A Quirky Pile-Splitting Puzzle

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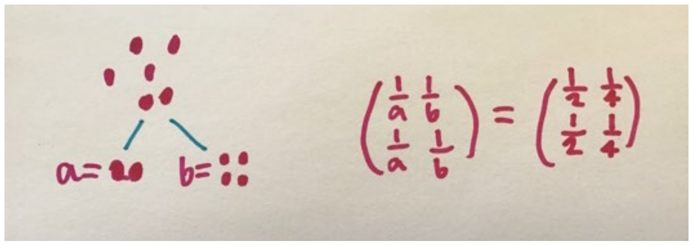
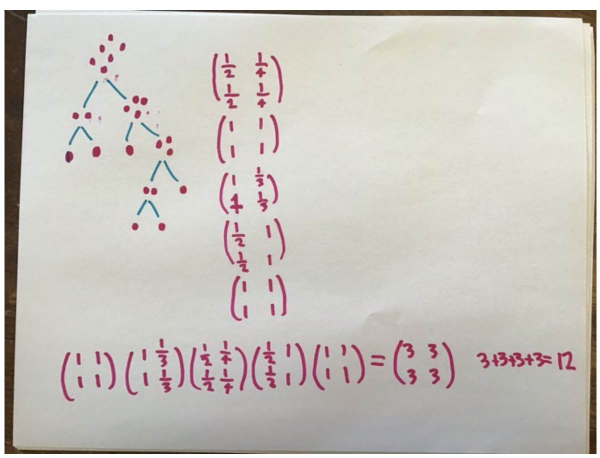
Here's a puzzle. Start with a pile of 6 objects: pebbles, beads, coins, whatever you like. Then split your pile into two smaller piles of sizes a and b, say. Off to the side, write down a 2x2 matrix with each row $(\frac{1}{a}, \frac{1}{b})$.

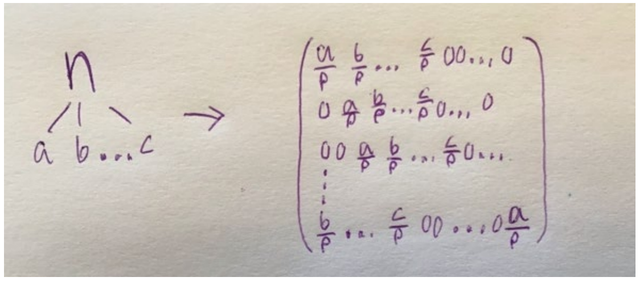


Repeat this process of splitting the piles and writing down the associated matrices until you only have 6 single pebble piles remaining (which, of course, cannot be split any further.) Now, take all of the matrices, in any order (it does not have to be the order in which you wrote them down), and multiply them all together. I'm willing to bet that no matter which order you multiplied the matrices, the sum of all of the entries of the product matrix—its "grand sum"—is sure to equal 12!



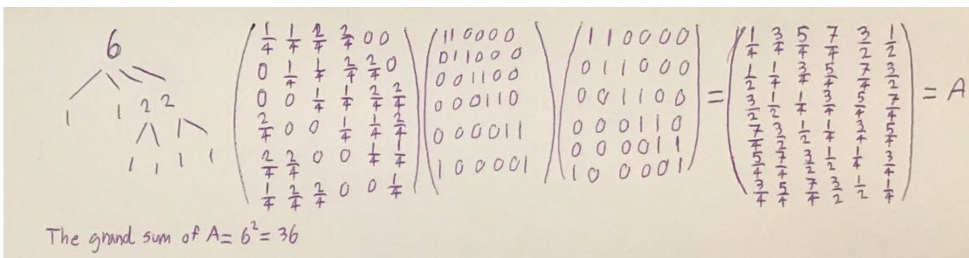
Try this! Go through this same pile splitting process, but make different splitting choices along the way. Does the grand sum still equal 12? What about if you change the order in which you multiply the matrices? Is this phenomenon unique to 6 pebbles?

This is kind of cool… but let's take this even further! Start with a pile of n objects, and split the pile into k smaller piles of sizes $a, b, \ldots, c$, say, where k is any number $\leq n$. Let $p = a \bullet b \bullet \ldots \bullet c$ be the product of the pile sizes. This time, write down an nxn matrix, with the entries in each row $(\frac{a}{p}, \frac{b}{p}, \ldots, \frac{c}{p})$ with zeros filling any excess entries (which will be the case if $k < n$). But, instead of keeping all of the rows the same, "shift" each entry one space to the right from one row to another with wrap around.



Again, repeat this process (changing up the value of k as you go), until you only have n single piles left. Multiply all of the matrices, in any order, and the grand sum of the resulting matrix is sure to equal n^2.

Here's an example with $n = 6$:



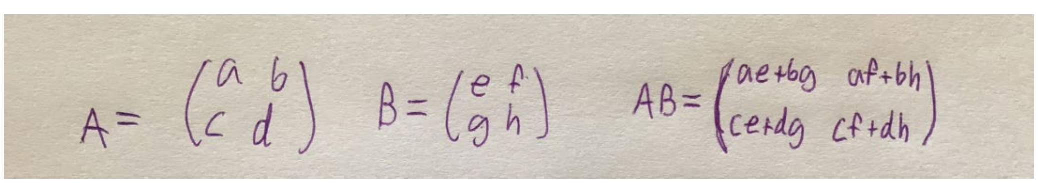
What is going on here?

Exploring why this works

(Make sure you've had some fun trying to explain this on your own, before I give it all away!)

Part 1: Matrices

Let's start by looking at that first puzzle, with the 2x2 matrices. More specifically, let's look at the bit where we take the product of all of the special matrices we create. If you did try multiplying the string of matrices which you got in several different orders, you probably noticed that the entries of the final product matrix would change, but the grand sum was always equal to 12. So, something in the specific structure of the matrices must accommodate a sort of "commutativity." Ok, I'm not entirely sure what that means yet, so let's play with it. Let's multiply two matrices, with random entries, and take things from there.



Ok, the grand sum of AB is $ae + bg + af + bh + ce + dg + cf + dh$. Is this related in some way to $(a + b + c + d)(e + f + g + h)$, the product of the grand sums of the original matrices?

Well, in our puzzle, the rows of our matrices are identical. So, let's presume $a+b=c+d$ and $e+f=g+h$. Now the rather nasty $ae+bg+af+bh+ce+dg+cf+dh$ simplifies to $a(e+f)+b(g+h)+c(e+f)+d(g+h)=a(e+f)+b(e+f)+c(e+f)+d(e+f)=(a+b)(e+f)+(c+d)(e+f)=2(a+b)(e+f)$! (Not a factorial, just an expression of excitement.) This is half the product of the grand sums of the two matrices. Or, it can be viewed as twice the product of the sum of the entries in either row of A and sum of entries of either row of B. Either way, it's the individual row sums at play here.

We have:

> If A and B are two-by-two matrices and the entries in each row of A sum to α and the entries in each row of B sum to β , then the entries in each row of AB sum to $\alpha\beta$.

Mathematicians call a square matrix whose entries in each row and each column sum to the same value *semi-magic*. Let's call a square matrix whose entries in just each row sum to a common value *demi-semi-magic* and let's call that common sum its *row sum*.

Now, before we move on to explain the pile splitting bit of all of this, let's first ensure that the above rule is true for arbitrarily sized demi-semi-magic matrices, and not just two-by-two ones. Actually, there's a nice slick way to do this: Let $\mathbf{1}$ be the nxn matrix with all entries equal to 1. A matrix, A, is demi-semi-magic if, and only if, $A\mathbf{1}=\alpha\mathbf{1}$ for some value of α (which is the row sum of A). So, if A and B are both demi-semi-magic, $AB\mathbf{1}=A(\beta\mathbf{1})=\beta(A\mathbf{1})=\beta\alpha\mathbf{1}=\alpha\beta\mathbf{1}$, showing that AB is demi-semi-magic with row sum $\alpha\beta$. So too is BA, with the same row sum $\alpha\beta$, following the same work. The order in which one multiplies demi-semi-matrices does not affect the row sum of the demi-semi-matrix that results. Neat!

> If A, B , .., C are demi-semi magic n x n matrices with row sums α,β, …,γ, then their product (in any order) is again demi-semi-magic, with row sum $\alpha\beta\cdots\gamma$.

Part 2: Pile Splitting

Now we just need to explain why, no matter which splitting choices we made along the way, the product of the row sums of the matrices was always the same. In each pile splitting puzzle, if we start with a pile of *n* items and split it into piles of *a, b*, …, *c* items (in our 2x2 matrices puzzle, only piles a and b), we write a demi-semi-matrix with row sum $\frac{a}{ab\cdots c}+\frac{b}{ab\cdots c}+\cdots\frac{c}{ab\cdots c}=\frac{n}{ab\cdots c}$. When we later split the pile of *a* items into piles of *d*, *e*, …, *f* items, we write a matrix with row sum $\frac{a}{ed\cdots f}$. And so on. The final step produces a demi-semi-magic matrix with row sum the product of all these row sums.

$$\frac{n}{ab\cdots c}\times\cdots\times\frac{a}{de\cdots f}\times\cdots\times\frac{b}{gh\ldots i}\times\cdots$$

As we can see, every pile that gets split will end up in the numerator of a fraction, and every pile that is the result of a split will end up in the denominator of a fraction. That is, every pile, except for our initial n and our final n 1s, will appear both in the numerator and denominator of a fraction, and will cancel. We see then that this product equals $\frac{n}{1\cdot 1\cdots\cdot 1}=n$.

For the first puzzle we start with $n = 6$ and write demi-semi-magic 2x2 matrices. The resulting matrix after play of the game has two rows each of row sum 6, and so grand sum of 12.

For the second puzzle we start with n objects and write demi-semi magic n x n matrices. The final matrix has n rows each of row sum n and hence grand sum of n^2.

Reference:

There's actually a more general way to express this, which the brilliant James Tanton came up with in his pile splitting puzzles (you can find them here: https://www.jamestanton.com/wp-content/uploads/2010/12/Pile-Splitting1.pdf), which gave me the inspiration to play with all of this. I have no idea how he thought of it, but here it is: Let $A(n)$ equal the nth term of a sequence (such as the counting numbers, or square numbers, or triangular numbers, etc.) With a split of n into a and b associate the fraction $\frac{A(n)}{A(a)A(b)}$. When we multiply all of these fraction at the end, all of the "middle stuff" will cancel, leaving $\frac{A(n)}{A(1)A(1)\ldots A(1)} = \frac{A(n)}{A(1)^n}$. I was then able to expand this a smidgen, to show that we can associate the fraction $\frac{A(n)}{A(a)A(b)\ldots A(c)}$ with a split of n into k piles, $a, b, \ldots, c$ (where k need not be the same value every time). As before, every pile, except for $A(n)$ and the n $A(1)$s, will end up in the denominator of one fraction, and the numerator of another, so when we take the product of all of these, almost everything will cancel, leaving us, again, with $\frac{A(n)}{A(1)^n}$. And as you can see from above, our two puzzles use $A(n) = n$.

One can also conduct an addictive version of pile-splitting by associating with each split $k \to a + b + \cdots + c$ the term $A(k) - A(a) - A(b) - \cdots - A(c)$. The sum of all such terms after pile-splitting is sure to be $A(n) - nA(1)$.

Some final questions:

What other pile splitting puzzles can you come up with using matrices? Is there a way to do something with logarithms? Trigonometric function? Modular arithmetic? And are there any other interesting puzzles which one could do with demi-semi-magic matrices? Could one use pile splitting to express every demi-semi-magic matrix as a product of other demi-semi-magic matrices? What if $A(n)$ were, itself, a matrix, how could that work in the context of matrix pile splitting?

There's so much left to explore here, and who knows what one might find!