Logarithms for Humans

Their Math, Their Meaning,

and Why We Care.

Content for an Online Module
(following a module on Exponents)

Section 13 here https://gdaymath.com/courses/gmp/

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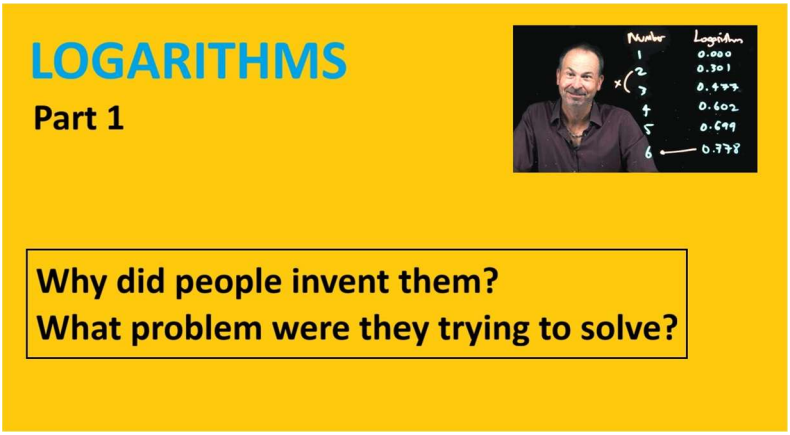
Logarithms for Humans

PART 1

What made people want them?
What problem were people trying to solve?



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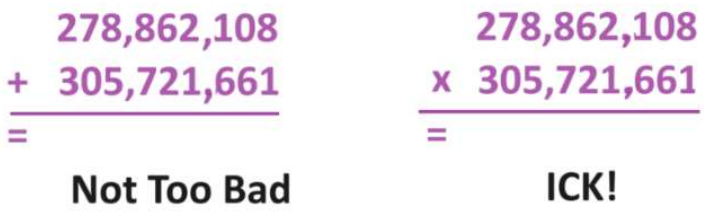
A Scientific Crisis – and its Resolution

There was a scientific crisis in Europe during the Renaissance, brought on simply by the task of having to do arithmetic by hand.

The 1400s and 1500s saw Western scholars make new advances in the arts and sciences, leading to new understandings of the natural world. The invention of Galileo's telescope, which he called a *perspicillum*, opened up the workings of the heavens too to fuel extraordinarily rapid progress in astronomy.

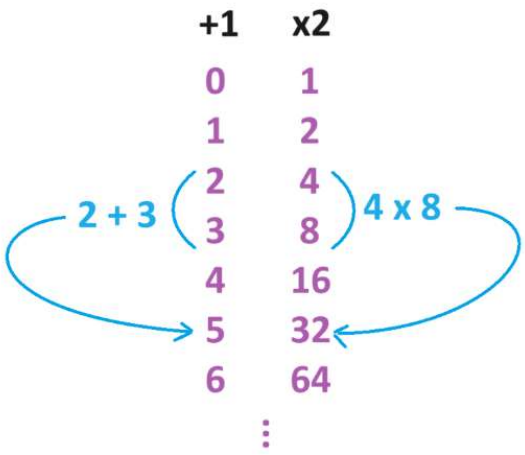
As data gathering methods became more precise, astronomers found themselves burdened by the simple process of arithmetic when performing calculations. While addition was manageable with chalk on slates or with quill on paper, conducting multiplication was onerous.



Astronomers used extensive tables of angular measures in their work with numbers adjusted by a factor of 10,000,000 to avoid dealing with fractions (decimals and decimal notation was not yet in use). This left them regularly conducting calculations with seven-, eight-, nine-digit numbers and they were severely hampered when it came to multiplying and dividing numbers or finding their square and cube roots.

Progress in science was severely held back.

But there was some hope. Mathematicians noticed an interplay between multiplication and simpler addition. For example, in 1544 German mathematician Michael Stifel illustrated such a connection with the counting numbers (constructed by repeatedly adding 1 to zero) and the doubling numbers (constructed by repeatedly multiplying the number 1 by two).



Computing 4×8 by aligning it with $2 + 3$ on the left.
The value 5 on the left matches the answer to 4×8 on the right.

The trouble was that scholars did not have a clear understanding of exponents beyond whole-number exponents. This meant that they could not, for instance, provide values between the lines in Stifel's table and conduct a wider range of products by matching them with addition problems.

In the 1590s, Scottish mathematician John Napier decided to tackle this very problem. He later wrote in reflection in 1614:

> *Seeing there is nothing that is so troublesome to mathematical practice, nor that doth more molest and hinder calculators, than the multiplications, divisions, square and cubical extractions of great numbers ... I began therefore to consider in my mind by what certain and ready art I might remove those hindrances.*

He succeeded. Napier's approach might seem strange and curious from our modern perspective. But we must remember that an understanding of exponents (yet alone a notation for them) was not available to him. He had to be innovative.

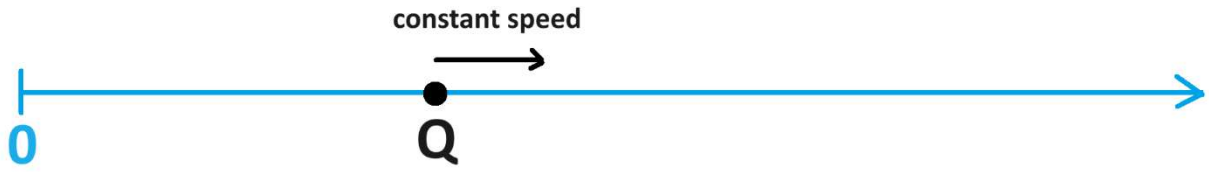
Napier took a kinematic approach.

He imagined a particle P moving along a line segment of length $r = 10{,}000{,}000$, starting at 0 on the left and moving to the right.

The particle starts out at a speed of r units per second, but it gets slower and slower as it moves to the right. In fact, the speed of particle is given by how much further it needs to travel: if the particle has x more units to cover, it now moves at a speed of x units per second.

Question 1
Will the particle ever reach the rightmost point of the segment?

Next, Napier imagined a second particle Q moving along a line, starting at 0 on the left, but always moving the constant speed of r units per second. This particle will move infinitely far to the right.



Napier was able to show that distance particle P has yet to travel and the distance Q has already traveled have the same sort of multiplicative/additive relationship akin the one Stifel had pointed out with the doubling numbers and the counting numbers.

Moreover, Napier had figured out a computational method that allowed him to approximate the distances covered by the particles every thousandth of a second, which gave him a table like Stifel's, but with the "in-between" numbers.

And it was this table that saved science! He showed the world how to reduce complicated arithmetical operations to ones of manageable addition and subtraction.

Napier coined the name **logarithm** for the numbers in his table, coming from the Greek words *logos* (ratio) and *arithmos* (number) to represent how he was comparing the values of the distances in the motions of the two particles.

Napier published his 90-page table of logarithm values in 1614.

It wasn't until more than a century that mathematicians properly understood exponents and could see Napier's work as theory of general exponents. But by then the name *logarithm* was firmly entrenched in scientific community's vernacular, and the name stays with us to this day.

And that's a fine thing. We should continue to honor Napier's magnificent achievement that saved scientific progress at a crucial time.

Logarithm Tables

Using my modern understanding of exponents (and my calculator!), I am able to do what Stifel and Napier could not initially do, namely, accurately fill in some values between the lines of Stifel's table.

I will also call these values *logarithms*, but they are not the logarithmic values Napier computed and what is shown here is not part of the table he presented to the world. But we can use this table to illustrate, nonetheless, how scholars converted multiplication problems into addition problems via Napier's work.

number	Stifel logarithm
1	0
2	1
3	1.585
4	2
5	2.322
6	2.585
7	2.807
8	3
9	3.170
10	3.322
11	3.459
12	3.585
13	3.700
14	3.807
15	3.907
16	4

Do you see the doubling numbers (this time to the left) and the matching counting numbers to the right, as per Stifel's table?

The decimal values you see have all been rounded to three decimal places. (So, any discrepancies we might encounter will be because of this rounding in the thousandth place. But this should not be too problematic for us.)

Also, I've only given logarithm values for the first sixteen numbers, so the multiplication problems we'll illustrate here will be small!

Suppose we wanted to work our 3×5 using the table. (Of course, we know the answer is 15.)

Here's the method:

1. The logarithmic value for $\mathbf{3}$ is 1.585.
2. The logarithmic value for $\mathbf{5}$ is 2.322.
3. The sum of these two values is 3.907.
4. Looking back at the table, the number with logarithmic value 3.907 is $\mathbf{15}$.

$$\begin{array}{r} 3 \\ \times\ 5 \\ \hline = \end{array} \qquad \begin{array}{r} 1.585 \\ +\ 2.322 \\ \hline = 3.907 \end{array}$$

In summary …

To multiply two numbers: Look up their logarithmic values. Add those values and then see which number has logarithmic value equal to the sum. That number is the product of the original two numbers.

Question 2

Use the table to compute each of these products.

3×4 $\quad$ 7×2 $\quad$ 3×3

Question 3

a) Knowing that $18 = 3 \times 6$, what is the logarithmic value that goes with the number 18?

b) Do you get the same value you think of 18 as 2×9 instead?

c) Make a guess as to logarithmic value that goes with the number 17.

Question 4

Might you care to research the details of Napier's "ratio of distances" approach to create logarithmic tables that solved the multiplication problem for science?

Logarithms for Humans

SELF CHECK 1:

Here's a frivolous quiz.

There is no need to answer any of the questions,
and if you do, the points you earn mean nothing!

(Feeling curious to give it a try anyway?)

SELF CHECK 1

Question 1 [20 points]

Which of these arithmetic problems would you prefer to solve with pencil and paper?
(*Please choose the third and fourth options*!)

a) $45{,}980{,}878 + 98{,}758{,}463$

b) $45{,}980{,}878 \times 98{,}758{,}463$

c) Neither of them

d) But if I were forced to do one, I'd choose the addition problem.

Question 2 [40 points]

Henry Briggs, a colleague of John Napier, helped create a table of logarithms aligned to the powers of ten: $1, 10, 100, \ldots$

number	Briggsian logarithm
1	0.000
2	0.301
3	0.477
4	0.602
5	0.699
6	0.778
7	0.845
8	0.903
9	0.954
10	1.000
11	1.041
12	1.079

Which of the following summations would allow you to compute 2 x 6 using Briggsian logarithms?
(Let's pretend we don't already know the answer is 12!)?

a) $0.301 + 0.477$

b) $0.301 + 1.079$

c) $0.301 + 0.778$

d) What are you even talking about? You can't use addition to solve a multiplication problem!

Question 3 [40 points]

What is the etymology of the word *logarithm*?

a) It comes from the Latin words for "ratio" and "arithmetic values" as Scottish mathematician John Napier was working with ratios of arithmetic values.

b) It originates from Latin words associated with snakes living under fallen trees: *logs* and *adders*.

c) It is French for "an exceptionally polite giraffe."

d) I don't know what "etymology" means and cannot guess its meaning from the context of this question.

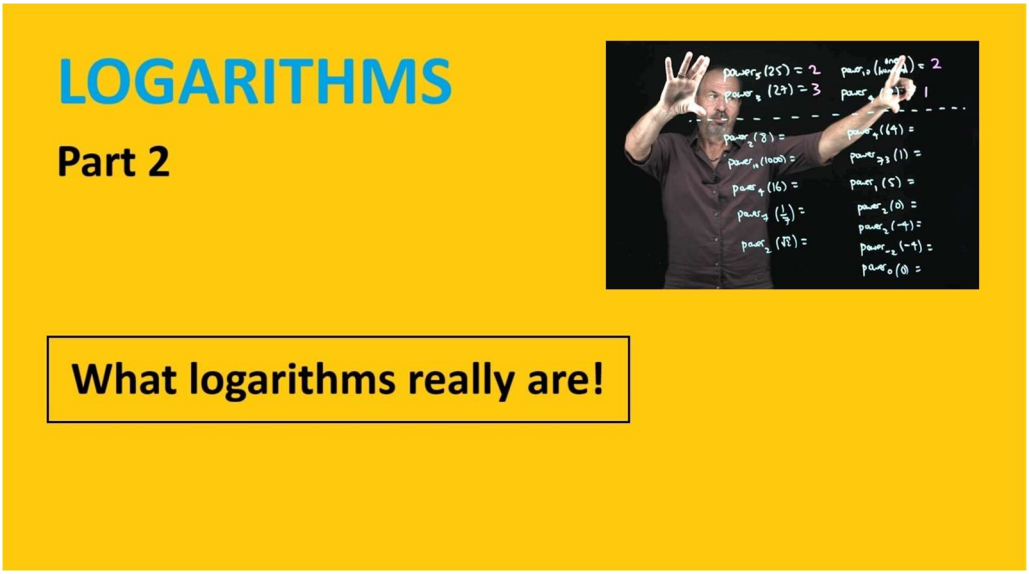
Answers:

1. c) d) 2. c) 3. a) or maybe d)

Logarithms for Humans

PART 2

What logarithms really are!



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Getting Straight Into it!

Without any introduction, discussion, or fuss, can you figure out what is being asked of you in this "worksheet"? Are you able to complete it?

Give this serious consideration before reading on.

$\text{power}_5(25) =$ 2 $\quad\quad \text{power}_{10}(1000) =$ 3

$\text{power}_3(27) =$ 3 $\quad\quad \text{power}_4(4) =$ 1

$\text{power}_2(8) =$ ____ $\quad\quad \text{power}_2(\sqrt{2}) =$ ____ $\quad\quad \text{power}_1(5) =$ ____

$\text{power}_{10}(100) =$ ____ $\quad\quad \text{power}_{10}(\text{million}) =$ ____ $\quad\quad \text{power}_2(0) =$ ____

$\text{power}_4(16) =$ ____ $\quad\quad \text{power}_{73}(1) =$ ____ $\quad\quad \text{power}_{-2}(-8) =$ ____

$\text{power}_4(64) =$ ____ $\quad\quad \text{power}_{100}(0.1) =$ ____ $\quad\quad \text{power}_{-2}(8) =$ ____

$\text{power}_7\left(\frac{1}{7}\right) =$ ____ $\quad\quad \text{power}_{\sqrt{6}}\left(\frac{1}{36}\right) =$ ____ $\quad\quad \text{power}_0(0) =$ ____

The first example

$$\text{power}_5(25) = \underline{2}$$

states:

The power of 5 *that gives the answer* 25 *is* 2

or, phrased backwards:

2 *is the power of* 5 *that gives* 25

Did you come to this understanding?

Filling out the table is a task of figuring out the correct power of a give base number to produce a stated answer. I could complete this for all but four examples.

$\text{power}_5(25) =$	2	$\text{power}_{10}(1000) =$	3		
$\text{power}_3(27) =$	3	$\text{power}_4(4) =$	1		
$\text{power}_2(8) =$	3	$\text{power}_2(\sqrt{2}) =$	$\frac{1}{2}$	$\text{power}_1(5) =$	?
$\text{power}_{10}(100) =$	2	$\text{power}_{10}(\text{million}) =$	6	$\text{power}_2(0) =$	?
$\text{power}_4(16) =$	2	$\text{power}_{73}(1) =$	0	$\text{power}_{-2}(-8) =$	3
$\text{power}_4(64) =$	3	$\text{power}_{100}(0.1) =$	$\frac{-1}{2}$	$\text{power}_{-2}(8) =$	?
$\text{power}_7(\frac{1}{7}) =$	-1	$\text{power}_{\sqrt{6}}(\frac{1}{36}) =$	-4	$\text{power}_0(0) =$	?

Comment: I computed the power of 100 that gives the answer 0.1 by writing $100^a = \frac{1}{10}$ and seeing that this is the equivalent to writing $10^{2a} = 10^{-1}$.

I computed the power of $\sqrt{6}$ that gives $\frac{1}{36}$ by looking at $\left(6^{\frac{1}{2}}\right)^b = 6^{-2}$.

Here's why those four examples are troublesome:

$\text{power}_1(5) =$?

$\text{power}_2(0) =$?

$\text{power}_{-2}(8) =$?

$\text{power}_0(0) =$?

- Every power of 1 is 1.

  $1^x = 1$ for all numbers x

  There is no power of 1 that gives the answer 5.

- Every power of 2 is a positive number. From our course on exponents, we know:

  For a positive base number a, the value of a^x is a positive number for all numbers x

  Thus, there is no power of 2 that gives the answer 0.

- We saw in our last course, powers of negative numbers are dangerous!
  I can't think of a power of -2 that gives the answer 8.

- There are too many powers of 0 that give the answer 0.
  For example, $0^2 = 0 \times 0 = 0$ and $0^5 = 0 \times 0 \times 0 \times 0 \times 0 = 0$.
  There are infinitely many possible answers in this case.

$\text{power}_1(5) =$ does not exist

$\text{power}_2(0) =$ does not exist

$\text{power}_{-2}(8) =$ does not exist

$\text{power}_0(0) =$ indeterminant

Every other example in the worksheet is fine. They each ask for a power of a positive number that produces a positive result. As we saw in our last course on exponents, working with powers of positive numbers is safe.

Let's Rephrase

Now, let's cross out each instance of the word "power" in our worksheet and replace it with the letters *log*, short for **logarithm**.

$\log_5(25) = 2$ $\quad\quad$ $\log_{10}(1000) = 3$

$\log_3(27) = 3$ $\quad\quad$ $\log_4(4) = 1$

$\log_2(8) = 3$ $\quad\quad$ $\log_{\sqrt{2}}(\sqrt{2}) = \frac{1}{2}$ $\quad\quad$ $\log_1(5) =$ does not exist

$\log_{10}(100) = 2$ $\quad\quad$ $\log_{10}(\text{million}) = 6$ $\quad\quad$ $\log_2(0) =$ does not exist

$\log_4(16) = 2$ $\quad\quad$ $\log_{73}(1) = 0$ $\quad\quad$ $\log_{-2}(-8) = 3$

$\log_4(64) = 3$ $\quad\quad$ $\log_{100}(0.1) = \frac{-1}{2}$ $\quad\quad$ $\log_{-2}(8) =$ does not exist

$\log_7(\frac{1}{7}) = -1$ $\quad\quad$ $\log_{\sqrt{6}}(\frac{1}{36}) = -4$ $\quad\quad$ $\log_0(0) =$ indeterminant

And that's it! That's what logarithms are.
They are just powers!

It took mathematicians over 100 years to realize that Napier's complicated work was just the theory of exponents in disguise. But to be fair, it rightly took many decades for scholars to even make sense of exponents b^x beyond just whole number values for x. (We explored some of these complications in the last course on exponents.)

By the mid-1700s, clarity finally emerged. The superscript notation for exponents was introduced around this time too.

But after more than a century of using the term *logarithm*, the name stuck. We still use it today. Unfortunately, this word often intimidates students because it's typically introduced without context or a story to ground it.

If we simply said **logarithms are powers**, the topic would feel far less scary. (After all, you managed to do the opening worksheet seeing only the word "power.")

When you see the word logarithm, think "power."

Definition of a Logarithm

Here's the formal definition of a logarithm in its full scariness—almost. (There is one complication we'll attend to in a moment.)

> *For a number b (called the **base**) and a number N, the base b **logarithm** on N,*
>
> $$log_b(N)$$
>
> *is the power of b that gives the value N.*

For example,

$\mathbf{log_4(16)}$ is the power of 4 that gives the value 16.

$$\log_4(16) = 2$$

$\mathbf{log_{10}(million)}$ is the power of 10 that gives the value one million.

$$\log_{10}(million) = 6$$

$\mathbf{log_{\sqrt{6}}(\frac{1}{36})}$ is the power of $\sqrt{6}$ that gives the value $\frac{1}{36}$.

$$\log_{\sqrt{6}}(\frac{1}{36}) = -4$$

Whenever you see the word *logarithm*, cross it out, and replace it with the word *power*.

power ~~log~~$_b(N)$ = the power of *b* that gives the answer *N*

Question 1 Please complete the following worksheet.

$\log_2(8) =$ _____ $\log_{0.1}(100) =$ _____

$\log_5(125) =$ _____ $\log_{402}(1) =$ _____

$\log_{125}(5) =$ _____ $\log_1(8) =$ _____

$\log_{\frac{1}{4}}(4) =$ _____ $\log_2(-4) =$ _____

a

We've seen that a base number of $b = 1$ is problematic, and negative bases are often problematic too. To avoid woes that could arise from them, people avoid them! They add some caveats to the formal definition of a logarithm.

For positive numbers b and N, with $b \neq 1$, the base b **logarithm** on N,

$$\log_b(N)$$

is the power of b that gives the value N.

We have

$\log_4(1024) = 5$ because $4^5 = 1024$

$\log_{\frac{1}{3}}\left(\frac{1}{9}\right) = 2$ because $\left(\frac{1}{3}\right)^2 = \frac{1}{9}$

$\log_{10}(0.001) = -3$ because $10^{-3} = \frac{1}{1000} = 0.001$

Question 2 The third example shows that a logarithm of a number can give a negative value as its answer. Does this violate the caveats of the formal definition?

Logarithms for Humans

SELF CHECK 2

Here's another frivolous quiz.

Again, the points mean nothing.

Enjoy!

SELF CHECK 2

Question 1 [10 points]

To what power do you need to raise the number 56 to get the answer 56?
(*Select all three correct answers.)*

a) 1

b) 56

c) It's 1.

d) Seriously, it's 1.

Question 2 [20 points]

What is the value of $\log_{56}(56)$?

a) 1

b) 56

c) It's 1.

d) Seriously, it's 1.

Question 3 [20 points]

Which of the following describes $\log_3(81)$?
(*Select all that apply*.)

a) It's the power to which the number 3 must be raised to give the value 81.

b) It's the power to which the number 81 must be raised to give the value 3.

c) 4

d) $\text{power}_3(81)$

e) It's just idle scribble with no mathematical meaning.

Question 4 [20 points]

We know that $2^{10} = 1024$.
Which of the following statements also expresses this fact?

a) $\log_2(1024) = 10$

b) $\log_{10}(1024) = 2$

c) $\log_{1024}(2) = 10$

d) $\log_{1024}(10) = 2$

Question 5 [30 points]

What is the value of $\log_{hundr} \quad (million)$?

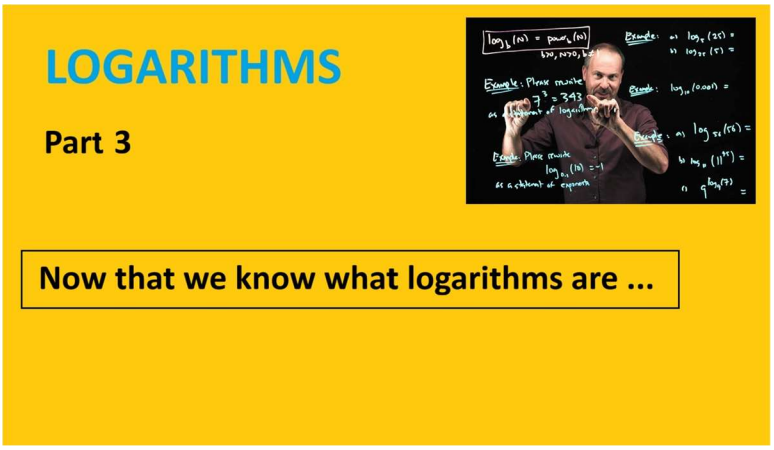
Answers:

1. a), c), d) **2.** a), c), d) **3.** a), c), d) **4.** a) **5.** 3

Logarithms for Humans

PART 3

Now that we know what logarithms are …



https://youtu.be/UQwTM8cGoXU

Logarithms are powers.

$$b^x = N$$

x is the power of b that gives the answer N

$$x = \log_b(N)$$

Example: Please rewrite $7^3 = 343$ as a logarithmic statement.

Answer: We see that 3 is the power of 7 giving the answer 343. Consequently:

$$3 = \log_7(343)$$

Example: Please rewrite $\log_{0.5}(16) = -4$ as a direct statement about exponents.

Answer: We are being told that -4 is the power of 0.5 that give this the value 16. That is,

$$(0.5)^{-4} = 16$$

Question 1: Show that $(0.5)^{-4} = 16$ is a true statement!

Question 2: Rewrite $\log_{0.1}(10) = -1$ as a statement about powers.

Question 3: Give an example of two positive numbers b and N such that

$$\log_b(N) = -3$$

Schoolbooks are chock full of icky-looking practice examples. Why such examples would ever appear in real life beats me!

Example: Please evaluate $\log_{\frac{1}{a^2}}\left(a^{\frac{2}{3}}\right)$.

First Answer: No!

Second Answer: Oh. Okay.

Deep breath.

We're looking for the power of $\frac{1}{a^2}$ that gives $a^{\frac{2}{3}}$.

That is, we are looking for the value x so that

$$\left(\frac{1}{a^2}\right)^x = a^{\frac{2}{3}}$$

Following all our rules of exponents, this reads as:

$$a^{-2x} = a^{\frac{2}{3}}$$

We can now see that $x = -\frac{1}{3}$ does the trick.

So,

$$\log_{\frac{1}{a^2}}\left(a^{\frac{2}{3}}\right) = -\frac{1}{3}$$

Question 4: Please evaluate $\log_{(b^{0.5})}\left(\frac{1}{\sqrt{b}}\right)$.

(Feel free to say NO!)

What's more fun are questions that are like logic puzzles.

Question 5:

a) What is the value of $\log_{56}(56)$?

b) In general, what can you say about the value of

$$\log_b(b)$$

for a positive number b?

Question 6:

a) What is the value of $\log_7(1)$?

b) In general, what can you say about the value of

$$\log_b(1)$$

for a positive number b?

Now things get tautological!

Question 7: Who is Lucy's daughter's mother?

Question 8:

a) What is the value of $\log_{11}(11^{4505})$?

b) In general, what can you say about the value of

$$\log_b(b^x)$$

for a positive number b and real number x?

Question 9:

a) What is the value of $9^{\log_9(777)}$?

b) In general, what can you say about the value of

$$b^{\log_b(N)}$$

for positive numbers b and N?

Playing with the definition of $\log_b(N)$ for positive numbers b and N is all well and good, but two deep questions about them are hanging over our heads.

1. **Who cares about logarithms today?**

2. Even if I imagine that I live in the 1600s and care about logarithms, what has our definition of logarithms got to do with Napier's work from last lesson?
   How do logarithms convert multiplication problems into addition problems?

We shall attend to both questions.

If you'd like to practice the mechanics of logarithms, try these textbook problems.

Practice 10 Please rewrite each of these statements in logarithmic form.

a) $3^5 = 243$ b) $625^{\frac{1}{4}} = 5$ c) $2^0 = 1$

Practice 11 Please rewrite each of these statements as a statement about powers.

a) $2 = \log_7(49)$ b) $\frac{1}{2} = \log_5(\sqrt{5})$ c) $a = \log_3(b)$

Practice 12 Kindly evaluate each of these expressions.

a) $\log_2(8^{303})$ b) $\log_{\frac{1}{2}}\left(\frac{1}{16}\right)$ c) $\log_m((m^3)^2)$

Practice 13 If you are still here doing practice problems … For each statement, kindly find the value of b that makes the statement true.

a) $\log_b(125) = 3$ b) $\log_b(125) = -3$ c) $\log_b(8) = -0.25$

Practice 14 Since you have come this far … For each statement, please consider finding a value of x that makes the statement true.

a) $\log_3(2x + 1) = 4$ b) $\log_{10}(x + 2) = -2$ c) $\log_x(4) = x$

Logarithms for Humans

SELF CHECK 3

Another self-check for your own amusement!

SELF CHECK 3

Question 1 [10 points]

Writing $p = \log_5(w)$ is equivalent to writing:

a) $5^w = p$ b) $5^p = w$ c) $p^5 = w$ d) $p^w = 5$

e) $w^5 = p$ f) $w^p = 5$ g) My brain hurts!

Question 2 [20 points]

Could you explain to a friend why $\log_5(25^7)$ equals 14?

a) Yes! With confidence!

b) Yes-ish

c) Nope!

d) What sort of friend would put up with me talking about this?

Question 3 [30 points]

i) Who is the person who has Alice's Social Security Number?

a) Alice b) Brian c) Corey d) It is not possible to know.

ii) What is the value of $\log_{13}(13^{8045})$?

a) 13 b) 8045 c) It is not possible to know.

d) What has this question got to do with part i)?

iii) What is the value of $8^{\log_8(605)}$?

a) 8 b) 605 c) It is not possible to know.

d) What has this question got to do with part i)?

Question 4 [40 points]

What value for b makes the following statement true?

$$\log_b(32) = 5$$

a) 2 b) 3 c) 4 d) 5 e) Something else

Question 5 [50 points]

What value for b makes the following statement true?

$$\log_5(b) = -1$$

Question 6 [There's a pattern to the number of points]

What value of x makes the following statement true?

$$\log_4(100 - 2x) = 3$$

(Again, why equations like this would ever appear in the real-world beats me!)

Answers:

1. b) **2.** What's your answer? (Noticing $25^7 = (5^2)^7 = 5^{14}$ helps.
3. i) a) ii. iii) b). [Questions ii) and iii) each have the same logical feel as answering i).]
4. a) **5.** $\frac{1}{5}$ **6.** 18

Logarithms for Humans

PART 4

Napier’s Dream

https://youtu.be/drUoCsQUbwU

My calculator says

$$10^{3.581} \approx 3810$$

and so

$$\log_{10}(3810) \approx 3.581$$

Comment: The symbol ≈ means "approximately equal to." We have some rounding error.

Napier claimed that knowing logarithmic values provides a means to transform a multiplication problem into an addition problem. Of course, he was right. But why is this so?

Of course, Napier did not realize that logarithms are powers in disguise. If so, he may have started with the basic property of exponents that underlies their theory.

$$b^n \times b^m = b^{n+m}$$

What is this saying?

$$n + m \textit{ is the power of } b \textit{ that gives the answer } b^n \times b^m$$

That is, we have:

$$\log_b(b^n \times b^m) = \log_b(b^{n+m}) = n + m$$

We're seeing that a logarithm has converted a product of two numbers into a sum of two numbers.

Here's how Napier's logarithm tables work:

Example: Let's compute $3810 \times 10{,}304$.
(Of course, on a modern-day calculator I see that the answer is $39{,}258{,}240$.)

The Approach:
Imagine each number to be power of some number base number b.
It became common practice to use $b = 10$.

$$3810 = 10^n$$

$$10304 = 10^m$$

To find the values n and m, notice:

$$n = \log_{10}(3810)$$

$$m = \log_{10}(10304)$$

According to a table of **common logarithms**, that is, base-10 logarithms, we have

$$n = \log_{10}(3810) \approx 3.581$$

$$m = \log_{10}(10304) \approx 4.013$$

(In truth, I used my calculator! If you see a "log" button without a subscript on your calculator, you can safely assume it is a base- 10 logarithm.)

The statement

$$\log_{10}(10^n \times 10^m) = n + m$$

translates as:

$$\log_{10}(3810 \times 10304) \approx 3.581 + 4.013 = 7.594$$

So, 7.594 is the power of 10 that gives the answer to 3810×10304.

On a calculator I see that $10^{7.594} \approx 39{,}264{,}494$ (and in the 1600s people would see from a table that the number $39{,}264{,}494$ has log value close to 7.594).

Thus

$$3810 \times 10{,}304 \approx 39{,}264{,}494$$

Question 1: The work here has rounding errors. How close is this answer to the true answer? Working with log values rounded to a larger number of decimal places reduces errors.

Question 2:

a) Check that $\log(100)$ is 2 on your calculator. (Why is this correct?)
b) On your calculator, what is the value of $n = \log(387)$?
c) On your calculator, what is the value of $m = \log(5092)$?
d) What is the value of $n + m$? (You should get a number close to 6.2946.)
e) What value V has $\log_{10}(V) = 6.2946$? (Think about how you can use your calculator to find this.)
f) Does V match the product 387×5092?

Schoolbooks present the observation

$$\log_b(b^n \times b^m) = n + m$$

in terms of the numbers $N = b^n$ and $M = b^m$ and so write

$$\log_b(N \times M) = \log_b(N) + \log_b(M)$$

The log of a product is the sum of the logs.

This summarizes Napier's dream.

Question 3:
a) Why should the value of $\log_{10}(7500)$ be 2 larger than the value of $\log_{10}(75)$?
(Is it? Feel free to check this on a calculator.)

b) What do you predict for the value of $\log_{10}(0.75)$ compared to that of $\log_{10}(75)$? Why?

Question 4:

a) Explain why the following is true.

$$\log_b\left(\frac{b^n}{b^m}\right) = n - m$$

b) Explain why it leads to the log rule

$$\log_b\left(\frac{N}{M}\right) = \log_b(N) - \log_b(M)$$

This fourth example shows that logarithms convert division problems into subtraction problems.

Logarithms for Humans

SELF CHECK 4

They keep coming!

SELF CHECK 4

Question 1 [$10 = 10^1$ points]

According to your calculator, which number is closest to the value of $10^{0.903}$?

a) 5 b) 7 c) 8 d) 12

Question 2 [$100 = 10^2$ points]

According to your calculator, which number is closest to the value of $\log_{10}(38)$?

a) 1.580 b) one million c) one millionth

Question 3 [$1,000 = 10^3$ points]

Match each item on the left with an item on the right.

$\log_b(b)$	0
$\log_b(1)$	1
$\log_b(b^2 \times b^5)$	2
$\log_b(\frac{b^9}{b^2})$	7
$\log_b(\frac{b^7 \times b^{-2}}{b^6 \times \frac{1}{b}})$	9

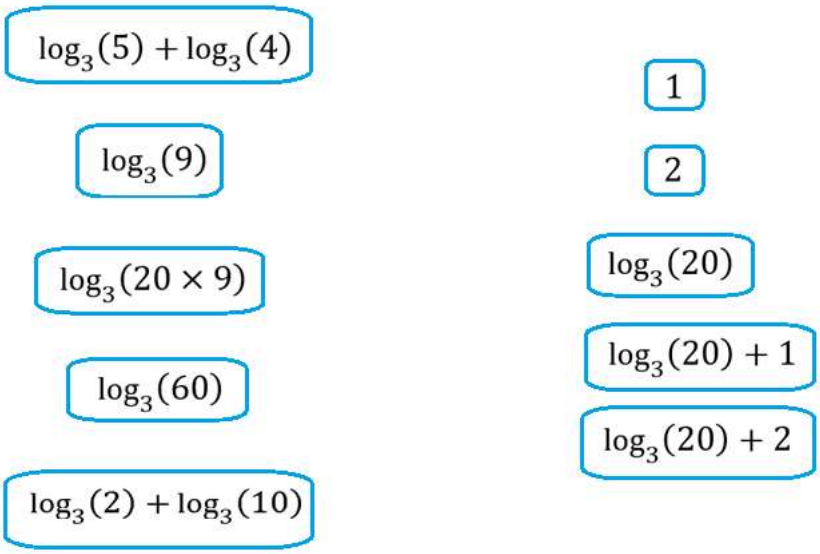
Question 4 [$10,000 = 10^4$ points]

Knowing that $\log_{10}(677) \approx 2.831$, do we know the value of $10^{2.831}$?

a) Yes, we do! It is exactly 677.
b) Almost! It is a value close 677.
c) Yes, we do! But it is a value that has nothing to do with the value 677.
d) It is impossible to say anything about its value.

Question 5 [$100,000 = 10^5$points]

Match each item on the left with an item on the right.



Answers:

1. c) 2. a)

3.

4. b) 5.

Logarithms for Humans

PART 5

Why do we care about logarithms today?

https://youtu.be/qRn9H7bfIhA

We now know the 17th-century reason why people cared about logarithms: they convert multiplication problems into addition problems:

$$\log_b(N \times M) = \log_b(N) + \log_b(M)$$

They also convert division problems into subtraction problems:

$$\log_b\left(\frac{N}{M}\right) = \log_b(N) - \log_b(M)$$

Question 1: Please remind me how we derived these two formulas. I remember we started by first writing N and M each as a power of b.

But why do people care about logarithms today?

It's because of this next property of exponents and how it translates into a statement about logarithms.

We have:

$$(b^n)^m = b^{n \times m}$$

This is saying

$n \times m$ is the power of b that gives the answer$(b^n)^m$.

$$\log_b((b^n)^m) = \log_b(b^{n \times m}) = n \times m$$

Let's rewrite this in the schoolbook way.

Start by rewriting $N = b^n$. This means that $n = \log_b(N)$.

Then our statement reads

$$\log_b(N^m) = \log_b(N) \times m$$

Because people like to use the letter x in algebra class, textbooks will write this as:

$$\log_b(N^x) = \log_b(N) \times x$$

We have:

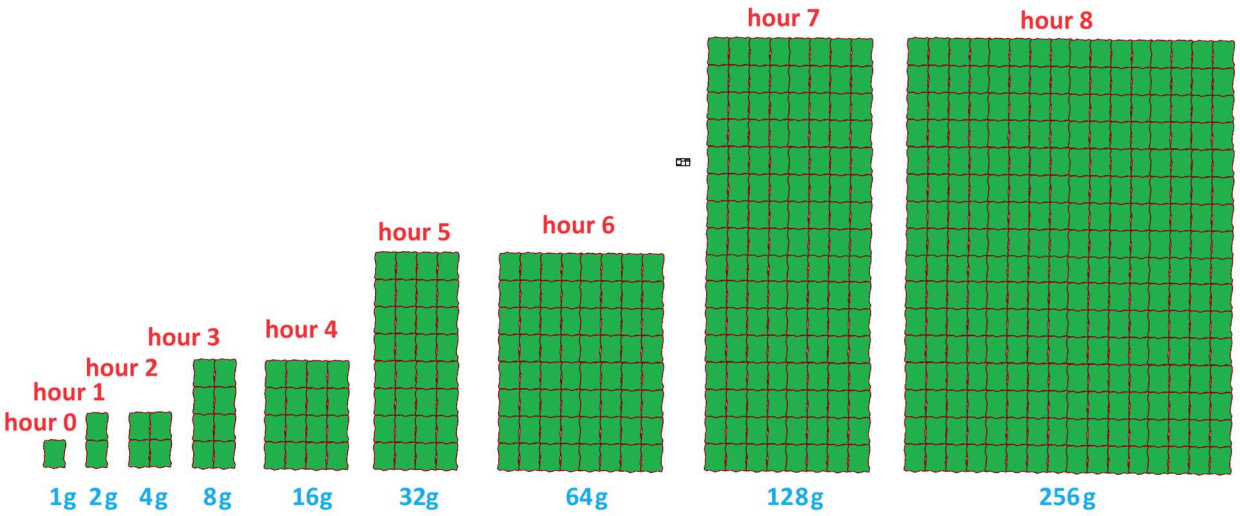
$$\log_b(N^x) = x \log_b(N)$$

Hit an exponential expression with a log and shake down the exponent!

Here's why people like this property: it's hard to solve equations if the variable is "trapped upstairs" as an exponent.

Logarithms provide a means to bring those variables down to a level were we can work with them.

As an example, consider a mass of green, gooey biological growing continuously in such way that it doubles in mass every hour. Let's ass time 0 hours we start with 1 gram of culture.



As we saw in our course on exponents, we know that the mass of the culture at time t hours will be 2^t grams.

Question 2: What will the malls of the goo be at time $t = 2.5$ hours? (Can you figure out how to use the x^y button on a calculator?)

Here's a challenge for us:

Example: At what time will we have 1.5 kg of goo?

To answer this, we need to find a time t such that

$$2^t = 1500$$

It is not clear how to solve this on a calculator. (We could say that $t = \log_2(1500)$, but there is no $\log_2$ on a calculator!)

We have a variable "trapped upstairs."

But we know that hitting an exponential expression with a log—any log!—shakes down the exponent. So, let's hit each side of this equation with a base-10 log, which does exist on my

calculator.

$$\log_{10}(2^t) = \log_{10}(1500)$$

Shaking down the exponent:

$$t\log_{10}(2) = \log_{10}(1500)$$

We can work out approximate values with the calculator

$$t \times 0.301 \approx 3.176$$

and so

$$t \approx \frac{3.176}{0.301} \approx 10.551$$

We have 1500 grams of culture at about time $t = 10.6$ hours.

Comment: If we avoid doing all the numerical calculations we can give an exact answer to this problem. From $t\log_{10}(2) = \log_{10}(1500)$ we see that

$$t = \frac{\log_{10}(1500)}{\log_{10}(2)}$$

This is exact number of hours solving this problem (though I don't have a feel for what this number is!)

However, writing exact answers and leaving all numerical calculations as the final step reduces the amount of rounding error.

Here's a more textbook-like example. (I have no idea why an equation like this to solve might appear in real life.)

Example: Kindly solve $2^{x+1} = 3^x$.

Answer: We have an exponent of $x + 1$ trapped upstairs on the left and an exponent of just x trapped upstairs on the right.

My calculator has a "log" button without a subscript, but it means a base-10 logarithm. Let me hit both sides of the equation with a logarithm and "shake down" each of those exponents.

$$\log(2^{x+1}) = \log(3^x)$$

$$(x + 1)\log(2) = x\log(3)$$

(Notice how I was careful to make clear that "$x + 1$" as a group was shaken down.)

According to my calculator, we now have (up to rounding):

$$(x + 1) \times 0.301 \approx x \times 0.477$$

$$x \times 0.301 + 0.301 \approx x \times 0.477$$

$$0.301 \approx 0.176x$$

$$x \approx 1.710$$

Question 3: a) Show that the exact answer is

$$x = \frac{\log(2)}{\log(3) - \log(2)}$$

b) Some people might rewrite this answer as

$$x = \frac{\log(2)}{\log(1.5)}$$

Why is this an equivalent answer?

c) Is this answer the same as $x = \log(\frac{2}{1.5})$?

Here's a slightly more complicated textbook problem:

Example: Please solve $5 \cdot 3^x = 7 \cdot 4^{x+1}$

Answer: The unknowns are all stuck upstairs.
To bring them down, let's hit each side of this equation with a log.

$$\log_{10}(5 \cdot 3^x) = \log_{10}(7 \cdot 4^{x+1})$$

We have the logarithm of products here. Recall: **The log of a product is the sum of the logs.**

$$\log_{10}(5) + \log_{10}(3^x) = \log_{10}(7) + \log_{10}(4^{x+1})$$

Now we're ready to **shake down exponents** (again catching that one of our exponents is all of $x + 1$).

$$\log_{10}(5) + x\log_{10}(3) = \log_{10}(7) + (x+1)\log_{10}(4)$$

This is visually confusing, but most everything here is just a number. According to my calculator this equation is (approximately):

$$0.699 + 0.477x = 0.845 + 0.602(x+1)$$

I am going to multiply through by 1000 to avoid all the decimals.

$$699 + 477x = 845 + 602(x+1)$$

$$699 + 477x = 845 + 602x + 602$$

$$125x = -748$$

$$x = -\frac{748}{125} \approx -5.984$$

Icky!

Question 4 Show that the exact solution to the previous example is

$$x = \frac{\log_{10}(5) - \log_{10}(7) - \log_{10}(4)}{\log_{10}(4) - \log_{10}(3)}$$

Show that this can also be written as

$$x = \frac{\log_{10}(\frac{5}{28})}{\log_{10}(\frac{4}{3})}$$

Question 5 Kindly solve $2^x \cdot 3^x = 4 \cdot 5^x$.
(If you are game, feel free to give the exact solution as well as an approximate one.)

Question 6

a) Please solve $\left(\frac{6}{5}\right)^x = 4$ with your answer in terms of base ten logarithms.

b) Why is the solution to this problem exactly the same as the solution to question 5?

This next example shows why calculators don't have logarithm buttons for every possible base.

Example: Compute the value of $\log_3(7)$ with a calculator.

Answer: The key is to give the quantity a name. Let's call it F for Frederica.

$$F = \log_3(7)$$

So, Frederica is the power of 3 that gives the answer 7.

$$3^F = 7$$

Let's hit this with our $\log_{10}$ button.

$$\log_{10}(3^F) = \log_{10}(7)$$

$$F\log_{10}(3) = \log_{10}(7)$$

$$F = \frac{\log_{10}(7)}{\log_{10}(3)} \approx \frac{0.845}{0.477} \approx 1.771$$

That's it:

$$\log_3(7) \approx 1.771$$

Question 7 a) Compute $\log_{37}(500)$ if doing so seems fun to you.

b) If you are game, show that $\log_b(N)$ can be computed as

$$\frac{\log_{10}(N)}{\log_{10}(b)}$$

Some curriculums want students to know this **change of base formula**.

Summary of Textbook Expectations on Logarithms

Curriculum can really "dig deep" into properties of logarithms and ask students to derive—and memorize—all sorts of complicated formulas.

In the end, each formula is just an application of three basics.

1. **$\log_b(N)$ is the power of b that gives the answer N.**

$$x = \log_b(N) \quad \text{means} \quad b^x = N$$

(Replace the word "log" with the word "power")

2. **The log of a product is the sum of the logs**.

$$\log_b(N \times M) = \log_b(N) + \log_b(M)$$

3. **Hitting an exponential expression with a log shakes down the exponent.**

$$\log_b(N^x) = x \log_b(N)$$

For instance, the additional rule

$$\log_b\left(\frac{N}{M}\right) = \log_b(N) - \log_b(M)$$

can be recreated by thinking of $\frac{N}{M}$ as $N \cdot \frac{1}{M} = N \cdot M^{-1}$.
This leads to:

$$\begin{aligned}\log_b\left(\frac{N}{M}\right) &= \log_b(N \cdot M^{-1}) \\ &= \log_b(N) + \log_b(M^{-1}) \\ &= \log_b(N) + (-1)\log_b(M) \\ &= \log_b(N) - \log_b(M)\end{aligned}$$

Question 8 Explain why $\log_b(\frac{1}{N})$ equals $-\log_b(N)$.

If you enjoyed question 8, try this one too. (If you didn't like it, don't do question 9.)

Question 9 Explain why $\log_{\frac{1}{b}}(N)$ equals $-\log_b(N)$.

Here's an annoying, made-up question just for the sake of testing students on logarithms and quadratics at the same time. My advice is to ignore this question.

Question 10 What value(s) of x make this equation true?

$$(\log_{15}(x))^2 + \log_{15}(x^2) = 15$$

Logarithms for Humans

SELF CHECK 5

It’s another one!

SELF CHECK 5

Question 1 [Some points]

Here are some logarithmic expressions:

$\log_4(16)$	$\log_5(625)$	$\log_9(9)$	$\log_{96}(96^{0.5})$	$\log_3(\frac{1}{9})$
$\log_{\frac{1}{2}}(\frac{1}{4})$	$\log_{\frac{1}{2}}(\frac{1}{2})$	$\log_{\frac{1}{2}}(1)$	$\log_{\frac{1}{2}}(4)$	$\log_{\frac{1}{2}}(2)$

a) How many of these expressions evaluate to a positive number?

b) How many of these expressions evaluate to a whole number, either positive, negative, or zero?

Question 2 [Some more points]

Solving $2^x + 2^{x+1} = 24$ is the same as solving which of these equations

a) $2^{2x+1} = 24$ (Don't choose this option.)

b) $2^x + 2 \cdot 2^x = 24$ (This answer is correct. And so is another option below.)

c) $2^x = 24$

d) $2^x = 8$

Question 3 [20 points. (Why 20? No reason!)]

The exact solution to $3^x + 3^{x+1} = 360$ is ….

a) $x = \frac{\log_{10}(36)}{\log_3(3)}$ b) $x = \frac{\log_{10}(360) - \log_{10}(10)}{\log_3(3)}$ c) Both of these d) Neither of these

Question 4 [200 points]

These self-check questions are feeling ….

a) harder b) easier c) whatever

Question 5 [Any number of points you like]

The statement "logarithms turn multiplication problems into addition problems" is referring to which of the following mathematical statements? (And it is the only statement that is correct!)

a) $\log_b(N + M) = \log_b(N) + \log_b(M)$
b) $\log_b(N + M) = \log_b(N) \times \log_b(M)$
c) $\log_b(N \times M) = \log_b(N) + \log_b(M)$
d) $\log_b(N \times M) = \log_b(N) \times \log_b(M)$

Question 6 [3 points—the usual number of points for a question on an exam]

What is the exact solution to $5 \cdot 3^w = 2 \cdot 5^w$?

a) $w = \frac{\log_{10}(5) - \log_{10}(3)}{\log_{10}(5) - \log_{10}(2)}$

b) $w = \frac{\log_{10}(5) - \log_{10}(3)}{\log_{10}(2) - \log_{10}(5)}$

c) $w = \frac{\log_{10}(3) - \log_{10}(5)}{\log_{10}(5) - \log_{10}(2)}$

d) Egad! All these options look scary and horrible!

Answers:

1. a) 6 **b)** 9 **2.** b), d) **3.** c) **4.** They feel harder to me! **5.** c) **6.** a) and d)!

Logarithms for Humans

PART 6

Practical Example:
Compounding Growth

The human population is growing.

According to an internet search at the time of me writing this, we seem to be averaging about 17.3 births for every 1000 people each year, and 8.9 deaths for every 1000 people each year.

So, we're growing at a rate of an increase of 8.4 additional Earth inhabitants for every 1000 people per annum.

> **Queston 1** These figures are continually changing. What are global birth and mortality rates at the time of you reading this?
>
> Can you find some data showing how these rates seem to have changed over the decades?
>
> **Comment**: The data I found is presented in terms of average counts per 1000 people. Some data might be presented instead as a percentage, an average count per 100 people.
>
> For example, the fraction $\frac{17.3}{1000}$ is the same as $\frac{1.73}{100} = 1.73\%$, so our data corresponds to a birth rate of 1.73%.

Just to be clear, we're speaking of growth *rates*, not absolute figures. The total numbers of actual births and deaths depends on the total number of people there are. We've just been told, for every group of 1000, we can expect an average number of 17.3 births, and so on.

For instance, my internet search also tells me there are currently about 8 billion people on the planet. If we trust the figures I've shared, over the coming year there will be

$$8{,}000{,}000{,}000 \times \frac{17.3}{1000} = 138{,}400{,}000 \text{ babies born}$$

$$8{,}000{,}000{,}000 \times \frac{8.9}{1000} = 71{,}200{,}000 \text{ deaths}$$

for a total increase of $67{,}200{,}000$ inhabitants (which is indeed 0.84% of $8{,}000{,}000{,}000$).

We're thus predicting a new human population figure of $8{,}067{,}000{,}000$ by the end of one year.

Question 2 Suppose the birth rate of 1.73% and death rate of 0.89% stay the same for next few years. What do you predict then for the Earth's population after a second year? We'll grow from 8.067 billion humans to how many?

And after a third year? A fourth?

We're seeing that if the Earth's population is a count of P people, then after a year there will be

$P \times 0.0173$ babies born

$P \times 0.0089$ deaths

for an increase $P \times 0.0173 - P \times 0.0089 = P \times 0.0084$ in the count of people on the planet.

Over a year, the human population changes from P people to $P + P \times 0.0084$ people.
We have

$$P \ + \ P \times 0.0084 = P \times (1 + 0.0084)$$
$$= P \times 1.0084$$

According to our overly basic analysis, the Earth's population changes by a factor of $\mathbf{1.084}$ from year to year.

Let's play with this.

We started with a population of $P = 8{,}000{,}000{,}000$ people.

After 1 year the population grows to

$$P_1 = 8{,}000{,}000{,}000 \times 1.084 = 8{,}067{,}000{,}000$$

After another year, by year 2, the population grows to

$$P_2 = 8{,}000{,}000{,}000 \times 1.084 \times 1.084 = 8{,}744{,}628{,}000$$

By year 3 the population is

$$P_3 = 8{,}000{,}000{,}000 \times 1.084 \times 1.084 \times 1.084 = 9{,}479{,}176{,}752$$

By year 4 the population is

$$P_4 = 8{,}000{,}000{,}000 \times 1.084 \times 1.084 \times 1.084 \times 1.084 = 11{,}046{,}052{,}825$$

If P_n denotes the population by the end of year n, then our (likely unrealistic) reasoning predicts that the Earth's population will be given by

$$P_n = 8{,}000{,}000{,}000 \times (1.084)^n$$

Question 3 Let's continue to believe that the Earth's population will grow without bound at a constant rate of 0.84% per annum.

According to our model:

a) What will the Earth's population by the end of ten years?

b) In how many years will the Earth's population be one-hundred billion?

Question 4 Assume that the birth and death rates of fruit flies living in my compost bin are each constant. Yesterday there were 160 fruit flies in the bin. Today there are 200.

Create simple mathematical model that predicts the number of fruit flies I can expect living in my compost bin each day.

On which day will I have just over a million fruit flies according to your model?

Question 5 There are fruit flies in my recycling bin too, but the population of them there is not thriving. I've observed they are breeding at a constant birth rate of 10% per day, but dying at a rate of 12% per day.

There are currently 100 fruit flies in my recycling bin.

Create simple mathematical model that predicts the number of fruit flies I can expect living in my recycling bin each day.

According to your model, on which day will I have just one lonely fruit fly in my bin?

Logarithms for Humans

PART 7

Practical Example:
Banking

If you think about it for a moment, it seems somewhat curious that banks pay you for the honor of conducting a service for you, namely, to securely house your money. Ten thousand dollars in cash is safer in the protection of a bank than in your pockets or under your mattress.

And how do banks pay you? By giving you interest on the money you have stored with them.

Of course, banks do make a profit despite regularly adding money to your and all their customers' accounts. But with the large cash sums they accrue from having many customers, banks can invest in high-paying financial opportunities beyond what any one individual can typically do. They make a profit and thank you for this by sharing some of the profit with you.

The total amount of money you earn from an interest payment of course depends on the amount of money you have in your account. An interest payment of 2% say on a balance of $\$10{,}000$ gives you $\frac{2}{100} \times 10{,}000 = \$200$, but on a balance of $\$1{,}000{,}000$ it gives you $\frac{2}{100} \times 1{,}000{,}000 = \$20{,}000$.

If you let a balance sit in an account untouched, you'll also earn interest on the interest payments awarded to you, repeatedly, and your account balance will grow.

> **Example**: I decide to invest $\$1200$ with *Grimy Hands Money Market.* They offer 3.5% annual interest calculated at the end of each calendar year.
>
> It's the start of the year, and I am willing to keep my money in the account untouched for many years.
>
> a) What will my balance be after ten years?
>
> b) I'd like to be a millionaire. By which year will I have a million dollars?

Answer: My starting balance is

$$B = 1200$$

dollars.

At the end of the first year, my balance will grow to

$$B_1 = 1200 \ + \ \frac{3.5}{100} \times 1200 = 1242$$

dollars. But let's write this answer as

$$B_1 = 1200 \times 1.035$$

At the end of the second year my balance will grow to

$$B_2 = 1242 \ + \ 1242 \times 0.035 = 1285.47$$

dollars. But this really

$$\begin{aligned} B_2 &= 1242 \times 1.035 \\ &= 1200 \times 1.035 \times 1.035 = 1200 \times (1.035)^2 \end{aligned}$$

Each year my balance grows by a factor of 1.035 and after n years will have

$$B_n = 1200 \times (1.035)^n$$

dollars.

a) By the end of ten years my balance is

$$B_{10} = 1200 \times (1.035)^{10} \approx 1692.71$$

(Grimy Hands always rounds down to the nearest penny!)

b) To be a millionaire I need

$$1200 \times (1.035)^n = 1{,}000{,}000$$

So,

$$1.035^n = \frac{1000000}{1200} = \frac{2500}{3}$$

"Hitting with a log" gives

$$n \log_{10}(1.035) = \log_{10}\left(\frac{2500}{3}\right)$$

And so

$$n = \frac{\log_{10}\left(\frac{2500}{3}\right)}{\log_{10}(1.035)}$$

According to my calculator, this gives $n \approx 195.5$. I'll need to wait 196 years to be a millionaire if I start with $\$1200$ and just sit back! (Maybe I should be a little more pro-active if this truly is my goal!)

The truth is banks don't assign interest just once a year, they assign it closer to every instant!

Let's make sense of this intriguing claim.

Example: I have $\$10{,}000$ I'd like to invest for a year. Two respectable institutions are offering 5% interest per annum but calculated over smaller time periods.

- *Buckets-o-Cash Bank* calculates their interest payments monthly, meaning that they assign $\frac{5}{12} \approx 0.417\%$ of interest each month for 12 months.
- *Cash Flux Bank* calculates their interest payments weekly, meaning that they assign $\frac{5}{52} \approx 0.096\%$ of interest each week for 52 weeks.

Which bank will leave me with the bigger balance by the end of the year?

Answer: Let's examine each bank in turn.

Buckets-o-Cash: They pay $\frac{5}{12} \approx 0.417\%$ interest each month. So, a balance of B dollars at the start of a month becomes

$$B + \frac{0.417}{100} \times B = B \times (1 + 0.00417) = B \times (1.00417)$$

dollars at the end of the month. That is, my balance grows by a factor of approximately 1.00417 each month.

Let's write out a "spread-sheet" of my balance month-per-month.

Start: $\$10{,}000$

Month 1: $\$10{,}000 \times (1.00417)$

Month 2: $\$10{,}000 \times (1.004167) \times (1.004167)$
$= \$10{,}000 \times (1.004167)^2$

Month 3: $\$10{,}000 \times (1.004167)^3$

⋮

Month 12: $\$10{,}000 \times (1.004167)^{12}$

My calculator says this final balance amount is $\$\mathbf{10,511.62}$.

Cash Flux Bank: They pay $\frac{5}{52} = 0.0961\%$ interest each week. So, a balance of B dollars at the start of a week becomes

$$B + \frac{0.0961}{100} \times B = B \times (1 + 0.000961) = B \times (1.000961)$$

dollars at the end of the week. That is, my balance grows by a factor of $1.000961\ldots$ each week.

Start: $\$10{,}000$

Week 1: $\$10{,}000 \times (1.000961)$

Week 2: $\$10{,}000 \times (1.000961)^2$

Week 3: $\$10{,}000 \times (1.000961)^3$

⋮

Week 52: $\$10{,}000 \times (1.000961)^{52}$

My calculator says this final balance amount is $\$\mathbf{10,512,46.}$

That's 84 cents better! I'll go with *Cash Flux Bank*.

> **Example Continued:** I have since learned of some more banks paying the same interest rate per annum but calculated over even shorter time periods.
>
> - *Swimming-In-It Bank* calculates their interest payments daily, meaning that they assign $\frac{5}{365}\%$ of interest each day for365 days.
> - *Cash Galore Bank* calculates their interest payments every hour, meaning that they spread the 5% interest payment over each and every hour of the year.
> - *Bank Bonanza* calculates their interest payments every minute, meaning that they spread the 5% interest payment over each and every minute of the year.
> - *No-Messing-About-Bank* calculates their interest payments every second, meaning that they spread the 5% interest payment over each and every second of the year.
>
> And there are additional banks that spread the 5% interest payment over every milli-second of the year, over every nano-second of the year, and so on.
>
> For the four banks named, what would my end-of-year balance be if I invested my $\$10{,}000$ with each of them?

We saw the following general structure from the first example.

> If a bank pays $r\%$ interest each period and my balance at the beginning of a period is B dollars, then my new balance at the end of that period will be $B(1+\frac{r}{100})$ dollars. (This is $B + \frac{r}{100} \times B$.)
>
> After n such periods my balance will be
>
> $$B\left(1+\frac{r}{100}\right)^n$$
>
> dollars.

Let's now analyze the next four banks.

Answer:

Swimming-In-It Bank: There are 365 days in a year. We have $r = \frac{5}{365} \approx 0.0137\%$ amount of interest paid each day for 365 days. My balance after a year will be

$$10{,}000 \times (1.0001367)^{365} \approx \$\mathbf{10,512.67}$$

Cash Galore Bank: There are $365 \times 24 = 8{,}760$ hours in a year.

Now $r = \frac{5}{8760} \approx 0.00057\%$ and we have 8760 periods.
My balance after a year will be

$$10{,}000 \times (1.0000057)^{8760} \approx \$\mathbf{10,512.70}$$

Bank Bonanza: There are $365 \times 24 \times 60 = 525{,}600$ minutes in a year.

Now $r = \frac{5}{525600} \approx 0.00000951\%$ and we have $525{,}600$ periods.
My balance after a year will be

$$10{,}000 \times (1.0000000951 \ldots)^{525600} \approx \$\mathbf{10,512.71}$$

No-Messing-About Bank: There are $365 \times 24 \times 60 \times 60 = 31{,}536{,}000$ seconds in a year.

Now $r = \frac{5}{31{,}536{,}000} \approx 0.000000158\%$ and we have $31{,}536{,}000$ periods.
My balance after a year will be

$$10{,}000 \times (1.00000000158 \ldots)^{31536000} \approx \$\mathbf{10,512.71}$$

Computing interest over finer and finer time periods at this point seems to create balance increases only in fractions of pennies. I don't think it is worth going through the work of calculating final balances for banks that compute interest every micro-second or finer. We won't see the effect at the levels of pennies.

But we do see that there does seem to be some kind of "ultimate" balance value if banks were to approach computing interest for us at each and every instant.

Question 1: When I conducted some internet research, I developed the impression that banks typically compute interest daily. (Check me on this.) But that led me to the question: do banks recognize leap years? Do they handle leap years differently from regular years?
What can you find out?

People don't typically conduct more than one bank transaction in a day, so computing interest daily is likely in line with the typical practices of customers. (It would seem unfair if interest were calculated monthly say, and you happened to make a large withdrawal just before "interest day.")

Question 2 Suppose I kept my account open with *Swimming-In-It Bank* for double the amount of time, 2 years instead of just 1.

What would my balance be after two years?

(After one year I will earn $512.67 in interest, but over two years my interest earned will be more than double this. This is because during the second year I'll be earning interest on even higher interest payments.)

Logarithms for Humans

A TYPICAL (unimaginative) EXAM

Here is an example of a math exam with questions typical for many curriculums. It is designed to resemble the style, tone, and structure you might encounter in a formal classroom setting.

A Few Notes to Keep in Mind:

Authoritative Tone:

This exam uses a formal, commanding tone. There's no use of words like "please," "kindly," or "if you will." While this can seem harsh, it's a common style in mathematics exams. Try not to let it unsettle you—it's just the convention!

Ambiguous Terms:

Words such as *simplify* and *expand* often appear in exams, even though their meanings can be subjective. What seems "simple" in one context might feel awkward in another. As you work through the exam, make your best effort to interpret what the examiner is asking for.

Unfamiliar Material:

It's possible the exam might include a question or concept that feels unfamiliar or wasn't explicitly taught. Don't panic! Take a deep breath and use the context of the question, along with your mathematical reasoning, to deduce what's being asked. You have the tools and knowledge to approach any question thoughtfully.

Logarithms Exam

Time: 60 minutes
Total Marks: 50

Section A: Basic Logarithms (10 marks)

1. (2 marks) Evaluate the following logarithms:

   a) $\log_2(16)$
   b) $\log_5(125)$

2. (2 marks) Rewrite the following expressions in exponential form:

   a) $\log_3(81) = 4$
   b) $\log_{10}(0.01) = -2$

3. (2 marks) Rewrite the following expressions in logarithmic form:

   a) $4^3 = 64$
   b) $10^{-3} = 0.001$

4. (4 marks) Simplify using the properties of logarithms:

   a) $\log_3(3^5)$
   b) $\log_4(80) - \log_4(20)$
   c) $2\log_5(25)$
   d) $\log_7(49 \cdot 7^x)$

Section B: Solving Equations (15 marks)

5. (3 marks) Solve for x:
   a) $\log_2(x) = 6$
   b) $\log_4(x) = \frac{1}{2}$
   c) $10^{x+1} = 1000$

6. (3 marks) Solve for x in the following:
   a) $\log_3(x) + \log_3(4) = 2$
   b) $\log_5(x) - \log_5(2) = 1$
   c) $\log_2(x^2) = 6$

7. (3 marks) Use the **change of base formula** to calculate $\log_6(50)$ to two decimal places. Show your work.

Editorial Comment: It is possible to figure out the answer to this question without having memorized the "change of base formula. (I certainly don't have it in my head!)

8. (6 marks) Expand or condense the following logarithmic expressions:

   a) Expand: $\log_2(16x^3)$
   b) Expand: $\log_5\left(\frac{x^2y}{z^3}\right)$
   c) Condense: $\frac{1}{2}\log_4(A) - \log_4(B) - \log_4(C)$

Section C: Applications (25 marks)

9. **Richter Scale** (6 marks)

The Richter scale measures the magnitude of an earthquake using the formula:

$$M = \log_{10}\left(\frac{A}{A_0}\right)$$

where A is the amplitude of the earthquake's waves and A_0 is a reference amplitude.

a) If the amplitude of an earthquake is 1000 times the reference amplitude, what is the magnitude M? (2 marks)

b) If an earthquake has a magnitude$M = 5,$ how many times larger is its amplitude than the reference amplitude? (4 marks)

10. **Population Growth** (6 marks)

A city's population grows according to the formula:

$$P(t) = P_0 \cdot b^t$$

where P_0 is the initial population, b is the growth factor, and t is time in years.

a) Solve for t in terms of $P(t)$, P_0, and b. (2 marks)
b) If $b = 2$, how many years will it take for the population to triple? (4 marks)

11. **Decibel Scale** (5 marks)

The loudness of a sound is measured in decibels using the formula:

$$L = \log_{10}\left(\frac{I}{I_0}\right)$$

where I is the intensity of the sound and I_0 is the reference intensity.

a) If a sound is 100 times more intense than the reference intensity, what is its decibel level? (2 marks)
b) A sound is measured at 4 decibels. How many times more intense is it compared to the reference intensity? (3 marks)

12. **Cooling Formula** (8 marks)

The temperature of a cooling object is modeled by:

$$T(t) = T_0 \cdot b^t$$

where $T(t)$T is the temperature at time t, T_0 is the initial temperature, and b is the cooling factor.

An object starts at 200°C, cools to 50°C in 2 hours.

a) What is the cooling factor b? (2 marks)
b) How many more hours will it take for the temperature of the object to drop to 10°C?

Answer Key

Section A: Basic Logarithms

1. a) 4 b) 3
2. a) $3^4 = 81$ b) $10^{-2} = 0.01$
3. a) $\log_4(64) = 3$ b) $\log_{10}(0.001) = -3$
4. a) 5 b) 1 c) 4 d) $x + 2$

Section B: Solving Equations

5. a) $x = 64$ b) $x = 2$ c) $x = 2$
6. a) $x = 2.25$ b) $x = 10$ c) $x = 8$ or -8
7. Let's give $\log_6(50)$ the name "G."

$$\log_6(50) = G$$

$$6^G = 50$$

$$\log_{10}(6^G) = \log_{10}(50)$$
$$G \cdot \log_{10}(6) = \log_{10}(50)$$

$$G = \frac{\log_{10}(50)}{\log_{10}(6)} \approx 2.183$$

8. a) $\log_2(16x^3) = \log_2(16) + \log_2(x^3) = 4 + 3\log_2(x)$
   b) $2\log_5(x) + \log_5(y) - 3\log_5(z)$
   c) $\log_4\left(\frac{\sqrt{A}}{BC}\right)$

Section C: Applications

9. a) $A = 1000A_0$ and so $M = \log_{10}(1000) = 3$
   b) $5 = \log_{10}\left(\frac{A}{A_0}\right)$ and so $A = 10{,}000 \times A_0$. It is $10{,}000$ times larger.

10. a) We have

$$P(t) = P_0 \cdot b^t$$

$$b^t = \frac{P(t)}{P_0}$$

$$t \log_{10}(b) = \log_{10}\left(\frac{P(t)}{P_0}\right)$$

$$t = \frac{\log_{10}\left(\frac{P(t)}{P_0}\right)}{\log_{10}(b)}$$

c) We want the time for which $P(t) = 3 \cdot P_0$. This occurs at time

$$t = \frac{\log_{10}(3)}{\log_{10}(2)} \approx 1.6$$

The population will triple in about 1 year and 7 months.

11. a) 2 Decibels

b) We have

$$4 = \log_{10}\left(\frac{I}{I_0}\right)$$

and so $\frac{I}{I_0} = 10^4$. It is $10{,}000$ times as intense.

12. a) $b = \frac{1}{2}$

b) Now with an initial temperature of 50°C we need to solve for t in

$$10 = 50 \cdot \left(\frac{1}{2}\right)^t$$

that is,

$$0.2 = (0.5)^t$$

We obtain :

$$t = \frac{\log_{10}(0.2)}{\log_{10}(0.5)} \approx 2.3$$

It reaches 10°C in a further 2.3 hours.

Logarithms for Humans

Practice Problems Galore

Question 1: Please evaluate each of the following:

a) $\log_2(32)$ b) $\log_{32}(2)$ c) $\log_{10}(million)$ d) $\log_3(243)$

e) $\log_5(1)$ f) $\log_{20}(400^2)$ g) $\log_{36}(6)$ h) $\log_{61}(61)$

Question 2: Please evaluate each of the following:

a) $\log_5(5^{37})$ b) $7^{\log_7(15)}$

Question 3: Rewrite $\log_{0.1}(10) = -1$ as a statement about powers.

Question 4: Please compute $\log_7(10)$ to three decimal places.

Question 5: Please find the value of the unknown in each of these equations.

a) $\log_2(x) = 5$ b) $\log_b(125) = 3$ c) $\log_b(125) = -3$

d) $\log_4(x) + \log_4(16) = 3$ e) $\log_x(1) = 0$

Question 6: Please explain why $\log_{100}(N)$ equals $\frac{1}{2}\log_{10}(N)$.

Question 7: Please solve x in each of these equations

a) $5 \cdot 4^x = 3^x$

b) $2^x + 2^{x+1} = 24$

c) $2^x + 2^{x+1} = 25$

d) $10 = (1.05)^x$

Question 8: Give an example of two positive numbers b and N such that

$$\log_b(N) = -billion$$

Question 9: What value of x makes this statement true?

$$\log_x(7) = 3$$

Question 10: Please condense

$$5\log_3(a) - \frac{1}{2}\log_3(b) + 100\log_3(c)$$

into a single logarithm.

Question 11: Please solve for N:

a) $\log_{10}(N) - \log_{10}(14.73) = 2$
b) $\log_{10}(N) - \log_{10}(14.73) = -2$

Question 12: Some biological gloop grows continuously in such a manner that its mass doubles every hour. How long does it take for the mass to triple?

Question 13: I would like to open a bank that makes every customer a millionaire by the end of ten years if they invest just $\$1{,}000$ with us. I'll offer an annual interest rate of $r\%$ per year for some value r and pay $\frac{r}{12}\%$ interest at the end of each month for 120 months.

What value for r should I offer?

OPTIONAL CHALLENGE JUST FOR FUN:

I borrowed $\$15{,}600$ to buy a new car, promising to pay $1,200 each month towards completing the loan.

The bank charges me 9% interest per annum, but charges it monthly ($\frac{9}{12}\%$ each month) and applies that interest charge just before my $\$1{,}200$ payment is made for that month.

How many months will it take me to pay off the loan?
How much will I have actually paid at the end of this loan?

Hint: Write out a "spread sheet" for this.

Logarithms for Humans

ALL SOLUTIONS

(To be completed)