Logarithms for Humans

PART 7

Practical Example:
Banking

If you think about it for a moment, it seems somewhat curious that banks pay you for the honor of conducting a service for you, namely, to securely house your money. Ten thousand dollars in cash is safer in the protection of a bank than in your pockets or under your mattress.

And how do banks pay you? By giving you interest on the money you have stored with them.

Of course, banks do make a profit despite regularly adding money to your and all their customers' accounts. But with the large cash sums they accrue from having many customers, banks can invest in high-paying financial opportunities beyond what any one individual can typically do. They make a profit and thank you for this by sharing some of the profit with you.

The total amount of money you earn from an interest payment of course depends on the amount of money you have in your account. An interest payment of 2% say on a balance of $\$10{,}000$ gives you $\frac{2}{100} \times 10{,}000 = \$200$, but on a balance of $\$1{,}000{,}000$ it gives you $\frac{2}{100} \times 1{,}000{,}000 = \$20{,}000$.

If you let a balance sit in an account untouched, you'll also earn interest on the interest payments awarded to you, repeatedly, and your account balance will grow.

Example: I decide to invest $\$1200$ with *Grimy Hands Money Market.* They offer 3.5% annual interest calculated at the end of each calendar year.

It's the start of the year, and I am willing to keep my money in the account untouched for many years.

a) What will my balance be after ten years?

b) I'd like to be a millionaire. By which year will I have a million dollars?

Answer: My starting balance is

$$B = 1200$$

dollars.

At the end of the first year, my balance will grow to

$$B_1 = 1200 \ + \ \frac{3.5}{100} \times 1200 = 1242$$

dollars. But let's write this answer as

$$B_1 = 1200 \times 1.035$$

At the end of the second year my balance will grow to

$$B_2 = 1242 \quad + \quad 1242 \times 0.035 = 1285.47$$

dollars. But this really

$$\begin{aligned} B_2 &= 1242 \times 1.035 \\ &= 1200 \times 1.035 \times 1.035 = 1200 \times (1.035)^2 \end{aligned}$$

Each year my balance grows by a factor of 1.035 and after n years will have

$$B_n = 1200 \times (1.035)^n$$

dollars.

a) By the end of ten years my balance is

$$B_{10} = 1200 \times (1.035)^{10} \approx 1692.71$$

(Grimy Hands always rounds <u>down</u> to the nearest penny!)

b) To be a millionaire I need

$$1200 \times (1.035)^n = 1{,}000{,}000$$

So,

$$1.035^n = \frac{1000000}{1200} = \frac{2500}{3}$$

"Hitting with a log" gives

$$n \log_{10}(1.035) = \log_{10}\left(\frac{2500}{3}\right)$$

And so

$$n = \frac{\log_{10}\left(\frac{2500}{3}\right)}{\log_{10}(1.035)}$$

According to my calculator, this gives $n \approx 195.5$. I'll need to wait 196 years to be a millionaire if I start with $\$1200$ and just sit back! (Maybe I should be a little more pro-active if this truly is my goal!)

The truth is banks don't assign interest just once a year, they assign it closer to every instant!

Let's make sense of this intriguing claim.

Example: I have $\$10{,}000$ I'd like to invest for a year. Two respectable institutions are offering 5% interest per annum but calculated over smaller time periods.

- *Buckets-o-Cash Bank* calculates their interest payments monthly, meaning that they assign $\frac{5}{12} \approx 0.417\%$ of interest each month for 12 months.
- *Cash Flux Bank* calculates their interest payments weekly, meaning that they assign $\frac{5}{52} \approx 0.096\%$ of interest each week for 52 weeks.

Which bank will leave me with the bigger balance by the end of the year?

Answer: Let's examine each bank in turn.

Buckets-o-Cash: They pay $\frac{5}{12} \approx 0.417\%$ interest each month. So, a balance of B dollars at the start of a month becomes

$$B \; + \; \frac{0.417}{100} \times B = B \times (1 + 0.00417) = B \times (1.00417)$$

dollars at the end of the month. That is, my balance grows by a factor of approximately 1.00417 each month.

Let's write out a "spread-sheet" of my balance month-per-month.

Start: $\$10{,}000$

Month 1: $\$10{,}000 \times (1.00417)$

Month 2: $\$10{,}000 \times (1.004167) \times (1.004167)$
$= \$10{,}000 \times (1.004167)^2$

Month 3: $\$10{,}000 \times (1.004167)^3$

⋮

Month 12: $\$10{,}000 \times (1.004167)^{12}$

My calculator says this final balance amount is $\$\mathbf{10,511,62.}$

Cash Flux Bank: They pay $\frac{5}{52} = 0.0961\%$ interest each week. So, a balance of B dollars at the start of a week becomes

$$B \ + \ \frac{0.0961}{100} \times B = B \times (1 + 0.000961) = B \times (1.000961)$$

dollars at the end of the week. That is, my balance grows by a factor of 1.000961 ... each week.

Start: $\$10{,}000$

Week 1: $\$10{,}000 \times (1.000961)$

Week 2: $\$10{,}000 \times (1.000961)^2$

Week 3: $\$10{,}000 \times (1.000961)^3$

⋮

Week 52: $\$10{,}000 \times (1.000961)^{52}$

My calculator says this final balance amount is $\$\mathbf{10,512,46.}$

That's 84 cents better! I'll go with *Cash Flux Bank*.

Example Continued: I have since learned of some more banks paying the same interest rate per annum but calculated over even shorter time periods.

- *Swimming-In-It Bank* calculates their interest payments daily, meaning that they assign $\frac{5}{365}\%$ of interest each day for365 days.

- *Cash Galore Bank* calculates their interest payments every hour, meaning that they spread the 5% interest payment over each and every hour of the year.

- *Bank Bonanza* calculates their interest payments every minute, meaning that they spread the 5% interest payment over each and every minute of the year.

- *No-Messing-About-Bank* calculates their interest payments every second, meaning that they spread the 5% interest payment over each and every second of the year.

And there are additional banks that spread the 5% interest payment over every milli-second of the year, over every nano-second of the year, and so on.

For the four banks named, what would my end-of-year balance be if I invested my $\$10{,}000$ with each of them?

We saw the following general structure from the first example.

If a bank pays $r\%$ interest each period and my balance at the beginning of a period is B dollars, then my new balance at the end of that period will be $B(1+\frac{r}{100})$ dollars. (This is $B + \frac{r}{100} \times B$.)

After n such periods my balance will be

$$B\left(1+\frac{r}{100}\right)^{n}$$

dollars.

Let's now analyze the next four banks.

Answer:

Swimming-In-It Bank: There are 365 days in a year. We have $r = \frac{5}{365} \approx 0.0137\%$ amount of interest paid each day for 365 days. My balance after a year will be

$$10{,}000 \times (1.0001367)^{365} \approx \$\mathbf{10,512.67}$$

Cash Galore Bank: There are $365 \times 24 = 8{,}760$ hours in a year.

Now $r = \frac{5}{8760} \approx 0.00057\%$ and we have 8760 periods.
My balance after a year will be

$$10{,}000 \times (1.0000057)^{8760} \approx \$\mathbf{10,512.70}$$

Bank Bonanza: There are $365 \times 24 \times 60 = 525{,}600$ minutes in a year.

Now $r = \frac{5}{525600} \approx 0.00000951\%$ and we have 525,600 periods.
My balance after a year will be

$$10{,}000 \times (1.0000000951 \ldots)^{525600} \approx \$\mathbf{10,512.71}$$

No-Messing-About Bank: There are $365 \times 24 \times 60 \times 60 = 31{,}536{,}000$ seconds in a year.

Now $r = \frac{5}{31{,}536{,}000} \approx 0.000000158\%$ and we have 31,536,000 periods.
My balance after a year will be

$$10{,}000 \times (1.00000000158 \ldots)^{31536000} \approx \$\mathbf{10,512.71}$$

Computing interest over finer and finer time periods at this point seems to create balance increases only in fractions of pennies. I don't think it is worth going through the work of calculating final balances for banks that compute interest every micro-second or finer. We won't see the effect at the levels of pennies.

But we do see that there does seem to be some kind of "ultimate" balance value if banks were to approach computing interest for us at each and every instant.

Question 1: When I conducted some internet research, I developed the impression that banks typically compute interest daily. (Check me on this.) But that led me to the question: do banks recognize leap years? Do they handle leap years differently from regular years?
What can you find out?

People don't typically conduct more than one bank transaction in a day, so computing interest daily is likely in line with the typical practices of customers. (It would seem unfair if interest were calculated monthly say, and you happened to make a large withdrawal just before "interest day.")

Question 2 Suppose I kept my account open with *Swimming-In-It Bank* for double the amount of time, 2 years instead of just 1.

What would my balance be after two years?

(After one year I will earn $\$512.67$ in interest, but over two years my interest earned will be more than double this. This is because during the second year I'll be earning interest on even higher interest payments.)