Solving Systems of Linear Equations via Algebra

ANNOTATED WORKSHEET FOR TEACHER REFERENCE
(Student version follows)

Math classes often give puzzles like this one.

> *A certain family has some cats (and no other pets). Each cat in the household has four legs and two ears and one nose, and each human in the household has two legs and two ears and one nose.*
>
> *In the house, there are a total of 34 legs and 22 ears.*
>
> *How many noses are there in the household?*
> *How many of those noses are human noses?*

Question 1: What is your typical emotional reaction to puzzles like these?

STEP 1: Be your honest human self. Acknowledge your genuine emotional reaction.
Deep breath ...
STEP 2: Do something! ANYTHING!

Nervous I feel like puzzles are set up to trick me. Distrustful.

Kinda intrigued. Scared I hate word problems

This is childish. It's quirky and fun. I feel pulled in.

Question 2: Take a deep breath.
Can you solve the puzzle, or some part of it, using nothing but common sense?

Common Sense is great!

Okay. There are 22 ears in the household. Each entity—a cat or a human—has two ears, so this means that there are 11 entities in the house.

The total number of noses is 11.

The number of legs in the house is 34 and the number of ears is 22. These numbers differ by 12. Since each cat has two more legs than humans, this means there must be 6 cats.

The number of human noses is 5.

Here's another puzzle that is likely to induce an emotional reaction.

We seek values a, b, c, d, e, f, and z that make each of these two equations a true statement about numbers.

$$3a + 8b + 2c + 7d + 6e + 5f + 11z = 103$$
$$3a + 8b + 2c + 7d + 6e + 5f + 12z = 105$$

Can you make a start?

Question 3: Okay. What is your emotional reaction to this?

STEP 1: Be your honest human self. Acknowledge your genuine emotional reaction.

Deep breath ...

STEP 2: Do something! ANYTHING!

This is horrific. Are you kidding me? Ugh! Really?

Who in their right mind wants to do this?

It only says to make a start – not actually do it. Hang on! I see something.

Question 4: When Sona saw this question she said that she had no idea what a, b, c, d, e, and f could be, but she could see right away that z would have to be 2. That's a start! How did Sona see, correctly, that z has to be 2?

PROMPT: Ask students what they notice about the two equations. What's the same about them? What is different?

The left side of each equation is same except the second one "goes up one more z" than the first. Yet the values of the left sides increase by 2, from 103 to 105.

So, an increase of one z corresponds to an increase of 2. The value of z has to be 2.

Question 5: Solve the following system of equations.

$$\begin{aligned} 3x + y &= 5 \\ 3x + 2y &= 7 \end{aligned}$$

(That is, find values for x and y that make each of the equations true statements about numbers simultaneously.)

The equations show that an increase of one y gives an increase in value of 2.
So, y must be 2.

Knowing that, the first equation says that $3x + 2 = 5$ and so x is 1.

We see that $x = 1$, $y = 2$ make both equations give true statements about numbers.

Question 6: Solve the following system of equations.

$$\begin{aligned} 3x + 5y &= 5 \\ 3x + 10y &= 25 \end{aligned}$$

We see that an increase of $5y$ corresponds to an increase of 20. So $y = 4$.
The first equation then gives $3x + 20 = 5$, and so $3x = -15$ showing that $x = -5$.

PROMPT: Could we have gone to the second equation instead of the first to find a value for *x*?

The equations have the simultaneous solution $x = -5$, $y = 4$.

Question 7: Solve the following system of equations.

$$\begin{aligned} 3x + 5y &= 60 \\ 3x - 2y &= 18 \end{aligned}$$

A decrease of $7y$ corresponds to a decrease of 42. Thus, we have $y = 6$.
The first equation then gives $3x + 30 = 60$ and so $x = 10$.

HOUSTON:
We have a problem!

Question 8: Solve the following system of equations.

$$\begin{aligned} x + y &= 3 \\ 3x + 4y &= 11 \end{aligned}$$

PROMPT: We've liked having the *x*s the same in both of our equations.

I want the x s to be the same. Let's create "$3x$" in the first equation by multiplying through by 3. (Everything will have to be multiplied.)

LIFE LESSON:
If there is something in life you want, make it happen! (And deal with the consequences.)

$$\begin{aligned} 3x + 3y &= 9 \\ 3x + 4y &= 11 \end{aligned}$$

Now we see that $y = 2$ and then that $x = 1$.

Question 9: Solve the following system of equations.

$$\begin{aligned} x + y &= 3 \\ 3x + 4y &= 8 \end{aligned}$$

To make the x s the same, let's multiply the first equation through by 3.

$$\begin{aligned} 3x + 3y &= 9 \\ 3x + 4y &= 8 \end{aligned}$$

We see that an <u>increase</u> of one y corresponds to a <u>decrease</u> of 1. We must have $y = -1$.

The first equation then shows $x = 4$.

PROMPT: Another way to say this is that an increase of one *y* corresponds to an increase of -1.

Question 10: Solve the following system of equations.

$$\begin{aligned} a + 4b &= -29 \\ 2a - 3b &= 8 \end{aligned}$$

Let's multiply the first equation through by 2.

$$\begin{aligned} 2a + 8b &= -58 \\ 2a - 3b &= 8 \end{aligned}$$

We see that a decrease of $11b$ matches an <u>in</u>crease of 66. Whoa!
We must have $b = -6$.

The first equation at the very beginning then gives $a - 24 = -29$ and so $a = -5$.

Question 11: Questions 9 and 10 were starting to get brain-hurty!

Relieve a headache

What do you think of the following piece of logic? Is it right?

> If $A = B$ is a true statement about numbers and $C = D$ is a second true statement about numbers, then $A - C = B - D$ must also be a true statement about numbers.

Could you use this piece of logic to help make matters less brain-hurty? How could you use it in question 9? In question 10?

PROMPT: If *x* and *y* are numbers that make each of these statements true,

$$3x + 3y = 9 \qquad A = B$$
$$3x + 4y = 8 \qquad C = D$$

then our logic says that we can "subtract" the equations to get another statement that must be true.

$$\begin{array}{r} 3x + 3y = 9 \\ 3x + 4y = 8 \\ \hline -y = 1 \end{array} \qquad \begin{array}{c} A = B \\ C = D \\ \hline A - C = B - D \end{array}$$

We must have *y* = -1.

HOUSTON: We have a problem!

Question 12: Solve the following system of equations.

$$3x - 4y = 13$$
$$5x + 2y = 13$$

PROMPT: Can we make the *x*s the same?

LIFE LESSON:
If there is something in life you want, make it happen! (And deal with the consequences.)

We can make both the x terms the same by multiplying the first equation through by 5 and the second through by 3.

$$15x - 20y = 65$$
$$15x + 6y = 39$$

We see that $-26y = 26$ and so $y = -1$.
From the original first equation we have that $3x + 4 = 13$, and so $x = 3$.

Question 13: Kyle said we could have solved the previous problem by making the y s the same instead: just multiply the second equation through by -2.

Umm. Is Kyle right? Can you do that? And if you can, does it yield the same solution $x=3$, $y=-1$?

Well, surely there is nothing special about making the x s the same. Let's make the y s the same as he suggests.

$$\begin{aligned} 3x - 4y &= \quad 13 \\ -10x - 4y &= -26 \end{aligned}$$

We see that a decrease of $13x$ corresponds so a decrease of 39. So, $x=3$.
The original first equation then gives $9-4y=13$, and so $-4y=4$ and $y=1$.

Yep! Focusing on making the y values the same is fine too!

Question 14: Solve the following system of equations.

$$\begin{aligned} 5m - 0.4n &= 1.0 \\ 0.5m + 0.1n &= 1.5 \end{aligned}$$

Let's multiply the second equation through by 10. Then we have

$$\begin{aligned} 5m - 0.4n &= 1.0 \\ 5m + \quad n &= 15 \end{aligned}$$

An increase of $1.4n$ corresponds to an increase of 14. So, $1.4n = 14$ and so $n=10$.

From the first equation we then get that $5m-4=1$ and so $m=1$.

Question 15:
Jio has 2400 coins on a table. Some are pennies (1 cent coins) and the rest of nickels (5 cent coins). Their total value is $58.80.

How many coins of each type are on the table?

Let p be the number of pennies and n the number of nickels on the table.
We have

$$\begin{aligned} p + n &= 2400 \\ p + 5n &= 5880 \end{aligned}$$

So, $4n = 5880 - 2400 = 3480$ giving $n = 870$.
Thus $p = 2400 - 870 = 1530$.

PROMPT: Is there a "common sense" way to solve this problem?

There are 1530 pennies and 870 nickels.

NATURAL QUESTIONS TO BE EXPLORED NEXT TIME

Every example of a pair of linear equations in this worksheet had a solution.

Is it possible for a system of two equations in two unknowns to have no solution?
(Would you like to try to create an example of such system right now?
Or maybe you think every system of equations is sure to have at least one solution?)

Is it possible for a pair of linear equations to have more than solution?
Could any of the examples in this worksheet have a second solution you didn't happen to find?

Bonus: Can you find a set of values for a, b, c, d, e, f, and z that make each of these two equations a true statement about numbers?

$$3a + 8b + 2c + 7d + 6e + 5f + 11z = 103$$
$$3a + 8b + 2c + 7d + 6e + 5f + 12z = 105$$

Can you find more than one set of values that work?

From Sona's thinking we know that z must be 2.

This then leaves us with two identical equations to think about

$$3a + 8b + 2c + 7d + 6e + 5f = 81$$
$$3a + 8b + 2c + 7d + 6e + 5f = 81$$

I can see that setting $a = 27$, $b = 0$, $c = 0$, $d = 0$, $e = 0$, $f = 0$ makes the equation(s) true.

So too does $a = 0$, $b = 0$, $c = 0$, $d = 0$, $e = 1$, $f = 15$.

I am sure there are plenty of more solutions than just these two!

Solving Systems of Linear Equations via Algebra

WORKSHEET

Math classes often give puzzles like this one.

> *A certain family has some cats (and no other pets). Each cat in the household has four legs and two ears and one nose, and each human in the household has two legs and two ears and one nose.*
>
> *In the house, there are a total of 34 legs and 22 ears.*
>
> *How many noses are there in the household?*
> *How many of those noses are human noses?*

Question 1: What is your typical emotional reaction to puzzles like these?

Question 2: Take a deep breath.
Can you solve the puzzle, or some part of it, using nothing but common sense?

Here's another puzzle that is likely to induce an emotional reaction.

We seek values a, b, c, d, e, f, and z that make each of these two equations a true statement about numbers.

$$3a + 8b + 2c + 7d + 6e + 5f + 11z = 103$$
$$3a + 8b + 2c + 7d + 6e + 5f + 12z = 105$$

Can you make a start?

Question 3: Okay. What is your emotional reaction to this?

Question 4: When Sona saw this question she said that she had no idea what a, b, c, d, e, and f could be, but she could see right away that z would have to be 2. That's a start! How did Sona see, correctly, that z has to be 2?

Question 5: Solve the following system of equations.

$$3x + y = 5$$
$$3x + 2y = 7$$

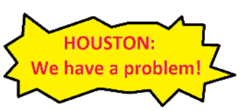
(That is, find values for x and y that make each of the equations true statements about numbers simultaneously.)

Question 6: Solve the following system of equations.

$$\begin{aligned} 3x + 5y &= 5 \\ 3x + 10y &= 25 \end{aligned}$$

Question 7: Solve the following system of equations.

$$\begin{aligned} 3x + 5y &= 60 \\ 3x - 2y &= 18 \end{aligned}$$



Question 8: Solve the following system of equations.

$$\begin{aligned} x + y &= 3 \\ 3x + 4y &= 11 \end{aligned}$$

Question 9: Solve the following system of equations.

$$\begin{aligned} x + \quad y &= 3 \\ 3x + 4y &= 8 \end{aligned}$$

Question 10: Solve the following system of equations.

$$\begin{aligned} a + 4b &= -29 \\ 2a - 3b &= 8 \end{aligned}$$

Question 11: Questions 9 and 10 were starting to get brain-hurty!

What do you think of the following piece of logic? Is it right?

> If $A = B$ is a true statement about numbers and $C = D$ is a second true statement about numbers, then $A - C = B - D$ must also be a true statement about numbers.

Could you use this piece of logic to help make matters less brain-hurty? How could you use it in question 9? In question 10?

HOUSTON:
We have a problem!

Question 12: Solve the following system of equations.

$$3x - 4y = 13$$
$$5x + 2y = 13$$

Question 13: Kyle said we could have solved the previous problem by making the y s the same instead: just multiply the second equation through by -2.

Umm. Is Kyle right? Can you do that? And if you can, does it yield the same solution $x = 3$, $y = -1$?

Question 14: Solve the following system of equations.

$$5m - 0.4n = 1.0$$
$$0.5m + 0.1n = 1.5$$

Question 15:
Jio has 2400 coins on a table. Some are pennies (1 cent coins) and the rest of nickels (5 cent coins). Their total value is $58.80.

How many coins of each type are on the table?

NATURAL QUESTIONS TO BE EXPLORED NEXT TIME

Every example of a pair of linear equations in this worksheet had a solution.

Is it possible for a system of two equations in two unknowns to have no solution?
(Would you like to try to create an example of such system right now?
Or maybe you think every system of equations is sure to have at least one solution?)

Is it possible for a pair of linear equations to have more than solution?
Could any of the examples in this worksheet have a second solution you didn't happen to find?

Bonus: Can you find a set of values for a, b, c, d, e, f, and z that make each of these two equations a true statement about numbers?

$$3a + 8b + 2c + 7d + 6e + 5f + 11z = 103$$
$$3a + 8b + 2c + 7d + 6e + 5f + 12z = 105$$

Can you find more than one set of values that work?