Those Thorny Math Questions!

The truth about arithmetic and the answers to those
“elementary” math questions you secretly hope
no student will ask you.

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Chapter 17: Comparing Fractions

Thorny Question Addressed:

- How can -307 be considered “smaller” than 3?

Introduction:

What is the true math mathematics behind the comparison of fractions—or the comparison of numbers in general?

Let's find out!

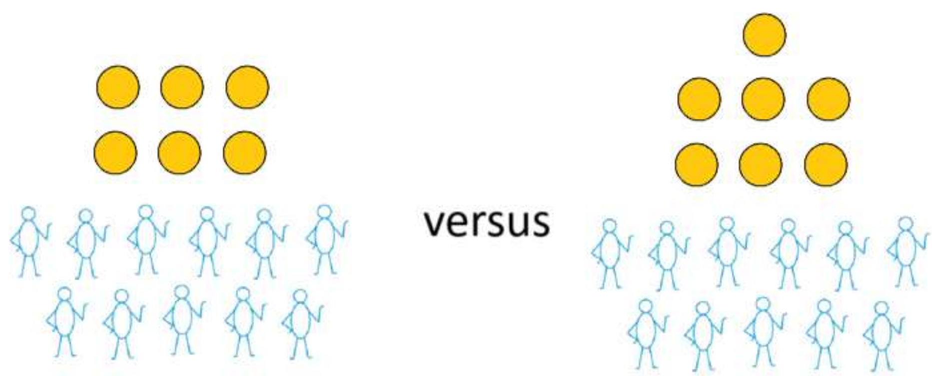
Motivation from the Real World

Fractions are the answers to division problems (see Section 13) and there are three ways to conceptualize division in the real world (see Section 10): counting groups of a given size within a set, sharing an object or objects equally among several people, or simply viewing it as multiplication in reverse.

Let's focus on the equal-sharing approach.

Example: Which is larger, $\frac{6}{11}$ or $\frac{7}{11}$?

Answer: Common sense tells us that the more pie you have to share among a fixed number of people, the more pie each person receives.



Therefore $\frac{7}{11}$ is larger than $\frac{6}{11}$.

In general, this equal-sharing model indicates that for counting numbers a, b, and n:

If b is bigger than a, then $\frac{b}{n}$ is bigger than $\frac{a}{n}$.

We can compare fractions with unlike denominators by rewriting them with a common denominator.

Example: Which is larger, $\mathbf{\frac{6}{11}}$ or $\mathbf{\frac{5}{9}}$?

Answer: Write each of the fractions with a denominator of 99.

$$\frac{6}{11} = \frac{6 \times 9}{11 \times 9} = \frac{54}{99}$$

$$\frac{5}{9} = \frac{5 \times 11}{9 \times 11} = \frac{55}{99}$$

We see now that $\frac{5}{9}$ is larger.

Example: Which is larger, $\mathbf{\frac{19}{6}}$ or $\mathbf{3}$?

Answer: We can think of 3 as the fraction $\frac{3}{1}$.

To obtain a common denominator of 6, we can rewrite this as

$$\frac{3}{1} = \frac{3 \times 6}{1 \times 6} = \frac{18}{6}$$

We see that $\frac{19}{6}$ is larger.

Math's Take-Away

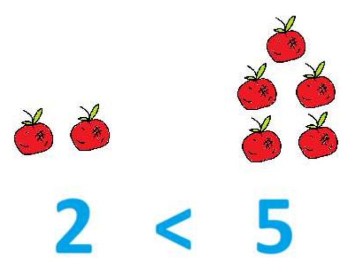
We can arrange the counting numbers, their opposites, and zero in an ordered list. We use the symbol $<$ to indicate that one number is to the left of the other in this list.

$$\cdots < -4 < -3 < -2 < -1 < 0 < 1 < 2 < 3 < 4 < \cdots$$

For instance, we have $4 < 5$ and $1 < 4$ and $-3 < 2$ and $-17 < -5$ and $-1 < 850$.

We say that numbers further to the right are "greater than" the numbers further to the left. Specifically, if a and b are two numbers in this list with $a < b$, then we say that b is **greater than or larger than** a, or that a is **less than or smaller than** b.

While it may be unclear what it physically means to say that 3 is greater than -307, this "greater than" language makes intuitive sense at least among the positive counting numbers. For example, eating five apples results in a fuller stomach than eating just two apples.



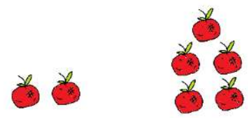
The takeaway here is that since the real-world suggests we can compare fractions—to say that one is larger than or greater than another—there should be a means to insert fractions into this ordered list of numbers.

Math's Boldness

We haven't given a mathematical definition to what it means to say that one number is larger or greater than another.

What exactly makes 5 bigger than 2, and 3 allegedly bigger than -307?

This picture provides a clue.



2 < 5

We see we need to add three more apples to the pair of apples to obtain a pile that matches the pile of five apples. That is, we need to add a positive quantity to the number 2 to have it equal 5.

Let's make that our definition:

> Among the counting numbers, their opposites, and zero, we say
>
> $$a < b$$
>
> (which we read as "$\boldsymbol{a}$ **is less than** $\boldsymbol{b}$" or as "$\boldsymbol{b}$ **is greater than** $\boldsymbol{a}$")
> if there is a positive number n such that $a + n = b$.
>
> (Loosely speaking, "we need to add something to a to get to b.")

Now this diagram makes sense!

⋯ < -4 < -3 < -2 < -1 < 0 < 1 < 2 < 3 < 4 < ⋯

- $4 < 5$ because $4 + \mathbf{1} = 5$
- $1 < 4$ because $1 + \mathbf{3} = 4$
- $-3 < 2$ because $-3 + \mathbf{5} = -2$
- $-17 < -5$ because $-17 + \mathbf{12} = -5$
- $-1 < 850$ because $-1 + \mathbf{851} = 850$
- $-307 < 3$ because $-307 + \mathbf{310} = 3$

and so on.

Let's declare a fraction to be **positive** if it is equivalent to a fraction of the form $\frac{a}{b}$ with a and b being positive counting numbers.

For instance,

- $\frac{2}{3}$ is a positive fraction
- $\frac{-2}{-3}$ is a positive fraction because it is equivalent to $\frac{(-2)\times(-1)}{(-3)\times(-1)} = \frac{2}{3}$
- 3 is a positive fraction because it is equivalent to $\frac{3}{1}$

However, $-\frac{2}{3} = \frac{(-2)}{3}$ and $0 = \frac{0}{9}$ are not positive fractions.

We can now extend our definition of "greater than" to fractions:

For two fractions $\frac{a}{b}$ and $\frac{c}{d}$, we say

$$\frac{a}{b} < \frac{c}{d}$$

if there is a positive fraction $\frac{m}{n}$ such that

$$\frac{a}{b} + \frac{m}{n} = \frac{c}{d}$$

Example: Which is larger, $\frac{6}{11}$ or $\frac{7}{11}$?

Answer: We have $\frac{6}{11} + \frac{1}{11} = \frac{7}{11}$ and $\frac{1}{11}$ is a positive fraction. So, $\frac{6}{11} < \frac{7}{11}$.

Example: Which is larger, $\frac{5}{23}$ or $\frac{19}{23}$?

Answer: We have that

$$5 + \mathbf{14} = 19$$

and, consequently,

$$\frac{5}{23} + \frac{\mathbf{14}}{\mathbf{23}} = \frac{19}{23}$$

with $\frac{14}{23}$ a positive fraction. Thus,

$$\frac{5}{23} < \frac{19}{23}$$

This example illustrates the general principle:

If b is greater than a, then $\frac{b}{n}$ is greater than $\frac{a}{n}$.

just as our real-world thinking suggested.

Math and real-world intuition are aligned here.

Example: Which is larger, $\frac{6}{11}$ or $\frac{5}{9}$?

Answer: We have

$$\frac{6}{11} = \frac{6 \times 9}{11 \times 9} = \frac{54}{99}$$

and

$$\frac{5}{9} = \frac{5 \times 11}{9 \times 11} = \frac{55}{99}$$

Now we see that

$$\frac{54}{99} + \frac{\mathbf{1}}{\mathbf{99}} = \frac{55}{99}$$

showing that $\frac{5}{9}$ is bigger.

Example: **Which is "smaller,"** $-\frac{5}{7\frac{1}{2}}$ **or** $-\frac{3}{5}$**?**

Answer: Let's make the first fraction look friendlier:

$$-\frac{5}{7\frac{1}{2}} = \frac{-5}{7+\frac{1}{2}} = \frac{(-5)\times 2}{\left(7+\frac{1}{2}\right)\times 2} = \frac{-10}{14+1} = \frac{-10}{15}$$

Looking for a common denominator, we can write the second fraction as:

$$-\frac{3}{5} = \frac{-3}{5} = \frac{(-3)\times 3}{5\times 3} = \frac{-9}{15}$$

Now we have

$$\frac{-10}{15} + \mathbf{\frac{1}{15}} = \frac{-9}{15}$$

showing that

$$-\frac{5}{7\frac{1}{2}} < -\frac{3}{5}$$

How to Use this Knowledge in the Classroom

Many textbooks distinguish between a fraction and a rational number:

> A **fraction** is any number equivalent to one of the form $\frac{a}{b}$, where a and b are positive whole numbers (and consequently b is not zero).
>
> A **rational number** is any number equivalent to one of the form $\frac{a}{b}$, where a and b are whole numbers—positive, negative, or perhaps with a equals to zero (but b cannot be zero).

For instance, $\frac{2}{3}$ and $\frac{4/7}{1/2} = \frac{8}{7}$ and $\frac{(-3)}{(-9)} = \frac{1}{3}$ are fractions, while $-\frac{2}{3} = \frac{(-2)}{3}$ and $0 = \frac{0}{1}$, are rational numbers but not fractions in this classification. Every fraction is a rational number, but not every rational number is a fraction.

This distinction is not considered significantly important in the broader mathematics community. For instance, people will say "the fraction $-\frac{3}{8}$."

Most textbooks simply rely on real-world models to generate two rules for comparing fractions (those comprise of positive whole numbers):

- If two fractions have the same denominator, the one with the bigger numerator is the greater.
- If two fractions have the same numerator, the one with the smaller denominator is the greater.

This second rule can also be "explained" through the equal-sharing model: sharing six pies equally among ten people gives more pie per person that sharing six pies equally among eleven people.

$$\frac{6}{11} < \frac{6}{10}$$

And we can also prove this is so with the mathematical definition. We have:

$$\frac{6}{11} = \frac{60}{110}$$

and

$$\frac{6}{10} = \frac{66}{110}$$

Thus,

$$\frac{6}{11} + \mathbf{\frac{6}{110}} = \frac{6}{10}$$

with $\frac{6}{110}$ being a positive fraction.

I have worked though the details of this chapter with high-school students, but only when the topic of comparing fractions arose. It felt helpful in those moments to take a half-class period to dig into matters.

THORNY QUESTION:
How can $-\mathbf{307}$ be considered "smaller" than $\mathbf{3}$?

If I am standing on the number line and take a step 307 units leftwards in the direction of the negatives, it is certainly a much larger step than stepping 3 units rightwards towards the positives. In terms of their magnitude, -307 is actually greater than 3.

It is unfortunate that we use the language "greater than" for our ordering of numbers on the number line.

… < -4 < -3 < -2 < -1 < 0 < 1 < 2 < 3 < 4 < …

It would be less confusing if we assumed the symbol $<$ come from the letter "L" for "to the left of" and read

$$-307 < 3$$

simply as "-307 is to the left of 3."

Among the positive numbers, if a is to the left of b, then a is, for certain, smaller in magnitude than b. However, this alignment of "to the left of" and "is smaller than" can only be guaranteed to hold in the positive section of the number line. The terms "smaller than" and "greater than" lose their real-world meaning when comparing numbers from other part of the number line.