# Logarithms for Humans

**PART 4**

Napier’s Dream

My calculator says

$$10^{3.581} \approx 3810$$

and so

$$\log_{10}(3810) \approx 3.581$$

**Comment:** The symbol ≈ means "approximately equal to." We have some rounding error.

Napier claimed that knowing logarithmic values provides a means to transform a multiplication problem into an addition problem. Of course, he was right. But why is this so?

Of course, Napier did not realize that logarithms are powers in disguise. If so, he may have started with the basic property of exponents that underlies their theory.

$$b^n \times b^m = b^{n+m}$$

What is this saying?

$$n + m \textit{ is the power of } b \textit{ that gives the answer } b^n \times b^m$$

That is, we have:

$$\log_b(b^n \times b^m) = \log_b(b^{n+m}) = n + m$$

We're seeing that a logarithm has converted a product of two numbers into a sum of two numbers.

Here's how Napier's logarithm tables work:

**Example:** Let's compute $3810 \times 10{,}304$.
(Of course, on a modern-day calculator I see that the answer is $39{,}258{,}240$.)

**The Approach:**
Imagine each number to be power of some number base number $b$.
It became common practice to use $b = 10$.

$$3810 = 10^n$$

$$10304 = 10^m$$

To find the values $n$ and $m$, notice:

$$n = \log_{10}(3810)$$

$$m = \log_{10}(10304)$$

According to a table of **common logarithms**, that is, base-10 logarithms, we have

$$n = \log_{10}(3810) \approx 3.581$$

$$m = \log_{10}(10304) \approx 4.013$$

(In truth, I used my calculator! If you see a "log" button without a subscript on your calculator, you can safely assume it is a base- 10 logarithm.)

The statement

$$\log_{10}(10^n \times 10^m) = n + m$$

translates as:

$$\log_{10}(3810 \times 10304) \approx 3.581 + 4.013 = 7.594$$

So, $7.594$ is the power of $10$ that gives the answer to $3810 \times 10304$.

On a calculator I see that $10^{7.594} \approx 39{,}264{,}494$ (and in the 1600s people would see from a table that the number $39{,}264{,}494$ has log value close to $7.594$).

Thus

$$3810 \times 10{,}304 \approx 39{,}264{,}494$$

**Question 1:** The work here has rounding errors. How close is this answer to the true answer? Working with log values rounded to a larger number of decimal places reduces errors.

**Question 2:**

a) Check that $\log(100)$ is 2 on your calculator. (Why is this correct?)
b) On your calculator, what is the value of $n = \log(387)$?
c) On your calculator, what is the value of $m = \log(5092)$?
d) What is the value of $n + m$? (You should get a number close to $6.2946$.)
e) What value $V$ has $\log_{10}(V) = 6.2946$? (Think about how you can use your calculator to find this.)
f) Does $V$ match the product $387 \times 5092$?

Schoolbooks present the observation

$$\log_b(b^n \times b^m) = n + m$$

in terms of the numbers $N = b^n$ and $M = b^m$ and so write

$$\log_b(N \times M) = \log_b(N) + \log_b(M)$$

The log of a product is the sum of the logs.

This summarizes Napier's dream.

**Question 3:**

a) Why should the value of $\log_{10}(7500)$ be 2 larger than the value of $\log_{10}(75)$?
(Is it? Feel free to check this on a calculator.)

b) What do you predict for the value of $\log_{10}(0.75)$ compared to that of $\log_{10}(75)$? Why?

**Question 4:**

a) Explain why the following is true.

$$\log_b\left(\frac{b^n}{b^m}\right) = n - m$$

b) Explain why it leads to the log rule

$$\log_b\left(\frac{N}{M}\right) = \log_b(N) - \log_b(M)$$

This fourth example shows that logarithms convert division problems into subtraction problems.