MIND THE GAP

Celebrating the space between
school's real-world portrayal of math
and
math's ability to soar beyond those confines.

James Tanton

Introduction: JUST NOTES FOR NOW

> *So much mathematics is motivated by real-world scenarios. But, in the end, mathematics is bigger and bolder than any one real-world context—even the one that motivated it in the first place!*

This is a fundamental paradox for curriculum writers.

Mathematics excels at describing and pinpointing the nuances of real-world examples; however, no singular real-world model can fully encapsulate, and thus explain, mathematics. (Yet school curricula give the impression that real-world models should.)

This book offers a series of essays that explore the space between real-world motivation and the broader scope of mathematics. Each essay examines a standard school math concept, starting with its real-world roots and ending in unexpected—and hopefully delightful—ways.

Math is bold, beautiful, and deeply human, yet it transcends our humanness.

Join me in celebrating this paradox of mathematics.

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Note to self: This is probably around 180 pages.

1
COUNTING

Motivation from the Real World

Counting is a fundamental part of being human. We can't imagine living in a world without at least some counting: none, one, two, many!

Young children find joy in counting—whether they're counting as they climb a set of stairs or counting again as they come back down. If they get different numbers for these two counts, they might think something about it, or they might not.

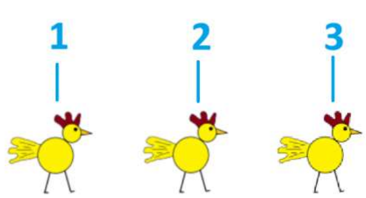
However, counting is actually quite complex. It involves memorizing a list of symbols and their names, as well as having a shared understanding of how to extend this list indefinitely.

As a community, we've agreed on this list, which we say represents the **counting numbers**.

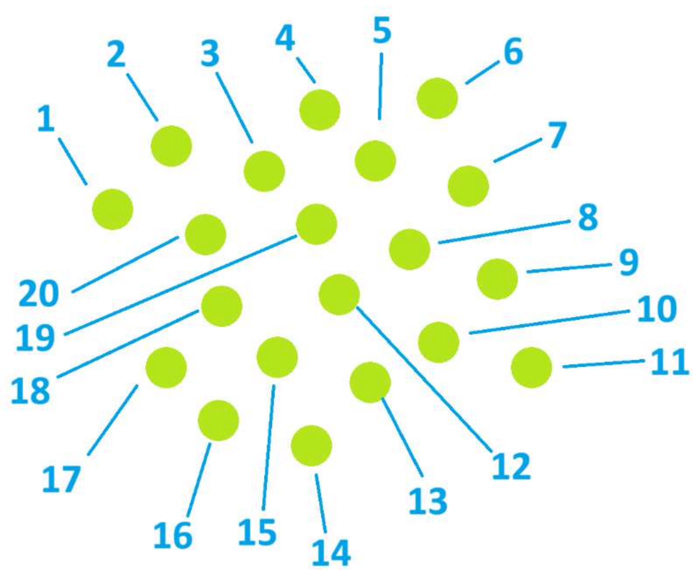
1 2 3 4 5 6 7 8 9 10 11 12 ...

We know how to continue this list: after 12 (twelve) comes 13 (thirteen), after 99 (ninety-nine) comes 100 (one hundred), and after 8,672,098,037 (I won't even attempt to say that one!) comes 8,672,098,038, and so on.

To **count** a set of objects, we assign each object a number from this list in order, starting with 1 (one). When we finish assigning numbers, the last number we reach tells us how many objects are in the set.



Three chickens

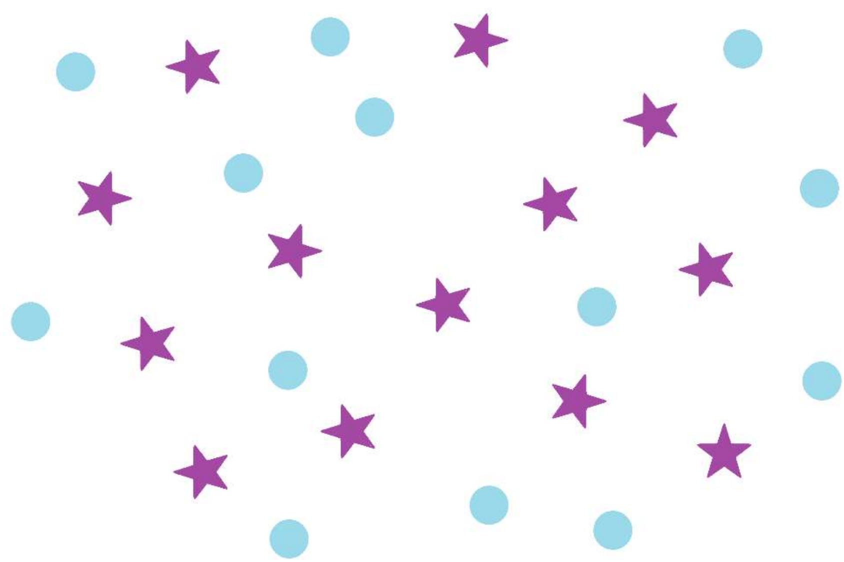


Twenty dots

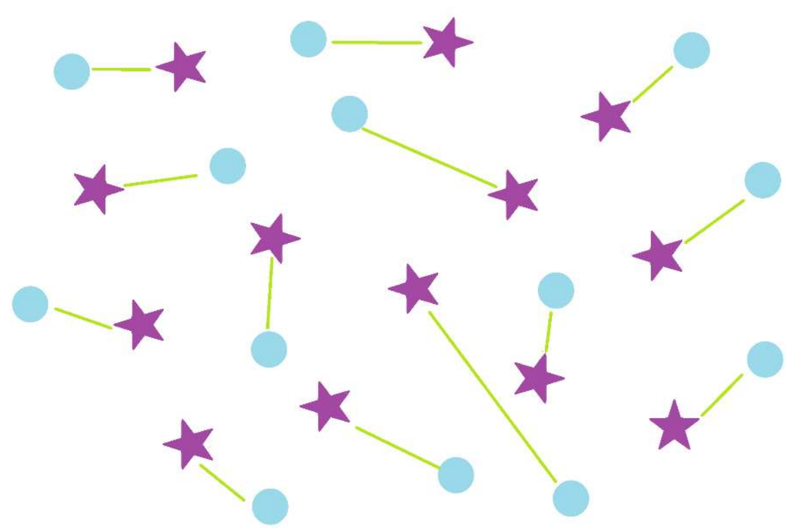
Math's Take Away

Drawing "leashes" between objects in sets provides a way to declare that two sets are "the same size."

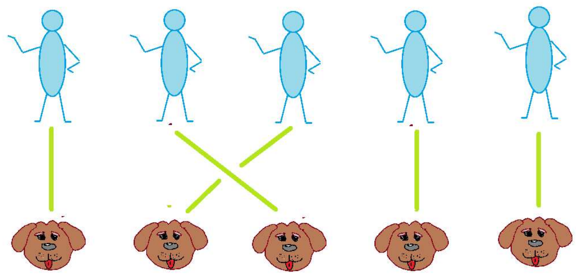
For example, here's a collection of dots and stars. It's hard to tell if there is an equal number of each.



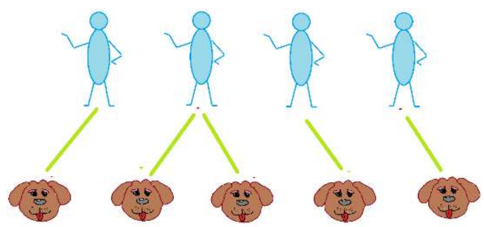
But this picture makes it clear that there is an equal number of them (and we can say this without ever counting *thirteen*.)



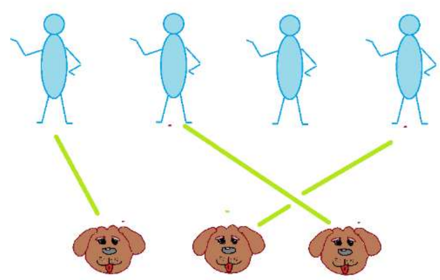
Going with the idea of leashes, this picture shows some people, some dogs, and some leashes. Each dog is leashed to one person and each person is leashed to one dog. Again, we can see that the set of people and the set of dogs are each the same size.



In this picture a person is leashed to more than one dog. Our intuition says that the set of people and the set of dogs are not this same size in this case.



And we'd agree matters are also problematic if a person (or a dog) is skipped in a leashing pattern.



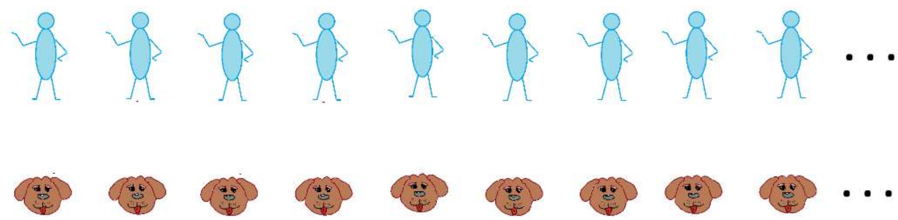
Mathematics' Take Away:

Two sets of objects are declared to be the ***same size*** *if it is possible to describe a leashing pattern between objects so that each item of the first set is leashed to exactly one item of the second set and each item of the second set is leashed to exactly one item of the first set.*

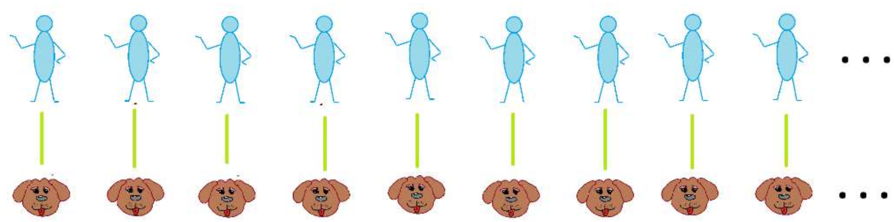
Math's Boldness

We can push this idea of leashing!

Here's an infinite line of people going infinitely far to the right and an infinite line of dogs also going infinitely far to the right. Are the set of people and the set of dogs in this picture "the same size"?



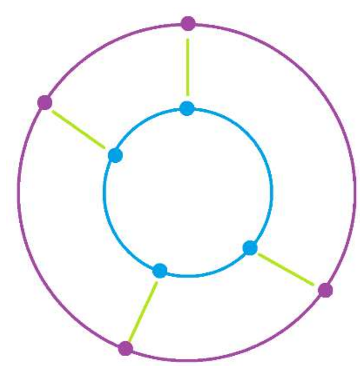
The answer is yes if we follow this leashing idea. We can certainly envision a way to draw leashes so that each dog is leashed to a person and each person is leashed to a dog.



Even with the "...", we can see how this leashing pattern will continue: the tenth dog and the tenth person will be leashed together, the thousandth dog and the thousandth person will be leashed together, the ten-millionth dog and the ten-millionth person will be leashed together, and so on.

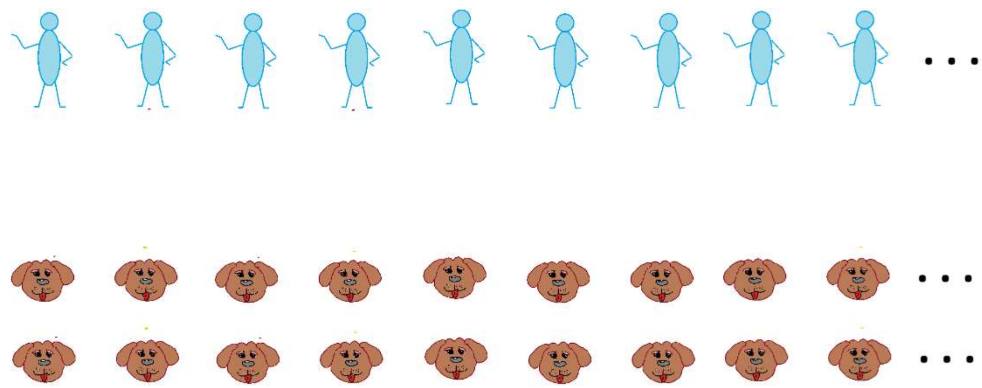
Describing, or just visually exhibiting a leashing pattern that can be continued, is enough to say that two different sets are the same size.

Four hundred years ago, Italian scientist and mathematician Galileo Galilei (1564-1642) observed that there "just as many" points on a small circle as there are on a bigger circle. He imagined leashes between points on the circles.



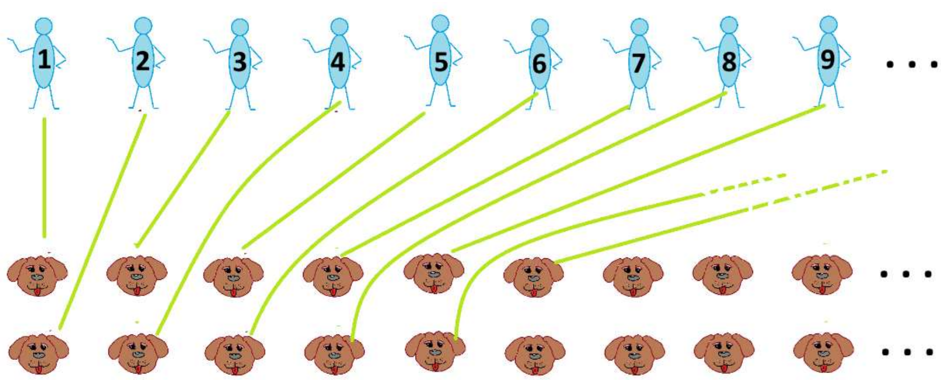
Two hundred years later, German mathematician Georg Cantor (1845-1918) took this play of infinite leash patterns to astounding heights.

Here's an infinite row of people and two infinite rows of dogs.



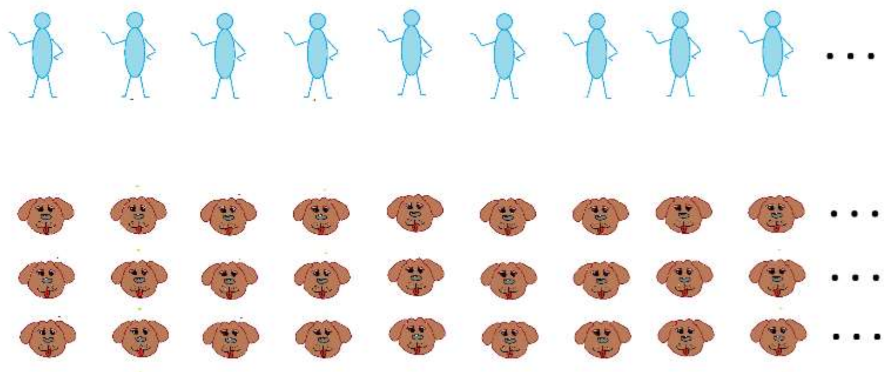
Are these two sets the same size?
Or is the count of dogs "double the infinity" of the count of people?

Surprisingly, our leashing idea shows that there are just as many people as there are dogs! Here's a way to show a valid leashing pattern that can clearly be extended.



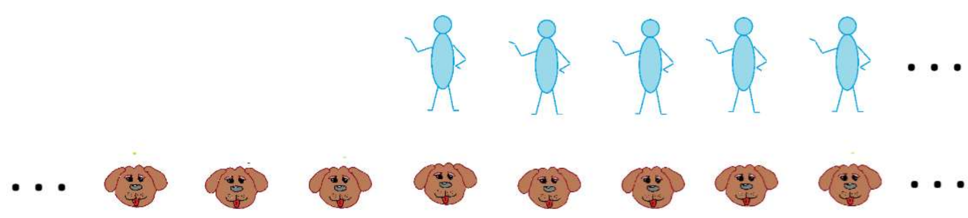
If we number the people 1, 2, 3, 4, 5, …, we can match people 1, 3, 5, 7, … with the dogs in the first row and people 2, 4, 6, 8, with the dogs in the second row.

Can you now show that a “triple infinity” of dogs is the same size as a “single infinity” of people?



How about a single infinity of people and a “double-ended infinity” of dogs?

Are these two sets the same size?

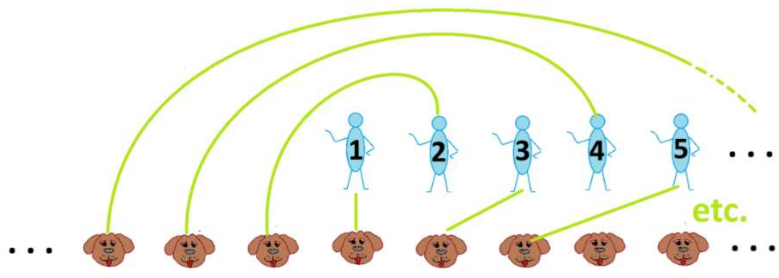


What do you think?

What does your instinct tell you?

Is there a leashing pattern between people and dogs that works, or is there no possible pattern?

Again, perhaps surprisingly, there is a leashing pattern that shows a "single infinity" and a "double-ended infinity" being the same size. Number the people and match person number 1 to a dog, and then people 3, 5, 7, 9, … to dogs to its right and people 2, 4, 6, 8,… to dogs to its the left.

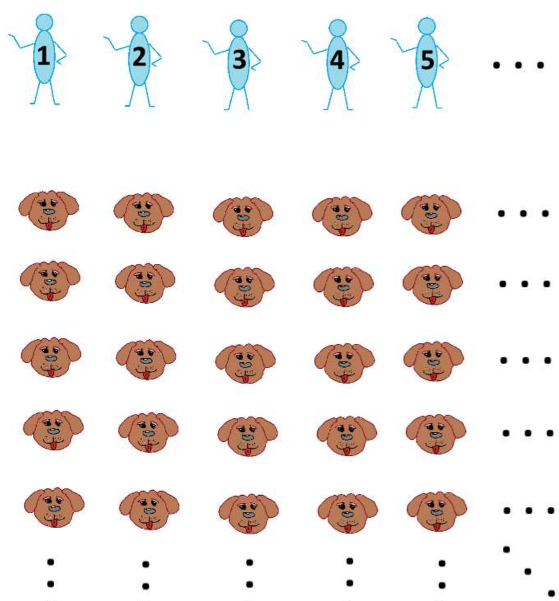


We've now shown that double- and triple- and double-ended-infinitely big sets of dogs are all the same size as a single infinity of people numbered 1, 2, 3, 4, … .

People call these infinite sets **countably infinite** because we are matching elements of those sets (the dogs) with the set of counting numbers (labeled people). All the infinite sets we've seen thus far are the same, countably infinite size.

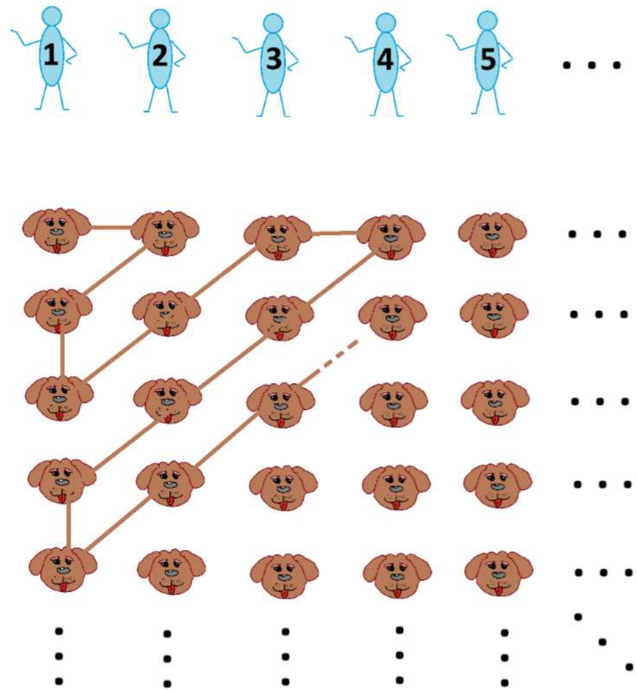
But surely a two-dimensional array of dogs—infinitely many rows of infinitely many dogs—is "more infinite" that than a single countable infinity? There just can't be a leashing pattern between the people and dogs in this picture.

What do you think?

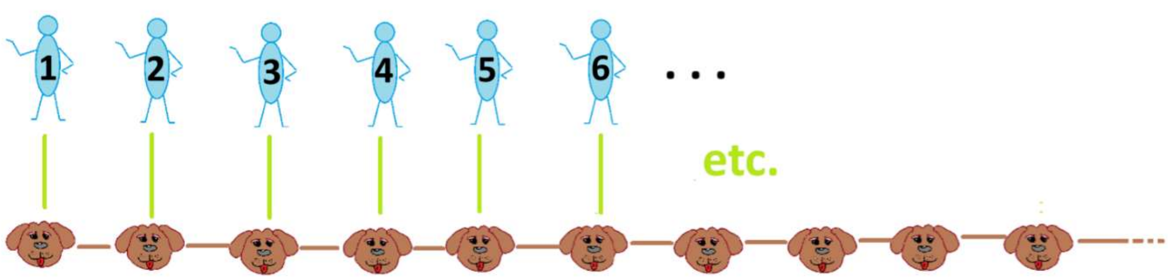


You might be surprised to see that there is a leashing pattern that works!

Start by drawing a zig-zag line that weaves through the whole two-dimensional array of dogs.

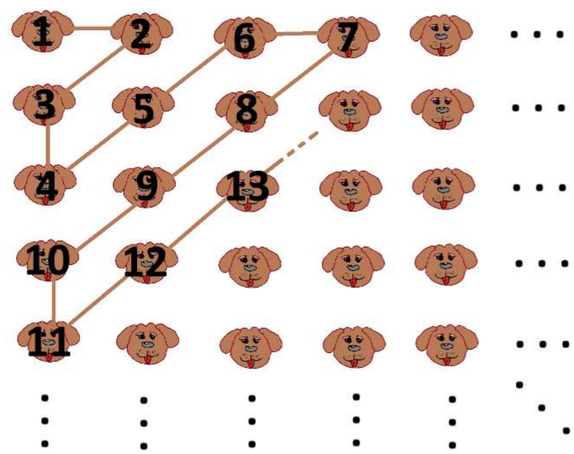


Then we can straighten out that line of dogs and display a leashing pattern.

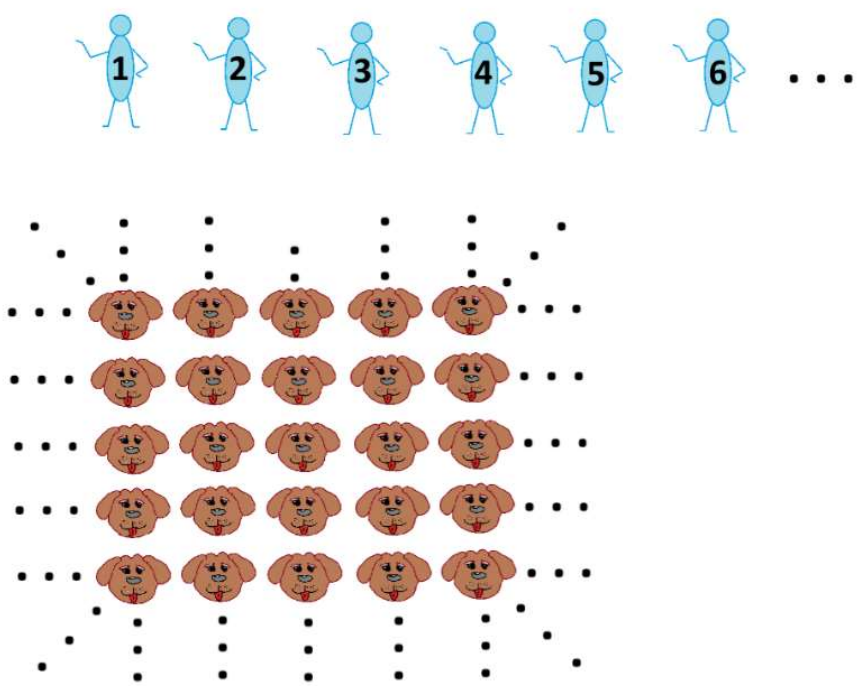


Actually … simply demonstrating a counting scheme 1, 2, 3, 4, … that weaves its way through an infinite set without ever missing an element is enough to illustrate a leashing pattern.

For example, this picture tells us to which person each dog is leashed.



Perhaps you can now envision a leashing pattern that shows that a full two-dimensional array of dogs is the same size as a "single infinity" of people. (Imagine a spiral pattern.)

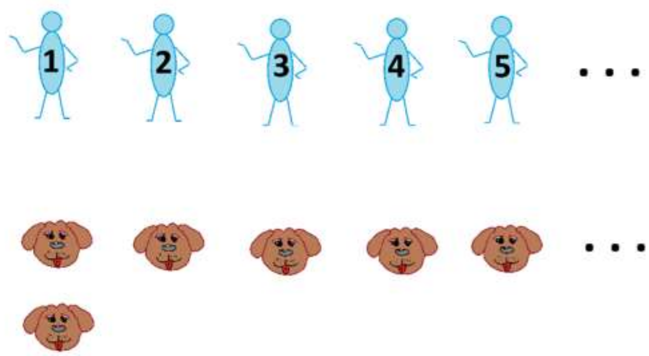


It seems that all the infinitely large sets we can imagine are the same size: “infinity is infinity is infinity.”

This makes this following school-yard exchange moot.

Kid A: **“I infinity dare you.”**
Kid B: **“I infinity plus one dare you!”**

(Indeed, show that this set of people and set of dogs are the same size.)



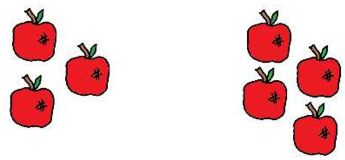
But before we might fall into a sense of complacency with counting, Cantor turned this notion on its head. He presented an example of a set that is truly "more infinite" than any set of people or dogs we've encountered in this essay. He proved that there is, in fact, more than one type of infinity!

If you’re feeling mathematically adventurous, turn to Appendix 1 for the details of this next bold turn.

2
ADDITION

Motivation from the Real World

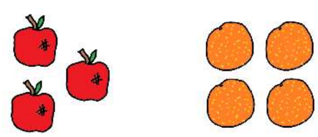
Here's a picture of some apples.



You might count 3 apples and 4 apples for a total of 7 apples.
You might even say that this picture represents the addition statement:

$$3 + 4 = 7$$

But what if we mixed apples and oranges?

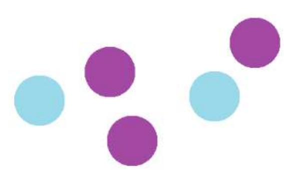


You can count 3 apples and you can count 4 oranges. But can we combine apples and oranges and say we have 7? Seven of what?

The way through this mental snag is to let go of differences and think simply "fruit": 3 pieces of fruit and 4 pieces of fruit make 7 pieces of fruit. A change of perspective brings addition back into play.

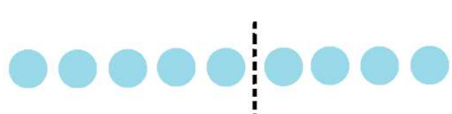
Addition *is the act of letting go of differences and recounting.*

For example, noticing and then letting go of color differences, this picture of dots represents the statement $2 + 3 = 5$.



2 + 3 = 5

Noticing dots on the left and dots on the right and then letting go of spatial differences, this picture represents $5 + 4 = 9$.



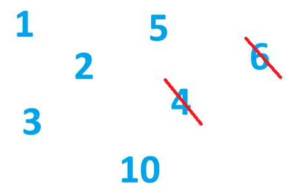
5 + 4 = 9

A Game of Solitaire

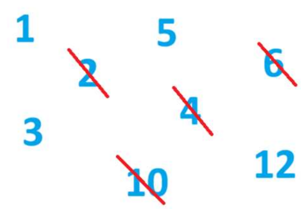
Write the numbers 1, 2, 3, 4, 5, and 6 on a page.

A “move” in this game of solitaire involves erasing two numbers and replacing them with their sum.

For example, in crossing out 4 and 6, you write 10 leaving 1, 2, 3, 5, and 10 to work with.



In next crossing out 10 and 2, you write 12 now leaving numbers 1, 3, 5, and 12.



And so on.

Each move has you erase two numbers and write one number in their stead. The count of numbers on the page steadily decreases and the goal of the game is to end with the number **21**.

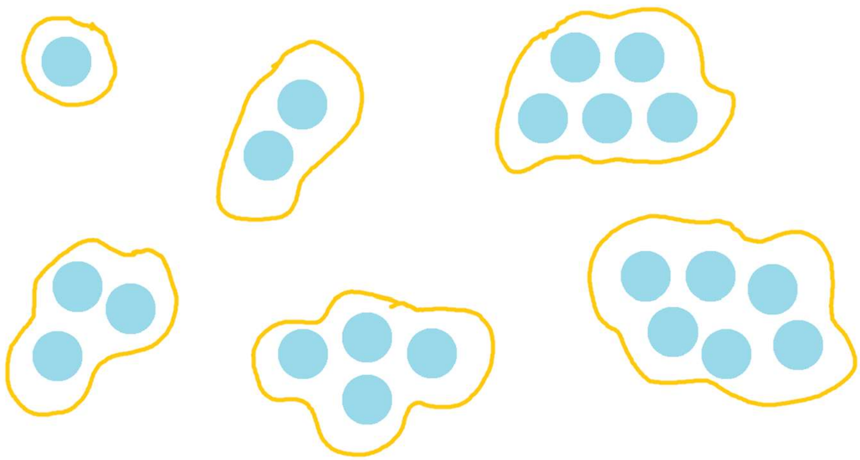
Try the game. Can you win?
I bet you can get 21 when you try.

I bet you can get 21 again playing a second time but making different choices along the way.

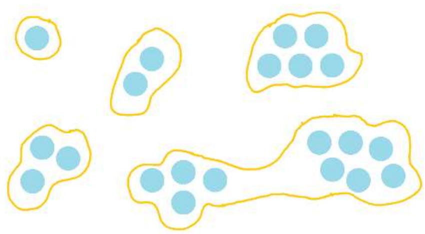
Next challenge: Play the game a third time and try to **not** get the answer 21!

This game of solitaire reveals something profound about addition.

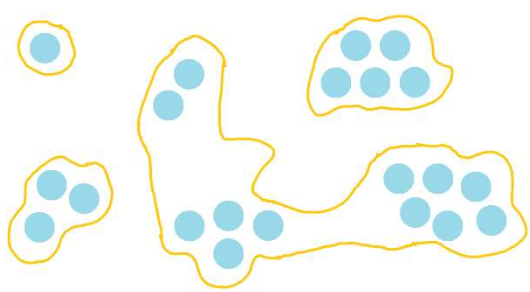
If we replace each number with a set of objects—I'll draw dots—we can imagine the start of the game as a collection of clusters of dots.



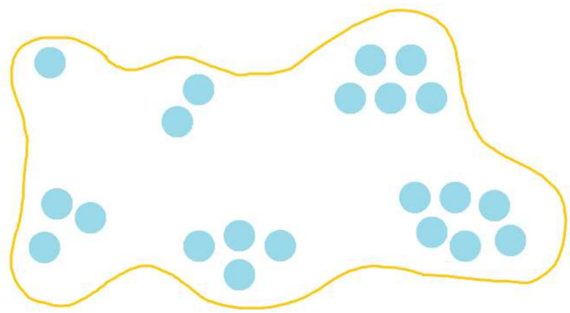
Addition is the act of ignoring differences. So, when we add "4" and "6", say, we simply ignore the fact that we have dots in different clusters and just regard them as one cluster.



Adding "10" and "2" is again a matter of ignoring cluster details.



No matter the choices one makes along the way, the end result is sure to be all dots regarded as sitting in one single cluster. The final sum is sure to match the total count of dots that were in the picture to begin with, which was 21.



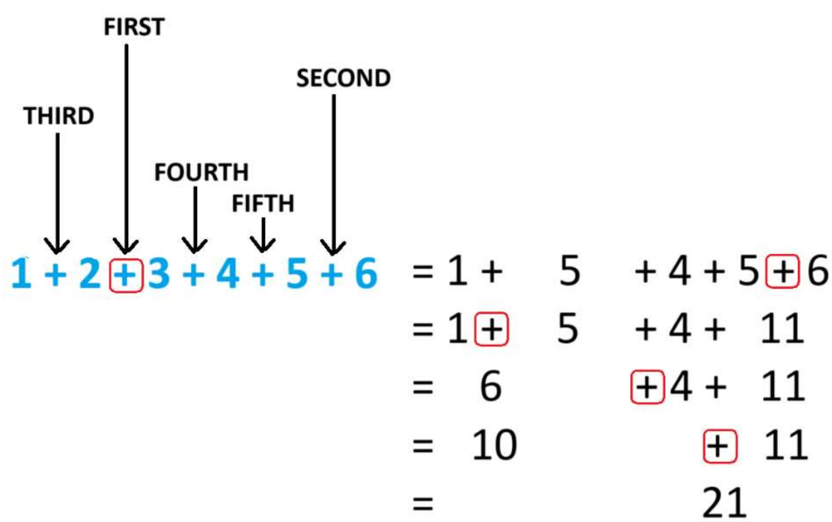
This game of solitaire shows that “order does not matter” when perform additions.

The game started with the long sum

$$1 + 2 + 3 + 4 + 5 + 6$$

But we’ve seen we need not add 1 and 2, and then 3, then 4, 5, and 6 in that order. We can add the individual numbers together in any order and the result shall be 21 each time.

Also, the order in which we conduct individual additions is irrelevant.



Upshot: The order in which one performs addition—be it the order of the individual numbers or the order of the individual additions you choose to follow—is immaterial.

The Number Zero

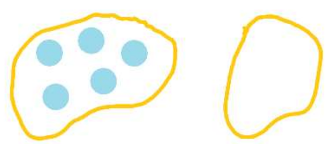
Humans have mused over the meaning of zero, 0, for millennia, wondering if the concept of zero pertains to counting or not.

Think about it.

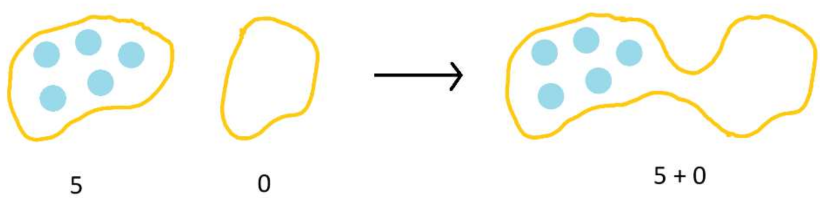
If I say that there are zero sparkly purple giraffes in the room I am sitting in right now as I type this very sentence (and it is true, there are no sparkly purple giraffes here with me), do you think it is because I actually *counted* zero sparkly purple giraffes, or did I not count and just *observe* a lack of sparkly purple giraffes? That is, does one **count** zero or does one just **observe** zero?

We can engage in this debate too if we like. But over the centuries, scholars have come to accept that zero has real-world relevance—to represent a lack of a quantity—and, moreover, that it has a place in arithmetic. (Seventh-century Indian mathematician and astronomer Brahmagupta laid out mathematical rules for working with zero and helped show the world that arithmetic remains logically consistent by incorporating zero.)

For example, here's a picture of 5 dots and 0 dots.



If we ignore the clustering differences, we obtain a picture of $5 + 0$ and see it is again a cluster of 5 dots.



We have

$$5 + 0 = 0$$

Since order does not matter when performing additions, we also have

$$0 + 5 = 5$$

And there is nothing special about the number 5 here. We also have $7 + 0 = 7$ and $0 + 93 = 93$ and even $0 + 0 = 0$.

Math’s Take Away

We have the set of counting numbers and the new number 0 (which might or might be regarded as counting something!).

0 1 2 3 4 5 6 7 8 9 10 11 12 ...

Moreover, there is an operation on these numbers called **addition**, which takes any two of these numbers a and b and produces from them a third number written “$a + b$”.

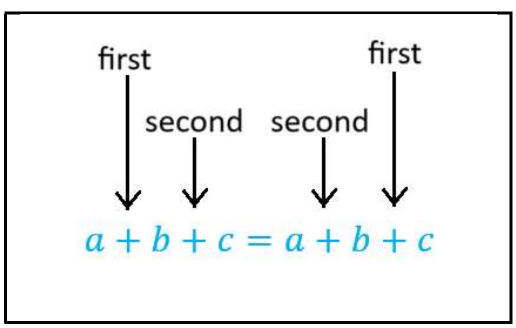
Addition: Pairs of Numbers ⟶ Numbers
a,b ⟶ a+b

This operation behaves the following way.

- **The order in which one adds individual items does not matter.**
  For example, $3 + 5$ and $5 + 3$ have the same value.

$$a + b = b + a$$

- **The order in which one performs individual additions does not matter.**
  For example. $2 + 3 + 4$ can be computed as $5 + 4$ to get 9, or as $2 + 7$ to then get 9.
  The same answer is sure to result.



- **Adding 0 to an item does not change its value.**

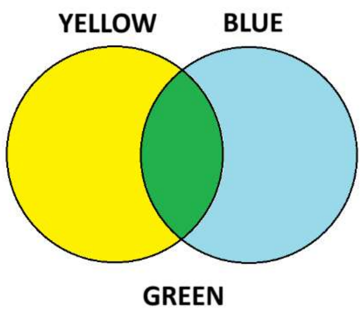
$$a + 0 = a$$

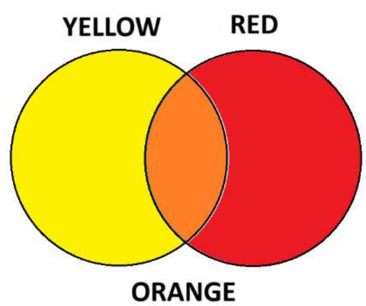
Math's Boldness

The key takeaway is that addition is now being defined by **how it behaves** and not by what it is. Thus, any operation that exhibits this behavior has the right to be called "addition."

For example, let's go back to art class and mix colors.

We observe that mixing yellow and blue makes green and mixing yellow and red produces orange.





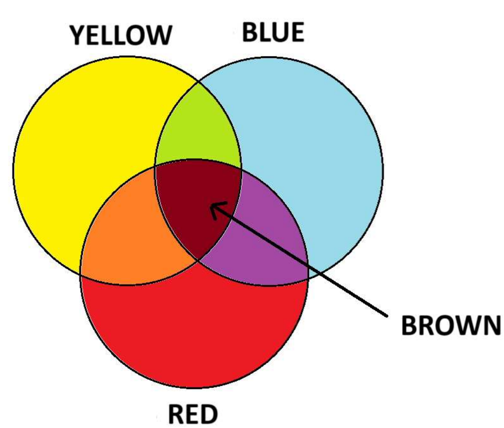
We have a means to take two colors and produce from them a third color. We even might be tempted to write mathematical statements.

$$YELLOW + BLUE = GREEN$$

$$YELLOW + RED = ORANGE$$

We can also combine more than one color.

$$YELLOW + BLUE + RED = BROWN$$



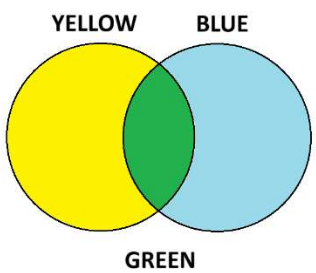
Is it legitimate to be thinking of color-mixing as addition? Does mixing paints behave like addition? Do we have the right to use the + sign?

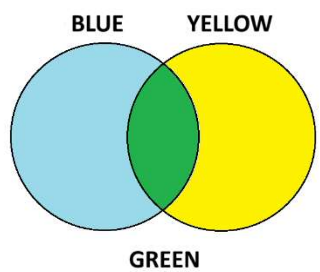
Let's check.

1. **The order in which one adds individual items does not matter.**

$$a + b = b + a$$

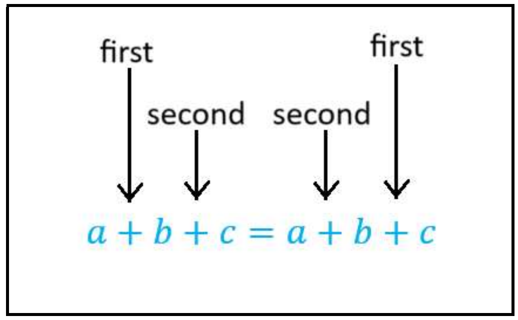
It is clear from our experience that the order in which we mix individual colors is not a concern: adding blue to yellow, or adding yellow to blue, for instance, each produce green.



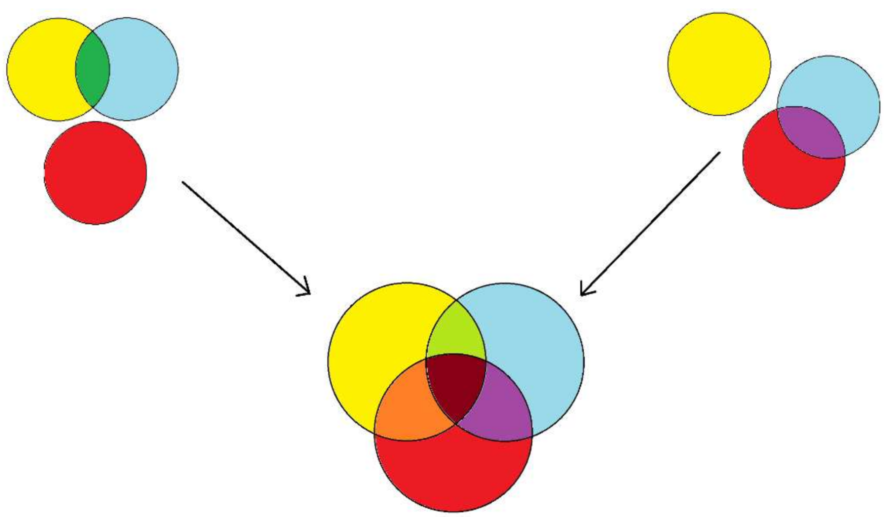


Color-mixing exhibits the first required behavior.

2. **The order in which one performs individual additions does not matter.**



In $YELLOW + BLUE + RED$ we can combine yellow and blue first, and then combine with the red. Or we could combine blue and red first, and then bring in yellow. Experience tells us that the order in which we mix pairs of colors does not matter. In the end, all three colors will combine to produce the same final color, brown in this case.



Color-mixing exhibits the second required behavior.

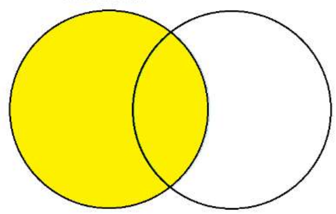
3. Adding 0 to an item does not change its value.

$$a + 0 = a$$

What's "zero" in this color-mixing game?

I've been drawing glass discs in my diagrams. Now imagine a clear glass disc. Combining it with a yellow disc produces no change in color: we still have yellow.

$$YELLOW + CLEAR = YELLOW$$



In the same way, $BLUE + CLEAR = BLUE$ and $BROWN + CLEAR = BROWN$. Adding $CLEAR$ to any color does not change the color. And so "$CLEAR$" is behaving like zero in this system.

Color-mixing exhibits the third required behavior.

We have thereby established a mathematical arithmetic for art class: mixing colors follows the behavior of "addition" and so deserves to be called addition. (We are also justified using the $+$ symbol in our work.)

Color-mixing might not be a profound example, but it illustrates a profound process. Mathematics has homed in on the key behavior of addition in one real-world context and used that behavior as the basis of more general meaning.

By stepping back from trying to define what addition **is** (a process that will forever keep us locked in one real-world context to the next), we define addition by how it **behaves**. This then encourages us to look for and recognize common structure in many different contexts all at once.

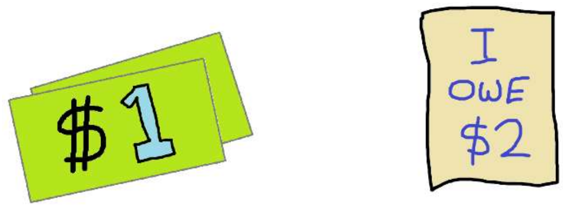
This approach is freeing, broad, and powerful.

3

NEGATIVE NUMBERS

Motivation from the Real World

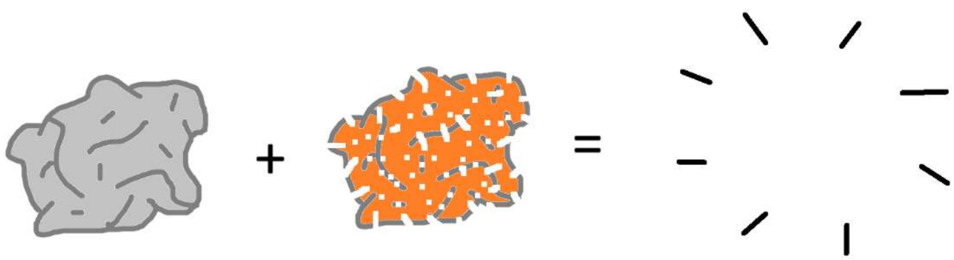
In my left pocket I have two dollars cash and in my right pocket a note to remind me that I owe Jake two dollars.



What happens if I bring these two together? Well, I can use the two dollars to pay my debt.
As a result, I'd be left with nothing: zero cash and zero debt.

The real world suggests that it could be handy to have the notion of "opposite" quantities, those that combine to return us to a zero state.

> If I take three steps forward and then three steps back, I've made no progress whatsoever.
>
> Consuming 500 calories and burning 500 calories through exercise results in no calorie gain.
>
> According to science fiction, equal amounts matter and anti-matter annihilate one another to leave no matter of any kind.



Let's say for number such as 5 or 17 there is an "opposite number" opp 5 and opp 17 such that

$$5 + opp\ 5 = 0$$

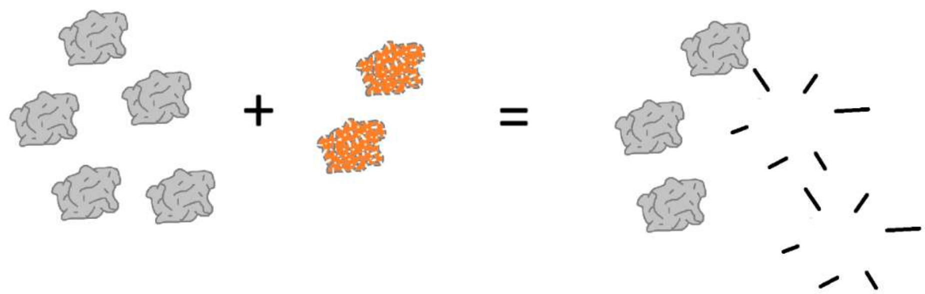
$$17 + opp\ 17 = 0$$

and so on.

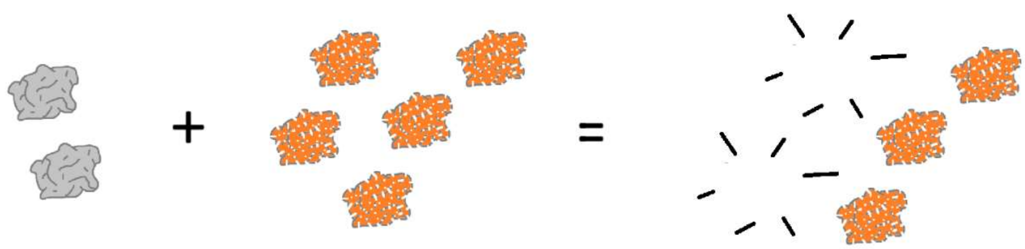
But rather than write "opp" to indicate an opposite number, people prefer to use a little dash "$-$" and write -5 and -17. They call these numbers "negative five" and "negative seventeen," respectively. (People outside of the U.S. call them "minus five" and "minus seventeen.")

Numbers that are not described as negative are often said to be "positive."

Playing with lumps of matter and antimatter, I can see that $5 + \ -2$ is 3,



and that $2 + \ -5$ is -3.



We can do arithmetic with positive and negative numbers.

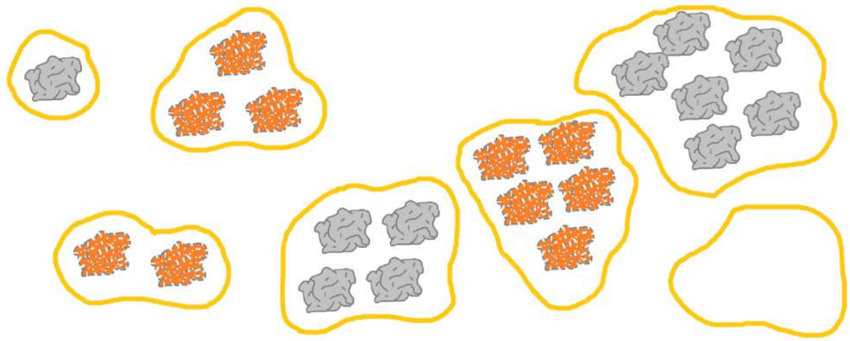
Solitaire Again

Let's play solitaire again, this time with these numbers.

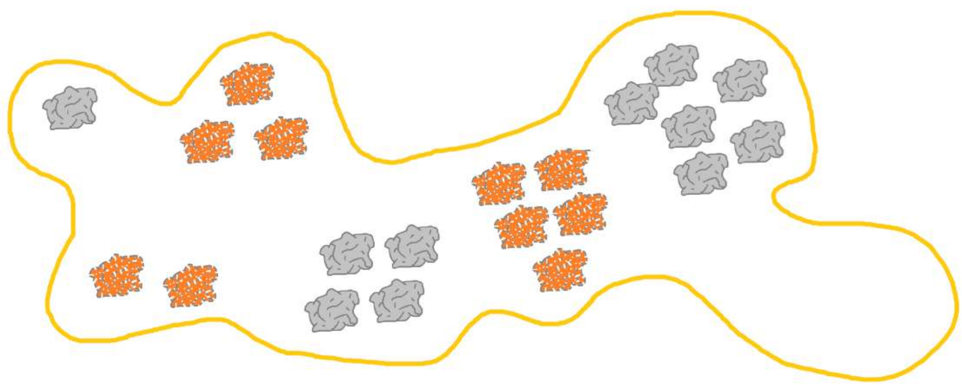
1   -3   6   -5   -2   4   0

Recall, a move consists of deleting two numbers and replacing them with their sum.
When you play, what final single number is sure to result?

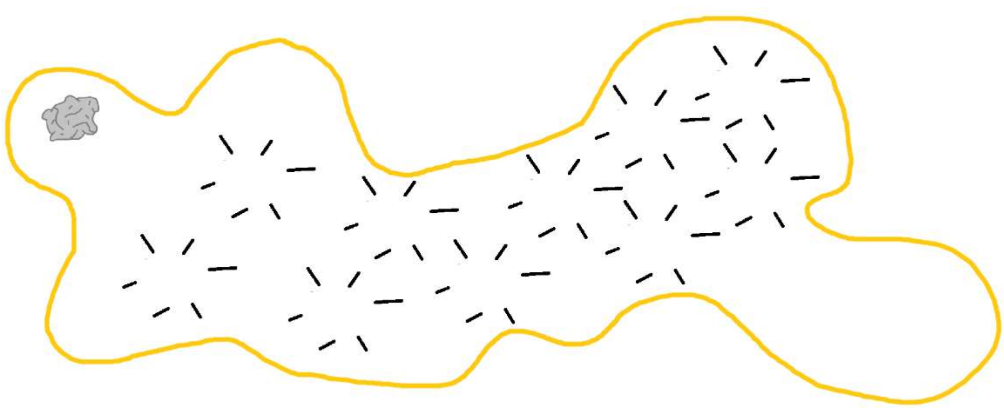
Here’s a visual representation of the start of the game In terms of clusters of matter and of amtimatter.



And here’s the end result of each and every game no matter the choices made along the way.



With all the annihilations, ten of them, we are left with just one piece of matter.



The solitaire game is sure to end with the number 1.

Math's Take Away

In addition to the counting numbers and zero we are now positing the existence of "opposite numbers." We declare:

- **For each number a, there is a number which we write as $-a$ with the property that**

$$a + -a = 0$$

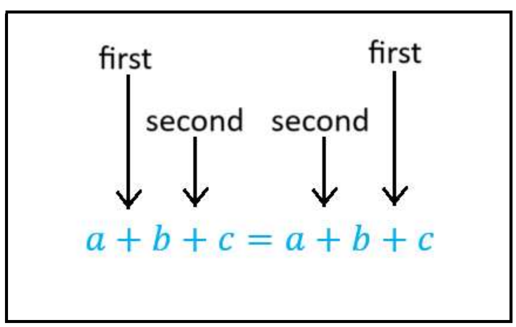
(Adding together a number and its opposite gives zero.)

These opposite numbers obey the rules of arithmetic we've established so far, namely:

- **The order in which one adds individual items still does not matter – even if some of the numbers are negative.**

$$a + b = b + a$$

- **The order in which one performs individual additions still does not matter—even if some of the numbers are negative.**



- **Adding 0 to a number does not change its value—even if that number is negative.**

$$a + 0 = a$$

Math's Boldness

We have four rules of arithmetic on the following set of numbers: the counting numbers, zero, and the opposite numbers.

	1	2	3	4	5	...
0						
	-1	-2	-3	-4	-5	...

There must be logical consequences of those rules!

Consequence 1: There is only one number that deserves to be called -5.

By its very definition, "-5" is that which you add to 5 to get the answer 0.

$$5 + \ -5 = 0$$

Could there be another weird number, let's call it w for "weird," that behaves this way too?

$$5 + \ w = 0$$

Let's show that w and -5 must be the same number.

One way to get there is to look at the triple sum

$$w \ + \ 5 \ + -5$$

It is one sum, and so has one answer. But it is possible to compute this sum two different ways.

One the one hand, we can compute $5 + \ -5$ first and see the answer w.

first ↓

$$w \ + \ 5 \ + -5 \ = \ w \ + \ 0 \ = w$$

Alternatively, we can compute $w + 5$ first, recognizing that this must much $5 + w$, which is 0.

first

$$w \overset{\downarrow}{+} 5 + -5 = 0 + -5 = -5$$

Again, one single sum cannot have two different answers. It must be that w and -5 are the same number.

We've established that there is only one number that behaves like the opposite to 5.
Of course, there is nothing special about the number 5 here. We can see, in general, that

The opposite of each number is unique to that number.

That is, for each number a there is only one number that fills this blank.

$$a + \boxed{} = 0$$

Consequence 2: The opposite of zero is zero:

$$\mathbf{-0 = 0}$$

By its very definition, -0 is the number you add to zero to get zero.

$$0 + \boxed{-0} = 0$$

But we can add zero to zero to get zero.

$$0 + \boxed{0} = 0$$

Each number has only one opposite. It must be that -0 and 0 are the same number!

Comment: The number zero has the curious status of its negative version being the same as its non-negative version. People say that "zero is neither positive nor negative."

Consequence 3: The opposite of the opposite of 5 is … 5!

$$\mathbf{- - 5 = 5}$$

By its definition, -5 is that which you add to 5 to get zero.

$$5 + -5 = 0$$

But we want to consider the opposite of -5.

By its definition, the opposite of -5 is that which you add of -5 to get zero.

$$-5 + \boxed{- -5} = 0$$

But the number 5 also fits the bill!

$$-5 + \boxed{5} = 0$$

Each number has only one opposite. It must be that $- - 5$ and 5 are the same number.

There is nothing special about the number 5 here. In general, for each number a we have

$$\boldsymbol{- - a = a}$$

We can have fun with this and deduce that

$$- - -5 = -5$$

$$- - - - 5 = - - 5 = 5$$

$$- - - - -5 = - - -5 = -5$$

and $- -5 = -5.$

4

DISTRIBUTING THE NEGATIVE SIGN

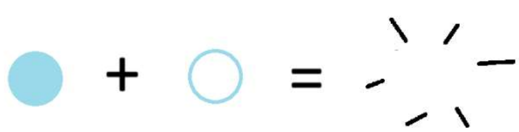
Motivation from the Real World

The real world has profit and loss that annihilate one another: one dollar in cash and a debt of one dollar result in zero net wealth.

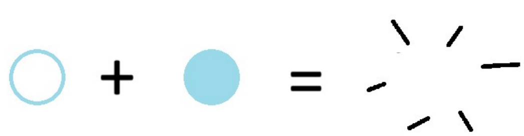
Science fiction has matter and antimatter that annihilate one another: one lump of matter and one lump of antimatter annihilate to leave no matter.

For ease of language and ease of drawing, let's draw dots and antidots to represent positive numbers and negative numbers, respectively.

A dot and an antidot annihilate one another. In this sense, an antidot is the opposite of a dot.



And what is the opposite of an antidot? That is, what annihilates an antidot? A dot.



It's helpful to insert parentheses into mathematical expressions. For example, it is a little easier to write the statement "the opposite of an antidot (-1) is a dot (1)" as

$$-(-1) = 1$$

Parenthesis help us parse statements.

The opposite of two dots and three dots altogether can be expressed as

$$-(2+3)$$

and the opposite of 4 dots and 7 antidots as a group can be expressed as

$$-(4+\ -7)$$

Question: What is the opposite of 2 **dots** and 3 **dots** as a group?

Answer: That would be 2 **antidots** dots and 3 **antidots**.
That is,

$$-(2+3)=-2+\ -3$$

Question: What is the opposite of 4 **dots** and 7 **antidots** as a group?

Answer: That would be 4 **antidots** dots and 7 **dots**.
That is,

$$-(4+\ -7)=-4+\ 7$$

Example: What is the opposite of "a dots and b antidots and 2 dots"?
That is, what is $-(a+-b+2)$?

Answer: That would be a antidots and b dots and 2 antidots.
Thus, the expression $-(a+-b+2)$ can be rewritten as $-a+b+-2$

Example: What is

$$-(x+-y)$$

expressed in words? How can the expression be rewritten?

Answer: The expression represents the opposite of x dots and y antidots as a group. This would be x antidots and y dots. Thus, we have

$$-(x+\ -y)=-x\ +\ \ y$$

People call the game we're playing **distributing the negative sign**.

This game is just a matter of identifying the opposites of everything presented to us.

Math's Take Away

There is another logical consequence to our rules of arithmetic.

Continuing our list …

Consequence 4: We have

$$-(\mathbf{2}+\mathbf{3}) = -\mathbf{2}\ +\ -\mathbf{3}$$

By its very definition, $-(2+3)$ is the number you add to $2+3$ to get zero.

$$2+3 \quad + \quad \boxed{-(2+3)} = \quad 0$$

But does $-2\ +\ -3$ also fill in the box? If I add it to $2+3$, do I also get zero?
Let's check by finding the value of

$$2+3+\ -2\ +\ -3$$

We see $2+\ -2$ in this sum, which is 0, and we see $3+\ -3$, which is also zero.
Consequently

$$2+3+\ -2\ +\ -3\ =\ \ 0\ +\ 0\ =\ 0$$

Yes! We also have

$$2+3 \quad + \quad \boxed{-2\ +-3} = \quad 0$$

But there is only one number that fills the box. It must be that $-(2+3)$ and $2+3$ are the same number.

In general, we have that

$$-(\boldsymbol{a}+\boldsymbol{b}) = -\boldsymbol{a}\ +\ -\boldsymbol{b}$$

no matter which numbers a and b represent. We can "distribute the negative sign."

We can also confirm that 1 antidot 7 antidots make 8 antidots, for instance, by “undistributing” the negative sign:

$$-1 + \ -7 = -(1 + 7) = -8$$

Also,

$$-3 + \ -2 = -(3 + 2) = -5$$

We can distribute the negative signs over negative numbers too.

For example,

$$-(4 + \ -7)$$

equals

$$-4 + \ -(-7)$$

By Consequence 3, this is

$$-4 \ + \ 7$$

That is,

$$-(4 + \ -7) = -4 \ + \ 7$$

matching our intuition that the opposite of 4 dots and 7 antidots is 4 antidots and 7 dots.

Math's Boldness

We are recognizing when it is appropriate to say that the "opposite" of a collection of objects is just the collection of individual opposites.

One can apply this principle to settings beyond just play with numbers.

For example, in the classic milk and soda puzzle, Penelope has one glass of soda and one glass of milk.

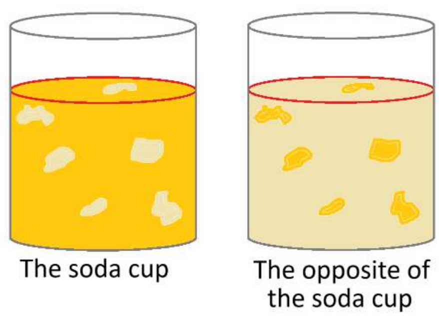
She takes a tablespoon of soda from the soda glass and haphazardly stirs it into the milk. She then takes a tablespoon of the milk/soda mixture and transfers it to the soda. Both drinks are now "contaminated."

Here's the question: *Which drink has more foreign substance?*

Is there more foreign milk in the soda than foreign soda in the milk? Or is it the other way round? Or is it impossible to say as it depends on how much or how little mixing took place?

One way to think about this is to focus on the contaminated soda cup. I've drawn it on the left.

Regarding soda and milk as "opposites," we can draw a picture of the opposite of this mixture. It is comprised of the opposite of the individual components.



We can see the soda missing from the soda cup in the picture of the opposite.
But in reality, the missing soda is in the physical milk cup.

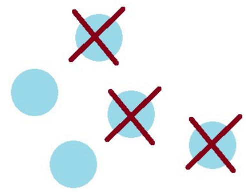
This picture of the opposite must match the reality of the milk cup, and now we see that the amount of foreign substance in each cup must be the same.

5

SUBTRACTION

Motivation from the Real World

If I have 5 dots and take 3 dots away from them, I'd be left with 2 dots.



We write

$$5 - 3 = 2$$

using—somewhat annoyingly—a little dash again to indicate "**take away**." We also read "5 − 3" as "five **subtract** three" or "five **minus** three."

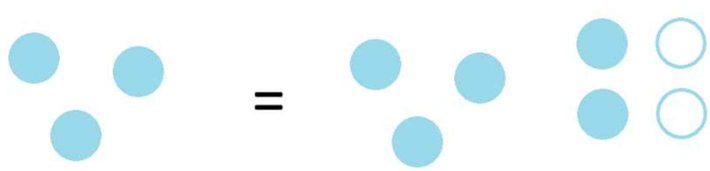
> **Comment:** In the U.S., people typically avoid calling −3 "minus three." The word "minus" is used solely as a verb for subtraction, not as an adjective to describe a negative number. In contrast, the rest of the world uses "minus" as both a verb and an adjective and relies on context to determine the meaning if its use.

Computing "three takeaway five,"

$$3 - 5$$

seems impossible.

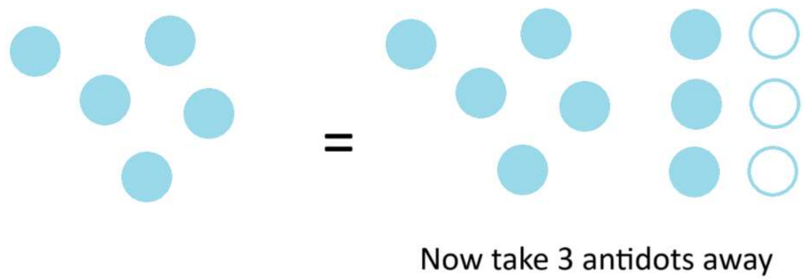
But adding some dots and antidots to a picture—to, in effect, add nothing—allows us to boost the number of dots we see to five, which can then take away. Two antidots remain.



$$3 - 5 = -2$$

Example: What's $5 - (-3)$?

Answer: To compute "five dots take away three antidots," add some dots and antidots to a picture of five dots to produce enough antidots to take away.

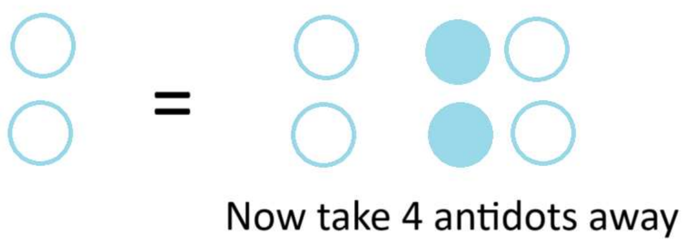


We see

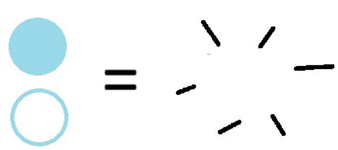
$$5 - (-3) = 8$$

Example: What's $-2 - (-4)$?

Answer: We see $-2 - (-4) = 2$.



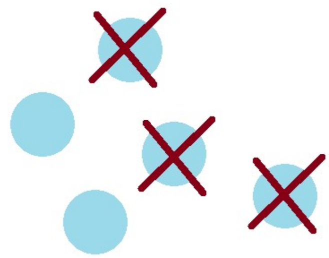
This idea of drawing in dot and antidot pairs, with each pair technically being "nothing" and so not affecting the value of a problem at hand, is a sneaky and helpful move! It creates items that you can then take away.



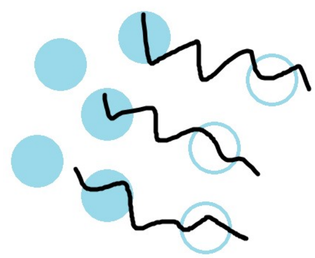
Math's Take Away

Humans want a notion of "take away" or "subtraction" as a new type of operation on numbers.

But math looks at our picture of "5 take away 3," say,



and says that the same task can be accomplished with the tools we already possess. To take away 3 dots, just add the opposite, 3 antidots. The annihilations that occur have the same effect as taking dots away.



That is, think of $5 - 3$ as $5 + -3$.

To be explicit about the annihilations that occur, write 5 as $2 + 3$. Then we see 3 and -3 annihilate.

$$5 + -3 \;=\; 2 + 3 + -3 \;=\; 2 + 0 \;=\; 2$$

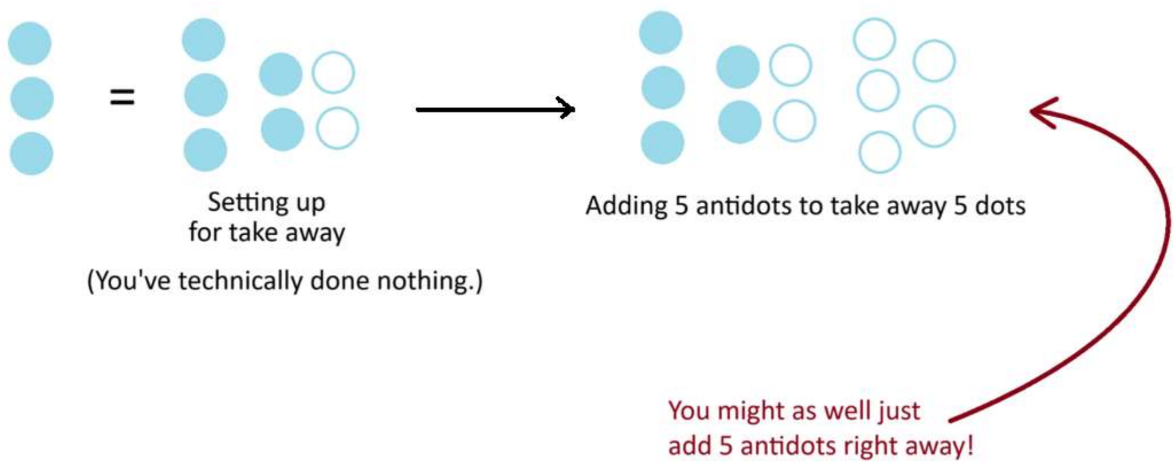
There is no need for a notion of "take away" or "subtraction."

Subtraction is just the <u>addition</u> of the opposite

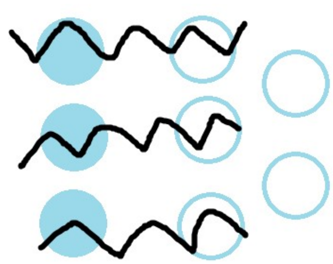
Example: Kindly compute

$$3 - 5$$

In the take-away mindset, this is "3 dots take away 5 dots." We can add some pairs of dots and antidots to create 5 dots to take away. And we can take those dots away be adding 5 antidots.



But math says you might as well add those 5 antidots to the 3 dots from the get-go!



Answer: There is no such thing as subtraction: $3 - 5$ is 3 plus the opposite of 5.

$$3 + \ -5$$

Thinking of -5 as $-3 + \ -2$ allows us to see the annihilations explicitly.

We get

$$
\begin{aligned}
3 + -5 &= 3 + -3 + -2 \\
&= 0 + -2 \\
&= -2
\end{aligned}
$$

Math and our intuition are aligned!

There is no such thing as subtraction.
Subtraction is just the addition of the opposite.

$\boldsymbol{a - b}$ **means** $\boldsymbol{a + -b}$

Example: Please compute $5 - (-3)$.

Answer: This is "5 plus the opposite of -3."

$$
\begin{aligned}
5 - (-3) &= 5 + -(-3) \\
&= 5 + 3 \\
&= 8
\end{aligned}
$$

Example: Please compute $-2 - (-4)$.

Answer: This is "-2 plus the opposite of -4."

$$-2 - (-4) = -2 + -(-4) = -2 + 4$$

Our intuition says that with 2 antidots and 4 dots two annihilations will occur leaving 2 dots. Math gives this too.

$$-2 + 4 = -2 + 2 + 2 = 0 + 2 = 2$$

Example: Please compute $6 - 1 + 3 - 7$.

Answer: There is no such thing as subtraction: think of this as addition of the opposite.

$$6 + \ -1 \ \ + 3 \ \ + \ -7$$

We can compute a summation in any order. So perhaps think of this as

$$9 + \ -8$$

Our intuition has us think of this as 9 dots and 8 antidots. There will be eight annihilations leaving 1 dot behind.

Math agrees!

$$9 + \ -8 \ \ = \ \ 1 + 8 \ + \ -8 \ \ = \ \ 1 + 0 \ \ = \ \ 1$$

Example: Kindly make $2 - (20 - x)$ look friendlier.

Answer: There is no such thing as subtraction. This is

$$2 + -(20 + -x)$$

which I read as

2 dots and the opposite of ... 20 dots and x antidots.

That would be

2 dots and 20 antidots and x dots,

giving x dots and 18 antidots.

Math agrees!

$$\begin{aligned} 2 + -(20 + -x) &= 2 + \ -20 + x \\ &= 2 + \ -2 + -18 + x \\ &= \ 0 + \ -18 + x \\ &= -18 + \ x \end{aligned}$$

Since the world likes subtraction, let's rewrite this as $x + \ -18$, which is

$$x - 18$$

Math is showing us that it is fully aligned with our dots and antidots intuition, and hence our "take away" thinking, and that we can rely on that imagery if we want.

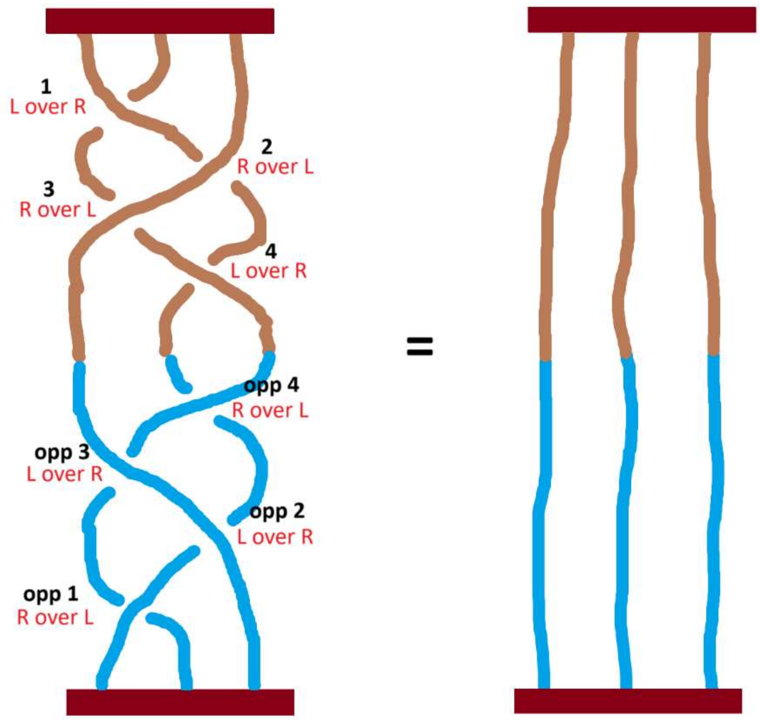
But in the end …

Subtraction is the addition of the opposite.

Math's Boldness

The idea of "adding the opposite" to take something away can be come in many guises.

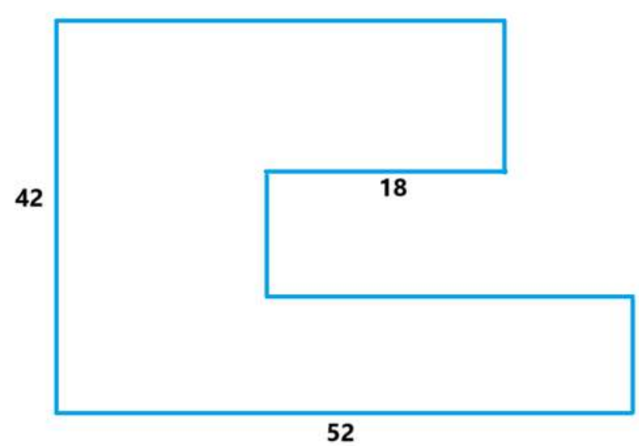
For example, to undo a braid we can add to it its "opposite."



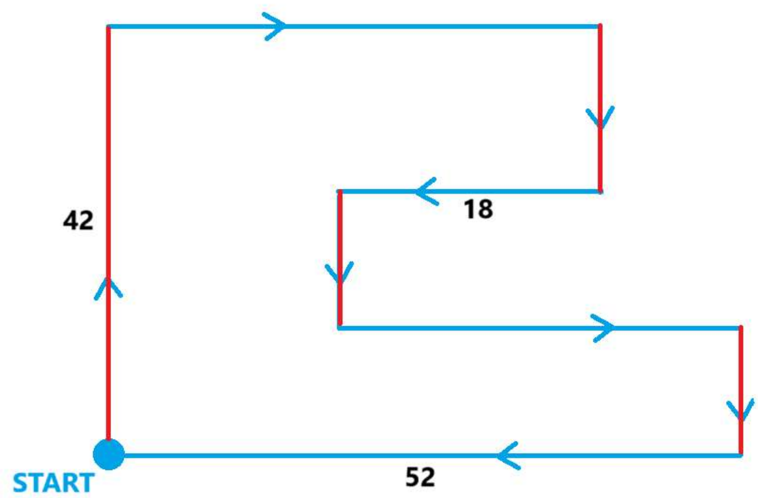
To "take away" the effect of walking some steps west and some north, just add to the journey the opposite: the same number of steps east as you took west and the same number of steps south as you took north.

This thinking allows you to solve a geometry puzzler.

Can you figure out the perimeter of this shape composed of right-angle corners? Three side lengths are given.



The key is to imagine starting at one location on the perimeter, say, at the bottom left corner, and walking along the perimeter. You return start. Since you walk 42 steps north, there must be 42 steps of southward motion. Also, $18 + 52 = 70$ steps of westward motion must be counteracted with 70 steps of eastward motion.



Any journey of 42 steps north, 42 steps south, 70 steps east, and 70 steps west is 224 steps long. The perimeter of the figure is 224 units.

But mathematics real boldness here is not “cute ways” ways to think about adding the opposites, but the efficiency of the structure of thought.

There is no need to create a new operation if the tools you already possess do the job for you.

We learn in school that there are four basic operations of arithmetic: additoin, subtraction, multiplication, and division.

We’ve cut that list down to three.

6

MULTIPLICATION