Logarithms for Humans

PART 2:

What logarithms really are!

Getting Straight Into it!

Without any introduction, discussion, or fuss, can you figure out what is being asked of you in this "worksheet"? Are you able to complete it?

Give this serious consideration before reading on.

$\text{power}_5(25) = \underline{\ 2\ }$ $\qquad$ $\text{power}_{10}(1000) = \underline{\ 3\ }$

$\text{power}_3(27) = \underline{\ 3\ }$ $\qquad$ $\text{power}_4(4) = \underline{\ 1\ }$

$\text{power}_2(8) =$ ____	$\text{power}_2(\sqrt{2}) =$ ____	$\text{power}_1(5) =$ ____
$\text{power}_{10}(100) =$ ____	$\text{power}_{10}(\text{million}) =$ ____	$\text{power}_2(0) =$ ____
$\text{power}_4(16) =$ ____	$\text{power}_{73}(1) =$ ____	$\text{power}_{-2}(-8) =$ ____
$\text{power}_4(64) =$ ____	$\text{power}_{100}(0.1) =$ ____	$\text{power}_{-2}(8) =$ ____
$\text{power}_7(\frac{1}{7}) =$ ____	$\text{power}_{\sqrt{6}}(\frac{1}{36}) =$ ____	$\text{power}_0(0) =$ ____

The first example

$$\text{power}_5\,(25) = \underline{\;2\;}$$

is stating that

the power of 5 *that gives the answer* 25 *is* 2

Or you might prefer to phrases this backwards as

2 *is the power of* 5 *that gives* 25

Did you come to this understanding?

Filling out the table is a task of figuring out the correct power of a give base number to give a certain answer for fifteen examples.

I could do this for all but four of them.

$\text{power}_5\,(25) = \underline{\;2\;}$	$\text{power}_{10}(1000) = \underline{\;3\;}$
$\text{power}_3\,(27) = \underline{\;3\;}$	$\text{power}_4\,(4) = \underline{\;1\;}$

$\text{power}_2\,(8) = \underline{\;3\;}$	$\text{power}_2\,(\sqrt{2}) = \underline{\;\frac{1}{2}\;}$	$\text{power}_1\,(5) = \underline{\;?\;}$
$\text{power}_{10}(100) = \underline{\;2\;}$	$\text{power}_{10}(\text{million}) = \underline{\;6\;}$	$\text{power}_2\,(0) = \underline{\;?\;}$
$\text{power}_4\,(16) = \underline{\;2\;}$	$\text{power}_{73}(1) = \underline{\;0\;}$	$\text{power}_{-2}(-8) = \underline{\;3\;}$
$\text{power}_4\,(64) = \underline{\;3\;}$	$\text{power}_{100}(0.1) = \underline{\;\frac{-1}{2}\;}$	$\text{power}_{-2}(8) = \underline{\;?\;}$
$\text{power}_7\,(\frac{1}{7}) = \underline{\;-1\;}$	$\text{power}_{\sqrt{6}}(\frac{1}{36}) = \underline{\;-4\;}$	$\text{power}_0\,(0) = \underline{\;?\;}$

Just as a side note:

I computed the power of 100 that gives the answer 0.1 by writing $100^a = \frac{1}{10}$ and seeing that this is the same as $10^{2a} = 10^{-1}$.

I computed the power of $\sqrt{6}$ that gives $\frac{1}{36}$ by looking at $\left(6^{\frac{1}{2}}\right)^b = 6^{-2}$.

Here are the four troublesome examples.

$$\text{power}_1(5) = \underline{\ ?\ }$$

$$\text{power}_2(0) = \underline{\ ?\ }$$

$$\text{power}_{-2}(8) = \underline{\ ?\ }$$

$$\text{power}_0(0) = \underline{\ ?\ }$$

Every power of 1 is 1.

$$\mathbf{1^x = 1 \text{ for all numbers } x}$$

There is no power of 1 that gives the answer 5.

Every power of 2 is a positive number. In fact, we saw in the course on exponents that

For a positive base number a, the value of a^x is a positive number for all numbers x

There is no power of 2 that gives the answer 0.

Again, as we saw in the last course, powers of negative numbers are dangerous! At the very least, I can't think of a power of -2 that gives the answer 8.

There are many powers of 0 that give the answer 0. For example, $0^2 = 0 \times 0 = 0$ and $0^5 = 0 \times 0 \times 0 \times 0 \times 0 = 0$. There are too many possible answers to the final example.

$$\text{power}_1(5) = \underline{\text{does not exist}}$$

$$\text{power}_2(0) = \underline{\text{does not exist}}$$

$$\text{power}_{-2}(8) = \underline{\text{does not exist}}$$

$$\text{power}_0(0) = \underline{\text{indeterminant}}$$

Every other example was fine: they each asked for a power of a positive number to give a positive number. As we saw last course, the mathematics of powers for positive values is all safe and secure.

Now let's do something very strange and cross out each instance of the word "power" in our worksheet and replace it with the letters *log*, short for **logarithm**.

Of course, "logarithm" is the word invented by Napier in the 1600s as the tool that saved scientific progress. With Napier's logarithms, scholars could conduct complicated arithmetic with the simpler task of addition.

log ~~power~~$_5$ (25) = 2	log ~~power~~$_{10}$ (1000) = 3
log ~~power~~$_3$ (27) = 3	log ~~power~~$_4$ (4) = 1

log ~~power~~$_2$ (8) = 3	log ~~power~~$_2$ ($\sqrt{2}$) = $\frac{1}{2}$	log ~~power~~$_1$ (5) = does not exist
log ~~power~~$_{10}$ (100) = 2	log ~~power~~$_{10}$ (million) = 6	log ~~power~~$_2$ (0) = does not exist
log ~~power~~$_4$ (16) = 2	log ~~power~~$_{73}$ (1) = 0	log ~~power~~$_{-2}$ (-8) = 3
log ~~power~~$_4$ (64) = 3	log ~~power~~$_{100}$ (0.1) = $\frac{-1}{2}$	log ~~power~~$_{-2}$ (8) = does not exist
log ~~power~~$_7$ ($\frac{1}{7}$) = -1	log ~~power~~$_{\sqrt{6}}$ ($\frac{1}{36}$) = -4	log ~~power~~$_0$ (0) = indeterminant

Why change the word "power" to "logarithm"? To bring us to the next mathematical breakthrough on this topic, which as this:

In the 1700s, mathematicians finally started making sense of exponents (powers) beyond just whole number exponents. They could see that, for any positive number a, it is possible start making sense of

$$a^x$$

for all real numbers x. (They developed the superscript notation for exponents at this time too.) Moreover, they saw that Napier's logarithms are really just powers in disguise! His method of converting an multiplication arithmetic problem into an addition problem as just a matter of working with powers, backwards!

The word *logarithm* is a scary word for many students, chiefly because the notion is typically presented in classrooms with no story or context. The school topic of logarithms would be far less scary if we used the word *power* for them. (After all, you managed to do the opening worksheet seeing only the work "power.")

But for historical reasons, the name *logarithm* stuck and we still use it to this day—even though we can see now what they are.

Here's the formal definition of a logarithm in its full scariness—almost. (The box at the bottom of this page is less scary.)

For a number b (called the **base**) and a number N, the base b **logarithm** on N,

$$\log_b(N)$$

is the power of b that gives the value N.

For example,

$\mathbf{log_4(16)}$ is the power of 4 that gives the answer 16.

$$\log_4(16) = 2$$

$\mathbf{log_{10}(\boldsymbol{million})}$ is the power of 10 that gives the answer one million.

$$\log_{10}(million) = 6$$

$\mathbf{log_{\sqrt{6}}(\frac{1}{36})}$ is the power of $\sqrt{6}$ that gives the answer $\frac{1}{36}$.

$$\log_{\sqrt{6}}(\frac{1}{36}) = -4$$

Whenever you see the word logarithm, I suggest crossing it out and writing the word power in its stead.

power ~~**log**~~$_b$**(N)** = the power of *b* that gives the answer *N*

Question 1 Please complete the following worksheet.

$\log_2(8) =$ _____

$\log_{0.1}(100) =$ _____

$\log_5(125) =$ _____

$\log_{402}(1) =$ _____

$\log_{125}(5) =$ _____

$\log_1(8) =$ _____

$\log_{\frac{1}{4}}(4) =$ _____

$\log_2(-4) =$ _____

a

Again, we are seeing that a base number of $b = 1$ is problematic, and negative values are usually problematic too. To obviate such woes, people usually add some caveats to the formal definition of a logarithm. (Now this is the FULL, and scary, definition.)

For positive numbers b and N, with $b \neq 1$, the base b **logarithm** on N,

$$\log_b(N)$$

is the power of b that gives the value N.

We have

$\log_4(1024) = 5$ because $4^5 = 1024$

$\log_{\frac{1}{3}}\left(\frac{1}{9}\right) = 2$ because $\left(\frac{1}{3}\right)^2 = \frac{1}{9}$

$\log_{10}(0.001) = -3$ because $10^{-3} = \frac{1}{1000} = 0.001$

Question 2 The third example shows that a logarithm of a number can give a negative value as its answer. Does this violate the caveats of the formal definition?