, consider this challenge.

Challenge: Show that $\frac{1}{2} \times \frac{1}{3}$ is really $\frac{1}{6}$ in disguise.

Answer: We need to show that $\frac{1}{2} \times \frac{1}{3}$ is the multiplicative inverse of 6.

That is, we need to show that $6 \times \frac{1}{2} \times \frac{1}{3}$ equals 1.

Here goes!

$$\begin{aligned} 6 \times \frac{1}{2} \times \frac{1}{3} &= 2 \times 3 \times \frac{1}{2} \times \frac{1}{3} \\ &= 2 \times \frac{1}{2} \times 3 \times \frac{1}{3} \\ &= 1 \times 1 \\ &= 1 \end{aligned}$$

We got it!

Practice 34.1 Copy this argument to show that $\frac{1}{k} \times \frac{1}{b}$ equals $\frac{1}{kb}$ for non-zero number k and b.

Here comes the fourth property.

We have that $\frac{ka}{kb}$ and $\frac{a}{b}$ are the same.

Reason:

Now $\frac{ka}{kb} = ka \times \frac{1}{kb}$.

And, with the previous practice problem, this is

$$k \times a \times \frac{1}{k} \times \frac{1}{b}$$

which can be computed as

$$k \times \frac{1}{k} \times a \times \frac{1}{b} \quad = 1 \times a \times \frac{1}{b} \quad = a \times \frac{1}{b}$$

And this answer is just $\frac{a}{b}$.

We started with $\frac{ka}{kb}$ and just showed that it is indeed the same as $\frac{a}{b}$.

And we already know that the fifth property is a logical consequence of all that we have done, but, for completeness, here it is again.

We have $b \times \frac{a}{b}$ is the same as a.

Reason: $b \times \frac{a}{b} = b \times a \times \frac{1}{b} = 1 \times a = a$.

That's it, we've got everything in Section 33.

Here are all the consequences (and see Section 33 for the reasoning).

Adding Fractions:

$$\frac{a}{N}+\frac{b}{N}=\frac{a+b}{N}$$

If the fractions don't have a common denominator, use the property $\frac{a}{N}=\frac{ka}{kN}$ to makes this so.

Negative Fractions:

$$-\frac{a}{b}=\frac{-a}{b}=\frac{a}{-b}$$

Subtracting Fractions:

Add the opposite.

$$\frac{a}{N}-\frac{b}{N}=\frac{a}{N}+\frac{-b}{N}=\frac{a-b}{N}$$

Multiplying Fractions:

$$\frac{a}{b}\times\frac{c}{d}=\frac{ac}{bd}$$

Dividing Fractions:

Make $\frac{a/b}{c/d}$ friendlier by multiplying top and bottom each by b and then each by d.

MUSINGS

Musing 34.2 If you are reading this question, I am guessing you read this entire section. Is that correct? (If so, which parts of this section made reasonable sense to you?)

Musing 34.3 In sections 33 we, essentially, said that "a fraction is an answer to a division problem." For example, $\frac{6}{3}$ is the answer to the problem $6 \div 3$, the result of sharing pie equally among students.

Many people like to think of division as "the reverse of multiplication."
For example, that $3 \times 2 = 6$ means $\frac{6}{3} = 2$.

Can we establish that the mathematics we have in this section is in agreement with this "reverse multiplication" thinking? The answers is: YES!

Here's what we need to show:

If $a \times b = n$, then it must be that $\frac{n}{a} = b$.

Let's do it!

a) What is $\frac{n}{a}$ shorthand for?

b) Remember, $n = a \times b$ here. So, what is $n \times \frac{1}{a}$?

c) Are we done?

Here's the backwards version of our thinking.

If $\frac{n}{a} = b$, then it must be that $n = a \times b$.

d) Establish that this backwards statement is also correct.

Comment: We used the idea of this question in section 33.9. In the section, I was careful to use the phrase "looks like it is correct." From this question, we know we can say each example is actually correct!

35. All the Rules of Arithmetic in One Place

Here they are!

Numbers come with two operations, addition and multiplication, which satisfy:

Rule 1: For any two numbers a and b we have $a+b=b+a$.

Rule 2: There is a number 0 such that $a+0=0$ and $0+a=0$ for all numbers a.

Rule 3: In a string of additions, it does not matter in which order one conducts individual additions.

Rule 4: For any two numbers a and b we have $ab=ba$.

Rule 5: There is a number 1 such that $a\times 1=a$ and $1\times a=a$ for all numbers a.

Rule 6: In a string of multiplications, it does not matter in which order one conducts individual multiplications.

Rule 7: For any number a we have $a\times 0=0$ and $0\times a=0$.

Rule 8: "We can chop up rectangles from multiplication and add up the pieces."
(Simple version: $a(b+c)=ab+ac$ for all numbers a, b, and c.)

Rule 9: For each number a, there is one other number, $-a$ such that $a+-a=0$.

Logical Consequences 10:

i) $-0=0$

ii) $--a=a$ for all numbers a.

iii) $-(a+b)=-a+-b$ for all numbers a and b.

iv) $(-a)b=-ab$ and $a(-b)=-ab$ for all numbers a and b.

v) $(-1)\times a=-a$ for all numbers a.

Rule 11: For each non-zero number a there is one number, $\frac{1}{a}$ such that $a\times\frac{1}{a}=1$.

Convention 12: For a number a and a non-zero number b, the notation $\frac{a}{b}$ is shorthand for $a\times\frac{1}{b}$.

Logical Consequences 13:

i) $\frac{a}{1} = a$

ii) $\frac{a}{a} = 1$ for a non-zero

iii) $k \times \frac{a}{b} = \frac{ka}{b}$ for b non-zero

iv) $\frac{ka}{kb} = \frac{a}{b}$ for k and b non-zero

v) $b \times \frac{a}{b} = a$ for b non-zero

vi) $\frac{a}{N} + \frac{b}{N} = \frac{a+b}{N}$ for N non-zero

vii) $-\frac{a}{b} = \frac{-a}{b} = \frac{a}{-b}$ for b non-zero

viii) $\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$ for b and d non-zero

Informal Tips:
Rewriting fractions with a common denominator makes adding and subtracting them easier.

To divide fractions, use Consequence 13iv) multiple times.

Consequence 13viii) shows how to multiply fractions.

Again, there is nothing to memorize here. All you were taught in school is mathematically sound!

36. So … What is a Fraction?

We've developed a system of arithmetic that is based on the counting numbers 1, 2, 3, 4, … and 0 and extends this basic system to include numbers denoted $-a$ and $\frac{a}{b}$ (with b not zero).

In our system we can create all sorts of "crazy" quantities of the form $\frac{a}{b}$ —for example,

$$\frac{2\frac{1}{1-\frac{1}{5}}}{1+\frac{1}{3}-\frac{1+\frac{1}{3}}{1-\frac{6}{5}}}$$

is crazy. But we like to say that a quantity $\frac{a}{b}$, with a and b each a counting number, is "truly" a fraction.

Okay, let's use Logical Consequence 13 iv), which says

$$\frac{ka}{kb}=\frac{a}{b}$$

for any non-zero number k. It allows us to write fractional quantities in alternative forms.

So, here is my answer to "What is a fraction?"

> A **fraction** is any quantity in our number system that, with the help of Logical Consequence 13 iv) (and perhaps other consequences too) , can be written in the form in the form $\frac{a}{b}$ or $-\frac{a}{b}$, with a and b counting numbers. Here a is allowed to be zero, but b is not.

For example, $\frac{1}{1/2}$ is a fraction because we can rewrite it as $\frac{2\times 1}{2\times\frac{1}{2}}=\frac{2}{1}$.

The integer 7 is a fraction as this can be written $\frac{7}{1}$.

The integer -3 is a fraction as this can be written $-\frac{3}{1}$.

Practice 36.1 Show that 0 is a fraction.

Practice 36.2 Show that $1+\frac{1}{2}$ is a fraction.

Practice 36.3 Show that $\frac{1-\frac{4}{3}}{1+\frac{2}{5}}$ is a fraction.

Practice 36.4 (DO NOT DO THIS!) Show that the "crazy example" on the previous page is a fraction.

Getting ahead of ourselves …

The quantity $\frac{1.2}{3}$ is a fraction as it is equivalent to $\frac{10\times 1.2}{10\times 3}=\frac{12}{30}$.

The quantity $\frac{2\sqrt{5}}{3\sqrt{5}}$ is a fraction as it is equivalent to $\frac{2}{3}$.

It was a real shocker some 2300 years ago when scholars realized that not all numbers that arise in the "real world" are fractions! We'll realize that too next chapter.

MUSINGS

Musing 36.5
Is the sum of two fractions sure to be a fraction?
Is the product of two fractions certain to be a fraction?
What if I divide one fraction by another? Must the result be another fraction?