The Truth About Arithmetic

and how to discuss and answer those thorny "elementary" questions you secretly hope no student will ask you.

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Chapter 10: DIVISION

Thorny Questions Addressed:

- What is the right way to think about division?
- What is the value of $8 \div 2(2+2)$?
- Why can't we divide by zero?

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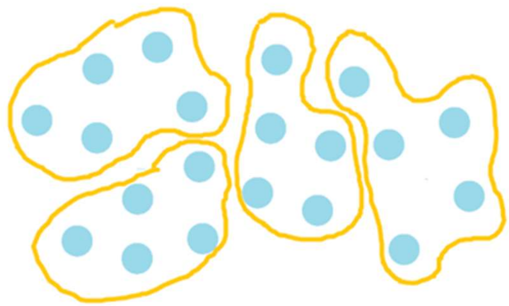
Introduction:

Early-grade curricula typically introduce three interpretations of division: counting groups, equal sharing, and multiplication in reverse. These concepts are often presented separately, leaving it unclear that they are logically equivalent. Instead, this equivalence is usually taken for granted.

Let's sort matters out!

Motivation from the Real World

Here's a picture.



It shows 4 groups of 5, making a total of 20 dots, and illustrates the multiplication statement

$$\boxed{4} \times 5 = 20$$

The picture also answers a question about multiplication in reverse:

How many groups of 5 *can be found in a collection of* 20 *dots?*

There are 4 such groups.

This process of "reverse multiplication" thinking is called **division**. The symbol $\div$ is used to denote it.

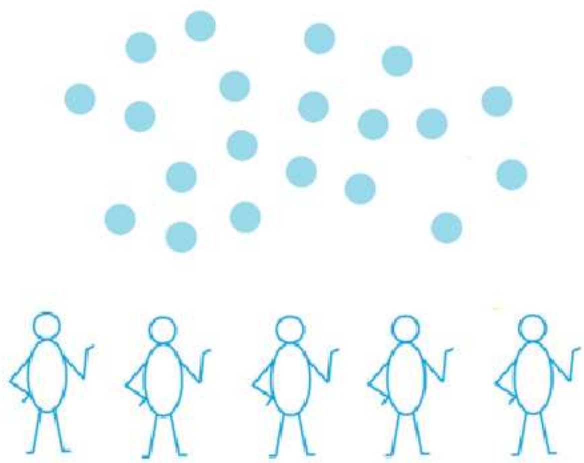
$$20 \div 5 = \boxed{4}$$

Aside: The **obelus** ($\div$) was originally used in analyzing ancient texts to mark words or passages that may be incorrect or obsolete. In 1659, Swiss mathematician Johann Rahn began using the symbol in mathematics to represent division.

Equal Sharing

Here's a question whose answer—suspiciously—is also 4.

> *I have 20 pies to share equally among 5 students.*
> *How many pies will each student receive?*

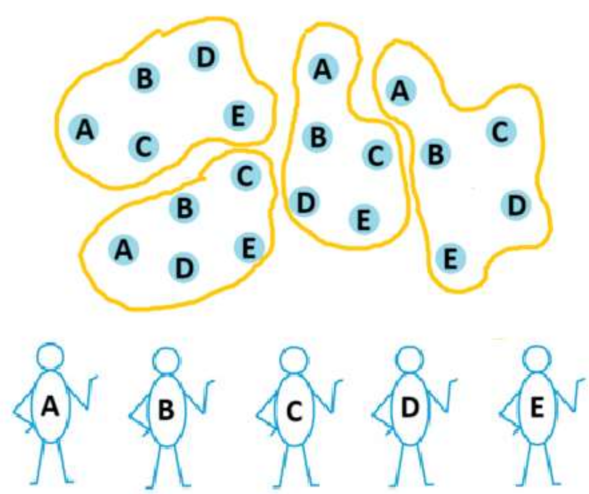


One can imagine physically completing this task by handing out one pie to the first student, one to the second student, one to the third student, one to the fourth student, and one to the fifth student. Then, repeat the process, giving a second pie to each student in the same order, and so forth. While this methos ensures the pies are distributed equally (assuming there are no leftover pies), it can be hard to predict how many pies each student will end up with.

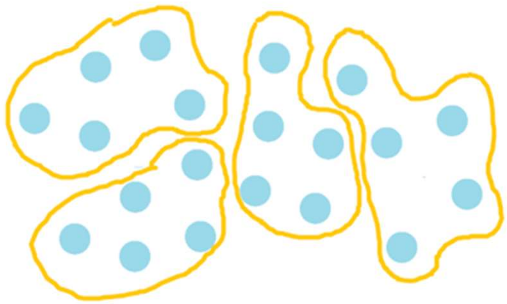
Alternatively, one can take a more organized approach to the task, allowing for anticipation of how many pies each student will receive.

> *Since there are five students, arrange the pies into groups of 5. Then, hand each student one pie from each group.*

With 4 groups of 5 among 20 pies, each student receives four pies.



This one picture can thus be interpreted in three ways:



- It illustrates multiplication:

$$\boxed{4} \times 5 = 20$$

4 *groups of* 5 *makes* 20

- It illustrates division as multiplication in reverse:

$$20 \div 5 = \boxed{4}$$

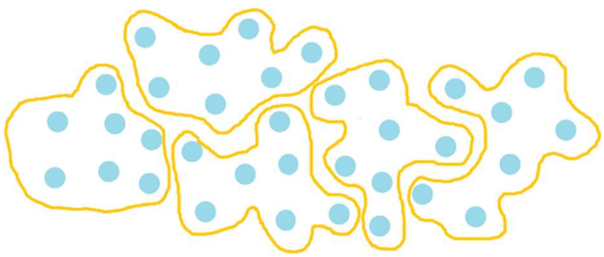
Dividing 20 *objects into groups of* 5 *yields* 4 *groups.*

- It illustrates organized sharing:

$$20 \div 5 = \boxed{4}$$

Sharing 20 *objects equally among* 5 *people gives* 4 *objects per person.*

In the same way, this picture illustrates three concepts as well:



- 6 groups of 7 make 42
- 42 divided into groups of 7 yields 6 groups
- 42 objects shared equally among 7 people yields 6 objetcs per person.

The numbers that fill in each of these blanks is the same:

$$42 \div 7 = \blacksquare$$

$$\blacksquare \times 7 = 42$$

Sharing 42 avocados equally among 7 bonobos yields ■ avocados per bonobo.

Math's Take-Away

It seems that the real world finds it helpful to designate multiplication-in-reverse as its own recognized operation: division.

$$a \div b = \blacksquare$$

means

$$\blacksquare \times b = a$$

The answer to $a \div b$ is the number that multiplies by b to give a.

There are two equivalent real-world interpretations of this operation of reverse multiplication:

- $a \div b$ is the number of groups of size b to be found among a objects.
- $a \div b$ is the number of objects each person receives when a objects are shared equally among b people.

Math's Boldness

There is no need to restrict "reverse multiplication" to just positive numbers. After all, we can conduct multiplication with positive numbers, negative numbers, and zero.

0
1 2 3 4 5 6 7 8 9 10 11 12 ...
-1 -2 -3 -4 -5 -6 -7 -8 -9 -10 -11 -12 ...

For instance,

- $(-12) \div 2 = -6$ is correct because $(-6) \times 2 = -12$.
- $20 \div (-5) = -4$ is correct because $(-4) \times (-5) = 20$.
- $(-60) \div (-4) = 15$ is correct because $15 \times (-4) = -60$
- $20 \div 1 = 20$ is correct because $20 \times 1 = 20$

Working with Zero

The statement

$$\mathbf{0 \div 5 = 0}$$

is correct because $0 \times 5 = 0$.

However, the statement $5 \div 0 = 5$ is not correct because 5×0 is not 5.

Similarly,

$5 \div 0 = 3$ is not correct because 3×0 is not 5.

$5 \div 0 = 7$ is not correct because 7×0 is not 5.

$5 \div 0 = 32$ is not correct because 32×0 is not 5.

$5 \div 0 = 978$ is not correct because 978×0 is not 5.

There is no number N to make the statement $5 \div 0 = N$ correct because $N \times 0$ will always be zero, not 5.

Curiously …

The statement

$0 \div 0 = 3$

appears to be correct because 3×0 does equal 0.

But we also have:

$0 \div 0 = 7$ appears to be correct because 7×0 does equal 0.

$0 \div 0 = 82$ appears to be correct because 82×0 does equal 0.

$0 \div 0 = 3002$ appears to be correct because 3002×0 does equal 0.

Since $N \times 0 = 0$, every number N seems to make the statement $0 \div 0 = N$ valid.

The quantity $5 \div 0$ is **undefined**, as no value N can make $5 \div 0 = N$ true.

The quantity $0 \div 0$ is **indeterminate**, as every number N appears to satisfy $0 \div 0 = N$.

Mathematics has no need to classify "reverse multiplication" as a separate operation. However, if one chooses to do so, it is essential to a**void division by zero**.

The equation

$$\blacksquare \times 0 = 5$$

has no solutions, and the equation

$$\blacksquare \times 0 = 0$$

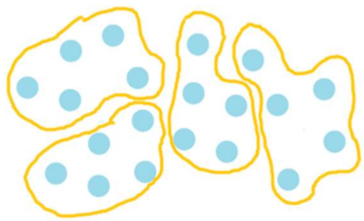
has too many solutions.

How to Use this Knowledge in the Classroom

Early-grade curricula discuss three interpretations of division – division as counting groups, division as equal sharing, and division as multiplication in reverse—usually separately, and it is often left unclear that the three approaches are logically equivalent. Instead, their equivalence is simply presumed.

Explicitly addressing this concern with students is essential.

In my experience, middle school and high school students are ready to examine a picture like this one, explore its different interpretations, and justify their equivalency, just as discussed in this chapter.



Math education specialists refer to "division by counting groups" **partitive division** and "division by equal sharing" as **quotative division**, but there is no need to use these (awkward) terms with students. The key point to share with middle school and high school students is that there are two real-world ways to interpret multiplication-in-reverse, and that reverse multiplication is the underlying mathematics behind them both.

Have students practice reading and creating statements such as following to bring this point home.

- $30 \div 6 = 5$ is correct because $5 \times 6 = 30$
- $24 \div 7 = 3$ is not correct because 3×7 is not 24.

Division by Zero

Some curricula rely on real-world intuition to "explain" why divisions by zero is impossible by highlighting the absurdity of the questions that arise:

- I want share 5 pies equally among zero students. How many pies does each student receive?
- I have 5 objects. How many groups of nothing can you count among them?
- I want share 0 pies equally among zero students. How many pies does each student receive?
- I have 0 objects. How many groups of nothing can you count among them?

While these scenarios are quirky and fun to contemplate, students deserve to understand the true mathematical reason for avoiding division by zero. The reverse-multiplication approach presented in this chapter resonates well with students.

THORNY QUESTION:
What is the right way to think of division?

Here's the bottom line:

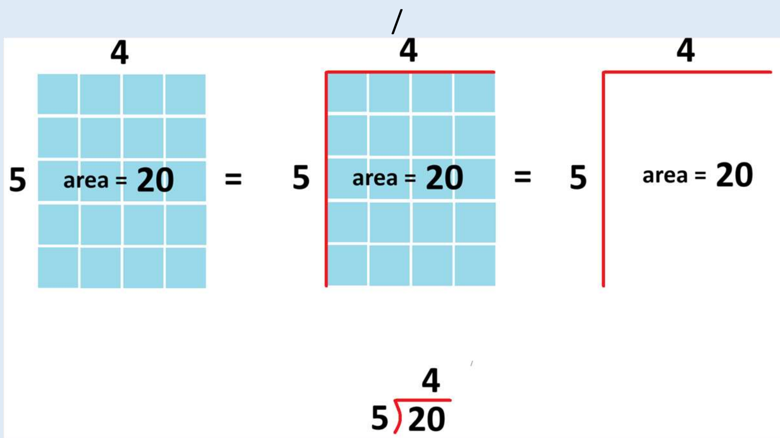
$a \div b = \blacksquare$

means

$\blacksquare \times b = a$

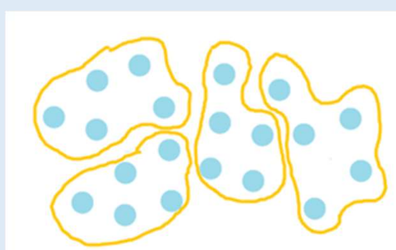
Division is the act of performing multiplication in reverse.

To say that $20 \div 5 = 4$, for instance, is to say that $4 \times 5 = 20$.



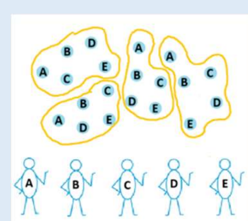
A delightful--and confusing—aspect of this reverse multiplication is that it often appears in two seemingly unrelated real-world contexts.

For example, here's a picture representing 4×5, four groups of five.



We can interpret this picture in three ways:

- It illustrates that 4×5 equals 20.
- It shows that there are 4 groups of size 5 among 20 objects.
- It demonstrates that sharing 20 objects equally among 4 people yields 5 objects per person.



This means that if we find ourselves counting groups of equal size in a given scenario, or if we are engaging in equal sharing, we can confidently state that we are effectively conducting multiplication in reverse and can refer to the process as “division.”

Ultimately, mathematics defines division as reverse multiplication, and all mathematical properties of division must be justified within that framework.

Acknowledgement: Thank you, Tierney Kennedy, for the visual imagery of transforming a rectangle illustrating multiplication to the division symbol.

THORNY QUESTION:
What is the value of $\mathbf{8 \div 2(2+2)}$?

Every few months, a math challenge circulates on social media asking for the value of this expression:

$$8 \div 2(2+2)$$

Some people argue vehemently that it equals 1. Can you see how they might arrive at that conclusion?

Others insist that it equals 16. Can you see how they might come to that value?

According to the conventions of arithmetic, we should compute the value of the expression inside the parentheses first. Thus, the expression is equivalent to

$$8 \div 2 \times 4$$

Is this equal 1, or is this 16?

The confusion arises because there is no clear convention for handling division and multiplication together in an expression—hence the debate!

We can eliminate this confusion by using more parentheses. Writing $8 \div (2(2+2))$ ensures that the expression evaluates to 1. Conversely, writing $(8 \div 2)(2+2)$ guarantees the result is 16.

The original expression is an example of intentionally ambiguous writing. No reputable math author would write it!

By the way, some textbooks state that ÷ and × are "equally powerful" operations. If you encounter an expression with a series of them in a row, such as $8 \div 2 \times 4$, they suggest reading the expression left to right and evaluating it in that order (which yields $8 \div 2 \times 4 = 4 \times 4 = 16$.) However, this is not a standard convention among all textbook authors, nor is it widely accepted in the general mathematics community.

For Fun …

Ask your students to create some deliberately ambiguous sentences, whether mathematical or otherwise.

Here are a few examples.

- I painted the room with the lights off.
- I saw a man with binoculars.
- She spoke to her friend with an accent.

THORNY QUESTION:
Why can't you divide by zero?

One can verify whether a statement of division is correct by checking with reverse multiplication.

For example,

- $12 \div 2 = 6$ is correct because $6 \times 2 = 12$.
- $20 \div 5 = 4$ is correct because $4 \times 5 = 20$.
- $0 \div 5 = 0$ is correct because $0 \times 5 = 0$.
- $60 \div 4 = 10$ is not correct because 10×4 is not 60.
- $8 \div 1 = 4$ is not correct because 4×1 is not 8.

Let's examine

$$5 \div 0$$

What value might it have?

- $5 \div 0 = 3$ is not correct because 3×0 is <u>not</u> 5.
- $5 \div 0 = 7$ is not correct because 7×0 is <u>not</u> 5.
- $5 \div 0 = 32$ is not correct because 32×0 is <u>not</u> 5.
- $5 \div 0 = 978$ is not correct because 978×0 is <u>not</u> 5.

There is no number N to make the statement $5 \div 0 = N$ correct because $N \times 0$ will always be zero, not 5.

Now let's examine

$$0 \div 0$$

It "suffers" from a different kind of problem.

For instance,

- $0 \div 0 = 3$ appears to be correct because 3×0 does equal 0.

Likewise,

- $0 \div 0 = 7$ appears to be correct because 7×0 does equal 0.
- $0 \div 0 = 82$ appears to be correct because 82×0 does equal 0.
- $0 \div 0 = 3002$ appears to be correct because 3002×0 does equal 0.

Since $N \times 0 = 0$, every number N seems to make the statement $0 \div 0 = N$ valid.

The quantity $5 \div 0$ is problematic because no value N can make $5 \div 0 = N$ true; it has no possible value.

The quantity $0 \div 0$ is problematic because every number N appears to satisfy $0 \div 0 = N$; it has too many possible values.

A quantity divided by zero simply cannot be defined. We must avoid dividing by zero.