Logarithms for Humans

PART 5

Why do we care about logarithms today?

We now know the 17th-century reason why people cared about logarithms: they convert multiplication problems into addition problems:

$$\log_b(N \times M) = \log_b(N) + \log_b(M)$$

They also convert division problems into subtraction problems:

$$\log_b\left(\frac{N}{M}\right) = \log_b(N) - \log_b(M)$$

Question 1: Please remind me how we derived these two formulas. I remember we started by first writing N and M each as a power of b.

But why do people care about logarithms today?

It's because of this next property of exponents and how it translates into a statement about logarithms.

We have:

$$(b^n)^m = b^{n\times m}$$

This is saying

$n \times m$ is the power of b that gives the answer$(b^n)^m$.

$$\log_b((b^n)^m) = \log_b(b^{n\times m}) = n \times m$$

Let's rewrite this in the schoolbook way.

Start by rewriting $N = b^n$. This means that $n = \log_b(N)$.

Then our statement reads

$$\log_b(N^m) = \log_b(N) \times m$$

Because people like to use the letter x in algebra class, textbooks will write this as:

$$\log_b(N^x) = \log_b(N) \times x$$

We have:

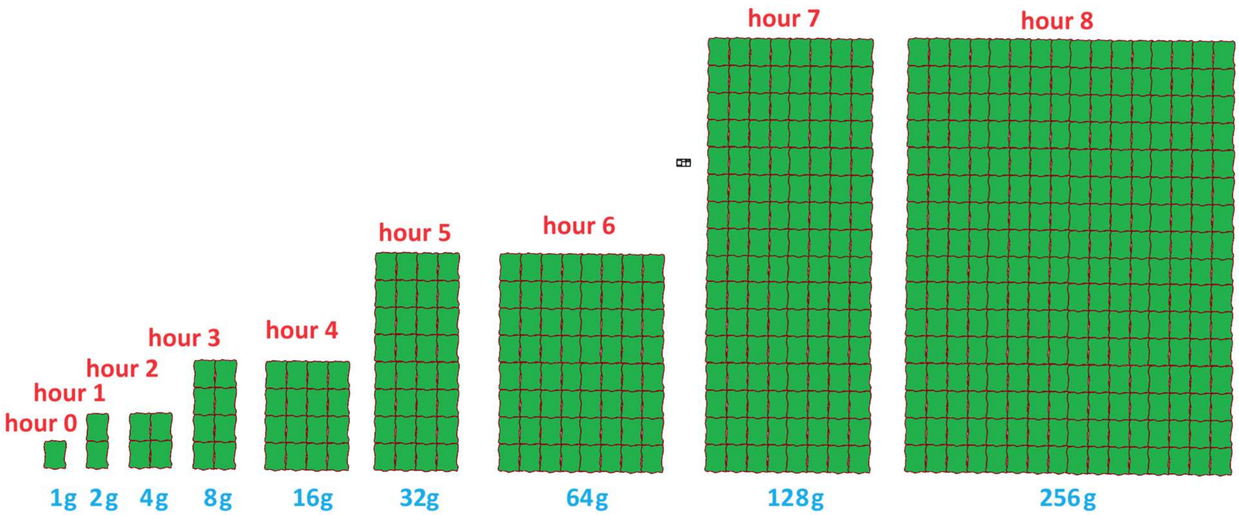
$$\log_b(N^x) = x \log_b(N)$$

Hit an exponential expression with a log and shake down the exponent!

Here’s why people like this property: it’s hard to solve equations if the variable is “trapped upstairs” as an exponent.

Logarithms provide a means to bring those variables down to a level were we can work with them.

As an example, consider a mass of green, gooey biological growing continuously in such way that it doubles in mass every hour. Let's ass time 0 hours we start with 1 gram of culture.



As we saw in our course on exponents, we know that the mass of the culture at time t hours will be 2^t grams.

Question 2: What will the malls of the goo be at time $t = 2.5$ hours? (Can you figure out how to use the x^y button on a calculator?)

Here's a challenge for us:

Example: At what time will we have 1.5 kg of goo?

To answer this, we need to find a time t such that

$$2^t = 1500$$

It is not clear how to solve this on a calculator. (We could say that $t = \log_2(1500)$, but there is no $\log_2$ on a calculator!)

We have a variable "trapped upstairs."

But we know that hitting an exponential expression with a log—any log!—shakes down the exponent. So, let's hit each side of this equation with a base-10 log, which does exist on my calculator.

$$\log_{10}(2^t) = \log_{10}(1500)$$

Shaking down the exponent:

$$t \log_{10}(2) = \log_{10}(1500)$$

We can work out approximate values with the calculator

$$t \times 0.301 \approx 3.176$$

and so

$$t \approx \frac{3.176}{0.301} \approx 10.551$$

We have 1500 grams of culture at about time $t = 10.6$ hours.

Comment: If we avoid doing all the numerical calculations we can give an exact answer to this problem. From $t \log_{10}(2) = \log_{10}(1500)$ we see that

$$t = \frac{\log_{10}(1500)}{\log_{10}(2)}$$

This is exact number of hours solving this problem (though I don't have a feel for what this number is!)

However, writing exact answers and leaving all numerical calculations as the final step reduces the amount of rounding error.

Here's a more textbook-like example. (I have no idea why an equation like this to solve might appear in real life.)

Example: Kindly solve $2^{x+1} = 3^x$.

Answer: We have an exponent of $x + 1$ trapped upstairs on the left and an exponent of just x trapped upstairs on the right.

My calculator has a "log" button without a subscript, but it means a base-10 logarithm. Let me hit both sides of the equation with a logarithm and "shake down" each of those exponents.

$$\log(2^{x+1}) = \log(3^x)$$

$$(x+1)\log(2) = x\log(3)$$

(Notice how I was careful to make clear that "$x + 1$" as a group was shaken down.)

According to my calculator, we now have (up to rounding):

$$(x+1) \times 0.301 \approx x \times 0.477$$

$$x \times 0.301 + 0.301 \approx x \times 0.477$$

$$0.301 \approx 0.176x$$

$$x \approx 1.710$$

Question 3: a) Show that the exact answer is

$$x = \frac{\log(2)}{\log(3) - \log(2)}$$

b) Some people might rewrite this answer as

$$x = \frac{\log(2)}{\log(1.5)}$$

Why is this an equivalent answer?

c) Is this answer the same as $x = \log(\frac{2}{1.5})$?

Here's a slightly more complicated textbook problem:

Example: Please solve $5 \cdot 3^x = 7 \cdot 4^{x+1}$

Answer: The unknowns are all stuck upstairs.
To bring them down, let's hit each side of this equation with a log.

$$\log_{10}(5 \cdot 3^x) = \log_{10}(7 \cdot 4^{x+1})$$

We have the logarithm of products here. Recall: **The log of a product is the sum of the logs.**

$$\log_{10}(5) + \log_{10}(3^x) = \log_{10}(7) + \log_{10}(4^{x+1})$$

Now we're ready to **shake down exponents** (again catching that one of our exponents is all of $x + 1$).

$$\log_{10}(5) + x\log_{10}(3) = \log_{10}(7) + (x+1)\log_{10}(4)$$

This is visually confusing, but most everything here is just a number. According to my calculator this equation is (approximately):

$$0.699 + 0.477x = 0.845 + 0.602(x + 1)$$

I am going to multiply through by 1000 to avoid all the decimals.

$$699 + 477x = 845 + 602(x + 1)$$

$$699 + 477x = 845 + 602x + 602$$

$$125x = -748$$

$$x = -\frac{748}{125} \approx -5.984$$

Icky!

Question 4 Show that the exact solution to the previous example is

$$x = \frac{\log_{10}(5) - \log_{10}(7) - \log_{10}(4)}{\log_{10}(4) - \log_{10}(3)}$$

Show that this can also be written as

$$x = \frac{\log_{10}(\frac{5}{28})}{\log_{10}(\frac{4}{3})}$$

Question 5 Kindly solve $2^x \cdot 3^x = 4 \cdot 5^x$.
(If you are game, feel free to give the exact solution as well as an approximate one.)

Question 6

a) Please solve $\left(\frac{6}{5}\right)^x = 4$ with your answer in terms of base ten logarithms.

b) Why is the solution to this problem exactly the same as the solution to question 5?

This next example shows why calculators don't have logarithm buttons for every possible base.

Example: Compute the value of $\log_3(7)$ with a calculator.

Answer: The key is to give the quantity a name. Let's call it F for Frederica.

$$F = \log_3(7)$$

So, Frederica is the power of 3 that gives the answer 7.

$$3^F = 7$$

Let's hit this with our $\log_{10}$ button.

$$\log_{10}(3^F) = \log_{10}(7)$$

$$F\log_{10}(3) = \log_{10}(7)$$

$$F = \frac{\log_{10}(7)}{\log_{10}(3)} \approx \frac{0.845}{0.477} \approx 1.771$$

That's it:

$$\log_3(7) \approx 1.771$$

Question 7 a) Compute $\log_{37}(500)$ if doing so seems fun to you.

b) If you are game, show that $\log_b(N)$ can be computed as

$$\frac{\log_{10}(N)}{\log_{10}(b)}$$

Some curriculums want students to know this **change of base formula**.

Summary of Textbook Expectations on Logarithms

Curriculum can really "dig deep" into properties of logarithms and ask students to derive—and memorize—all sorts of complicated formulas.

In the end, each formula is just an application of three basics.

1. **$\log_b(N)$ is the power of b that gives the answer N.**

$x = \log_b(N)$ means $b^x = N$

(Replace the word "log" with the word "power")

2. **The log of a product is the sum of the logs**.

$$\log_b(N \times M) = \log_b(N) + \log_b(M)$$

3. **Hitting an exponential expression with a log shakes down the exponent.**

$$\log_b(N^x) = x \log_b(N)$$

For instance, the additional rule

$$\log_b\left(\frac{N}{M}\right) = \log_b(N) - \log_b(M)$$

can be recreated by thinking of $\frac{N}{M}$ as $N \cdot \frac{1}{M} = N \cdot M^{-1}$.
This leads to:

$$\begin{aligned}\log_b\left(\frac{N}{M}\right) &= \log_b(N \cdot M^{-1}) \\ &= \log_b(N) + \log_b(M^{-1}) \\ &= \log_b(N) + (-1)\log_b(M) \\ &= \log_b(N) - \log_b(M)\end{aligned}$$

Question 8 Explain why $\log_b(\frac{1}{N})$ equals $-\log_b(N)$.

If you enjoyed question 8, try this one too. (If you didn't like it, don't do question 9.)

Question 9 Explain why $\log_{\frac{1}{b}}(N)$ equals $-\log_b(N)$.

Here's an annoying, made-up question just for the sake of testing students on logarithms and quadratics at the same time. My advice is to ignore this question.

Question 10 What value(s) of x make this equation true?

$$(\log_{15}(x))^2 + \log_{15}(x^2) = 15$$