Straight-Up Logarithms

Here it is. Straight into some math with zero explanation of what we are doing and why we are doing it! Can you do this anyway?

Question 1: Fill in the blanks.

$$power_{10}\left(million\right) = 6 \qquad power_{10}\left(billion\right) = \_\_\_\_$$

$$power_{2}\left(8\right) = 3 \qquad power_{10}\left(0.01\right) = \_\_\_\_$$

$$power_{5}\left(25\right) = 2 \qquad power_{73}\left(1\right) = \_\_\_\_$$

$$power_{2}\left(16\right) = \_\_\_\_ \qquad power_{82}\left(82\right) = \_\_\_\_$$

$$power_{4}\left(16\right) = \_\_\_\_ \qquad power_{0.01}\left(1000\right) = \_\_\_\_$$

$$power_{4}\left(64\right) = \_\_\_\_ \qquad power_{100}\left(0.1\right) = \_\_\_\_$$

$$power_{3}(81) = \_\_\_\_ \qquad power_{2}\left(2^{12} \cdot 2^{51}\right) = \_\_\_\_$$

$$power_{10}\left(10^{903}\right) = \_\_\_\_ \qquad power_{\sqrt{6}}\left(36\right) = \_\_\_\_$$

$$power_{7}\left(\frac{1}{7}\right) = \_\_\_\_ \qquad power_{-2}\left(4\right) = \_\_\_\_$$

$$power_{3}\left(\sqrt{3}\right) = \_\_\_\_ \qquad power_{-2}\left(8\right) = \_\_\_\_$$

$$power_{\frac{1}{5}}\left(\frac{1}{25}\right) = \_\_\_\_ \qquad power_{1}\left(5\right) = \_\_\_\_$$

$$power_{\frac{1}{3}}\left(9\right) = \_\_\_\_ \qquad power_{0}\left(5\right) = \_\_\_\_$$

Can you add another example to this list that has no answer?

Can you add an example to this list that has more than one possible answer?

Question 2: How would you describe your emotional state right now?
What question is in your mind at present? (Is it "Why on Earth are we doing this?")

Question 3: (Well, it's just an activity.)

Please go through the examples in the previous question and cross out the word "power" in all the places it appears. Replace it with the letters "log," which is short for *logarithm*.

$\overset{\text{log}}{\cancel{power}}_{10}(million) = 6$	$power_{10}(billion) = \_\_\_\_$
$\overset{\text{log}}{\cancel{power}}_{2}(8) = 3$	$power_{10}(0.01) = \_\_\_\_$
$\overset{\text{log}}{\cancel{power}}_{5}(25) = 2$	$power_{73}(1) = \_\_\_\_$
$power_{2}(16) = \_\_\_\_$	$power_{82}(82) = \_\_\_\_$
$power_{4}(16) = \_\_\_\_$	$power_{0.01}(1000) = \_\_\_\_$
$power_{4}(64) = \_\_\_\_$	$power_{100}(0.1) = \_\_\_\_$ etc.

Great! We've just learned logarithms!

Question 4: How is your emotional state now?
What question is in your mind at present? (Is it still "Why on Earth are we doing this?")

Question 5: Most school textbooks start a chapter on logarithms with no explanation too, but with a scary statement like this.

The expression $\log_b(x) = a$ is to be read as "the power of b that gives the answer x is a." That is,

$$\boxed{\text{The statement } \log_b(x) = a \text{ means } b^a = x.}$$

Even though we don't know why we are doing this, can you at least follow the definition and see that it matches what you did in question 1?

Question 6: Actually, the definition that appears in textbooks has caveats. (What does the word "caveat" mean?)

For $b > 0$ and $b \neq 1$, $\log_b(x) = a$ means that "a is the power of b that gives the answer x."

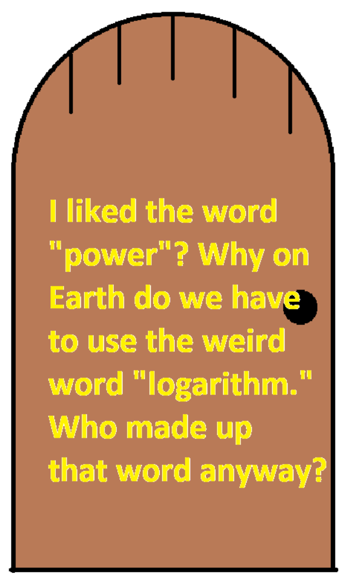
Why do you think school textbooks also insist on disallowing b to be equal to 0 or to be equal to 1 or to be negative?

Question 7: What's your burning question right now?

CHOOSE YOUR OWN ADVENTURE(S)

Here are two burning questions others have asked. Choose one (or both!) and follow the corridors of intrigue that lie behind each door.

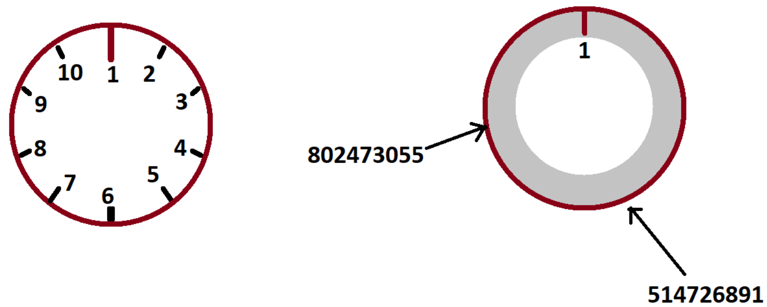
DOOR 1
Turn to page 4



DOOR 2
Turn to page 14

DOOR 1: WHY DO PEOPLE CARE ABOUT POWERS?

Question 1: Audio equipment like radios and technical instruments in general used to, and sometimes sill do, have dials.



Is it easier to work with a dial labeled with the numbers 1 through 10, or with one with numbers labeled 1 through a billion?

Have you ever heard someone describe a number by the count of digits it has?

Question 2: Arjun would like to make a “six-figure salary” one day. What does that mean? What’s the smallest amount of money he needs to earn each year to fulfil his wish?

Becky *is* currently making a six-figure salary. And if she earned just one dollar more each year, she’ll be earning a seven-figure salary! What’s her current yearly salary?

Question 3: The truth is, I have a five-digit salary. Arjun's salary is an order of magnitude bigger than mine. And if Becky earns that extra dollar, her salary will be two orders of magnitude bigger than mine.

One often hears people speak this way comparing "orders of magnitude" of two big numbers. Do you have a sense of what it could mean to say that one number is one order of magnitude larger than another? Two orders of magnitude?

Content Burst

Some people say that the "order of magnitude" of a positive integer is the number of digits it possesses. For example, 423 has, with this definition, order of magnitude 3; and a million has order of magnitude 7.

That's a fine definition. But surely, we want a sense of the "order of magnitude" of non-integer numbers too? What's the order of magnitude of the number 7823.073? Of 0.001?

We could just ignore the decimal part of any given positive number and simply count the number of digits in what remains. (So 7823.073 has order of magnitude 4 and 0.001 has order of magnitude 1, maybe? Or 0 maybe? Hmm.)

If we do this, then 0.001 (a thousandth) and 0.01 (a hundredth) and 0.1 (a tenth) would all be dubbed as having the same order of magnitude. But a thousandth is ten times smaller than a hundredth, and a hundredth is ten times smaller than a tenth, and so on. These quantities feel like they have different "orders of magnitude," at least in a "getting significantly smaller" sense.

It seems we need a better definition. What do you think of this one?

If a positive number M sits between 10^n and 10^{n+1} (possibly equaling 10^n) with n an integer

$$10^n \le M < 10^{n+1}$$

call n the **order of magnitude** of the number M .

For example, 423 sits between 10^2 and 10^3 and so, with this definition, has order of magnitude 2. (All three-digit numbers have magnitude 2.) The number 7823.073 has order of magnitude 3, and 100000.0245 has order of magnitude 5, and 0.001 has order of magnitude -3. (Why?)

Question 4: According to the new definition …

a) What's the order of magnitude of $533{,}973{,}983{,}004{,}428\frac{1}{2}$?
b) What's the smallest number with order of magnitude 8?
c) What's the smallest eight-digit number and what's its order of magnitude?
d) If M has order of magnitude N, what is the magnitude of $10 \times M$?
e) Give an example of a number with order of magnitude 0.
f) Give an example of a number with order of magnitude -10.

Have you noticed the mismatch between the first definition of "order of magnitude" and the second one?

Question 5: A number sits between 10^{15} and 10^{16}. How many digits does it have? What is its order of magnitude?

Question 6:

a) A positive integer has order of magnitude 173. How many digits does the number have?
b) A positive number has order of magnitude -173. How many digits are there to the right of its decimal point?

Just to be clear:

> **From now on we will only work with the second definition for the order of magnitude of a positive number.**

Content Burst

One can hear a pin drop – just – and one has no choice but to hear the roar of the engine as a rocket takes off at Cape Canaveral if you are standing near it. In fact, the sound intensity of a NASA Saturn V rocket launch, for instance, is about $100{,}000{,}000{,}000{,}000{,}000{,}000$ times more intense than the sound of a pin drop.

Rather measure sound intensity on a scale from single digits to numbers in the billions, it is easier for us to have a scale based on the order of magnitude of these measurements, giving a scale of 1 to 20, say. That seems more manageable.

The Bel scale for sound intensity essentially does this. For example, the sound intensity of a pin drop measures 1 Bel and the sound intensity of the Saturn V rocket launch is 20 Bel. (It has become standard to speak in terms tenths of Bels—*decibels*. A pin drop is 10 dB and a rocket launch 200 dB.)

The Richter scale for earthquake intensity is also a scale based on order of magnitude. An earthquake of measure 6 on the Richter scale is one unit of magnitude up from one of measure 5. That is, the earthquake is ten times as strong.

> **Question 7:** I said, "the sound intensity of a NASA Saturn V rocket launch … is about $100{,}000{,}000{,}000{,}000{,}000{,}000$ times more intense than the sound of a pin drop."
>
> Given that, and the fact that the sound intensity of a pin drop records as 1 dB, do I have the sound intensity of the rocket launch, at 20 dB, correct?

Do you now see what I asked question 1?

Question 8: Just to be confusing, there is actually a third (and different) definition of the "order of magnitude" of a positive number that scientists (especially astronomers) often use. It is based on thinking of all numbers as powers of ten in their own right.

Observe that

$$10^6 = 1,000,000$$

and

$$10^7 = 10,000,000$$

Now $3,300,000$ is a value between one million and ten million so it is possible to imagine that there is a power of ten, between 6 and 7, that gives this value. That is, we need to find:

$$power_{10}(3,300,000) = \_\_\_\_$$

Experimentation with a calculator shows that $10^{6.519}$ is about $3,300,000$. (Check this.)

In the same way, we see $3,100,000$ seems to be about $10^{6.491}$.

Here's the third definition of order of magnitude of a positive value.

> *Write the given number as a power of ten, most likely with a decimal as the power:* 10^r. *Then round* r *up or down to the nearest integer. Call that rounded value the order of magnitude of the number.*

Examples:

$3,300,000 \approx 10^{6.519}$ and 6.519 rounds to 7. So $3,300,000$ has order of magnitude 7 in this setting.

$3,100,000 \approx 10^{6.491}$ and 6.491 rounds to 6. So $3,100,000$ has order of magnitude 6 in this setting.

(Both of these numbers have order of magnitude 6 according to our second definition.)

Is this confusing?

CONTINUING THE ADVENTURE

Thinking about orders of magnitude immediately has us thinking about powers of ten. This is one reason why many people find understanding powers very helpful. After all, in the final question we saw the desire to solve

$$power_{10}(3,300,000) = \underline{\qquad\quad} .$$

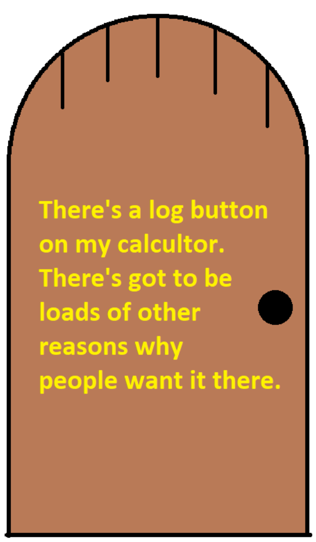
Do you remember that we crossed out the word “power” and replaced it with “log” for the word *logarithm*? (Why? Check out Door 2 on page XX if you haven’t already.)

There are more reasons as to why the school math curriculum wants students to know about logarithms. And we will need to get to those. But we can hold off on them a bit if you want.

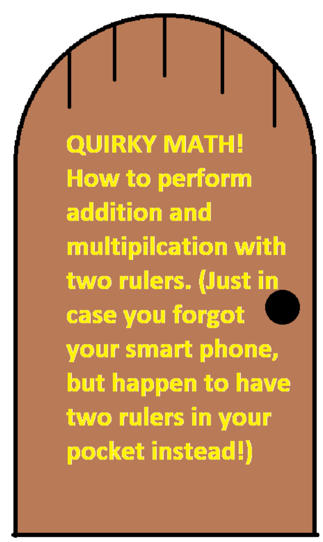
So, here’s our next choice:

> Go to the first door to dig into some traditional school math on this topic right away
> or
> Hold off on that for bit and do something a little quirky.

(Or … do both!)



Door 5
Turn to page 10

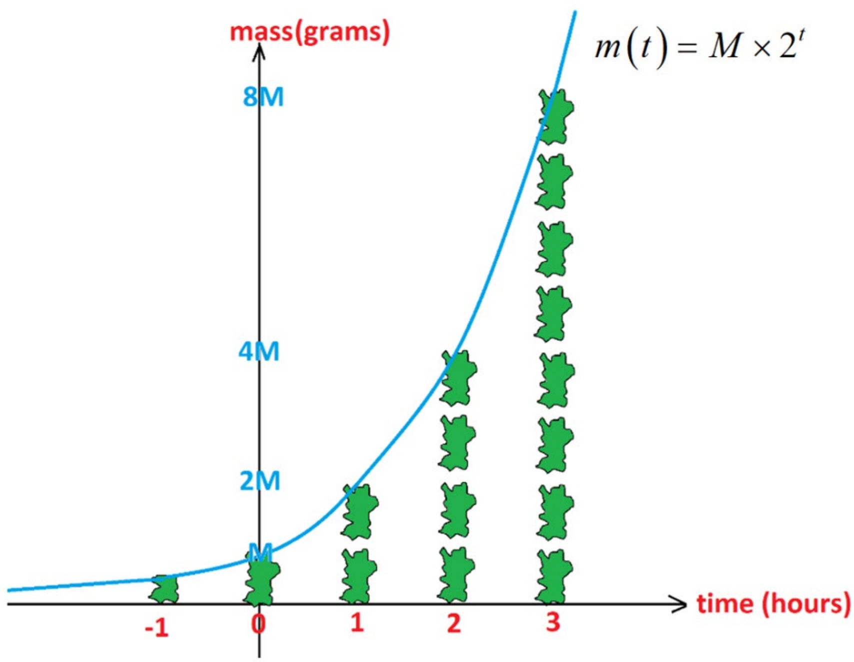


Door 6
Turn to page XX

DOOR 5: A MODERN REASON TO CARE ABOUT LOGARITHMS

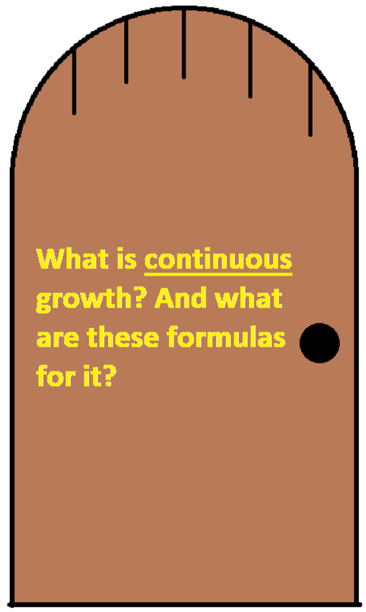
Here's a contrived scenario that mimics genuine population growth.

> Suppose a culture of algae grows continuously to double in mass every hour. Suppose it's mass at the start time $t = 0$ hours is M grams. (In my picture, each green blob represents M grams of algae.) A basic formula for the mass m of the culture at any time t is $m(t) = M \times 2^t$.



Question 1: Does this formula make any sense at all? What are the values of $m(1)$, $m(2)$, and $m(3)$? Do they match what you see in the picture? (Does my picture make any sense?) Also, what is the value of $m(-1)$ and what does it mean?

Question 2: Do we need a door right now? If so, feel free to follow it now.



DOOR 8

Turn to page XX

According to the graph, it looks like at some time between hours $t = 1$ and $t = 2$ the mass of the culture is $3M$ grams. ("Three blobs' worth.")

A first guess is that a mass of $3M$ grams appears at time one-and-a-half hours: $t = 1.5$ hours.

Question 2: What is the value of $m(1.5)$? Is it $3M$ grams?

It isn't!

So, this leaves us with the task of trying to find the value t such that $M \times 2^t = 3M$, or, equivalently, $2^t = 3$.

This is awkward. It is hard to solve an equation that has the unknown "stuck upstairs" as a power.

Question 3: Experiment on a calculator for three minutes. Try to find a value for t, to two decimal places, for which 2^t is really close to the value 3.

There's got to be a way to avoid this tedious work!

Let's think about what are we trying to do here. We're hoping to compute

$$\log_2(3) = \_\_\_\_ .$$

(That is, we're trying to compute $power_2(3)$.)
If we had a general mathematical theory about how powers/logarithms work, then maybe we can start solving mathematical equations about powers with some ease?

The modern reason for caring about logarithms is that a general theory about them will help us solve equations that have the variable "stuck upstairs" as a power.

Question 4: Most scenarios that model real life situations involve numbers that are far from friendly. What might the best way to handle a more realistic equation like this one?

$$1.076\times(2.647)^{t-3.244} = 15.000$$

a) Type the equation into a computer algebra software and press enter.
b) Develop a general theory of powers/logarithms first and then solve this by hand.
c) Guess and check with a calculator.
d) Don't! Leave your work blank in defiance!

Of course, we live in the 21st century and have technology that will solve an equation like this with ease. That it is the smartest thing to use to get the answer.

But if you are curious about what the technology is doing (and if ever you want to program future technology) it could be good to understand the mathematics behind it all.

To set this idea up for students, textbooks often give a series of carefully constructed (but unrealistic) equations like the following to solve.

Question 5: Solve for the unknown in as many of the following as you have the patience for.

a) $2^t = 32$ b) $2^{t+3} = 32$ c) $2^{3t-1} = 32$ d) $2^{-t} = \dfrac{1}{32}$ e) $8 \cdot 2^t = 2^{4t}$

f) $9 \cdot 3^{x-2} = 27^{x-1}$ g) $5 \cdot 5^x \cdot 5^{2x} = 1$ h) $\dfrac{1}{7^{2-a}} = 49$ i) $(0.01)^{8n} = (10^{7n-15})^2$

CONTINUING THE ADVENTURE

Okay, we have the door from question 2 to follow. Or perhaps you would like this door next? (After all, this essay never talked about the "log" button on our calculators, which was part of the original question!)

DOOR 3

Turn to page XX

DOOR 2: WHO CHOSE THE WORD "LOGARITHM"?

During the Renaissance, science blossomed and flourished in Europe. (Err. When was the Renaissance? Which years/centuries are we talking about?) Scholars were collecting data and working with data to understand the world around them. And they had to repeatedly perform arithmetic computations on large lists of numbers to perform statistics, to formulate and analyze equations, and so on. And all this arithmetic had to be done by hand, with pencil and paper.

Question 1: Here are five data values from some scientific experiment:

3.17 2.98 3.02 2.47 3.28

For one analysis of them, you might want to add together all five values. (Perhaps you are computing their average value, that is, their *arithmetic mean*.)

In a different analysis, you might need to multiply together all five numbers. (There are possible reasons for this. The "multiplicative" version of an average is called a *geometric mean*.)

Now imagine you are back in the 1500s.

Which is easier to do by hand: adding together these five numbers or multiplying together these five numbers?

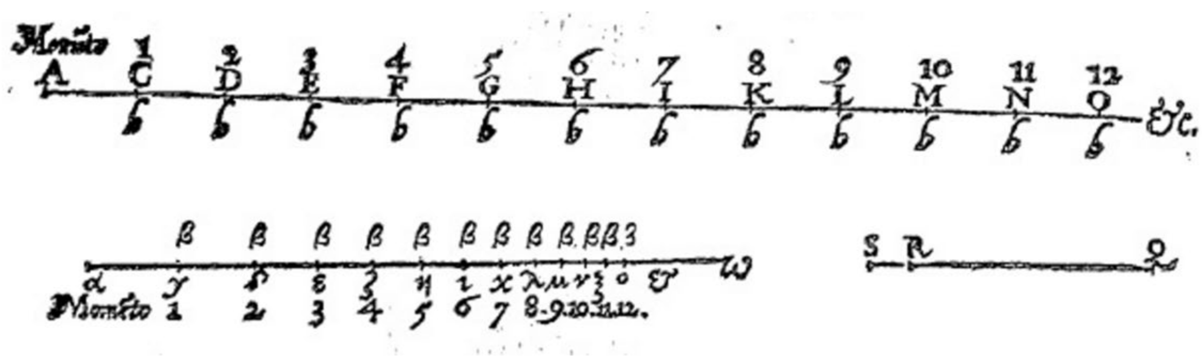
Here's my answer to question 1:

3.17	**3.17**
+ 2.98	**x 2.98**
+ 3.02	**x 3.02**
+ 2.47	**x 2.47**
+ 3.28	**x 3.28**
=	**=**
Not fun, but doable	**HORRID!**

The fact that multiplying numbers together by hand is so hard and so tedious actually held back scientific progress all through the 1400s and 1500s!

For this reason, a Scottish mathematician by the name of John Napier (1550 – 1617) set out to ease the tremendous woe of all science and create a method that would turn multiplication problems into addition problems.

Napier was a mighty inventive fellow. After much toying and playing, he came up with a very involved method that did the trick. To multiply to numbers, say M and N, Napier imagined two particles each moving along a number line, one a line of infinite length and one of finite length. The first particle moved at a uniform speed that was related to the number M, and the second at a speed that was related to the number N and varied according to the distance it still had to traverse across the line segment. (Did I say Napier was creative?)



Napier found that computing the ratio of the velocities of the two particles was a procedure that essentially turned the computation of $M \times N$ into an addition problem. It was complicated, and strange, but it worked! Napier also inserted a factor of 10,000,000 into all his computations to help out geoscientists who were working with large measurement across the Earth.

Napier invented a name for his method based on the Greek word *logos* for ratio and *arithmos* for number. His word was *logarithm*.

No one really understood his method at the time. With the help of a colleague, Henry Briggs (1561 – 1630) he decided to create tables of values – log tables – that scientists could simply refer to, without knowing the details behind his method, to convert multiplication problems into addition problems so that they could be computed with relative ease.

value	logarithm
1	0
2	.301
3	.477
4	.602
5	.699
6	.778
7	.845
8	.903
9	.954
10	1

To compute 2x3

log(2) = .301
log(3) = .477

Add .778

We see this matches the answer 6.

Napier's logarithms literally saved the progress of science.

It wasn't until another 150 years or so before scholar realized that Napier's logarithms were, essentially, just "powers" in disguise. But by then – and now another three hundred years later – the name *logarithm* had stuck. We still use the word logarithm to this day!

If you are curious about Napier's thinking ad methods, check out this piece from the Mathematical Association of America. http://www.maa.org/press/periodicals/convergence/logarithms-the-early-history-of-a-familiar-function-john-napier-introduces-logarithms